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S.L. CHEN †
Research Engineer, LTAS

J.C. Golinval ‡
Charge de cours, LTAS

Michel GERADIN *
Professor

Dynamique des Constructions Mécaniques

† e-mail : *chen@atlas.ltas.ulg.ac.be*

‡ e-mail : *golinval@atlas.ltas.ulg.ac.be*

* e-mail : *geradin@cupidon.ltas.ulg.ac.be*

tel.: 32(0)41 669165
fax : 32(0)41 521049

Université de Liège

A Dynamic Stiffness-Transfer Matrix Method for Forced Response Analysis of One-Dimensional Structures

S.L. Chen, J.C. Golinval, M. Géradin

Aerospace Laboratory (LTAS), University of Liège
21, Rue E. Solvay, 4000 Liège, Belgium

Abstract An approach is proposed to predict the forced response of one-dimensional structures under arbitrary excitation using the combined dynamic stiffness and transfer matrix method and fast Fourier transform. In this approach, the system is first modeled with arbitrary selected degrees of freedom to which the exciting forces are applied and the responses are to be calculated. The dynamic equilibrium equation in frequency domain is then solved by the fast Fourier transform. The procedure provides flexibility to consider any kind of excitations. Few numerical examples are given and the results are compared with those obtained by the Newmark direct time integration method.

1. Introduction

Various methods for the determination of transient response of one-dimensional vibration structures such as beam or shaft systems have been developed and widely used during the last decades. Among these techniques, the modal superposition method (MSM), the direct time integration method (DTIM) are widely used. Consider the matrix equations of motion for a given system:

$$[M]\ddot{x}(t) + [C]\dot{x}(t) + [K]x(t) = f(t) \tag{1}$$

where [M], [C], [K] are respectively the mass, damping and stiffness matrix obtained by a finite element discretization of the structure, $x(t)$ is the time response and $f(t)$ is the exciting force.

In the finite element method (FEM), the number of degrees of freedom (dof) is usually very large which can lead to matrices [M], [C], [K] of large dimension. The use of the MSM requires to solve first the corresponding free vibration problem in order to get the modal parameters. The Duhamel integration and the decomposition into generalized coordinates and forces are required. The effectiveness of the MSM is remarkable as long as fundamental modes are predominant in the response [1]. However, in the opposite case where the frequency spectrum requires to include a high number of modes so as to ensure good convergence, the procedure calculating the response by this method is often tedious.

The DTIM, which is based on finite time differences, allows to take care of high frequency components in a straightforward manner. It is also the only powerful method for nonlinear systems. However, the parameters of the time integration process are to be adjusted correctly according to the accuracy and the stability required. For large structures, this method can become very time consuming.

Recently, the combined finite element and transfer matrix method (FETM), which was first developed by Dokainish in 1972 [2] and extended by others [3-5], was further modified to transient dynamic analysis of linear and nonlinear rotors [6]. In this method, the finite element matrices is first transformed to discrete time transfer matrix. The response is then obtained by making use of appropriate time marching numerical integration algorithms. Obviously, for each time steps, one needs to form the transfer matrix and to solve a set of equations. Furthermore, because the basic matrices are obtained by the FEM, some errors in the response calculation cannot be avoided.

An improved dynamic stiffness matrix method (IDSM), which is based on the combination of dynamic stiffness and transfer matrices, has been developed recently for beam structures and rotor-bearing systems [7,8,9]. One of the most important characteristics is that the global dynamic stiffness matrix for a system or a subsystem is directly obtained from the corresponding transfer matrix which is very low in dimension. Because the element dynamic stiffness matrix is exact, the IDSM is also exact.

In this paper, the IDSM is further extended to the forced response analysis of the one-dimensional structures, such as beam or rotor-bearing systems. The frequency response function (FRF) matrix between any two points of the structures is first obtained by inverting the corresponding dynamic stiffness matrix. Because the dynamic stiffness matrix obtained by the IDSM is usually very low, exact FRF matrix may be obtained. In order to calculate the forced response, the dynamic equations in frequency domain are transformed to time domain by the inverse fast Fourier transform (IFFT). Few examples are given to show the procedures and the reliability of the method.

2. Theory

2.1. Element dynamic stiffness matrix

Consider a uniform beam shown in Fig.1. The exact dynamic stiffness matrix for the dampened beam is obtained by Leung [10]:

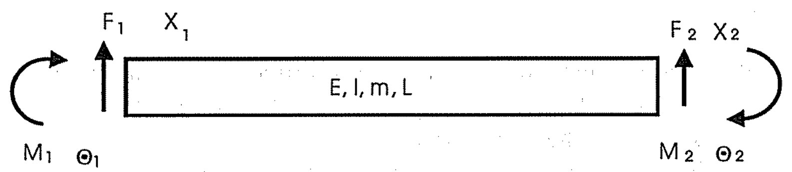

Fig.1: A beam element

$$\begin{pmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{pmatrix} = B \begin{pmatrix} Z_1 & Z_2 & Z_4 & -Z_5 \\ Z_2 & Z_3 & Z_5 & Z_6 \\ Z_4 & Z_5 & Z_1 & -Z_2 \\ -Z_5 & Z_6 & -Z_2 & Z_3 \end{pmatrix} \begin{pmatrix} X_1 \\ \theta_1 \\ X_2 \\ \theta_2 \end{pmatrix} \tag{2}$$

where

$B = EI/(1 - \cos aL \cosh aL)$

$Z_1 = a^3(\cos aL \sinh aL + \sin aL \cosh aL)$; $\quad Z_2 = a^2 \sin aL \sinh aL$;

$Z_3 = a(\sin aL \cosh aL - \cos aL \sinh aL)$; $\quad Z_4 = -a^3(\sin aL + \sinh aL)$;

$Z_5 = a^2(\cos aL - \cosh aL); \quad Z_6 = a(\sinh aL - \sin aL);$

$a = ((\omega^2 - 2i\delta\omega)\rho A/EI(1 + 2i\gamma\omega))^{1/4}$

with E being Young's modulus, L length, I area moment of inertia, ρ mass density and A cross section area. The additional damping constants γ and δ are due to stress rate and inertia, respectively.

2.2. Element transfer matrix

Equation (1) may be rewritten in submatrix form as follows:

$$\begin{pmatrix} F_l \\ F_r \end{pmatrix} = \begin{pmatrix} D_{11}^e & D_{12}^e \\ D_{21}^e & D_{22}^e \end{pmatrix} \begin{pmatrix} X_l \\ X_r \end{pmatrix} \tag{3}$$

where $F_l = [F_1, M_1]^t, \quad X_l = [X_1, \theta_1]^t, \quad F_r = [F_2, M_2]^t, \quad X_r = [X_2, \theta_2]^t.$

Equation (3) may be rewritten in the following transfer matrix form:

$$\begin{pmatrix} X_r \\ -F_r \end{pmatrix} = \begin{pmatrix} T_{11}^e & T_{12}^e \\ T_{21}^e & T_{22}^e \end{pmatrix} \begin{pmatrix} X_l \\ F_l \end{pmatrix} \tag{4}$$

with $T_{11}^e = -{D_{12}^e}^{-1} D_{11}^e, \quad T_{12}^e = {D_{12}^e}^{-1}$

$T_{21}^e = -D_{21}^e + D_{22}^e {D_{12}^e}^{-1} D_{11}^e, \quad T_{22}^e = -D_{22}^e {D_{12}^e}^{-1}$

The transfer matrix of a lumped system with stiffness k and damping c or a lumped system with mass m can be directly written respectively as:

$$[T_k] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ k + i\omega c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

$$[T_m] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -m\omega^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

2.3. Substructure transfer matrix

Shown in Fig.2 is a typical substructure which consists of N_k elastic supports and N_m rigid masses and N_b beam element.

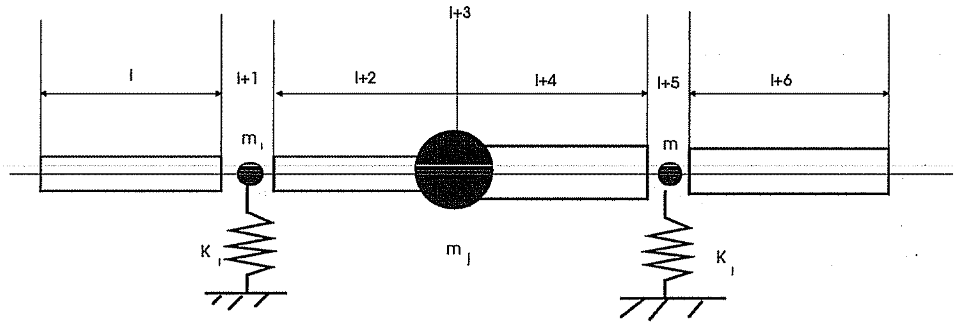

Fig.2: A substructure

The global transfer matrix for the substructure is:

$$[T] = [T_N][T_{N-1}]......[T_2][T_1] \tag{7}$$

where N is the total number of elements and $N = N_b + N_k + N_m$.

2.4. Substructure dynamic stiffness matrix

The global transfer matrix [T] of a substructure relates the forces and displacements at both ends of the substructure in the following way:

$$\begin{pmatrix} X_r \\ -F_r \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} X_l \\ F_l \end{pmatrix} \tag{8}$$

where F_l and X_l are the force and displacement vectors at the left end of the substructure, F_r and X_r are the same quantities at the right end. Equation (8) may be rewritten in the dynamic stiffness matrix form:

$$\begin{pmatrix} F_l \\ F_r \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} X_l \\ X_r \end{pmatrix} \tag{9}$$

with

$$D_{11} = -T_{12}{}^{-1}T_{11}$$

$$D_{12} = T_{12}^{-1}$$

$$D_{21} = -T_{21} + T_{22}T_{12}^{-1}T_{11}$$

$$D_{22} = -T_{22}T_{12}^{-1}$$

where [D] is the global dynamic stiffness matrix of the substructure whose elements are frequency dependent. Note that the global dynamic stiffness matrix of the substructure has the same dimension as the element dynamic stiffness matrix.

2.5 Global dynamic stiffness matrix

The global dynamic stiffness matrix for the entire structure can be assembled using the above dynamic stiffness matrices of all substructures. After introducing the boundary conditions at both ends, the dimension of the global stiffness matrix can further be decreased. The restrained global dynamic stiffness matrix is denoted by $D_g(\omega)$. The unwanted degrees of freedom such as rotational d.o.f. can be removed in an exact manner [7,8].

2.6 FRF matrix

The transfer function matrix $H(\omega)$ for the selected d.o.f. can be obtained by inverting the above global dynamic stiffness matrix. We have

$$[H(\omega)] = [D_g(\omega)]^{-1} \tag{10}$$

Because one can always establish a model using the improved dynamic stiffness matrix method with any two node's coordinates, the matrix $[D_g]$ is usually very low in dimension. For example, for axial, torsional and in-plane lateral (deleting the rotational d.o.f.) vibrations , $[D_g]$ is 2×2 in dimension and for lateral vibrations of rotating beam after deleting the rotational d.o.f., $[D_g]$ is 4×4 in dimension. Therefore, the inversion of matrix $[D_g]$ is very easy to perform.

2.7 Modal parameter evaluations

Once the global dynamic stiffness matrix $[D_g]$ of the system is obtained, the natural frequencies are those values of ω for which

$$[D_g(\omega)]X = 0 \tag{11}$$

where X is the mode shape. There are many ways to solve this nonlinear eigenvalue problems [11,12,13]. Because $[D_g]$ is very low in dimension, equation (11) can be solved by a straightforward procedure of calculating $det[D_g(\omega)]$ by inspection method. The mode shape at arbitrary locations may be recovered using the shape function [14].

The modal mass for n_{th} mode may be obtained by [15]:

$$M_s^n = X_n^T[M(\Omega_n)]X_n \tag{12}$$

2.8 Forced responses

(a.) Harmonic excitation

The response to harmonic excitation at any frequency ω can be obtained by solving equation (13).

$$X(\omega) = [H(\omega)]F(\omega) \tag{13}$$

(b.) Non-harmonic excitation

As stated before, there are two main ways to calculate the steady-state response under any type of excitation: modal superposition method (MSM) and direct time integration method

(DTIM). However, if the model is established by dynamic stiffness matrix method or by the improved dynamic stiffness matrix method as presented in this paper, both the MSM and the DTIM are not convenient due to the fact that the elements of matrix are frequency dependent. In this section, a FFT-based method is proposed.

Theoretically, the FRF matrix obtained in section 2.6 is valid not only for harmonic excitation but also for other type excitations. That is, equation (13) is still hold for any kind of excitation in which $X(\omega)$ and $F(\omega)$ are displacement and force vectors in frequency domain. Performing IFFT on equation (13), we have:

$$x(t) = ifft[H(\omega)F(\omega)] \tag{14}$$

where $x(t)$ is the displacement vector in time domain. In Equation (14), the FRF matrix $[H(\omega)]$ may be calculated in advance and stored in the computer for a given system because it depends only on the eigenproperties of the system. Therefore, for different types of excitations, it is not necessary to calculate the FRF matrix repeatly.

If the exciting force function has explicit form, its Fourier transform used in equation (14) can be easily obtained. In this case, the usuage of equation (14) is convenient. However, if the exciting force $f(t)$ has no explicit form or only available by experiment, equation (15) may be more convenient.

$$x(t) = [h(t)] * f(t) \tag{15}$$

where * denotes the convolution, matrix $[h(t)]$ is the IFFT of FRF matrix $[H(\omega)]$. In equation (15), the matrix $[h(t)]$ may be calculated in advance and stored in the computer for different type of excitations.

3. Numerical Examples

3.1. Force function

The objective here is to show the ability of the proposed method to perform a dynamic response calculation. For this purpose, four typical transient excitation forces are considered as the input.

Type 1: Exponentially varying harmonic excitation

$$f_1(t) = F_0 e^{-\beta t} sin(\omega_0 t) \tag{16}$$

where F_0, β and ω_0 are given real constants.

Type 2: Step excitation

$$f_2(t) = \begin{cases} F_0, & \text{if } t \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

where F_0 is a real constant.

Type 3: Impulse excitation

$$f_3(t) = \begin{cases} F_0, & \text{if } 0 \leq t \leq t_0; \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

where t_0 is a very small value, usually $t_0 \leq 0.1$.

Type 4: Sweep sine excitation

$$f_4(t) = F_0 \sin(bt^2) \tag{19}$$

where F_0 and b are real constant.

3.2 Example 1

Consider the six d.o.f. lumped mass-stiffness-damper system shown in Fig.3. The system is excited at the sixth mass. We want to know the response at mass six.

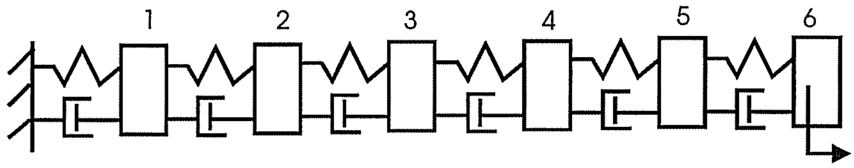

Fig.3: A six d.o.f. lumped system

Because all the internal d.o.f. are not needed, the system can be treated as one substructure. The global transfer matrix is:

$$T = T_m^6 T_{kc}^6 T_m^5 T_{kc}^5 ......T_m^1 T_{kc}^1 \tag{20}$$

where T_m^j and T_{kc}^j are defined as:

$$T_m^j = \begin{pmatrix} 1 & 0 \\ -m_j\omega^2 & 0 \end{pmatrix} \tag{21}$$

$$T_{kc}^j = \begin{pmatrix} 1 & 1/(k_j + i\omega c_j) \\ 0 & 1 \end{pmatrix} \tag{22}$$

where $i = \sqrt{-1}$

The global dynamic stiffness matrix can be obtained by rearranging the global transfer matrix T using equation (9). Introducing the boundary condition at the left end, the global dynamic stiffness matrix is reduced to a single complex number, $d(\omega)$, for a given frequency. The FRF matrix is $H(\omega) = 1/d(\omega)$.

Table 1: Eigenfrequencies (Hz) obtained by the different methods

Mode	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**
Analytical	0.7744	2.2423	3.4229	6.5947	7.0053	7.3986
Present	0.7744	2.2423	3.4229	6.5947	7.0053	7.3986

Table 1 lists the eigenfrequencies obtained by the present method and by the direct analytical methods. The results are exactly the same because both methods are exact method. The harmonic response is shown in Fig.4.

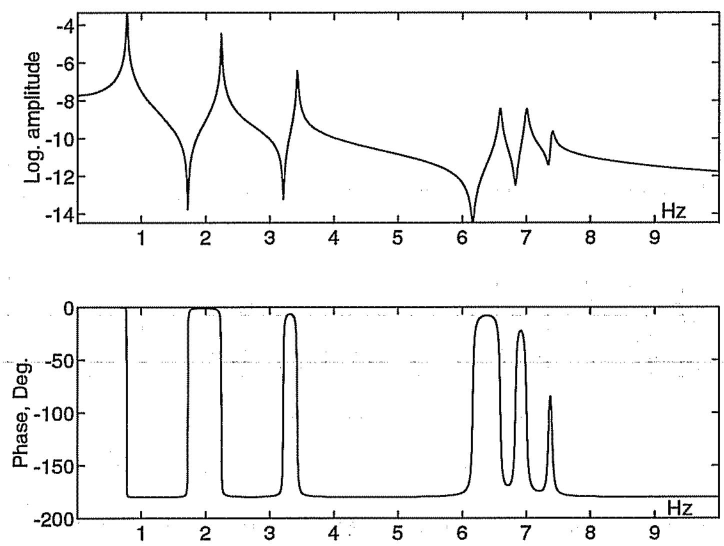

Fig.4: Harmonic response in frequency domain

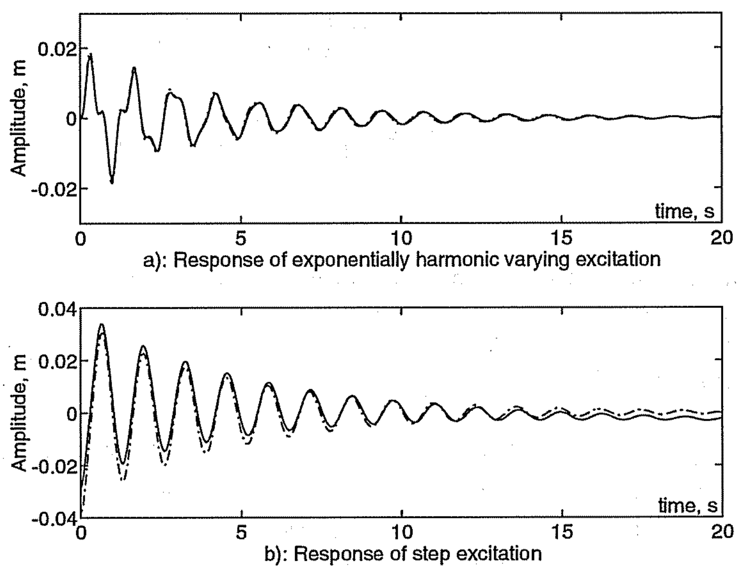

Fig.5: Forced responses in time domain
——: present method; -.-.-.: Newmark method

Fig.5 and Fig.6 depict the dynamic responses under the four types of excitation described above. The initial conditions are zero. For comparson, the responses obtained by the direct time integration method (Newmark algorithm) are also shown. In Fig.5b, the static response or constant component in the response has been removed. It is seen that the responses obtained by the two methods are very close.

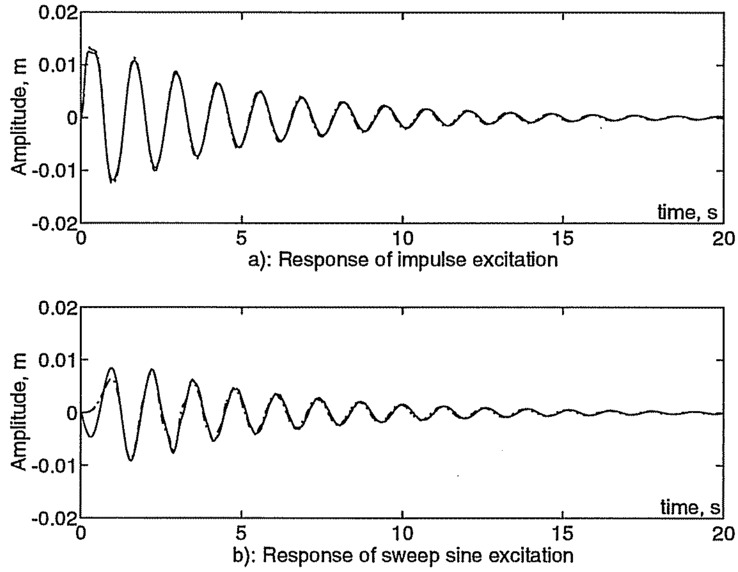

Fig.6: Forced responses in time domain
——: present method; -.-.-.: Newmark method

3.3 Example 2

A non-uniform cantilever beam shown in Fig.6a is studied as the second example. The beam is excited at the internal node 3. We want to know the responses at node 3 and node 6.

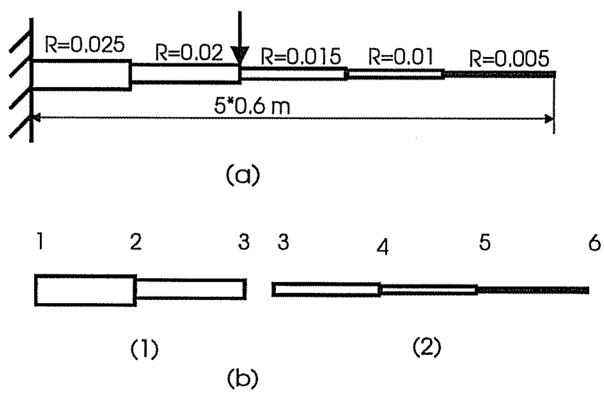

Fig.7: A non-uniform cantilever beam

To this end, the beam is first divided into two substructures as shown in Fig.6b. The transfer matrix for each substructure $[T_1]$ and $[T_2]$ can easily be obtained by multiplying the element transfer matrices. Their corresponding dynamic stiffness matrices $[D_1]$ and $[D_2]$ are obtained by rearranging the matrices $[T_1]$ and $[T_2]$ using equation (9), respectively.

The global dynamic stiffness matrix [D] can be obtained by assemblying the matrices $[D_1]$ and $[D_2]$ as done in the finite element method. After introducing the boundary conditions at the left end, one finally obtains the dynamic equilibrium equation in the frequency domain:

$$\begin{pmatrix} F_3(\omega) \\ M_3(\omega) \\ F_6(\omega) \\ M_6(\omega) \end{pmatrix} = [D(\omega)]_{4\times4} \begin{pmatrix} X_3(\omega) \\ \theta_3(\omega) \\ X_6(\omega) \\ \theta_6(\omega) \end{pmatrix} \tag{23}$$

The rotational d.o.f., θ_3, θ_6, can be removed from the above equation by setting the corresponding moments M_3 and M_6 to zero. We have finally:

$$\begin{pmatrix} F_3(\omega) \\ F_6(\omega) \end{pmatrix} = \begin{pmatrix} D_{11}(\omega) & D_{12}(\omega) \\ D_{21}(\omega) & D_{22}(\omega) \end{pmatrix} \begin{pmatrix} X_3(\omega) \\ X_6(\omega) \end{pmatrix} \tag{24}$$

By inverting the matrix $[D(\omega)]$, we obtain the FRF matrix $[H(\omega)] = [D(\omega)]^{-1}$. Equation (24) can be rewritten as:

$$\begin{pmatrix} X_3(\omega) \\ X_6(\omega) \end{pmatrix} = \begin{pmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{pmatrix} \begin{pmatrix} F_3(\omega) \\ F_6(\omega) \end{pmatrix} \tag{25}$$

Note that there is no difficulty to obtain $[D]^{-1}$ because [D] is 2×2 in dimension.

The eigenfrequencies, mode shapes, modal masses can be easily obtained from equation (24) according to section 2.7. The dynamic responses can be obtained by performing IFFT as described in section 2.8.

Table 2: Eigenfrequencies (Hz) obtained by the different methods

Mode	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**
FEM	8.2937	17.8431	35.3019	68.3871	118.2586	152.4120
Present	8.2935	17.8430	35.3013	68.3860	118.1380	152.2290

Table 2 lists the first six eigenfrequencies obtained by the present method and by FEM (Samcef, beam element type 1). The results obtained by the present method are theoretically exact values. The harmonic response is shown in Fig.8.

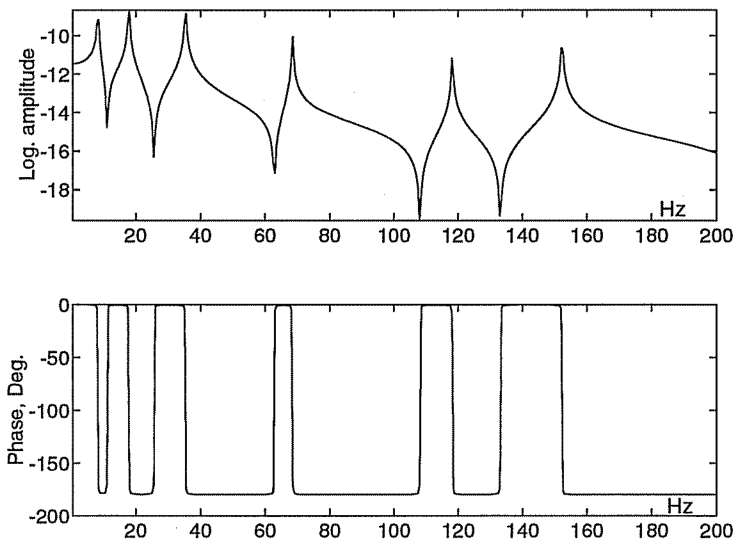

Fig.8: Harmonic response in frequency domain

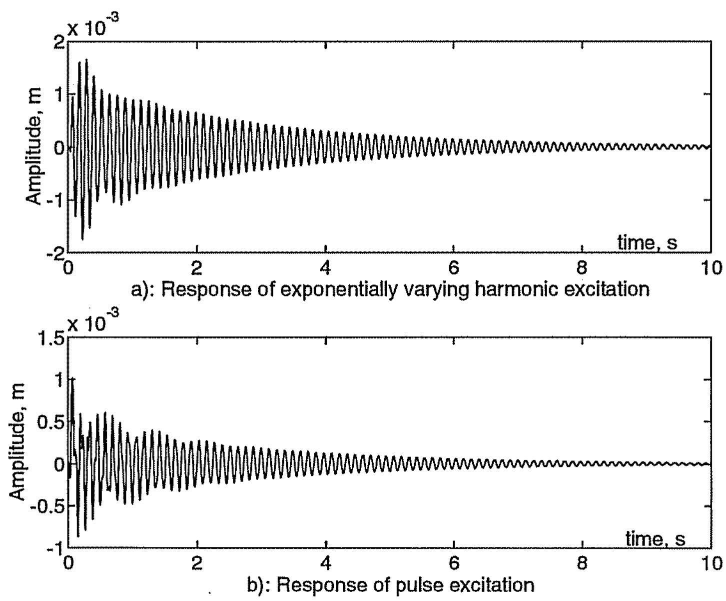

Fig.9: Forced responses in time domain

The forced dynamic responses at node 6 under different types of excitations are shown in Fig.9.

4. Conclusions

An approach to calculate forced dynamic response has been developed for one-dimensional vibrating structures based on the improved dynamic stiffness matrix method and fast Fourier transform.

Three outstanding advantages associated with this method are: (1) The system can be modeled exactly with arbitrary degrees of freedom, most of the internal degrees of freedom can be automatically avoided. This fact makes it possible to obtain directly the FRF matrix between any two points (one is exciting point and another is response point). (2) Any kind of exciting forces can be considered without difficult thanks to the introduction of the FFT. (3) The method handles only small matrices, hence, the computation effort is reduced considerably without losing any accuracy.

The simplicity and high efficiency of the method has been demonstrated by the numerical examples shown in the paper.

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