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Abstract

We explore the optimal fertility age-pattern in a four-period OLG economy with physical capital accumulation. For that purpose, we firstly compare the dynamics of two closed economies, Early and Late Islands, which differ only in the timing of births. On Early Island, children are born from parents in young adulthood, whereas, on Late Island, children are born from parents in older adulthood. We show that, unlike on Early Island, there exists no stable stationary equilibrium on Late Island, which exhibits cyclical dynamics. We also characterize the social optimum in each economy, and show that Samuelson's Serendipity Theorem still holds. Finally, we study the dynamics and social optimum of an economy with interior fertility rates during the reproduction period. It is shown that various fertility age-patterns are compatible with the social optimum, as long as these yield the optimal cohort growth rate. The Serendipity Theorem remains valid in that broader demographic environment.

Keywords: childbearing ages, early and late motherhoods, fertility, overlapping generations, social optimum.

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1 Introduction

The optimal fertility rate was firstly studied by Samuelson (1975) in a two-period overlapping generations (OLG) economy *à la* Diamond (1965). Samuelson showed that the optimal fertility rate is equal to the marginal rate of substitution between consumptions at successive ages of life, and equal to the marginal productivity of capital at the Golden Rule capital level. Samuelson emphasized also the capacity of the fertility rate to allow for the decentralization of the social optimum. This is the Serendipity Theorem: if there exists a unique stable stationary equilibrium, a perfectly competitive economy will converge towards the social optimum when the optimal fertility rate is imposed.

Following Samuelson's pioneer works, a particular attention has been paid to the characterization of the optimal fertility rate. Deardorff (1976) showed that an interior optimal fertility rate does not always exist in a two-period OLG economy with Cobb-Douglas production and utility functions. In a more general model with CES production function and preferences, Michel and Pestieau (1993) emphasized that an interior optimal fertility rate requires a sufficiently low substitutability between capital and labour in the production process, and between first- and second-period consumptions in utility functions.[1] Recently, Jaeger and Kuhle (2009) and de la Croix *et al* (2011) examined the robustness of the Serendipity Theorem to the introduction of debt and of risky lifetime.

Whereas the optimal (total) fertility rate has been widely studied, the optimal *timing* of births has received less attention. There exist some recent studies on childbearing ages, but these are descriptive rather than normative. In a pioneer study, Gustafsson (2001) examined the reasons why women have, over the last decades, postponed the time of the first birth, and reviewed some empirical and theoretical literature aimed at explaining that trend. More recently, Momota (2009) examined, in a three-period OLG model with fixed total fertility, the impact of changes in the timing of births on the dynamics of the economy. Furthermore, D'Albis *et al* (2010) studied, in a continuous time OLG model, the joint dynamics of demography and economy under endogenous childbearing ages, and proved that there exists a monetary steady-state if the average age of consumers is larger than the average age of producers.

The goal of the present paper is to complement that literature, by characterizing, in an OLG economy, the optimal timing of births. The motivation for shifting from the study of the optimal total fertility rate to the optimal fertility age-pattern is empirical. Actually, the timing of births has significantly changed during the last decades, with a rise of the average age of mothers at their first child.[2] For instance, in the United States, the average age of first-time mothers increased by 3.6 years between 1970 and 2006, from 21.4 to 25.0 years. That tendency is even stronger in Japan, where the average age of first-time mothers has grown from 25.6 to 29.2 years over the same period.

Those changes in the timing of births raise several questions, concerning

[1] Moreover, Abio (2003) and Abio *et al* (2004) complemented those papers by studying the optimal fertility rate under costly, endogenous fertility.

[2] Sources: NCHS (2009).

their causes, as well as their effects on the economy.[3] One may also, following Samuelson's (1975) normative works, wonder whether the tendency towards later motherhood is *optimal* from a social perspective. Is later motherhood socially optimal, or does this conflict with social welfare maximization in an intergenerational context? One can expect that, for a *given* total fertility rate (i.e. a given total number of children per women on her lifecycle), the timing of births may not be neutral at all for the dynamics of population and capital accumulation. If that conjecture is correct, the characterization of the social optimum requires not only a study of the optimal total fertility, but also of the optimal timing of births.

In order to explore the optimal timing of births, we will focus on a four-period OLG model with physical capital accumulation, and where the reproduction period covers the second and third periods of life. Moreover, for the sake of presentation, we will firstly compare two closed economies, Early and Late Islands, which differ only in the timing of births. On Early Island, children are born from parents in young adulthood (second period), whereas, on Late Island, children are born from parents in older adulthood (third period). We will compare the dynamics of those economies and their social optimum. Then, in a second stage, we will study a more general economy, with interior age-specific fertility rates during the reproduction period.

Anticipating our results, we first show, by comparing Early and Late Islands, that the timing of births matters strongly for the long-run dynamics. Whereas there exists, under mild conditions, a stable stationary equilibrium on Early Island, the same is not true on Late Island, which exhibits cyclical dynamics. We also characterize the (stationary) social optimum of Early Island, and the (non-stationary) social optimum on Late Island, and show that the Serendipity Theorem remains valid in those economies. Then, we study the general economy with interior age-specific fertility rates, and show that it admits, under mild conditions, a stable stationary equilibrium, so that the introduction of a - possibly low - strictly positive fertility at young adulthood prevents cyclical dynamics. We also show that the social optimum can be described in terms of the optimal cohort growth rate, which differs from the standard total fertility rate. Finally, it is shown that a perfectly competitive economy converges towards the social optimum provided the government imposes the optimal cohort growth rate, which can be obtained under various fertility age-patterns.

The rest of the paper is organized as follows. Section 2 presents the long-run dynamics of Early and Late Islands. We also characterize their social optimum, and examine whether the Serendipity Theorem still holds there. Then, Section 3 considers a more general economy where all age-specific fertility rates are positive during the reproduction period, and examines the social optimum and the Serendipity Theorem in that context. Conclusions are drawn in Section 4.

[3] Various factors were proposed to explain that trend, such as the better earning opportunities for women, and their better educational achievements (Ermisch and Ogawa 1994, Joshi 2002). Regarding its consequences, Ermisch and Pevalin (2005) showed that very early motherhood (teen births) worsens later outcomes on the marriage market.

2 A fable with two islands

This section compares two closed four-period OLG economies with distinct timings for births.

- In the first economy, *Early Island*, all children are born from young parents (i.e. living their second period). All children will, during their childhood, coexist with their (young) parents and grand-parents.
- In the second economy, *Late Island*, all children are born from older parents (i.e. living their third period). All children will, during their childhood, coexist with their old parents (their grand-parents being dead).

2.1 Demography

Throughout this paper, we assume initial conditions insuring that the two economies exhibit a non-zero number of births at any period: $N_{-1} > 0$, $N_0 > 0$, where N_t denotes the number of individuals born at period t.

On Early Island, all children are born from young parents, so that the number of births is (for $t > 0$):

$$N_t = n_1 N_{t-1} \tag{1}$$

where n_1 is the number of children born from young parents, or, alternatively, the fertility rate. The growth rate of cohort size, denoted by $g_t \equiv \frac{N_t}{N_{t-1}}$, is:

$$g_t \equiv \frac{N_t}{N_{t-1}} = n_1 \tag{2}$$

As $\frac{N_t}{N_{t-1}} = n_1$ for all t, the cohort growth rate g_t is constant over time.

All adult agents take part to the labour market during their second and third periods of life, so that the total labour force, denoted by L_t, is:

$$L_t = L_t^y + L_t^o$$

where L_t^y is the number of young workers, and L_t^o is the number of older workers. The total labour force on Early Island can be rewritten as:

$$L_t = N_{t-1} + N_{t-2} = n_1 N_{t-2} + n_1 N_{t-3}$$

Let us now consider Late Island. On Late Island, all children are born from old parents, so that the number of births at time t is (for $t > 0$):

$$N_t = n_2 N_{t-2} \tag{3}$$

where n_2 denotes the number of children born from old parents.[4] Unlike on Early Island, the cohort growth rate g_t is not, on Late Island, constant over

[4] Hence, given that $\frac{N_t}{N_{t-2}} = n_2 = \frac{N_t}{N_{t-1}} \times \frac{N_{t-1}}{N_{t-2}}$, the (geometric) average of the growth rate of cohort size g_t on Late Island, denoted by $\bar{g}_t$, is given by: $\bar{g}_t = \frac{N_t}{N_{t-1}} = \sqrt[2]{n_2}$.

time. To see this, note that:

$$\begin{aligned} g_1 &= \frac{N_1}{N_0} = \frac{n_2 N_{-1}}{N_0} = \frac{n_2}{g_0} \\ g_2 &= \frac{N_2}{N_1} = \frac{n_2 N_0}{N_1} = \frac{n_2}{g_1} \\ g_3 &= \frac{N_3}{N_2} = \frac{n_2 N_1}{N_2} = \frac{n_2}{g_2} \\ &\dots \\ g_t &= \frac{N_t}{N_{t-1}} = \frac{n_2 N_{t-2}}{N_{t-1}} = \frac{n_2}{g_{t-1}} \end{aligned}$$

Thus, for any two periods t and $t+1$, we have:

$$g_{t+1} = \frac{n_2}{g_t} \tag{4}$$

In the particular case where $N_{-1} = N_0 > 0$, the cohort growth rate follows a simple pattern, i.e. a two-period cycle:[5]

$$1, n_2, 1, n_2, 1, n_2...$$

Given that all adult agents take part to the labour market in their second and third periods of life, the labour force is:

$$L_t = N_{t-1} + N_{t-2} = n_2 N_{t-3} + n_2 N_{t-4}$$

2.2 Production

For simplicity, the economies of Early and Late Islands are supposed to be exactly identical on all dimensions except the timing of births. Thus production takes place exactly in the same way in the two islands.

In each economy, the production of an output Y_t involves capital K_t and labour L_t, according to the function:[6]

$$Y_t = F(K_t, L_t) = \bar{F}(K_t, L_t) + (1-\delta)K_t \tag{5}$$

where δ is the depreciation rate of capital. The production function $\bar{F}(K_t, L_t)$ is assumed to be homogeneous of degree one. Hence, the total production function $F(K_t, L_t)$ is also homogeneous of degree one, and, if one substitutes for the labour force L_t, the production process as a whole can be rewritten as:

$$Y_t = F\left(K_t, N_{t-1}\left(1 + \frac{N_{t-2}}{N_{t-1}}\right)\right)$$

The production process can also be rewritten in intensive terms as:

$$y_t = F\left(k_t, 1 + \frac{N_{t-2}}{N_{t-1}}\right) \tag{6}$$

[5] Indeed, when $N_{-1} = N_0$, $g_1 = 1$, $g_2 = n_2$, $g_3 = \frac{n_2}{n_2} = 1$, etc.

[6] It is assumed that the undepreciated units of capital are sold on the goods market.

where $y_t = \frac{Y_t}{L_t^y} = \frac{Y_t}{N_{t-1}}$ denotes output per young worker, and $k_t = \frac{K_t}{L_t^y} = \frac{K_t}{N_{t-1}}$ denotes the capital per young worker.

The resource constraint of the economy, which states that what is produced is either consumed or invested, is:

$$F(K_t, L_t) = c_t N_{t-1} + d_t N_{t-2} + b_t N_{t-3} + K_{t+1}$$

where c_t, d_t and b_t are first-, second- and third-period consumptions.

Dividing that constraint by the young labour force $L_t^y = N_{t-1}$, one gets:

$$F\left(k_t, 1 + \frac{N_{t-2}}{N_{t-1}}\right) = c_t + d_t \frac{N_{t-2}}{N_{t-1}} + b_t \frac{N_{t-3}}{N_{t-1}} + k_{t+1} \frac{N_t}{N_{t-1}} \tag{7}$$

That resource constraint will take different forms in the two economies under study, since the cohort size ratios are different, as we shall see.

2.3 The dynamics on Early Island

An adult agent living on Early Island at time t chooses first-, second- and third-period consumptions, denoted respectively by c_t, d_{t+1} and b_{t+2}, in such a way as to maximize his lifetime welfare subject to his budget constraint. We assume a standard time-additive lifetime welfare:

$$U(c_t, d_{t+1}, b_{t+2}) = u(c_t) + \beta u(d_{t+1}) + \beta^2 u(b_{t+2}) \tag{8}$$

where $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$, while β is a time preference factor ($0 < \beta < 1$).

The agent takes all prices, as well as the fertility rate, as given. The first-, second- and third-period consumptions must satisfy the following constraints:

$$\begin{aligned} c_t &= w_t - s_t \\ d_{t+1} &= w_{t+1} + R_{t+1} s_t - z_{t+1} \\ b_{t+2} &= R_{t+2} z_{t+1} \end{aligned}$$

where w_t and R_t denote, respectively, the wage and the return on savings at period t, whereas s_t and z_{t+1} denote, respectively, the second- and third-period savings. We consider a perfectly competitive economy, where factors are paid at their marginal productivities:

$$w_t = \left[F\left(k_t\left(, 1 + \frac{1}{n_1}\right)\right) - F_k\left(k_t, 1 + \frac{1}{n_1}\right)\right] \frac{n_1}{1 + n_1} \tag{9}$$

$$R_t = F_k\left(k_t, 1 + \frac{1}{n_1}\right) \tag{10}$$

The problem of the agent can be rewritten as:

$$\max_{c_t, d_{t+1}, b_{t+2}} u(c_t) + \beta u(d_{t+1}) + \beta^2 u(b_{t+2})$$

$$s.t. w_t + \frac{w_{t+1}}{R_{t+1}} = c_t + \frac{d_{t+1}}{R_{t+1}} + \frac{b_{t+2}}{R_{t+1} R_{t+2}}$$

Resolving that optimization problem allows us to derive some solutions for individual savings s_t and z_{t+1}. Under perfect foresight, those optimal savings can be rewritten as functions of current and future factor prices:[7]

$$\begin{aligned} s_t &\equiv s\left(R_{t+1}, R_{t+2}, w_t, w_{t+1}\right) \\ z_{t+1} &\equiv z\left(R_{t+1}, R_{t+2}, w_t, w_{t+1}\right) \end{aligned}$$

Backwarding the second equation by one period gives us z_t, i.e. the old worker's savings chosen at $t-1$. Then, substituting for s_t and z_t in the capital accumulation equation $K_{t+1} = N_{t-1}s_t + N_{t-2}z_t$ and dividing by the number of young workers $L^y_{t+1} = N_t$, yields:

$$k_{t+1} = \frac{s\left(R_{t+1}, R_{t+2}, w_t, w_{t+1}\right)}{g_t} + \frac{z\left(R_t, R_{t+1}, w_{t-1}, w_t\right)}{g_{t-1}g_t}$$

Given that $R_t = R(k_t)$ and $w_t = w(k_t)$, the dynamics of k_t is described by a difference equation of order 3. The highest-order term k_{t+2} comes from the interest factor at old adulthood for the young adult at t, i.e. R_{t+2}, whereas the lowest-order term k_{t-1} comes from the wage faced by old adults at t when being young workers at $t-1$, i.e. w_{t-1}.

As this was stressed by de la Croix and Michel (2001), the dynamics of capital under perfect foresight is quite complex when savings are made at several periods. There exist only a few ways to overcome that complexity. A first approach consists of imposing particular functional forms for the individual utility function and the production function. However, the robustness of the results to the specific functional forms chosen is not guaranteed, and this may strongly limitate the scope of the results. A second approach consists of keeping general functional forms, but of relaxing the perfect foresight assumption. If, for instance, one considers myopic anticipations, the number of lags in the laws describing the dynamics of capital can be reduced.[8] Such an assumption has a cost, since it presupposes that agents tend to make mistakes when the economy is not in a stationary equilibrium. Nevertheless, under that alternative assumption, the dynamics of the economy can be studied without having to rely on specific analytical examples. Therefore we will, when describing the dynamics of our economies, assume myopic anticipations about factor prices.[9]

Under myopic anticipations on factor prices, the optimal savings s_t and z_{t+1} depend only on the current level of capital per worker, i.e. k_t:

$$\begin{aligned} s_t &= s\left(R(k_t), R(k_t), w(k_t), w(k_t)\right) \equiv \sigma\left(k_t\right) \\ z_{t+1} &= z\left(R(k_t), R(k_t), w(k_t), w(k_t)\right) \equiv \zeta(k_t) \end{aligned}$$

[7] See de la Croix and Michel (2001, pp. 64-66).

[8] In our context, myopic anticipations mean that agents, when choosing their savings, take the *current* wages and interest rates as a proxy for future wages and interest rates.

[9] Such a reliance will have no effect on our normative analysis, which will rely on a stationary environment, at which agents with myopic anticipations make no mistake (see below).

Backwarding the second equation by one period and substituting for s_t and z_t in the capital accumulation equation yields:

$$k_{t+1} = \frac{\sigma(k_t)}{g_t} + \frac{\zeta(k_{t-1})}{g_{t-1}g_t} = \frac{\sigma(k_t)}{n_1} + \frac{\zeta(k_{t-1})}{(n_1)^2}$$

since the cohort growth rate g_t is a constant and equal to n_1.

For the sake of analytical tractability, let us now introduce the variable Ω_t:

$$\Omega_t \equiv \frac{\zeta(k_{t-1})}{(n_1)^2}$$

which is defined as the old workers's savings per young worker at time t. Forwarding that expression by one period allows us to represent the dynamics of the economy by the following two-dimensional system:

$$\begin{aligned} k_{t+1} &\equiv G(k_t, \Omega_t) = \frac{\sigma(k_t)}{n_1} + \Omega_t \\ \Omega_{t+1} &\equiv H(k_t) = \frac{\zeta(k_t)}{(n_1)^2} \end{aligned} \qquad (11)$$

Whether there exists a stationary equilibrium or not on Early Island depends on whether there exists a pair (k_t, Ω_t) such that $k_t = G(k_t, \Omega_t)$ and $\Omega_t = H(k_t)$.

Proposition 1 *Assume myopic anticipations about factor prices. Assume that* $\sigma(0) = 0$, $\sigma'(k_t) > 0$, $\zeta(0) = 0$ *and* $\zeta'(k_t) > 0$. *If* $\lim_{k_t \to 0} \left[1 - \frac{\sigma'(k_t)}{n_1}\right] < \lim_{k_t \to 0} \left[\frac{\zeta'(k_t)}{(n_1)^2}\right]$ *and* $\lim_{k_t \to \infty} \left[1 - \frac{\sigma'(k_t)}{n_1}\right] > \lim_{k_t \to \infty} \left[\frac{\zeta'(k_t)}{(n_1)^2}\right]$, *there exists, on Early Island, a locally stable stationary equilibrium* (k^*, Ω^*) *with* $\Omega^* = \frac{\zeta(k^*)}{(n_1)^2}$.

Proof. See the Appendix. ■

The conditions guaranteeing the existence and the stability of a stationary equilibrium have the following interpretation. These state that, at low capital levels (i.e. low wages and high interest rates), the third-period savings are more reactive to a rise in k_t than the second-period savings, whereas the opposite holds for high capital levels (i.e. high wages and low interest rates).

2.4 The dynamics on Late Island

Dividing the capital accumulation equation $K_{t+1} = N_{t-1}s_t + N_{t-2}z_t$ by the number of young workers, $L^y_{t+1} = N_t$, and assuming myopic anticipations, we can rewrite the capital accumulation as:

$$k_{t+1} = \frac{\sigma(k_t)}{g_t} + \frac{\zeta(k_{t-1})}{g_{t-1}g_t}$$

Contrary to what occurs on Early Island, the cohort growth rate is not a constant here, but follows the dynamic law: $g_{t+1} = \frac{n_2}{g_t}$, which implies that

$g_{t+1} \times g_t = n_2$ for any t and $t+1$. Hence, the capital accumulation equation is:

$$k_{t+1} = \frac{\sigma(k_t)}{g_t} + \frac{\zeta(k_{t-1})}{n_2}$$

As a consequence of the variation of the cohort growth rate over time, the analysis of the dynamics on Late Island requires, under myopic anticipations, to study the following 3-dimensional dynamic system:

$$\begin{aligned} k_{t+1} &\equiv G(k_t, \Omega_t, g_t) = \frac{\sigma(k_t)}{g_t} + \Omega_t \\ \Omega_{t+1} &\equiv H(k_t) = \frac{\zeta(k_t)}{n_2} \\ g_{t+1} &\equiv I(g_t) = \frac{n_2}{g_t} \end{aligned} \tag{12}$$

When the initial conditions are $N_{-1} = N_0 > 0$ (i.e. $g_0 = 1$) and when the fertility rate is equal to the replacement rate ($n_2 = 1$), Late Island exhibits, under mild conditions, a locally stable stationary equilibrium. However, in other cases, there exists no stable stationary equilibrium on Late Island.

Proposition 2 *Assume myopic anticipations about factor prices.*

- *When $N_{-1} = N_0 > 0$ and $n_2 = 1$, and provided $\sigma(0) = 0$, $\sigma'(k_t) > 0$, as well as $\zeta(0) = 0$, $\zeta'(k_t) > 0$, we have that, if $lim_{\ k_t \to 0}\ [1 - \sigma'(k_t)] < lim_{\ k_t \to 0}\ \zeta'(k_t)$ and $lim_{\ k_t \to \infty} 1 - \sigma'(k_t) > lim_{\ k_t \to \infty} \zeta'(k_t)$, there exists a locally stable stationary equilibrium $(k^*, \zeta(k^*), 1)$ on Late Island.*

- *When $N_{-1} \neq N_0$ or $n_2 \neq 1$, there is no stable stationary equilibrium on Late Island. Provided $\sigma(0) = 0$, $\sigma'(k_t) > 0$, as well as $\zeta(0) = 0$, $\zeta'(k_t) > 0$, we have that, if $lim_{k \to 0}\ 1 - \frac{\sigma'(k_t)}{\sqrt[2]{n_2}} < lim_{k \to 0}\ \frac{\zeta'(k_t)}{n_2}$ and $lim_{k \to +\infty}\ 1 - \frac{\sigma'(k_t)}{\sqrt[2]{n_2}} > lim_{k \to +\infty}\ \frac{\zeta'(k_t)}{n_2}$, there exists an unstable stationary equilibrium $\left(k^*, \zeta(k^*), \sqrt[2]{n_2}\right)$ on Late Island.*

Proof. See the Appendix. ■

When $g_0 = 1$ and $n_2 = 1$, the cohort growth rate g_t is constant and equal to unity, so that the size of cohorts is the same over time. Hence, the Late Island economy is not subject to fluctuations in the size of the labour force, and there exists a level of capital per young worker that can be reproduced over time.

If, on the contrary, $g_0 \neq 1$, the cohort growth rate fluctuates over time, with $g_{t+1} = \frac{n_2}{g_t}$. When $n_2 = 1$, the cohort size also fluctuates between two levels, since that fertility rate only insures the replacement of two successive cohorts of unequal sizes. The resulting fluctuations in the labour supply prevent the convergence towards a stationary equilibrium. When $n_2 \neq 1$, the fluctuations in g_t prevent also the convergence towards a steady-state: under mild conditions, Late Island is then characterized by long-run cycles of length 2.

Proposition 3 *Assume myopic anticipations about factor prices. Assume that* $N_{-1} > 0$, $N_0 > 0$ *and* $n_2 \neq 1$. *Assume that* $\sigma(0) = 0$, $\sigma'(k_t) > 0$, $\zeta(0) = 0$, *and* $\zeta'(k_t) > 0$ *for* $k_t > 0$. *Let us denote by* $\bar{\Omega}$ *the solution to* $\Omega_t = \frac{\zeta(\sigma(k_t)+\Omega_t)}{n_2}$ *when* $k_t \to +\infty$, *and by* $\tilde{\Omega}$ *the solution to* $\Omega_t = \frac{\zeta(\frac{\sigma(k_t)}{n_2}+\Omega_t)}{n_2}$ *when* $k_t \to +\infty$.

- *If* $lim_{k_t \to 0}\zeta'(k_t) > n_2$ *and* $lim_{k_t \to +\infty}\sigma^{-1}(n_2 k_t - \zeta(k_t)) - \sigma(k_t) > \bar{\Omega}$, *and if,* $lim_{k_t \to +\infty}\zeta'(k_t) < n_2$ *and* $lim_{k_t \to +\infty}\sigma^{-1}\left(k_t - \frac{\zeta(k_t)}{n_2}\right) - \frac{\sigma(k_t)}{n_2} > \tilde{\Omega}$, *then the long-run dynamics of Late Island consists of a two-period cycle* $\left((\hat{k}, \hat{\Omega}, 1), (\check{k}, \check{\Omega}, n_2)\right)$, *with* $\hat{\Omega} = \frac{\zeta(\check{k})}{n_2}$ *and* $\check{\Omega} = \frac{\zeta(\hat{k})}{n_2}$. *The equilibrium cycle* $\left((\hat{k}, \hat{\Omega}, 1), (\check{k}, \check{\Omega}, n_2)\right)$ *is unstable.*
- *Under* $N_{-1} = N_0 > 0$, *the Late Island economy can converge from initial conditions* $(k_0, \Omega_0, 1)$ *towards the long-run cycle* $\left((\hat{k}, \hat{\Omega}, 1), (\check{k}, \check{\Omega}, n_2)\right)$.

Proof. See the Appendix. ■

Proposition 3 describes conditions that are sufficient to guarantee, when $n_2 \neq 1$, the existence of a two-period cycle in Late Island. Note, however, that, if $N_{-1} \neq N_0 > 0$ (i.e. $g_0 \neq 1$), the equilibrium cycle is unstable. This is the reason why we will, in the rest of this section, focus on the case where initial conditions $N_{-1} = N_0 > 0$ allow the (conditional) convergence towards the equilibrium cycle.

To conclude, the major role played by the age-pattern of fertility along the lifecycle could hardly be overemphasized. Whether births are located at young adulthood (as on Early Island) or at old adulthood (as on Late Island) makes a large difference. In the former case, the long-run dynamics is, in general, *stationary*, whereas, in the latter case, the dynamics is *cyclical*. Hence, even under a given *total* fertility rate, i.e. $n_1 + n_2 = \bar{n}$, the two dynamics will differ strongly. Thus the timing of births matters a lot when studying economic dynamics, and the total fertility rate hides a big part of the picture.

2.5 The social optimum

Let us first characterize the social optimum on Early Island. For that purpose, we assume that there exists, on Early Island, a unique locally stable stationary equilibrium with a strictly positive capital level k.[10] The social planner's problem consists of choosing consumptions c, d and b, the capital k and the fertility rate n_1 in such a way as to maximize the lifetime welfare of a cohort alive at the steady-state, subject to the resource constraint of the economy:

$$\max_{c,d,b,k,n_1} \; u(c) + \beta u(d) + \beta^2 u(b)$$
$$s.t.\; F\left(k, 1 + \frac{1}{n_1}\right) = c + \frac{d}{n_1} + \frac{b}{(n_1)^2} + n_1 k$$

[10] Hence, we get rid of time indexes in the rest of this subsection.

An interior optimum $(c^*, d^*, b^*, k^*, n_1^*)$ satisfies the following FOCs:[11]

$$\begin{aligned}
\frac{u'(c^*)}{\beta u'(d^*)} &= \frac{u'(d^*)}{\beta u'(b^*)} = n_1^* \\
F_k(k^*, \cdot) &= n_1^* \\
k^* + F_L(k^*, \cdot)\frac{1}{(n_1^*)^2} &= \frac{1}{(n_1^*)^2}\left(d^* + \frac{2b^*}{n_1^*}\right)
\end{aligned}$$

The marginal rate of substitution between consumptions at two successive ages should be equal to the optimum fertility rate n_1^*. The second expression is the Golden Rule for optimal capital accumulation. The third FOC characterizes the optimal fertility rate n_1^*. A larger fertility has two negative effects (on the LHS), and one positive effect (on the RHS), which must be equal at the social optimum. The first negative effect is the capital widening effect: a larger fertility puts more pressure on capital accumulation. The second negative effect concerns the effect of additional workers on the marginal productivity of labour. On the other hand, a higher fertility tends also to relax the resource constraint, by making the consumption of the parent and grand-parents less constraining.

Proposition 4 *Assume that a unique locally stable stationary equilibrium exists on Early Island. The social optimum on Early Island* $(c^*, d^*, b^*, k^*, n_1^*)$ *is such that (1) the marginal rate of substitution between consumptions at two successive ages of life is equal to the optimal fertility rate* n_1^**; (2) the marginal productivity of capital* k^* *is equal to the optimal fertility rate* n_1^**; (3) the optimal fertility rate* n_1^* *equalizes the marginal welfare losses from capital widening and labour productivity loss and the marginal welfare gains from intergenerational transfers.*

Proof. See the above FOCs. ■

Regarding the social optimum on Late Island, we will first consider the case where $N_{-1} = N_0 > 0$ and $n_2 = 1$, under which a stable stationary equilibrium can exist (Proposition 2), and, then, the case where $N_{-1} = N_0 > 0$ and $n_2 \neq 1$, under which there is no stable stationary equilibrium (Proposition 3).

Considering the case where $N_{-1} = N_0 > 0$ and $n_2 = 1$, and assuming that a locally stable stationary equilibrium exists, the social planner's problem is:

$$\begin{aligned}
&\max_{c,d,b,k}\ u(c) + \beta u(d) + \beta^2 u(b) \\
&s.t.\ F(k_t, 1+1) = c + d + b + k
\end{aligned}$$

An interior optimum (c^*, d^*, b^*, k^*) satisfies the following FOCs:

$$\begin{aligned}
\frac{u'(c^*)}{\beta u'(d^*)} &= \frac{u'(d^*)}{\beta u'(b^*)} = 1 \\
F_k(k^*, \cdot) &= 1
\end{aligned}$$

[11] We assume that second-order conditions are satisfied. Note that, in the light of Deardorff (1976), this is not a weak assumption.

Thus, the social optimum on Late Island involves, under $n_2 = 1$, a constant marginal rate of substitution between the consumptions at different periods, and equal to the cohort growth rate $n_2 = 1$. Moreover, k should be such that the marginal productivity of capital equals the cohort growth rate.

Turning now to the general case where the optimum fertility differs from replacement fertility (i.e. $n_2 \neq 1$), and assuming that there exists a stable two-period equilibrium cycle $((\hat{k}, \hat{\Omega}, 1), (\check{k}, \check{\Omega}, n_2))$, the social optimum consists of the consumptions $(\hat{c}^*, \check{c}^*)$, $(\hat{d}^*, \check{d}^*)$ and $(\hat{b}^*, \check{b}^*)$, as well as capital $(\hat{k}^*, \check{k}^*)$ and fertility n_2^* that maximize the lifetime welfare along the equilibrium cycle. The resource constraint on Late Island is, in intensive terms:

$$F\left(k_t, 1 + \frac{1}{g_{t-1}}\right) = c_t + \frac{d_t}{g_{t-1}} + \frac{b_t}{g_{t-2}g_{t-1}} + k_{t+1}g_t$$

Given that the cohort size ratio g_t follows a two-period cycle $(n_2, 1)$, the resource constraint on Late Island is either:

$$F\left(k_t, 1 + \frac{1}{n_2}\right) = c_t + d_t\frac{1}{n_2} + b_t\frac{1}{n_2} + k_{t+1}$$

$$\text{or} \qquad F\left(k_t, 1 + 1\right) = c_t + d_t + b_t\frac{1}{n_2} + k_{t+1}n_2$$

depending on the location on the cohort size cycle, those two conditions imposing themselves successively.

Hence, the social planner's optimization problem on Early Island is:

$$\max_{\hat{c},\hat{d},\hat{b},\hat{k},\check{c},\check{d},\check{b},\check{k},n_2} \quad u(\hat{c}) + \beta u(\check{d}) + \beta^2 u(\hat{b}) + u(\check{c}) + \beta u(\hat{d}) + \beta^2 u(\check{b})$$

$$s.t.\ F\left(\hat{k}, 1+1\right) = \hat{c} + \hat{d} + \hat{b}\frac{1}{n_2} + \check{k}n_2$$

$$s.t.\ F\left(\check{k}, 1 + \frac{1}{n_2}\right) = \check{c} + \check{d}\frac{1}{n_2} + \check{b}\frac{1}{n_2} + \hat{k}$$

Denoting by λ and μ the Lagrange multipliers associated to the two resource constraints, the social optimum $\left(\hat{c}^*, \hat{d}^*, \hat{b}^*, \hat{k}^*, \check{c}^*, \check{d}^*, \check{b}^*, \check{k}^*, n_2^*\right)$ satisfies the following first-order conditions:

$$\begin{aligned}
u'(\hat{c}^*) &= \beta u'(\hat{d}^*) = n_2^*\beta^2 u'(\hat{b}^*) = \lambda \\
\lambda F_k\left(\hat{k}^*, 1+1\right) &= u'(\check{c}^*) = \mu \\
\beta u'(\check{d}^*) &= \beta^2 u'(\check{b}^*) = \frac{\mu}{n_2^*} \\
\mu F_k\left(\check{k}^*, 1 + \frac{1}{n_2^*}\right) &= \lambda n_2^* \\
\lambda\check{k}^* + \mu\frac{F_L\left(\check{k}^*, 1 + \frac{1}{n_2^*}\right)}{(n_2^*)^2} &= \lambda\frac{\hat{b}^*}{(n_2^*)^2} + \mu\frac{\check{d}^*}{(n_2^*)^2} + \mu\frac{\check{b}^*}{(n_2^*)^2}
\end{aligned}$$

Further simplifications yield, regarding consumptions:

$$\frac{u'(\hat{c}^*)}{\beta u'(\hat{d}^*)} = 1 = \frac{u'(\check{d}^*)}{\beta u'(\check{b}^*)}$$
$$\frac{u'(\hat{d}^*)}{\beta u'(\hat{b}^*)} = n_2^* = \frac{u'(\check{c}^*)}{\beta u'(\check{d}^*)}$$

Hence we have: $\hat{c}^* > \hat{d}^* \gtreqless \hat{b}^*$ and $\check{c}^* \lesseqgtr \check{d}^* > \check{b}^*$. Consumption profiles may not be monotonic on the lifecycle, because of the equality of the MRS between consumptions and the (fluctuating) cohort growth rate. The magnitude of consumption fluctuations depends on how much n_2^* differs from 1.[12]

Regarding the optimal capital levels, we obtain, from the two FOCs:

$$F_k\left(\hat{k}^*, 1+1\right) \times F_k\left(\check{k}^*, 1+\frac{1}{n_2^*}\right) = n_2^*$$

That condition, which differs from the standard Golden Rule, states that the product of the marginal productivities of capital at levels $\hat{k}^*$ and $\check{k}^*$ should be equal to n_2^*, which equals the product of the cohort growth rates 1 and n_2^*.[13]

Finally, regarding the optimal fertility rate n_2^*, we have, after substitutions:

$$\check{k}^* + \frac{u'(\check{c}^*)}{u'(\hat{c}^*)} \frac{F_L\left(\check{k}^*, 1+\frac{1}{n_2^*}\right)}{(n_2^*)^2} = \frac{1}{n_2^*}\left(\frac{u'(\check{d}^*)}{u'(\hat{d}^*)}\check{d}^* + \frac{\hat{b}^*}{n_2^*} + \frac{u'(\check{b}^*)}{u'(\hat{b}^*)}\frac{\check{b}^*}{n_2^*}\right)$$

The LHS is the marginal welfare loss from raising the fertility rate n_2^*. As usual, it includes the capital widening effect (first term), and the marginal loss in terms of the productivity of labour (second term). The RHS is the intergenerational transfer effect: a higher n_2^* reduces the pressure put by the elderly's consumption on resources.

The FOC for optimal fertility n_2^* depends on the shape of the temporal utility function, through the ratios $\frac{u'(\check{c}^*)}{u'(\hat{c}^*)}$, $\frac{u'(\check{d}^*)}{u'(\hat{d}^*)}$ and $\frac{u'(\check{b}^*)}{u'(\hat{b}^*)}$, which capture the sensitivity of welfare to the consumption cycle induced by the fertility cycle. To understand the impact of welfare sensitivity on optimal fertility, let us take two polar cases. Assume first that temporal utility is *linear*, so that lifetime welfare is not very sensitive to the cycle. In that case, the FOC becomes:

$$\check{k}^* + \frac{F_L\left(\check{k}^*, 1+\frac{1}{n_2^*}\right)}{(n_2^*)^2} = \frac{1}{n_2^*}\left(\check{d}^* + \frac{\hat{b}^* + \check{b}^*}{n_2^*}\right)$$

which is quite close to the FOC for optimal fertility on Early Island, but with $(n_1^*)^2 = n_2^*$. On the basis of that FOC, the optimal fertility rate on Late Island should be close to the square of the optimal fertility rate on Early Island.

[12] If n_2^* equals 1, we are back to a constant MRS, implying monotonic consumption paths.

[13] If n_2^* was equal to 1, the two resource constraints would coincide, and so would $\hat{k}^*$ and $\check{k}^*$. As a consequence, the socially optimal capital level k^* should then satisfy: $F_k(k^*, 1+1) = \sqrt[2]{n_2^*} = 1$ in conformity with the standard Golden Rule.

Alternatively, suppose that the temporal utility function is not linear, and that ratios of marginal utilities are strongly sensitive to little variations in consumption. Then, consumption fluctuations due to the demographic fluctuations have large welfare effects, and these affect the optimal fertility rate. To see this, take, for instance, the case where a larger fertility implies lower consumption at the young age, but higher consumptions at the old ages:

$$\check{k}^* + \underbrace{\frac{u'(\check{c}^*)}{u'(\hat{c}^*)}}_{>1} \frac{F_L\left(\check{k}^*, 1 + \frac{1}{n_2^*}\right)}{(n_2^*)^2} = \underbrace{\frac{u'(\check{d}^*)}{u'(\hat{d}^*)}}_{<1} \frac{\check{d}^*}{n_2^*} + \frac{\hat{b}^*}{(n_2^*)^2} + \underbrace{\frac{u'(\check{b}^*)}{u'(\hat{b}^*)}}_{<1} \frac{\check{b}^*}{(n_2^*)^2}$$

The second term of the LHS is increased, leading to a higher marginal welfare loss. The lower ratios of marginal utilities reduce two terms on the RHS, and, thus, reduce also the marginal welfare gains from a higher fertility. Hence, if the LHS is increased and the RHS reduced, the optimal fertility rate n_2^* must differ. We expect, in that case, that, *ceteris paribus*, the larger sensitivity of temporal welfare to the cycle supports the selection of a *lower* fertility rate: $n_2^* < (n_1^*)^2$.[14]

Proposition 5 *Assume that $N_{-1} = N_0 > 0$, and that a unique stable two-period cycle exists on Late Island. The social optimum on Late Island $\left(\hat{c}^*, \hat{d}^*, \hat{b}^*, \hat{k}^*, \check{c}^*, \check{d}^*, \check{b}^*, \check{k}^*, n_2^*\right)$ is such that (1) the marginal rate of substitution between consumptions at two successive ages of life is equal either to 1 or to the optimal fertility rate n_2^*; (2) the product of the marginal productivities of capital at $\hat{k}^*$ and $\check{k}^*$ equals the optimal fertility n_2^*; (3) the optimal fertility n_2^* equalizes the marginal welfare losses from capital widening and labour productivity loss, and the marginal welfare gains from intergenerational transfers.*

Proof. See the above FOCs. ■

Finally, let us make some welfare comparisons across islands. When $N_{-1} = N_0 > 0$, and when the optimal fertility rates are equal to the replacement ratio (i.e. $n_1^* = n_2^* = 1$), the two economies are equivalent in welfare terms. The reason is that, in that case, initial conditions guarantee that a constant number of children is born at any period on the two islands, implying a constant equal labour supply in both economies. Given that the two economies are similar on all other aspects, these must also be equivalent in welfare terms. In other cases, where $n_1^* \neq n_2^* \neq 1$, it is impossible to draw unambiguous welfare conclusions, because of the uncertain impact of fertility on welfare. Fertility dilutes capital and lowers the marginal productivity of labour, but this also weakens the resource constraint thanks to larger intergenerational transfers. Those three effects are at work in the determination of the optimal fertility rate on Early and Late Islands, and there is no unambiguous conclusion.[15]

[14] Note that this conclusion could vary if we made other assumptions on the relative consumption levels at the down and the top of the cycle.

[15] Actually, to be able to make further comparisons of the two islands in welfare terms, it would be necessary to impose functional forms for the production and utility functions.

Proposition 6 *Assume that* $N_{-1} = N_0 > 0$.

- *When the optimal fertility rate on Early Island* n_1^* *and on Late Island* n_2^* *are equal to* 1*, the lifetime welfare is, at the social optimum, exactly equal on the two islands.*
- *When the optimal fertility rates on the two islands differ, whether average lifetime welfare on one island is larger than on the other island is ambiguous, and depends on the welfare effects induced by cohort growth.*

Proof. See the above FOCs. The FOCs characterizing the social optimum on Late Island when $n_2^* = 1$ correspond exactly to the FOCs characterizing the social optimum on Early Island when $n_1^* = 1$. As a consequence, the maximum lifetime welfare is the same in the two economies. ■

2.6 The Serendipity Theorem

Let us now examine whether Samuelson's (1975) Serendipity Theorem remains valid in our environment. The question is the following. Assume that agents on Early and Late Islands behave like price-takers on competitive markets, and take fertility rates as given. Is the economy going to converge towards the social optimum when the optimum fertility is imposed?

Consider first an agent living on Early Island. That agent chooses second- and third-period savings, in such a way as to maximize his lifetime welfare. He takes factor prices and the fertility rate as given. Assuming a perfectly competitive economy, factors are paid at their marginal productivities, which, at the steady-state, can be written as:

$$\begin{aligned} w &= \left[F\left(k\left(,1+\frac{1}{n_1}\right)\right) - F_k\left(k, 1+\frac{1}{n_1}\right)\right]\frac{n_1}{1+n_1} \\ R &= F_k\left(k, 1+\frac{1}{n_1}\right) \end{aligned}$$

The problem of the agent is thus:

$$\max_{c,d,b} u(c) + \beta u(d) + \beta^2 u(b)$$

$$s.t.\ w + \frac{w}{R} = c + \frac{d}{R} + \frac{b}{R^2}$$

The FOCs are

$$\frac{u'(c)}{\beta u'(d)} = \frac{u'(d)}{\beta u'(b)} = R$$

Hence, if the social planner fixes the fertility rate n_1 at a level such that $n_1^* = F_k(k^*, \cdot)$ where k^* is the optimal level of capital per young worker, then individuals, being price-takers, will choose their savings optimally, since the two above FOCs will then coincide with the FOCs for optimal intergenerational allocations of resources. This is the Serendipity Theorem: provided the social planner imposes the optimal fertility rate, the perfectly competitive economy converges towards the social optimum (i.e. the *optimum optimorum*).

Turning now to agents living on Late Island, we need here to consider two cases: $n_2^* = 1$ and $n_2^* \neq 1$.[16] In the former case, the problem of the agent is

$$\max_{c,d,b} u(c) + \beta u(d) + \beta^2 u(b)$$
$$s.t.\ w + \frac{w}{R} = c + \frac{d}{R} + \frac{b}{R^2}$$

where

$$\begin{aligned} w &= [F(k, 1+1) - F_k(k, 1+1)]\frac{1}{2} \\ R &= F_k(k, 1+1) \end{aligned}$$

The FOCs are

$$\frac{u'(c)}{\beta u'(d)} = \frac{u'(d)}{\beta u'(b)} = R$$

Imposing $n_2 = n_2^* = 1 = F_k(k^*, \cdot)$ suffices to decentralize the social optimum, in conformity with Samuelson's Serendipity Theorem.

However, when n_2^* differs from 1, the Serendipity Theorem in its standard form cannot apply, since it requires the existence of a unique stable stationary equilibrium, which is not possible here (see Proposition 2). On Late Island, when $N_{-1} = N_0 > 0$, the economy will, under mild conditions, exhibit a two-period equilibrium cycle (see Proposition 3). Hence, we can investigate whether the decentralized Late Island economy converges towards its (non-stationary) social optimum when the optimal fertility n_2^* is imposed one period out of two.

When the economy exhibits a two-period cycle $((\hat{k}, \hat{\Omega}, 1), (\check{k}, \check{\Omega}, n_2))$, factor prices oscillate between two values. If we consider an individual who is a young adult at a period where $g_t = 1$, factor prices are

$$\begin{aligned} \hat{w} &= \left[F\left(\hat{k}\,(,1+1)\right) - \hat{k}F_k\left(\hat{k}, 1+1\right)\right]\frac{1}{2} \\ \hat{R} &= F_k\left(\hat{k}, 1+1\right) \end{aligned}$$

Whereas, when $g_t = n_2$, factor prices are:

$$\begin{aligned} \check{w} &= \left[F\left(\check{k}\left(,1+\frac{1}{n_2}\right)\right) - \check{k}F_k\left(\check{k}, 1+\frac{1}{n_2}\right)\right]\frac{n_2}{1+n_2} \\ \check{R} &= F_k\left(\check{k}, 1+\frac{1}{n_2}\right) \end{aligned}$$

[16] We restrict ourselves here to $N_{-1} = N_0 > 0$.

Hence the problem of an agent who is a young worker when $g_t = 1$ is:[17]

$$\max_{\hat{c},\check{d},\hat{b}} u(\hat{c}) + \beta u(\check{d}) + \beta^2 u(\hat{b})$$

$$s.t.\ \hat{w} + \frac{\check{w}}{\check{R}} = \hat{c} + \frac{\check{d}}{\check{R}} + \frac{\hat{b}}{\hat{R}\check{R}}$$

The FOCs are

$$\frac{u'(\hat{c})}{\beta u'(\check{d})} = \check{R} \text{ and } \frac{u'(\check{d})}{\beta u'(\hat{b})} = \hat{R}$$

The problem of the agent who is a young worker when $g_t = n_2$ is:[18]

$$\max_{\check{c},\hat{d},\check{b}} u(\check{c}) + \beta u(\hat{d}) + \beta^2 u(\check{b})$$

$$s.t. \check{w} + \frac{\hat{w}}{\hat{R}} = \check{c} + \frac{\hat{d}}{\hat{R}} + \frac{\check{b}}{\hat{R}\check{R}}$$

The FOCs are

$$\frac{u'(\check{c})}{\beta u'(\hat{d})} = \hat{R} \text{ and } \frac{u'(\hat{d})}{\beta u'(\check{b})} = \check{R}$$

Combining the two sets of FOCs, we have:

$$\frac{u'(\hat{c})}{u'(\check{d})} = \frac{u'(\hat{d})}{u'(\check{b})} \text{ and } \frac{u'(\check{d})}{u'(\hat{b})} = \frac{u'(\check{c})}{u'(\hat{d})}$$

whereas the social optimum requires:

$$\begin{aligned} \frac{u'(\hat{c}^*)}{u'(\hat{d}^*)} &= \frac{u'(\check{d}^*)}{u'(\check{b}^*)} \iff \frac{u'(\hat{c}^*)}{u'(\check{d}^*)} = \frac{u'(\hat{d}^*)}{u'(\check{b}^*)} \\ \frac{u'(\hat{d}^*)}{u'(\hat{b}^*)} &= \frac{u'(\check{c}^*)}{u'(\check{d}^*)} \iff \frac{u'(\check{d}^*)}{u'(\hat{b}^*)} = \frac{u'(\check{c}^*)}{u'(\hat{d}^*)} \end{aligned}$$

Hence the slopes of the consumption paths in the decentralized economy coincide with the one at the social optimum. Regarding the decentralization of the

[17] The intertemporal budget constraint is obtained from:

$$\begin{aligned} \hat{c} &= \hat{w} - \hat{s} \\ \check{d} &= \check{w} + \check{R}\hat{s} - \check{z} \\ \hat{b} &= \hat{R}\check{z} \end{aligned}$$

[18] The intertemporal budget constraint is obtained from:

$$\begin{aligned} \check{c} &= \check{w} - \check{s} \\ \hat{d} &= \hat{w} + \hat{R}\check{s} - \hat{z} \\ \check{b} &= \check{R}\check{z} \end{aligned}$$

optimal capital levels, we have, in the decentralized economy:

$$\frac{u'(\hat{c})}{\beta u'(\check{d})} = F_k\left(\check{k}, 1 + \frac{1}{n_2}\right) \text{ and } \frac{u'(\check{d})}{\beta u'(\hat{b})} = F_k\left(\hat{k}, 1+1\right)$$

whereas the social optimum requires:

$$\frac{u'(\hat{c}^*)}{\beta u'(\check{d}^*)} = 1$$

$$\frac{u'(\check{d}^*)}{\beta u'(\hat{b}^*)} = F_k\left(\hat{k}^*, 1+1\right) \times F_k\left(\check{k}^*, 1 + \frac{1}{n_2^*}\right)$$

We know that, at the optimum, $\frac{u'(\hat{c}^*)}{\beta u'(\check{d}^*)} = \frac{\lambda n_2^*}{\mu}$ and $\frac{\lambda}{\mu} = \frac{F_k\left(\check{k}^*, 1+\frac{1}{n_2^*}\right)}{n_2^*}$. Therefore $\frac{u'(\hat{c}^*)}{\beta u'(\check{d}^*)} = F_k\left(\check{k}^*, 1 + \frac{1}{n_2^*}\right)$. Moreover, we have $\frac{u'(\check{d}^*)}{\beta u'(\hat{b}^*)} = \frac{\mu}{\lambda} = F_k\left(\hat{k}^*, 1+1\right)$. Hence, we have, at the optimum:

$$\frac{u'(\hat{c}^*)}{\beta u'(\check{d}^*)} = F_k\left(\check{k}^*, 1 + \frac{1}{n_2^*}\right) \text{ and } \frac{u'(\check{d}^*)}{\beta u'(\hat{b}^*)} = F_k\left(\hat{k}^*, 1+1\right)$$

In the light of this, the Serendipity Theorem still holds when $n_2^* \neq 1$. Indeed, imposing the optimal fertility rate $n_2^* = F_k\left(\hat{k}^*, 1+1\right) \times F_k\left(\check{k}^*, 1 + \frac{1}{n_2^*}\right)$ allows us to deduce, from the FOCs describing the agents' savings decisions in a competitive economy, all FOCs characterizing the social optimum.

Proposition 7 *Assume that $N_{-1} = N_0 > 0$.*

1. *Assume that there exists a unique stable stationary equilibrium on Early Island. Then the perfectly competitive economy will converge towards the (stationary) social optimum if the optimal fertility rate n_1^* is imposed.*

2. *Assume that $n_2 = 1$ is the optimum fertility rate n_2^* on Late Island. Then the perfectly competitive economy will converge towards the (stationary) social optimum if the optimal fertility rate n_2^* is imposed.*

3. *Assume that $n_2 = 1$ is not the optimum fertility rate n_2^* on Late Island. Then the perfectly competitive economy will converge towards the (non-stationary) social optimum if the optimal fertility rate n_2^* is imposed.*

Proof. The proof follows from the comparison of FOCs for the social planner's optimization problem and for the individual's utility maximization problem. ■

Proposition 7 emphasizes the significant robustness of the Serendipity Theorem to assumptions on the *timing* of births. Samuelson's result is indeed robust to the number of time-periods *after* births (point 1.) or *before* births (points 2. and 3.). More importantly, point 3. suggests that the Serendipity Theorem applies not only to stationary environments (as in Samuelson's initial result), but also holds in the context of a non-stationary social optimum.

3 The general model

The fable with Early and Late Islands allowed us to contrast two economies that differ only regarding the timing of births, and to discuss the social optimum in that context. However, in reality, all economies exhibit some aspects of Early Island, and some aspects of Late Island, since some children are born from young parents, and others from older parents. Therefore we will consider here an economy where all age-specific fertility rates are strictly positive during the reproduction period ($n_1 > 0$ and $n_2 > 0$), to complement the cases of Early Island ($n_1 > 0$ and $n_2 = 0$) and Late Island ($n_1 = 0$ and $n_2 > 0$).

As above, we still consider here a four-period OLG model, where households supply one unit of labour in the second and in the third period. We assume, as usual, $N_{-1} > 0$ and $N_0 > 0$. The difference with respect to Section 2 is that individuals now have $n_1 > 0$ children in the second period, and $n_2 > 0$ children in the third period. Therefore, the total fertility rate is here equal to $n_1 + n_2$. Hence, the number of individuals born at time t is now:

$$N_t = n_1 N_{t-1} + n_2 N_{t-2} \tag{13}$$

where the first term of the RHS consists of children born from young parents, whereas the second term consists of children born from older parents. Hence the growth rate of cohort size g_t is given by:

$$g_t \equiv \frac{N_t}{N_{t-1}} = n_1 + n_2 \frac{N_{t-2}}{N_{t-1}}$$

The cohort growth rate g_t follows thus the dynamic law:

$$g_t = n_1 + \frac{n_2}{g_{t-1}} \tag{14}$$

That expression obviously encompasses the cases of Early Island (where $n_2 = 0$), and of Late Island (where $n_1 = 0$).

Given that all agents alive in the second and third periods take part to the labour market, the total labour force $L_t = N_{t-1} + N_{t-2}$ can be written as:

$$L_t = n_1 N_{t-2} + n_2 N_{t-3} + n_1 N_{t-3} + n_2 N_{t-4}$$

The first two terms consist of the young workers, who can have either young parents (first term) or older parent (second term), while the last two terms consist of the old workers, who can also have young or old parents. Thus the total labour force follows the dynamic law:

$$L_t = n_1 L_{t-1} + n_2 L_{t-2}$$

When all children are born from young parents (i.e. $n_2 = 0$), we are back to the standard dynamic law $L_t = n_1 L_{t-1}$ prevailing on Early Island. When all children are born from old parents (i.e. $n_1 = 0$), we get the law $L_t = n_2 L_{t-2}$, which prevails on Late Island. Note that the labour force growth rate is:

$$\frac{L_{t+1}}{L_t} = n_1 + n_2 \frac{L_{t-1}}{L_t}$$

Thus, in the case where all children are born from young parents, the labour force grows at a period rate n_1, whereas in the case where all children have old parents, the labour force grows at a rate n_2 over two periods.

The production process is the same as on Early and Late Islands, and the resource constraint of the economy is, in intensive terms:

$$F\left(k_t, 1 + \frac{N_{t-2}}{N_{t-1}}\right) = c_t + d_t \frac{N_{t-2}}{N_{t-1}} + b_t \frac{N_{t-3}}{N_{t-2}} + k_{t+1} \frac{N_t}{N_{t-1}} \tag{15}$$

3.1 The dynamics

Given that the capital stock comes from the savings of the adults in the second and third periods of their life, i.e. $K_{t+1} = N_{t-1} s_t + N_{t-2} z_t$, we obtain, by dividing this by the number of young workers, $L^y_{t+1} = N_t$:

$$k_{t+1} = \frac{s_t}{g_t} + \frac{z_t}{(g_{t-1})(g_t)}$$

We know that

$$g_t = n_1 + \frac{n_2}{g_{t-1}} \iff g_{t-1} = \frac{n_2}{g_t - n_1}$$

Hence the capital accumulation equation can be rewritten as:

$$k_{t+1} = \frac{s_t}{g_t} + \left(1 - \frac{n_1}{g_t}\right) \frac{z_t}{n_2}$$

As on Early and Late Islands, the savings s_t depends, under perfect foresight, on factor prices at periods t, $t+1$, and $t+2$, while the savings z_t depends, under perfect foresight, on factor prices at periods $t-1$, t and $t+1$. Such a high number of time lags makes the dynamic study quite complex. Hence, to study the long-run dynamics of the general economy, we will, as above, rely on myopic anticipations regarding factor prices. Under such myopic anticipations, second-period savings s_t are a mere function $\sigma(k_t)$, whereas the third-period savings z_t are a mere function $\zeta(k_{t-1})$. Hence the capital accumulation equation becomes:

$$k_{t+1} = \frac{\sigma(k_t)}{g_t} + \left(1 - \frac{n_1}{g_t}\right) \frac{\zeta(k_{t-1})}{n_2}$$

If one defines the old adults savings as the variable $\Omega_t \equiv \frac{\zeta(k_{t-1})}{n_2}$, the dynamics of the economy is summarized by the three-dimensional system:

$$\begin{aligned} k_{t+1} &\equiv G(k_t,\Omega_t,g_t)=\frac{\sigma\left(k_t\right)}{g_t}+\left(1-\frac{n_1}{g_t}\right)\Omega_t \\ \Omega_{t+1} &\equiv H\left(k_t\right)=\frac{\zeta(k_t)}{n_2} \\ g_{t+1} &\equiv I(g_t)=n_1+\frac{n_2}{g_t} \end{aligned} \tag{16}$$

Proposition 8 characterizes the long-run dynamics of that economy.

Proposition 8 *Assume $N_{-1}>0$ and $N_0>0$ and $n_1>0$ and $n_2>0$. Denote $\sqrt[2]{n_1^2+4n_2}$ by Ψ.*

- *If $\sigma(0)=0$, $\zeta(0)=0$, $lim_{k\to 0}\ \frac{\Psi+n_1}{\Psi-n_1}\left[1-\frac{2\sigma'(k_t)}{n_1+\Psi}\right]<lim_{k\to 0}\ \frac{\zeta'(k_t)}{n_2}$ and $lim_{k\to +\infty}\ \frac{\Psi+n_1}{\Psi-n_1}\left[1-\frac{2\sigma'(k_t)}{n_1+\Psi}\right]>lim_{k\to +\infty}\ \frac{\zeta'(k_t)}{n_2}$,*

  there exists a stationary equilibrium (k^*,Ω^*,g^*) with $\Omega^*=\frac{\zeta(k^*)}{n_2}$ and $g^*=\frac{n_1+\sqrt[2]{(n_1)^2+4n_2}}{2}$.

- *Provided the conditions: $\frac{4\zeta'(k^*)(\Psi-n_1)}{(\Psi+n_1)^3}<1$; $\frac{2\sigma'(k^*)-\frac{\zeta'(k^*)}{n_2}(\Psi-n_1)}{\Psi+n_1}>-1$ and $\frac{2\sigma'(k^*)+\frac{\zeta'(k^*)}{n_2}(\Psi-n_1)}{\Psi+n_1}<1$,*

  as well as: $1+\frac{\frac{\zeta'(k^*)}{n_2}(\Psi-n_1)}{(\Psi+n_1)}+\frac{8\sigma'(k^*)\left(n_2+\zeta'(k^*)\right)}{(\Psi+n_1)^3}+\frac{32\left(\zeta'(k^*)\right)^2 n_1(\Psi-n_1)}{(\Psi+n_1)^6}>16\zeta'\left(k^*\right)\frac{n_1\sigma'(k^*)+\zeta'(k^*)+n_2\frac{(\Psi-n_1)}{(\Psi+n_1)}}{(\Psi+n_1)^4}$

  are satisfied, that stationary equilibrium is locally stable. Starting from any initial conditions (k_0,Ω_0,g_0), the economy will converge non-monotonically towards the stationary equilibrium (k^*,Ω^*,g^*).

Proof. See the Appendix. ■

Hence, once we consider an economy with interior age-specific fertility rates during the reproduction period, there exists a locally stable stationary equilibrium under mild conditions. In the light of our study of the dynamics of Late Island (where $n_1=0$), we see how the introduction of early births, i.e. $n_1>0$, suffices to imply a stationary long-run dynamics, in contrast with the economy of Late Island where no stable equilibrium exists.

One could hardly overemphasize the distinct dynamic corollaries of early and late motherhoods. Some positive - even extremely small - fertility at young adulthood suffices to make the economy escape from the cyclical dynamics of Late Island. Once again, if we only look at the total fertility rate n_1+n_2, we miss a central dimension of the dynamics, since the qualitative properties of the dynamics of the economy - stationary or cyclical - depend on $n_1>0$ or $n_1=0$, whatever the total fertility n_1+n_2 is.

3.2 The social optimum

Let us now assume that the conditions of Proposition 8 are satisfied, so that there exists a unique stable stationary equilibrium. Given that, at the steady-state, the cohort growth factor g_t is constant and equal to $\frac{n_1+\sqrt[2]{n_1^2+4n_2}}{2}$, the resource constraint of the economy can be written as:

$$\begin{aligned} & F\left(k, \frac{n_1 + \sqrt[2]{n_1^2 + 4n_2} + 2}{n_1 + \sqrt[2]{n_1^2 + 4n_2}}\right) - k\frac{n_1 + \sqrt[2]{n_1^2 + 4n_2}}{2} \\ = \quad & c + d\frac{2}{n_1 + \sqrt[2]{n_1^2 + 4n_2}} + b\left(\frac{2}{n_1 + \sqrt[2]{n_1^2 + 4n_2}}\right)^2 \end{aligned}$$

where k denotes the steady-state capital level.

The social planner chooses all consumptions, the capital level and the age-specific fertility rates in such a way as to maximize the lifetime welfare of a cohort alive at the steady-state. The planner's optimization problem is thus:

$$\begin{aligned} & \max_{c,d,b,k,n_1,n_2} \quad u(c) + \beta u(d) + \beta^2 u(b) \\ s.t. \quad & F\left(k, \frac{n_1 + \sqrt[2]{n_1^2 + 4n_2} + 2}{n_1 + \sqrt[2]{n_1^2 + 4n_2}}\right) - k\frac{n_1 + \sqrt[2]{n_1^2 + 4n_2}}{2} \\ = \quad & c + d\frac{2}{n_1 + \sqrt[2]{n_1^2 + 4n_2}} + b\left(\frac{2}{n_1 + \sqrt[2]{n_1^2 + 4n_2}}\right)^2 \end{aligned}$$

An interior optimum $(c^*, d^*, b^*, k^*, n_1^*, n_2^*)$ satisfies the following FOCs:

$$\begin{aligned} \frac{u'(c^*)}{\beta u'(d^*)} &= \frac{u'(d^*)}{\beta u'(b^*)} = \frac{n_1^* + \sqrt[2]{n_1^{*2} + 4n_2^*}}{2} \\ F_k\left(k^*, \cdot\right) &= \frac{n_1^* + \sqrt[2]{n_1^{*2} + 4n_2^*}}{2} \end{aligned}$$

The first expression implies that the MRS between consumptions at two successive periods is, at the optimum, equal to the optimal cohort growth ratio, $(n_1^* + \sqrt[2]{n_1^{*2} + 4n_2^*})/2$. Thus, from the point of view of the optimal consumption profile, it is not the total fertility rate $n_1^* + n_2^*$ that matters, but the cohort growth rate.

The second expression is the Golden Rule: the optimal stock of capital per young worker k^* is such that the marginal productivity of capital is equal to the cohort growth rate. Here again, for a given total fertility rate $n_1^* + n_2^*$, the optimal capital will vary greatly with the optimal timing of births.

Regarding the FOCs for optimal n_1^* and n_2^*, we have:

$$F_L(k^*,\cdot)\,2\frac{1+n_1^*\left(n_1^{*2}+4n_2^*\right)^{-1/2}}{\left(n_1^*+\sqrt[2]{n_1^{*2}+4n_2^*}\right)^2}+k^*\frac{1+n_1^*\left(n_1^{*2}+4n_2^*\right)^{-1/2}}{2}$$

$$= 2\frac{1+n_1^*\left(n_1^{*2}+4n_2^*\right)^{-1/2}}{\left(n_1^*+\sqrt[2]{n_1^{*2}+4n_2^*}\right)^2}\left(d^*+\frac{4b^*}{n_1^*+\sqrt[2]{n_1^{*2}+4n_2^*}}\right)$$

and

$$F_L(k^*,\cdot)\,4\frac{\left(n_1^{*2}+4n_2^*\right)^{-1/2}}{\left(n_1^*+\sqrt[2]{n_1^{*2}+4n_2^*}\right)^2}+k^*\left(n_1^{*2}+4n_2^*\right)^{-1/2}$$

$$= 4\frac{\left(n_1^{*2}+4n_2^*\right)^{-1/2}}{\left(n_1^*+\sqrt[2]{n_1^{*2}+4n_2^*}\right)^2}\left(d^*+\frac{4b^*}{n_1^*+\sqrt[2]{n_1^{*2}+4n_2^*}}\right)$$

Those FOCs include the standard determinants of optimal fertility. On the LHS, we have the negative effects of fertility on the marginal productivity of labour (first term), as well as the capital widening effect (second term). On the RHS, we find the gains from intergenerational redistribution. To interpret those FOCs, note that these can be written as:

$$g_{n_1}^*\left[\frac{-F_L(k^*,\cdot)}{g^{*2}}-k^*+\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}\right] = 0$$

$$g_{n_2}^*\left[\frac{-F_L(k^*,\cdot)}{g^{*2}}-k^*+\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}\right] = 0$$

where $g_{n_1^*}^*$ and $g_{n_2^*}^*$ denote the derivatives of the optimal cohort growth rate with respect to the optimal age-specific fertility rates n_1^* and n_2^* respectively. Those derivatives are given by:

$$g_{n_1^*}^* = \frac{1+n_1^*\left(n_1^{*2}+4n_2^*\right)^{-1/2}}{2}=\frac{1}{2}+\frac{n_1^*}{2}\frac{1}{\sqrt[2]{n_1^{*2}+4n_2^*}}>0$$

$$g_{n_2^*}^* = \left(n_1^{*2}+4n_2^*\right)^{-1/2}=\frac{1}{\sqrt[2]{n_1^{*2}+4n_2^*}}>0$$

As we consider here the case of interior optimal age-specific fertility rates, i.e. $n_1^*>0$, $n_2^*>0$, it follows that $g_{n_1^*}^*$ is always different from 0. The same is also true for $g_{n_2^*}^*$. Therefore, in order to have the two above FOCs satisfied simultaneously, it must be the case that the following equality holds:

$$k^*+\frac{F_L(k^*,\cdot)}{g^{*2}}=\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}} \tag{17}$$

That equation characterizes the optimal cohort growth rate g^*. At the social optimum, the marginal welfare loss from a higher cohort growth rate (the LHS) are equal to the marginal welfare gain from a higher cohort growth rate (the RHS). The negative welfare effects due to a higher cohort growth rate are the capital widening effect (1st term of the LHS) and the negative productivity effect (2nd term of the LHS), whereas the positive welfare effects are the intergenerational redistribution effects (1st and 2nd terms of the RHS). That condition for optimal cohort growth rate can be rewritten as:

$$g^{*3} + \frac{(F_L(k^*, \cdot) - d^*)}{k^*} g^* - \frac{2b^*}{k^*} = 0$$

In the Appendix, we solve that cubic equation, and derive the optimal cohort growth rate g^*. That variable is the key variable characterizing the social optimum, since this determines both the optimal consumption paths and capital level k^*. Moreover, we know from above that, as long as g takes its optimum level g^*, the two FOCs characterizing the optimal age-specific fertility rates n_1^* and n_2^* are also satisfied. Hence the characterization of the social optimum requires, above all, a characterization of the optimal cohort growth rate g^*, rather than of the age-specific fertility rates, which affect optimal consumption paths and capital *only through* the optimal cohort growth rate.

Proposition 9 *Assume that a unique stable stationary equilibrium exists. The social optimum* $(c^*, d^*, b^*, k^*, n_1^*, n_2^*)$ *is such that:*

- *the marginal rate of substitution between consumptions at two successive ages of life is equal to the optimal cohort growth rate* g^**;*
- *the capital* k^* *is such that the marginal productivity is equal to the optimal cohort growth rate* g^**;*
- *the optimal age-specific fertility rates* n_1^* *and* n_2^* *are such that* $\frac{n_1^* + \sqrt[2]{n_1^{*2} + 4n_2^*}}{2}$ *is equal to the optimal cohort growth rate* g^**;*
- *the optimal cohort growth rate* g^* *is characterized as follows:*

  - if $\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}} > 0$ *at the optimum, we have:*

  $$g^* = \sqrt[3]{\frac{\frac{2b^*}{k^*} + \sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}}}{2}} + \sqrt[3]{\frac{\frac{2b^*}{k^*} - \sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}}}{2}}$$

  - if $\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}} = 0$ *at the optimum, we have:*

  $$g^* = \sqrt[3]{\frac{b^*}{k^*} - \sqrt[2]{\frac{b^{*2}}{k^{*2}} + \frac{\Phi^3}{27k^{*3}}}} - \frac{\Phi}{3k^* \sqrt[3]{\frac{b^*}{k^*} - \sqrt[2]{\frac{b^{*2}}{k^{*2}} + \frac{\Phi^3}{27k^{*3}}}}}$$

  - if $\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}} < 0$ *at the optimum, we have:*

  $$g^* = \sqrt[3]{\frac{-\frac{2b^*}{k^*} - \sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}} + \frac{2b^* i \sqrt[2]{3}}{k^*} + i\sqrt[2]{3}\sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}}}{4}}$$
  $$+ \sqrt[3]{\frac{-\frac{2b^*}{k^*} + \sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}} - \frac{2b^* i \sqrt[2]{3}}{k^*} + i\sqrt[2]{3}\sqrt[2]{\frac{4b^{*2}}{k^{*2}} + \frac{4\Phi^3}{27k^{*3}}}}{4}}$$

  where $\Phi \equiv (F_L(k^*, \cdot) - d^*)$*.*

Proof. See the Appendix. ■

The social optimum depends on age-specific fertility rates n_1^* and n_2^* *only insofar as* these must yield the optimal cohort growth rate g^*. However, as long as n_1^* and n_2^* are such that $(n_1^* + \sqrt[2]{n_1^{*2} + 4n_2^*})/2$ is equal to the optimal cohort growth rate g^*, the precise levels of n_1^* and n_2^* do not matter. It follows from this that there is not one, but *several* social optima $(c^*, d^*, b^*, k^*, n_1^*, n_2^*)$, since various pairs (n_1^*, n_2^*) can yield the optimal cohort growth rate g^*.

To illustrate this, suppose that the optimal cohort growth rate g^* equals 1 (i.e. the replacement rate). Then, the social optimum involves $\frac{u'(c^*)}{\beta u'(d^*)} = \frac{u'(d^*)}{\beta u'(b^*)} = 1$ and k^* such that $F_k(k^*, \cdot) = 1$. That social optimum only requires that n_1^* and n_2^* satisfy the condition: $(n_1^* + \sqrt[2]{n_1^{*2} + 4n_2^*})/2 = 1$, which can be true for various pairs (n_1^*, n_2^*).[19] Thus Proposition 9 states that, as long as the cohort growth rate takes its optimal level, the precise levels of age-specific fertility rates do not matter.[20]

When g^* equals 1, the formula $(n_1 + \sqrt[2]{n_1^2 + 4n_2})/2 = 1$ can be simplified to $n_1 + n_2 = 1$, so that the total fertility rate $n_1 + n_2$ is relevant from the point of view of social optimality. However, in all other cases where the optimal cohort growth rate g^* differs from 1, the total fertility rate $n_1 + n_2$ is irrelevant, since a pair (n_1, n_2) of age-specific fertility rates is optimal only insofar as these yield the optimal cohort growth rate g^*, which differs from the mere sum $n_1 + n_2$. Hence, here again, an exclusive emphasis on the total fertility rate that ignores the timing of births may be quite misleading. The timing of births definitely matters for the definition of the social optimum.

3.3 The Serendipity Theorem

To investigate whether the Serendipity Theorem still holds here, we will first consider the problem faced by an agent living at the steady-state, who chooses second- and third-period savings, in such a way as to maximize his lifetime welfare. He takes all prices and the age-specific fertility rates as given.

Assuming a perfectly competitive economy, we have:

$$\begin{aligned} w &= \left[F\left(k\left(, 1+\frac{1}{g}\right)\right) - F_k\left(k, 1+\frac{1}{g}\right)\right]\frac{g}{1+g} \\ R &= F_k\left(k, 1+\frac{1}{g}\right) \end{aligned}$$

where g denotes the steady-state cohort growth rate.

[19] We could thus have a total fertility equally spread on the reproduction period (i.e. $n_1^* = 0.5$ and $n_2^* = 0.5$), but, also, a higher early fertility and a lower late fertility (i.e. $n_1^* = 0.9$ and $n_2^* = 0.1$), or the opposite (i.e. $n_1^* = 0.1$ and $n_2^* = 0.9$).

[20] Note, however, that, although there exist various pairs (n_1^*, n_2^*) leading to the optimum cohort growth rate, it remains that the strength of the constraint $(n_1^* + \sqrt[2]{n_1^{*2} + 4n_2^*})/2 = g^*$ should not be underestimated, in particular when the optimum cohort growth rate g^* differs strongly from the replacement rate.

The problem of the agent can be written as:

$$\max_{c,d,b} u(c) + \beta u(d) + \beta^2 u(b)$$

$$s.t. w + \frac{w}{R} = c + \frac{d}{R} + \frac{b}{R^2}$$

The FOCs are

$$\frac{u'(c)}{\beta u'(d)} = \frac{u'(d)}{\beta u'(b)} = R$$

Hence, if the social planner fixes the fertility rates n_1 and n_2 at levels such that $F_k(k^*, \cdot) = (n_1 + \sqrt[2]{n_1^2 + 4n_2})/2 = g^*$ where k^* takes its socially optimal level, then individuals, being price-takers, will choose their savings optimally, since the above FOC will then coincide with the FOC for optimal intergenerational allocations of resources. Thus the Serendipity Theorem holds in that economy. Provided the social planner imposes the optimal cohort growth rate, the perfectly competitive economy will converge towards the social optimum.

Proposition 10 *Assume that there exists a unique stable stationary equilibrium. Then the perfectly competitive economy will converge towards the (stationary) social optimum provided the optimal cohort growth rate g^* is imposed. This amounts to impose age-specific fertility rates n_1 and n_2 such that $(n_1 + \sqrt[2]{n_1^2 + 4n_2})/2$ equals to optimal cohort growth rate g^*.*

Proof. The proof follows from comparing the FOCs of the agent's problem and of the social planner's problem. ■

Proposition 10 extends Samuelson's Serendipity Theorem to the case of a period of reproduction longer than one period. Actually, the "happy coincidence" result derived by Samuelson is robust to the introduction of different ages of motherhood. Provided the social planner can impose optimal fertility, all other variables will, in a perfectly competitive economy, take their optimal values at the steady-state. But Proposition 10 goes also beyond that result: having not one, but *two* possible periods of reproduction gives now *two* instruments - instead of one - to the social planner. The social planner has here an additional degree of freedom, since only the cohort growth rate matters for the decentralization of the social optimum, and age-specific fertility rates matter only insofar as these yield the optimal cohort growth.

Note also that, here again, the relevant fertility concept is not the *total* fertility rate $n_1 + n_2$, but the cohort growth rate. Hence given that what matters for the Serendipity result is to impose the cohort growth rate - whatever age-specific fertility rates n_1 and n_2 are - the happy coincidence result can be obtained under *various* total fertility rates, as long as these are compatible with the optimal cohort growth rate g^*.

4 Conclusions

Whereas the study of the optimal total fertility rate has received a lot of attention since Samuelson's (1975) pioneer work, the characterization of the optimal timing of births along the lifecycle has remained so far largely neglected. The goal of the present paper was to explore that issue. For that purpose, we focused on a four-period OLG economy with physical capital accumulation. We explored the long-run dynamics of economies with different age-patterns for fertility, and their consequences for the social optimum. We proceeded in two stages. We first looked at a fable with two islands - Early and Late Islands - which differ only in the timing of births (young adulthood and older adulthood).

We showed that, unlike on Early Island, there exists no stable stationary equilibrium on Late Island, which exhibits cyclical dynamics. Hence, even for a *given* total fertility rate ($n_1 = n_2$), the dynamics of the economy differs significantly, depending on the timing of births. Thus the timing of births is not a detail for long-run dynamics, but matters as much as the total number of births. We also characterized the social optimum on each economy, and showed that, on each island, Samuelson's Serendipity Theorem remains valid.

In a second stage, we considered a more general economy with interior fertility rates during the reproduction period. We showed that there exists, under mild conditions, a locally stable stationary equilibrium in that economy. We also characterized the associated social optimum, and showed that various age-specific fertility rates are compatible with the social optimum, as long as these yield the optimal cohort growth rate. We also showed that Samuelson's Serendipity Theorem remains valid in that broader demographic environment, but with more degrees of freedom for the social planner: as long as the age-specific fertility rates yield the optimal cohort growth rate, the perfectly competitive economy will converge towards the social optimum, whatever the precise pair of age-specific fertility rates is.

In sum, this paper highlights that, besides the total *number* of births, the *timing* of births is a major characteristic of an economy. Considering the total fertility rate is not sufficient for the study of the long-run dynamics, since the timing of births determines the form - stationary or cyclical - of long-run dynamics. Moreover, when characterizing the social optimum, the central variable is not the total fertility rate, but the cohort growth rate. Hence focusing only on the total fertility rate while ignoring the timing of births is also most damageable from a normative perspective.

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6 Appendix

6.1 Proposition 1

Existence of a stationary equilibrium The dynamics of Early Island economy can be summarized by the following two-dimensional dynamic system:

$$\begin{aligned} k_{t+1} &\equiv G(k_t, \Omega_t) = \frac{\sigma(k_t)}{n_1} + \Omega_t \\ \Omega_{t+1} &\equiv H(k_t) = \frac{\zeta(k_t)}{(n_1)^2} \end{aligned}$$

The existence of a stationary equilibrium amounts to find a pair (k^*, Ω^*) such that $k^* = G(k^*, \Omega^*)$ and $\Omega^* = H(k^*)$. Note that, provided $\zeta(0) = 0$ and $\zeta'(k_t) > 0$ for any $k_t > 0$, the second expression coincides with an increasing curve on the (k_t, Ω_t) space. Regarding the first expression, we know that, if a steady-state exists, it must be the case that:

$$\Omega_t = k_t - \frac{\sigma(k_t)}{n_1}$$

That expression describes the sustainable level of capital for a given Ω_t. The derivative of that expression with respect to k_t is:

$$\Omega'(k_t) = 1 - \frac{\sigma'(k_t)}{n_1} < 1$$

If one imposes that:

$$lim_{k \to 0}\, 1 - \frac{\sigma'(k_t)}{n_1} < lim_{k \to 0} \frac{\zeta'(k_t)}{(n_1)^2}$$

we then know, given $\sigma(0) = 0$ and $\sigma'(k_t) > 0$ and $\zeta(0) = 0$ and $\zeta'(k_t) > 0$ for $k_t > 0$, that the $G(k_t, \Omega_t)$ curve lies below the $H(k_t)$ curve in the neighbourhood of 0. Furthermore, if one assumes that:

$$lim_{k \to +\infty}\, 1 - \frac{\sigma'(k_t)}{n_1} > lim_{k \to +\infty} \frac{\zeta'(k_t)}{(n_1)^2}$$

then the $G(k_t, \Omega_t)$ curve lies above the $H(k_t)$ curve at high capital levels.

Thus, given that the $G(k_t, \Omega_t)$ curve lies below the $H(k_t)$ curve in the neighbourhood of 0, but lies above it for high capital levels, it must be the case, by continuity of $G(k_t, \Omega_t)$ and $H(k_t)$, that these intersect for some pair (k^*, Ω^*).[21]

Stability of a stationary equilibrium To study the stability of (k^*, Ω^*), let us look at the properties of the Jacobian matrix. The Jacobian matrix is:

$$J \equiv \begin{pmatrix} \frac{\partial G(k_t, \Omega_t)}{\partial k_t} & \frac{\partial G(k_t, \Omega_t)}{\partial \Omega_t} \\ \frac{\partial H(k_t)}{\partial k_t} & \frac{\partial H(k_t)}{\partial \Omega_t} \end{pmatrix}$$

[21] Note that the above existence proof relies on the limits when k_t takes extreme values. Hence, it says nothing on the number of intersections of the two curves $G(k_t, \Omega_t)$ and $H(k_t)$. Thus uniqueness is not guaranteed. Actually, given that $\sigma(0) = 0$ and $\zeta(0) = 0$, it is also the case that $(0, 0)$ is a stationary equilibrium. But even beyond that trivial steady-state, other intersections of the two curves $G(k_t, \Omega_t)$ and $H(k_t)$ may exist.

When computing the entries of that matrix, we have:

$$\begin{aligned}
\frac{\partial G(k_t,\Omega_t)}{\partial k_t} &= \frac{\sigma'(k_t)}{n_1} > 0 \\
\frac{\partial G(k_t,\Omega_t)}{\partial \Omega_t} &= 1 \\
\frac{\partial H(k_t)}{\partial k_t} &= \frac{\zeta'(k_t)}{(n_1)^2} > 0 \\
\frac{\partial H(k_t)}{\partial \Omega_t} &= 0
\end{aligned}$$

Thus the determinant and the trace of the Jacobian matrix are:

$$det(J) = -\frac{\zeta'(k_t)}{(n_1)^2} < 0;\ tr(J) = \frac{\sigma'(k_t)}{n_1} > 0$$

The characteristic equation associated to dynamic system is:

$$\lambda^2 - tr(J)\lambda + det(J) = 0$$

Note that $\Delta \equiv (tr(J))^2 - 4det(J)$ is here

$$\left(\frac{\sigma'(k_t)}{n_1}\right)^2 - 4\left(-\frac{\zeta'(k_t)}{(n_1)^2}\right) > 0$$

Thus the eigenvalues are:

$$\lambda_{1,2} = \frac{tr(J) \pm \sqrt[2]{(tr(J))^2 - 4det(J)}}{2}$$

Hence it is straightforward to deduce that the eigenvalues are:

$$\begin{aligned}
\lambda_1 &= \frac{\frac{\sigma'(k_t)}{n_1} + \sqrt[2]{\left(\frac{\sigma'(k_t)}{n_1}\right)^2 + 4\frac{\zeta'(k_t)}{(n_1)^2}}}{2} > 0 \\
\lambda_2 &= \frac{\frac{\sigma'(k_t)}{n_1} - \sqrt[2]{\left(\frac{\sigma'(k_t)}{n_1}\right)^2 + 4\frac{\zeta'(k_t)}{(n_1)^2}}}{2} < 0
\end{aligned}$$

We are thus in a case where $\Delta > 0$, and where the two eigenvalues are of opposite signs. Note that the hyperbolicity of the stationary equilibrium requires that all eigenvalues are smaller than 1 in modulo: $|\lambda_1| < 1$, $|\lambda_2| < 1$. Otherwise, the equilibrium is not hyperbolic, with the consequence that the local stability of the linearized system does not inform us about the local stability of the actual, non-linear, system.

The condition for $|\lambda_1| < 1$ is:

$$\sqrt[2]{\left(\frac{\sigma'(k_t)}{n_1}\right)^2 + 4\frac{\zeta'(k_t)}{(n_1)^2}} < 2 - \frac{\sigma'(k_t)}{n_1}$$

The above condition can be rewritten as:

$$\frac{\zeta'(k_t)}{(n_1)^2} < 1 - \frac{\sigma'(k_t)}{n_1}$$

We know that, at the stationary equilibrium, the $G(k_t, \Omega_t)$ curve intersects the $H(k_t)$ curve from below, which means that:

$$1 - \frac{\sigma'(k^*)}{n_1} > \frac{\zeta'(k^*)}{(n_1)^2}$$

Hence the condition for $|\lambda_1| < 1$ is satisfied at the stationary equilibrium.
Regarding the condition for $|\lambda_2| < 1$, this can be rewritten as:

$$\frac{\zeta'(k_t)}{(n_1)^2} < 1 + \frac{\sigma'(k_t)}{n_1}$$

Here again, given that, at the equilibrium, the $G(k_t, \Omega_t)$ curve intersects the $H(k_t)$ curve from below, we have that:

$$1 - \frac{\sigma'(k^*)}{n_1} > \frac{\zeta'(k^*)}{(n_1)^2}$$

Hence the condition for $|\lambda_2| < 1$ is also satisfied at our equilibrium. Note that, while stability is guaranteed, the convergence towards the stationary equilibrium takes a non-monotonic form, due to the opposite signs of the eigenvalues.

6.2 Proposition 2

Existence of a stationary equilibrium The dynamics of the Late Island economy can be summarized by the following three-dimensional system:

$$\begin{aligned} k_{t+1} &\equiv G(k_t, \Omega_t, g_t) = \frac{\sigma(k_t)}{g_t} + \Omega_t \\ \Omega_{t+1} &\equiv H(k_t) = \frac{\zeta(k_t)}{n_2} \\ g_{t+1} &\equiv I(g_t) = \frac{n_2}{g_t} \end{aligned}$$

Imposing $k_{t+1} = k_t$ in the former equation, and isolating Ω_t defines the kk locus, along which k_t is constant:

$$\Omega_t = k_t - \frac{\sigma(k_t)}{g_t}$$

Imposing $\Omega_{t+1} = \Omega_t$ in the second equation defines the $\Omega\Omega$ locus, along which Ω_t is constant:

$$\Omega_t = \frac{\zeta(k_t)}{n_2}$$

Imposing $g_{t+1} = g_t$ in the third equation, and isolating g_t defines the gg locus, along which the cohort growth rate is constant:

$$g = \sqrt[2]{n_2}$$

It is not obvious to study whether the three loci intersect. But the analysis is straightforward for the case where $N_0 = N_1 > 0$ and $n_2 = 1$. The reason is that, in that case, the cohort growth factor g_t is:

$$g_t = \frac{N_t}{N_{t-1}} = \frac{n_2 N_{t-2}}{N_{t-1}} = \frac{N_{t-2}}{N_{t-1}} = \frac{N_t}{N_{t+1}} = \frac{1}{g_{t+1}}$$

Hence, under initial conditions $N_0 = N_1 > 0$, it must be the case that the cohort growth factor is constant and equal to 1: $g_{t+1} = g_t = 1$. Hence, in that case, the dynamic system becomes two-dimensional, as on Early Island:

$$\begin{aligned} k_{t+1} &\equiv \bar{G}(k_t, \Omega_t) = \sigma(k_t) + \Omega_t \\ \Omega_{t+1} &\equiv H(k_t) = \frac{\zeta(k_t)}{n_2} \end{aligned}$$

where $\bar{G}(k_t, \Omega_t) \equiv G(k_t, \Omega_t, 1)$.

Hence, provided $\sigma(0) = 0$, $\sigma'(k_t) > 0$, as well as $\zeta(0) = 0$, $\zeta'(k_t) > 0$, and provided $lim_{k \to 0}\ 1 - \sigma'(k_t) < lim_{k \to 0} \frac{\zeta'(k_t)}{n_2}$ and $lim_{k \to +\infty}\ 1 - \sigma'(k_t) > lim_{k \to +\infty} \frac{\zeta'(k_t)}{n_2}$, the $\bar{G}(k_t, \Omega_t)$ curve lies below the $H(k_t)$ curve in the neighbourhood of 0, but lies above it for high capital levels, so that it must be the case, by continuity of $\bar{G}(k_t, \Omega_t)$ and $H(k_t)$, that these intersect for some pair (k^*, Ω^*). Hence, under $N_{-1} = N_0 > 0$, there exists a stationary equilibrium on Late Island. Moreover, given that $\bar{G}(k_t, \Omega_t)$ curve crosses the $H(k_t)$ from below, that stationary equilibrium is locally stable (the proof is similar to the proof of stability for the steady-state on Early Island).

However, if we abstract from that case, the dynamics of Late Island remains three-dimensional. We have that the gg locus is a horizontal plan at $g_t = \sqrt[2]{n_2}$ in the (k_t, Ω_t, g_t) space.

Let us consider the conditions under which a stationary equilibrium exists. That problem can be formulated as whether the kk locus and the $\Omega\Omega$ locus intersect at some point on the gg locus. On the gg locus, the kk locus can be rewritten as: $\Omega_t = k_t - \frac{\sigma(k_t)}{\sqrt[2]{n_2}}$. Moreover, the $\Omega\Omega$ locus is such that: $\Omega_t = \frac{\zeta(k_t)}{n_2}$. Hence, provided $lim_{k \to 0}\ 1 - \frac{\sigma'(k_t)}{\sqrt[2]{n_2}} < lim_{k \to 0} \frac{\zeta'(k_t)}{n_2}$ and $lim_{k \to +\infty}\ 1 - \frac{\sigma'(k_t)}{\sqrt[2]{n_2}} > lim_{k \to +\infty} \frac{\zeta'(k_t)}{n_2}$, the kk locus lies below the $\Omega\Omega$ locus for low capital levels, but lies above it for high capital level. Hence, by continuity, the two loci must intersect. That intersection is a stationary equilibrium (k^*, Ω^*, g^*), which can be rewritten as: $\left(k^*, \frac{\zeta(k^*)}{n_2}, \sqrt[2]{n_2}\right)$.

Stability of the stationary equilibrium Let us now show that this stationary equilibrium is unstable. The Jacobian matrix is:

$$J \equiv \begin{pmatrix} \frac{\partial G(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial G(k_t,\Omega_t,g_t)}{\partial \Omega_t} & \frac{\partial G(k_t,\Omega_t,g_t)}{\partial g_t} \\ \frac{\partial H(k_t)}{\partial k_t} & \frac{\partial H(k_t)}{\partial \Omega_t} & \frac{\partial H(k_t)}{\partial g_t} \\ \frac{\partial I(g_t)}{\partial k_t} & \frac{\partial I(g_t)}{\partial \Omega_t} & \frac{\partial I(g_t)}{\partial g_t} \end{pmatrix}$$

When computing the entries of that matrix, we have:

$$\begin{aligned} \frac{\partial G(k_t,\Omega_t,g_t)}{\partial k_t} &= \frac{\sigma'(k_t)}{g_t} > 0 \\ \frac{\partial G(k_t,\Omega_t,g_t)}{\partial \Omega_t} &= 1 \\ \frac{\partial G(k_t,\Omega_t,g_t)}{\partial g_t} &= \frac{-\sigma(k_t)}{(g_t)^2} < 0 \\ \frac{\partial H(k_t)}{\partial k_t} &= \frac{\zeta'(k_t)}{n_2} > 0 \\ \frac{\partial H(k_t)}{\partial \Omega_t} &= \frac{\partial H(k_t)}{\partial g_t} = \frac{\partial I(g_t)}{\partial k_t} = \frac{\partial I(g_t)}{\partial \Omega_t} = 0 \\ \frac{\partial I(g_t)}{\partial g_t} &= \frac{-n_2}{(g_t)^2} < 0 \end{aligned}$$

Thus the determinant and trace of the Jacobian matrix are:

$$det(J) = \frac{\zeta'(k_t)}{n_2}\frac{n_2}{(g_t)^2} = \frac{\zeta'(k_t)}{(g_t)^2} > 0; tr(J) = \frac{\sigma'(k_t)}{g_t} - \frac{n_2}{(g_t)^2} \gtrless 0$$

At the equilibrium, we have $g_t = \sqrt[2]{n_2}$, so that:

$$det(J) = \frac{\zeta'(k_t)}{n_2};\ tr(J) = \frac{\sigma'(k_t)}{\sqrt[2]{n_2}} - 1$$

Following Brooks's (2004) study of stability of first-order three-dimensional dynamic systems, we know that all eigenvalues of a 3x3 Jacobian matrix are lower than 1 in modulo (implying stability) provided the following three conditions are satisfied:

(i) $|\det(J)| < 1$

(ii) $1 > [\sum M_i(J)] - [tr(J)][\det(J)] + [\det(J)]^2$

(iii) $-[\sum M_i(J) + 1] < tr(J) + \det(J) < [\sum M_i(J) + 1]$

where $\det(J)$, $tr(J)$ and $\sum M_i(J)$ denote respectively the determinant, the trace and the sum of the principal minors of the Jacobian matrix. Those conditions are necessary and sufficient for the stability of the linearized dynamic system.

Condition (iii) can be written here as:

$$-1 + \frac{\zeta'(k_t)}{n_2} + \frac{\sigma'(k_t)}{\sqrt[2]{n_2}} < \frac{\sigma'(k_t)}{\sqrt[2]{n_2}} - 1 + \frac{\zeta'(k_t)}{n_2} < -\frac{\zeta'(k_t)}{n_2} - \frac{\sigma'(k_t)}{\sqrt[2]{n_2}} + 1$$

That condition is not satisfied here, as the $-\left[\sum M_i(J)+1\right]=tr(J)+\det(J)$. As a consequence, the stationary equilibrium on Late Island is not stable.

6.3 Proposition 3

Existence of cycles on Late Island To study the conditions under which a stable cycle arises, let us rewrite the variables as a function of their lagged past values. This gives us the following three-dimensional dynamic system:

$$\begin{aligned}
k_{t+2} &\equiv G(k_{t+1},\Omega_{t+1},g_{t+1}) = g_t\frac{\sigma\left(\frac{\sigma(k_t)}{g_t}+\Omega_t\right)}{n_2}+\frac{\zeta(k_t)}{n_2}\equiv\Gamma\left(k_t,\Omega_t,g_t\right)\\
\Omega_{t+2} &\equiv H(k_{t+1}) = \frac{\zeta(\frac{\sigma(k_t)}{g_t}+\Omega_t)}{n_2}\equiv\Theta\left(k_t,\Omega_t,g_t\right)\\
g_{t+2} &\equiv I(g_{t+1}) = \frac{n_2}{g_{t+1}} = g_t\equiv\Lambda\left(g_t\right)
\end{aligned}$$

Given that $g_{t+2}=g_t$ for all t, any level of g_t can constitute a stationary cohort growth rate for the two-lagged dynamic system. As such, the equilibrium is degenerate, as there is a continuum of equilibrium values for g_t. However, if we focus on the case where initial conditions are $N_{-1}=N_0>0$, so that $g_0=1$, we have that g_t can only take two values: 1 or n_2. As a consequence, the gg locus takes, in that case, the form of two horizontal planes in the (k_t,Ω_t,g_t) space, at $g_t=1$ and $g_t=n_2$.

The kk locus consists of all combinations (k_t,Ω_t) such that:

$$k_t=g_t\frac{\sigma\left(\frac{\sigma(k_t)}{g_t}+\Omega_t\right)}{n_2}+\frac{\zeta(k_t)}{n_2}$$

The $\Omega\Omega$ locus consists of all combinations (k_t,Ω_t) such that:

$$\Omega_t=\frac{\zeta(\frac{\sigma(k_t)}{g_t}+\Omega_t)}{n_2}$$

At a stationary equilibrium, it must be the case that those two loci intersect. Moreover, we know that the equilibrium cohort growth rate is either $g_t=1$ or $g_t=n_2$. Hence, at $g_t=1$, the two loci become:

$$\begin{aligned}
k_t &= \frac{\sigma\left(\sigma\left(k_t\right)+\Omega_t\right)}{n_2}+\frac{\zeta(k_t)}{n_2}\\
\Omega_t &= \frac{\zeta(\sigma\left(k_t\right)+\Omega_t)}{n_2}
\end{aligned}$$

On the basis of the first expression, we can isolate Ω_t and get the expression: $\Omega_t=\sigma^{-1}\left(n_2k_t-\zeta(k_t)\right)-\sigma\left(k_t\right)$. We know that, as k_t tends towards 0, we have that Ω_t tends also towards 0, since $\sigma\left(0\right)=0$ and $\sigma'\left(\cdot\right)>0$. Hence the kk locus goes through $(0,0)$ in the (k_t,Ω_t) space.

On the basis of the second expression, we know that, as k_t tends towards 0, Ω_t satisfies the expression $\Omega_t = \frac{\zeta(\sigma(0)+\Omega_t)}{n_2} = \frac{\zeta(\Omega_t)}{n_2}$. Assuming that $\zeta(0) = 0$ and $\zeta'(\cdot) > 0$, we know that, provided $lim_{\Omega \to 0} \zeta'(\Omega_t) > n_2$ and $lim_{\Omega \to +\infty} \zeta'(\Omega_t) < n_2$, there exists a solution Ω^+ to the equation $\Omega_t = \frac{\zeta(\Omega_t)}{n_2}$. Moreover, still given $\zeta'(\cdot) > 0$, we know that the solution Ω^{++} to $\Omega_t = \frac{\zeta(\sigma(k_t)+\Omega_t)}{n_2}$ is increasing with k_t. Indeed, once $k_t > 0$, it is impossible to have $\Omega^{++} = 0$, since $\frac{\zeta(\sigma(k_t)+0)}{n_2} > 0$ when $k_t > 0$. Thus Ω^{++} is increasing in k_t, so that the $\Omega\Omega$ locus is increasing in the (k_t, Ω_t) space.

If one denotes by $\bar{\Omega}$ the solution to $\Omega_t = \frac{\zeta(\sigma(k_t)+\Omega_t)}{n_2}$ when $k_t \to +\infty$, the condition

$$lim_{k_t \to +\infty} \sigma^{-1} \left(n_2 k_t - \zeta(k_t) \right) - \sigma \left(k_t \right) > \bar{\Omega}$$

insures that the kk locus lies above the $\Omega\Omega$ locus for high capital levels.

Hence, given that the kk locus lies strictly below the $\Omega\Omega$ locus for low levels of capital, but lies, under the above condition, strictly above the $\Omega\Omega$ locus, we know that the two loci must intersect. Hence, there must exist an intersection $\left(\hat{\Omega}, \hat{k}\right)$. Thus, taking the associated cohort growth rate $g = 1$, we have the equilibrium $\left(\hat{k}, \hat{\Omega}, 1\right)$.

Moreover, at the equilibrium cohort growth rate $g = n_2$, the two loci become:

$$\begin{aligned} k_t &= \sigma \left(\frac{\sigma(k_t)}{n_2} + \Omega_t \right) + \frac{\zeta(k_t)}{n_2} \\ \Omega_t &= \frac{\zeta(\frac{\sigma(k_t)}{n_2} + \Omega_t)}{n_2} \end{aligned}$$

On the basis of the first expression, we can isolate Ω_t and get the expression: $\Omega_t = \sigma^{-1} \left(k_t - \frac{\zeta(k_t)}{n_2} \right) - \frac{\sigma(k_t)}{n_2}$. We know that, as k_t tends towards 0, we have that Ω_t tends also towards 0, since $\sigma(0) = 0$ and $\sigma'(\cdot) > 0$. Hence the kk locus has $(0, 0)$ as a starting point in the (k_t, Ω_t) space.

On the basis of the second expression, we know that, as k_t tends towards 0, Ω_t satisfies the expression $\Omega_t = \frac{\zeta(\frac{\sigma(0)}{n_2}+\Omega_t)}{n_2} = \frac{\zeta(\Omega_t)}{n_2}$. Assuming that $\zeta(0) = 0$ and $\zeta'(\cdot) > 0$, we know that, provided $lim_{\Omega \to 0} \zeta'(\Omega_t) > n_2$ and $lim_{\Omega \to +\infty} \zeta'(\Omega_t) < n_2$, there exists a solution Ω^+ to the equation $\Omega_t = \frac{\zeta(\Omega_t)}{n_2}$. Moreover, still given $\zeta'(\cdot) > 0$, we know that the solution Ω^{++} to $\Omega_t = \frac{\zeta(\frac{\sigma(k_t)}{n_2}+\Omega_t)}{n_2}$ is increasing with k_t. Indeed, once $k_t > 0$, it is impossible to have $\Omega^{++} = 0$, since $\frac{\zeta(\frac{\sigma(k_t)}{n_2}+0)}{n_2} > 0$ when $k_t > 0$. Thus Ω^{++} is increasing in k_t, so that the $\Omega\Omega$ locus is increasing in the (k_t, Ω_t) space.

If one denotes by $\tilde{\Omega}$ the solution to $\Omega_t = \frac{\zeta(\frac{\sigma(k_t)}{n_2}+\Omega_t)}{n_2}$ when $k_t \to +\infty$, the condition

$$lim_{k_t \to +\infty} \sigma^{-1} \left(k_t - \frac{\zeta(k_t)}{n_2} \right) - \frac{\sigma(k_t)}{n_2} > \tilde{\Omega}$$

insures that the kk locus lies above the $\Omega\Omega$ locus for high capital levels.

Hence, given that the kk locus lies strictly below the $\Omega\Omega$ locus for low levels of capital, but lies, under the above condition, strictly above the $\Omega\Omega$ locus, we know that the two loci must intersect. Hence, there exists an intersection of the two loci at a point $\left(\check{\Omega}, \check{k}\right)$. This gives us another equilibrium, i.e. $\left(\check{k}, \check{\Omega}, n_2\right)$.

It follows from all this that the dynamic system with one additional lag admits two equilibria $\left(\hat{k}, \hat{\Omega}, 1\right)$ and $\left(\check{k}, \check{\Omega}, n_2\right)$.

Stability of the cycle Let us now consider whether the two equilibria $\left(\hat{k}, \hat{\Omega}, 1\right)$ and $\left(\check{k}, \check{\Omega}, n_2\right)$ are stable. The Jacobian matrix is:

$$J \equiv \begin{pmatrix} \frac{\partial \Gamma(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial \Gamma(k_t,\Omega_t,g_t)}{\partial \Omega_t} & \frac{\partial \Gamma(k_t,\Omega_t,g_t)}{\partial g_t} \\ \frac{\partial \Theta(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial \Theta(k_t,\Omega_t,g_t)}{\partial \Omega_t} & \frac{\partial \Theta(k_t,\Omega_t,g_t)}{\partial g_t} \\ \frac{\partial \Lambda(g_t)}{\partial k_t} & \frac{\partial \Lambda(g_t)}{\partial \Omega_t} & \frac{\partial \Lambda(g_t)}{\partial g_t} \end{pmatrix}$$

When computing the entries of that matrix, we have:

$$\begin{aligned}
\frac{\partial \Gamma(k_t,\Omega_t,g_t)}{\partial k_t} &= g_t \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)}{n_2} \frac{\sigma'(k_t)}{g_t} + \frac{\zeta'(k_t)}{n_2} > 0 \\
\frac{\partial \Gamma(k_t,\Omega_t,g_t)}{\partial \Omega_t} &= g_t \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)}{n_2} > 0 \\
\frac{\partial \Gamma(k_t,\Omega_t,g_t)}{\partial g_t} &= \frac{\sigma\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)}{n_2} + g_t \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)}{n_2} \frac{-\sigma(k_t)}{(g_t)^2} \gtrless 0 \\
\frac{\partial \Theta(k_t,\Omega_t,g_t)}{\partial k_t} &= \frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} \frac{\sigma'(k_t)}{g_t} > 0 \\
\frac{\partial \Theta(k_t,\Omega_t,g_t)}{\partial \Omega_t} &= \frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} > 0 \\
\frac{\partial \Theta(k_t,\Omega_t,g_t)}{\partial g_t} &= \frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} \frac{-\sigma(k_t)}{(g_t)^2} < 0 \\
\frac{\partial \Lambda(g_t)}{\partial k_t} &= \frac{\partial \Lambda(g_t)}{\partial \Omega_t} = 0 \\
\frac{\partial \Lambda(g_t)}{\partial g_t} &= 1
\end{aligned}$$

Thus the determinant of the Jacobian matrix is:

$$det(J) = \frac{\zeta'(k_t)}{n_2} \frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} > 0$$

while the trace of the Jacobian matrix is:

$$tr(J) = \left(g_t \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)}{n_2} \frac{\sigma'(k_t)}{g_t} + \frac{\zeta'(k_t)}{n_2} \right) + \frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} + 1$$

We now derive Brooks's (2004) conditions (i), (ii) and (iii), which are necessary and sufficient for the stability of the linearized dynamic system.

Condition (i) is satisfied only if $\frac{\zeta'(\frac{\sigma(k_t)}{g_t}+\Omega_t)\zeta'(k_t)}{(n_2)^2} < 1$.

Regarding condition (ii), note that $\sum M_i(J)$ is here

$$\frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} + \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)\sigma'(k_t)}{n_2} + \frac{\zeta'(k_t)}{n_2} + \frac{\zeta'(k_t)}{n_2}\frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2}$$

Hence condition (ii) requires:

$$\begin{aligned} 1 \quad > \quad & \frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2}\left(1 - \left(\frac{\zeta'(k_t)}{n_2}\right)^2\right) + \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)\sigma'(k_t)}{n_2} + \frac{\zeta'(k_t)}{n_2} \\ & - \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)\sigma'(k_t)\zeta'(k_t)\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{(n_2)^3} \\ & + \left[\frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2}\right]^2 \frac{\zeta'(k_t)}{n_2}\left[\frac{\zeta'(k_t)}{n_2} - 1\right] \end{aligned}$$

Condition (iii) requires:

$$\begin{aligned} & -\left[\frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} + \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)\sigma'(k_t)}{n_2} + \frac{\zeta'(k_t)}{n_2} + \frac{\zeta'(k_t)}{n_2}\frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} + 1\right] \\ < \quad & \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)\sigma'(k_t)}{n_2} + \frac{\zeta'(k_t)}{n_2} + \frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} + 1 + \frac{\zeta'(k_t)}{n_2}\frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} \\ < \quad & \left[\frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} + \frac{\sigma'\left(\frac{\sigma(k_t)}{g_t} + \Omega_t\right)\sigma'(k_t)}{n_2} + \frac{\zeta'(k_t)}{n_2} + \frac{\zeta'(k_t)}{n_2}\frac{\zeta'(\frac{\sigma(k_t)}{g_t} + \Omega_t)}{n_2} + 1\right] \end{aligned}$$

The first inequality is satisfied, but the second inequality is not, since here $tr(J) + \det(J) = [\sum M_i(J) + 1]$.

As a consequence of this, the two equilibria $\left(\hat{k}, \hat{\Omega}, 1\right)$ and $\left(\check{k}, \check{\Omega}, n_2\right)$ are not stable. Thus nothing insures, from any initial conditions (k_0, Ω_0, g_0), the convergence towards that two-period cycle.

That negative result motivates us to restrict the set of initial conditions, to explore whether stability could arise conditionally on *some* initial conditions. To do this, let us now assume that $N_{-1} = N_0 > 0$, so that $g_0 = 1$. Under that assumption, the initial cohort growth rate is already at an equilibrium value, and g_t will only exhibit two different values over time: either 1 or n_2. Hence we can use that information to derive some *conditional* stability results.

Let us now investigate whether, under initial conditions $(k_0, \Omega_0, 1)$, the convergence towards the equilibrium cycle $\left(\left(\hat{k}, \hat{\Omega}, 1\right), \left(\check{k}, \check{\Omega}, n_2\right)\right)$ arises. For that purpose, we will rely on geometric analysis and discuss the direction of dynamic arrows around the equilibrium cycle. Remind that the kk locus and $\Omega\Omega$ locus can be written as:

$$\begin{aligned} k_t &= \frac{\sigma\left(\sigma\left(k_t\right) + \Omega_t\right)}{n_2} + \frac{\zeta(k_t)}{n_2} \iff \frac{\sigma\left(\sigma\left(k_t\right) + \Omega_t\right)}{n_2} = k_t - \frac{\zeta(k_t)}{n_2} \\ \Omega_t &= \frac{\zeta(\sigma\left(k_t\right) + \Omega_t)}{n_2} \end{aligned}$$

For any k_t that lies below the level corresponding to the kk locus, we know, for a given Ω_t, that $k_t < \frac{\sigma(\sigma(k_t)+\Omega_t)}{n_2} + \frac{\zeta(k_t)}{n_2}$, so that k_t must grow over time, as $k_{t+1} > k_t$. Inversely, for any k_t that lies above the level corresponding to the kk locus, we know, for a given Ω_t, that $k_t > \frac{\sigma(\sigma(k_t)+\Omega_t)}{n_2} + \frac{\zeta(k_t)}{n_2}$, so that k_t must fall over time, as $k_{t+1} < k_t$. Regarding the $\Omega\Omega$ locus, we know that, for any given k_t, if Ω_t lies below the level defined by the $\Omega\Omega$ locus, we have $\Omega_t < \frac{\zeta(\sigma(k_t)+\Omega_t)}{n_2}$, so that Ω_t will tend to grow over time, as $\Omega_{t+1} > \Omega_t$. Inversely, for any given k_t, if Ω_t lies above the level defined by the $\Omega\Omega$ locus, we have $\Omega_t > \frac{\zeta(\sigma(k_t)+\Omega_t)}{n_2}$, so that Ω_t will tend to grow over time, as $\Omega_{t+1} < \Omega_t$.

From the above discussion, it appears that, if we focus on the (k_t, Ω_t) space under $g_t = 1$, the dynamic arrows that can be drawn on all sides of the two loci point towards the equilibrium $\left(\hat{k}, \hat{\Omega}\right)$. This is not a proof of convergence towards $\left(\hat{k}, \hat{\Omega}\right)$, but at least it is not inconceivable that, at each period when $g_t = 1$ (i.e. each period out of two), the economy converges towards $\left(\hat{k}, \hat{\Omega}, 1\right)$.

The same kind of argument can be developed on the basis of the kk locus and the $\Omega\Omega$ locus when g_t takes its other value: $g_t = n_2$. The two loci are:

$$\begin{aligned} k_t &= \sigma\left(\frac{\sigma\left(k_t\right)}{n_2} + \Omega_t\right) + \frac{\zeta(k_t)}{n_2} \iff \sigma\left(\frac{\sigma\left(k_t\right)}{n_2} + \Omega_t\right) = k_t - \frac{\zeta(k_t)}{n_2} \\ \Omega_t &= \frac{\zeta(\frac{\sigma(k_t)}{n_2} + \Omega_t)}{n_2} \end{aligned}$$

Here again, if we draw the dynamic arrows in the two-dimensional space (k_t, Ω_t), the convergence towards the equilibrium $\left(\check{k}, \check{\Omega}\right)$ is conceivable. Hence, here again, it is not impossible that, at each period when $g_t = n_2$ (i.e. each period out of two), the economy converges towards the equilibrium $\left(\check{k}, \check{\Omega}, n_2\right)$. Therefore, provided the economy starts initially with an equilibrium cohort growth rate (i.e. $g_0 = 1$), the convergence towards the two-period equilibrium cycle is not rejected by geometrical analysis.

6.4 Proposition 8

Existence of a stationary equilibrium The dynamic system:

$$\begin{aligned} k_{t+1} &\equiv G(k_t,\Omega_t,g_t)=\frac{\sigma\left(k_t\right)}{g_t}+\left(1-\frac{n_1}{g_t}\right)\Omega_t \\ \Omega_{t+1} &\equiv H\left(k_t\right)=\frac{\zeta(k_t)}{n_2} \\ g_{t+1} &\equiv I(g_t)=n_1+\frac{n_2}{g_t} \end{aligned}$$

From the first equation, we can define the kk locus, along which k_t is constant. Imposing $k_{t+1}=k_t$ yields:

$$\Omega_t=\left(\frac{g_t}{g_t-n_1}\right)\left(k_t-\frac{\sigma\left(k_t\right)}{g_t}\right)$$

From the second equation, we can define the $\Omega\Omega$ locus, along which Ω_t is constant:

$$\Omega_t=\frac{\zeta(k_t)}{n_2}$$

From the third equation, we can define the gg locus, along which g_t is constant. Actually, there is only one level of constant cohort growth rate:

$$g_t=n_1+\frac{n_2}{g_t}\iff g_t=\frac{n_1+\sqrt[2]{n_1^2+4n_2}}{2}$$

As a consequence, the gg locus is a horizontal plan in the (k_t,Ω_t,g_t) space, at a level $g_t=g^*=\frac{n_1+\sqrt[2]{n_1^2+4n_2}}{2}$.

Let us study under which conditions the kk locus and the $\Omega\Omega$ locus intersect with each others at the cohort growth rate $g^*=\frac{n_1+\sqrt[2]{n_1^2+4n_2}}{2}$.

The kk locus can, at that cohort growth rate, be rewritten as

$$\Omega_t=\left(\frac{n_1+\sqrt[2]{n_1^2+4n_2}}{\sqrt[2]{n_1^2+4n_2}-n_1}\right)\left[k_t-\frac{2\sigma\left(k_t\right)}{n_1+\sqrt[2]{n_1^2+4n_2}}\right]$$

The $\Omega\Omega$ locus can be rewritten as:

$$\Omega_t=\frac{\zeta(k_t)}{n_2}$$

Note that, as $\sigma(0)=0$ and $\zeta(0)=0$, the two loci intersect at $k_t=0$. Moreover, assuming that

$$lim_{k\to 0}\,\frac{n_1+\sqrt[2]{n_1^2+4n_2}}{\sqrt[2]{n_1^2+4n_2}-n_1}\left[1-\frac{2\sigma'\left(k_t\right)}{n_1+\sqrt[2]{n_1^2+4n_2}}\right]<lim_{k\to 0}\,\frac{\zeta'(k_t)}{n_2}$$

and

$$lim_{k\to+\infty}\frac{n_1+\sqrt[2]{n_1^2+4n_2}}{\sqrt[2]{n_1^2+4n_2}-n_1}\left[1-\frac{2\sigma'(k_t)}{n_1+\sqrt[2]{n_1^2+4n_2}}\right]>lim_{k\to+\infty}\frac{\zeta'(k_t)}{n_2}$$

we know that the kk locus lies below the $\Omega\Omega$ locus for low capital levels, but lies above it for high capital level. As a consequence, and by continuity, we know that the kk and $\Omega\Omega$ loci must necessarily intersect at some point along the gg locus. That intersection is a stationary equilibrium (k^*, Ω^*, g^*), which can be rewritten as: $\left(k^*, \frac{\zeta(k^*)}{n_2}, \frac{n_1+\sqrt[2]{n_1^2+4n_2}}{2}\right)$.

Stability of a stationary equilibrium The Jacobian matrix is:

$$J\equiv\begin{pmatrix}\frac{\partial G(k_t,\Omega_t,g_t)}{\partial k_t} & \frac{\partial G(k_t,\Omega_t,g_t)}{\partial \Omega_t} & \frac{\partial G(k_t,\Omega_t,g_t)}{\partial g_t}\\ \frac{\partial H(k_t)}{\partial k_t} & \frac{\partial H(k_t)}{\partial \Omega_t} & \frac{\partial H(k_t)}{\partial g_t}\\ \frac{\partial I(g_t)}{\partial k_t} & \frac{\partial I(g_t)}{\partial \Omega_t} & \frac{\partial I(g_t)}{\partial g_t}\end{pmatrix}$$

When computing the entries of that matrix, we have:

$$\begin{aligned}\frac{\partial G(k_t,\Omega_t,g_t)}{\partial k_t} &= \frac{\sigma'(k_t)}{g_t}>0\\ \frac{\partial G(k_t,\Omega_t,g_t)}{\partial \Omega_t} &= 1-\frac{n_1}{g_t}>0\\ \frac{\partial G(k_t,\Omega_t,g_t)}{\partial g_t} &= \frac{-\sigma(k_t)}{(g_t)^2}+\frac{n_1}{(g_t)^2}\Omega_t \lesseqgtr 0\\ \frac{\partial H(k_t)}{\partial k_t} &= \frac{\zeta'(k_t)}{n_2}>0\\ \frac{\partial H(k_t)}{\partial \Omega_t} &= \frac{\partial H(k_t)}{\partial g_t}=\frac{\partial I(g_t)}{\partial k_t}=\frac{\partial I(g_t)}{\partial \Omega_t}=0\\ \frac{\partial I(g_t)}{\partial g_t} &= \frac{-n_2}{(g_t)^2}<0\end{aligned}$$

Thus the determinant of the Jacobian matrix is:

$$det(J)=\left(1-\frac{n_1}{g_t}\right)\frac{\zeta'(k_t)}{(g_t)^2}>0$$

since at $g_t=g^*=\frac{n_1+\sqrt[2]{n_1^2+4n_2}}{2}>n_1$.

The trace of the Jacobian matrix is:

$$tr(J)=\frac{\sigma'(k_t)}{g_t}-\frac{n_2}{(g_t)^2}\gtreqless 0$$

In that economy, Brooks's (2004) conditions (i), (ii) and (iii) impose the following restrictions.

Condition (i) is satisfied only if

$$\frac{4\zeta'(k^*)\left(\sqrt[2]{n_1^2+4n_2}-n_1\right)}{\left(n_1+\sqrt[2]{n_1^2+4n_2}\right)^3}<1$$

Regarding Condition (ii), note that $\sum M_i(J)$ is here

$$-\frac{\zeta'(k_t)}{n_2}\left(1-\frac{n_1}{g_t}\right)-\frac{n_2\sigma'(k_t)}{(g_t)^3}$$

Hence condition (ii) amounts to:

$$\begin{aligned}&\left(\frac{g_t}{g_t-n_1}\right)^2+\frac{\zeta'(k_t)}{n_2}\left(\frac{g_t}{g_t-n_1}\right)+\frac{n_2\sigma'(k_t)}{(g_t)(g_t-n_1)^2}+\\&\frac{\zeta'(k_t)\sigma'(k_t)}{(g_t)^2(g_t-n_1)}-\frac{\zeta'(k_t)n_2}{(g_t)^3(g_t-n_1)}\\>\;&\frac{(\zeta'(k_t))^2}{(g_t)^4}\end{aligned}$$

That condition can, at the equilibrium, be rewritten as:

$$\begin{aligned}&1+\frac{\frac{\zeta'(k^*)}{n_2}\left(\sqrt[2]{n_1^2+4n_2}-n_1\right)}{\left(n_1+\sqrt[2]{n_1^2+4n_2}\right)}+\frac{8\sigma'(k^*)\left(n_2+\zeta'(k^*)\right)}{\left(n_1+\sqrt[2]{n_1^2+4n_2}\right)^3}+\frac{32\left(\zeta'(k^*)\right)^2 n_1\left(\sqrt[2]{n_1^2+4n_2}-n_1\right)}{\left(n_1+\sqrt[2]{n_1^2+4n_2}\right)^6}\\>\;&16\zeta'(k^*)\left[\frac{n_2\left(\sqrt[2]{n_1^2+4n_2}-n_1\right)+\left(n_1\sigma'(k^*)+\zeta'(k^*)\right)\left(n_1+\sqrt[2]{n_1^2+4n_2}\right)}{\left(n_1+\sqrt[2]{n_1^2+4n_2}\right)^5}\right]\end{aligned}$$

Condition (iii) amounts to:

$$\begin{aligned}&\left(1-\frac{n_1}{g_t}\right)\frac{\zeta'(k_t)}{n_2}+\frac{\sigma'(k_t)}{g_t}\frac{n_2}{(g_t)^2}-1\\<\;&\left(1-\frac{n_1}{g_t}\right)\frac{\zeta'(k_t)}{n_2}\frac{n_2}{(g_t)^2}+\frac{\sigma'(k_t)}{g_t}-\frac{n_2}{(g_t)^2}\\<\;&1-\left(1-\frac{n_1}{g_t}\right)\frac{\zeta'(k_t)}{n_2}-\frac{\sigma'(k_t)}{g_t}\frac{n_2}{(g_t)^2}\end{aligned}$$

The first inequality can, at the equilibrium, be rewritten as:

$$-1<\frac{2\sigma'(k^*)-\frac{\zeta'(k^*)}{n_2}\left(\sqrt[2]{n_1^2+4n_2}-n_1\right)}{n_1+\sqrt[2]{n_1^2+4n_2}}$$

The second inequality can, at the equilibrium, be rewritten as:

$$\frac{2\sigma'(k^*) + \frac{\zeta'(k^*)}{n_2}\left(\sqrt[2]{n_1^2+4n_2} - n_1\right)}{n_1 + \sqrt[2]{n_1^2+4n_2}} < 1$$

6.5 Proposition 9

As shown above, the equation:

$$g^3 + \frac{(F_L(k,\cdot) - d)}{k} g - \frac{2b}{k} = 0$$

characterizes the *interior* optimal cohort growth rate g. Given that this equation takes the form of a so-called "depressed cubic" equation, we can use the resolution method develop by Cardano (1545). That method consists in first introducing two new variables, whose sum equals g:

$$s + t = g$$

We substitute in the depressed cubic equation, and we get:

$$\begin{aligned} (s+t)^3 + (\frac{F_L(k,\cdot) - d}{k})(s+t) - \frac{2b}{k} &= 0 \\ s^3 + t^3 + (\frac{F_L(k,\cdot) - d}{k} + 3st)(s+t) - \frac{2b}{k} &= 0 \end{aligned}$$

Then, imposing the constraint

$$\frac{F_L(k,\cdot) - d}{k} + 3st = 0$$

we get:

$$\begin{aligned} s^3 + t^3 &= \frac{2b}{k} \\ st &= -\frac{F_L(k,\cdot) - d}{3k} \Longrightarrow s^3 t^3 = -\frac{(F_L(k,\cdot) - d)^3}{27k^3} \end{aligned}$$

Thus s^3 and t^3 are the roots of the equation:

$$m^2 + m\left(-\frac{2b}{k}\right) - \frac{(F_L(k,\cdot) - d)^3}{27k^3} = 0$$

Note that

$$\Delta \equiv \left(-\frac{2b}{k}\right)^2 + \frac{4\left(\frac{F_L(k,\cdot)-d}{k}\right)^3}{27} = \frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot) - d)^3}{27k^3} \gtrless 0$$

That equation can then be solved following the usual procedures for finding roots.

If $\Delta > 0$, we have the two roots:

$$m_1 = s^3 = \frac{\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{2}; m_2 = t^3 = \frac{\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{2}$$

Hence it follows that the optimal g is given by:

$$g = s + t = \sqrt[3]{\frac{\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{2}} + \sqrt[3]{\frac{\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{2}}$$

If $\Delta = 0$, we need to choose a cubic root for s^3. As there is no direct way to choose the corresponding cubic root of t^3, we need to use the relation $t = -\frac{F_L(k,\cdot)-d}{3ks}$, which yields:

$$s = \sqrt[3]{\frac{b}{k} - \sqrt[2]{\frac{b^2}{k^2} + \frac{(F_L(k,\cdot)-d)^3}{27k^3}}}$$

Hence it follows that the optimal cohort growth rate $g = s + t$ is:

$$g = \sqrt[3]{\frac{b}{k} - \sqrt[2]{\frac{b^2}{k^2} + \frac{(F_L(k,\cdot)-d)^3}{27k^3}}} - \frac{F_L(k,\cdot)-d}{3k\sqrt[3]{\frac{b}{k} - \sqrt[2]{\frac{b^2}{k^2} + \frac{(F_L(k,\cdot)-d)^3}{27k^3}}}}$$

Finally, if $\Delta < 0$, one can obtain the complex cubic roots by multiplying one of the two above cubic roots by $\frac{-1}{2} + i\frac{\sqrt[2]{3}}{2}$, and the other by $\frac{-1}{2} - i\frac{\sqrt[2]{3}}{2}$. This yields the two roots:

$$\begin{aligned} m_1 &= s^3 = \frac{-\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4} + i\sqrt[2]{3}\frac{\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4} \\ m_2 &= t^3 = \frac{-\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4} - i\sqrt[2]{3}\frac{\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4} \end{aligned}$$

Hence $g = s + t$ is:

$$\begin{aligned} g &= \sqrt[3]{\frac{-\frac{2b}{k} - \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}} + \frac{2bi\sqrt[2]{3}}{k} + i\sqrt[2]{3}\sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4}} \\ &+ \sqrt[3]{\frac{-\frac{2b}{k} + \sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}} - \frac{2bi\sqrt[2]{3}}{k} + i\sqrt[2]{3}\sqrt[2]{\frac{4b^2}{k^2} + \frac{4(F_L(k,\cdot)-d)^3}{27k^3}}}{4}} \end{aligned}$$