COMBINATORIAL PROPERTIES OF LAZY EXPANSIONS IN CANTOR REAL BASES

CÉLIA CISTERNINO

Abstract. The lazy algorithm for a real base β is generalized to the setting of Cantor bases $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ introduced recently by Charlier and the author. To do so, let $x_{\boldsymbol{\beta}}$ be the greatest real number that has a $\boldsymbol{\beta}$-representation $a_0a_1a_2\cdots$ such that each letter a_n belongs to $\{0,\ldots,\lceil\beta_n\rceil-1\}$. This paper is concerned with the combinatorial properties of the lazy $\boldsymbol{\beta}$-expansions, which are defined when $x_{\boldsymbol{\beta}} < +\infty$. As an illustration, Cantor bases following the Thue-Morse sequence are studied and a formula giving their corresponding value of $x_{\boldsymbol{\beta}}$ is proved. First, it is shown that the lazy $\boldsymbol{\beta}$-expansions are obtained by "flipping" the digits of the greedy $\boldsymbol{\beta}$-expansions. Next, a Parry-like criterion characterizing the sequences of non-negative integers that are the lazy $\boldsymbol{\beta}$-expansions of some real number in $(x_{\boldsymbol{\beta}}-1, x_{\boldsymbol{\beta}}]$ is proved. Moreover, the lazy $\boldsymbol{\beta}$-shift is studied and in the particular case of alternate bases, that is the periodic Cantor bases, an analogue of Bertrand-Mathis' theorem in the lazy framework is proved: the lazy $\boldsymbol{\beta}$-shift is sofic if and only if all quasi-lazy $\boldsymbol{\beta}^{(i)}$-expansions of $x_{\boldsymbol{\beta}^{(i)}}-1$ are ultimately periodic, where $\boldsymbol{\beta}^{(i)}$ is the i-th shift of the alternate base $\boldsymbol{\beta}$.

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1. Introduction

Two well-known generalizations of the integer base representations are the Cantor and real base representations. The former was introduced by Cantor in 1869 [3]. The Cantor representation of a real number x via a base sequence $(b_n)_{n\in\mathbb{N}} \in (\mathbb{N}_{\geq 2})^{\mathbb{N}}$ is an infinite sequence $a_0a_1a_2\cdots$ of non-negative integers such that

$$x = \sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{i=0}^{n} b_i}.$$

The latter was defined by Rényi in 1957 [11] and well understood since the pioneering work of Parry in 1960 [10]. A real base representation of a real number x via a real base $\beta > 1$ is an infinite sequence $a_0a_1a_2\cdots$ of non-negative integers such that

$$x = \sum_{n\in\mathbb{N}} \frac{a_n}{\beta^{i+1}}.$$

Gathering both, the notion of Cantor real bases was introduced by Charlier and the author in a recent work [4]. Note that these type of representations involving more than one base simultaneously and independently aroused the interest of mathematicians [2, 4, 8, 9, 12].

A *Cantor real base* is a sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 such that $\prod_{n\in\mathbb{N}} \beta_n = +\infty$. A representation of a real number x via a Cantor real base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ is an infinite sequence $a_0a_1a_2\cdots$ over $\mathbb{N}$ such that

$$x = \sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{i=0}^{n} \beta_i}.$$

The digits of a $\boldsymbol{\beta}$-representation can be chosen by using several appropriate algorithms. As in the real base theory, in order to represent non-negative real numbers smaller than or equal to $x_{\boldsymbol{\beta}}$ where

$$x_{\boldsymbol{\beta}} = \sum_{n\in\mathbb{N}} \frac{\lceil \beta_n \rceil - 1}{\prod_{i=0}^{n} \beta_i},$$

the most commonly used algorithms are the greedy and the lazy ones. In the greedy algorithm, each digit is chosen as the largest possible among $0, \ldots, \lceil \beta_n \rceil - 1$ at position n. At the other extreme, the lazy algorithm picks the least possible digit at each step. The so-obtained $\boldsymbol{\beta}$-representations are respectively called the greedy and lazy $\boldsymbol{\beta}$-expansions.

In the initial work [4], the combinatorial properties of the greedy $\boldsymbol{\beta}$-expansions of real numbers in $[0, 1)$ were investigated. In particular, generalizations of several combinatorial results of real base expansions were obtained, such as Parry's criterion for greedy expansions and, while considering periodic Cantor real bases, called *alternate bases*, Bertrand-Mathis' characterization of sofic shifts. Next, in [5], in the particular case of alternate bases, the lazy expansions were defined and both greedy and lazy expansions were studied in terms of dynamics. These results generalize the well-known ones from the theory of real base expansions (see [6, 7, 10, 11]). Note that the lazy real base expansions have been widely studied in terms of dynamics and, to the best of the author's knowledge, not in terms of combinatorics.

The goal of this paper is to study the combinatorial properties of the lazy expansions in Cantor real bases. In particular, the aim is to obtain a version of Parry's theorem [10] and Bertrand-Mathis' theorem [1] in the lazy Cantor real base framework.

This paper is organized as follows. First, the Cantor bases and the associated greedy and lazy algorithms are introduced in Section 2. Note that the lazy algorithm is defined when $x_{\boldsymbol{\beta}} < +\infty$ hence, this paper deals with Cantor bases such that $x_{\boldsymbol{\beta}} < +\infty$. As an illustration, in Section 2, the value of $x_{\boldsymbol{\beta}}$ is studied when $\boldsymbol{\beta}$ is a Cantor base defined thanks to the Thue-Morse sequence over an alphabet $\{\alpha, \beta\}$ with $\alpha, \beta > 1$, that is $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \cdots)$. Next, it is shown in Section 3 that the lazy $\boldsymbol{\beta}$-expansions are obtained by "flipping" the digits of the greedy $\boldsymbol{\beta}$-expansions. This allow us, to translate the greedy properties from [4] to their lazy analogues. Section 4 is then concerned by first few properties of lazy $\boldsymbol{\beta}$-expansions. Then, we define the quasi-lazy $\boldsymbol{\beta}$-expansions of $x_{\boldsymbol{\beta}} - 1$ in Section 5 and show that the same "flip" permits us to go from the quasi-greedy $\boldsymbol{\beta}$-expansion to the quasi-lazy one. Hence, in Section 6, the lazy $\boldsymbol{\beta}$-admissible sequences are studied and a Parry-like criterion characterizing the lazy $\boldsymbol{\beta}$-expansions is proved. Finally, in Section 7, the lazy $\boldsymbol{\beta}$-shift is studied and in the particular case of alternate bases, an analogue of Bertrand-Mathis' theorem in the lazy case is proved. That is, if $\boldsymbol{\beta}$ is an alternate base, we obtain that the lazy $\boldsymbol{\beta}$-shift is sofic if and only if all quasi-lazy $\boldsymbol{\beta}^{(i)}$-expansions of $x_{\boldsymbol{\beta}^{(i)}} - 1$ are ultimately periodic, where $\boldsymbol{\beta}^{(i)}$ is the i-th shift of the alternate base $\boldsymbol{\beta}$.

2. Cantor real bases

In this section, the needed definitions and conventions are given. Throughout this text, if a is an infinite word then for all $n \in \mathbb{N}$, a_n designates its letter indexed by n, so that $a = a_0a_1a_2\cdots$, that is the $(n+1)^{\text{st}}$ letter of a. Moreover, an interval of non-negative integers $\{i, \ldots, j\}$ with[1] $i \leq j$ is denoted $[\![i, j]\!]$ and $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ respectively denote the floor and ceiling functions.

A *Cantor real base*, or simply a *Cantor base*, is a sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 such that $\prod_{n\in\mathbb{N}} \beta_n = +\infty$. For instance, any sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 that takes only finitely many values is a Cantor base since in this case, the condition $\prod_{n\in\mathbb{N}} \beta_n = +\infty$ is trivially satisfied. In particular, if $\beta > 1$ then $\boldsymbol{\beta} = (\beta, \beta, \ldots)$ is a Cantor base and in this case, all notions coincides with the widely studied theory of β-expansions. An *alternate base* is a periodic Cantor base, that is a Cantor base for which there exists $p \in \mathbb{N}_{\geq 1}$ such that for all $n \in \mathbb{N}$, $\beta_n = \beta_{n+p}$. In this case, we simply write $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$ and the integer p is called the *length* of the alternate base $\boldsymbol{\beta}$.

Let $\boldsymbol{\beta}$ be a Cantor base. We define

$$\boldsymbol{\beta}^{(n)} = (\beta_n, \beta_{n+1}, \ldots) \quad \text{for all } n \in \mathbb{N}.$$

In particular $\boldsymbol{\beta}^{(0)} = \boldsymbol{\beta}$. The *$\boldsymbol{\beta}$-value (partial) map* $\mathrm{val}_{\boldsymbol{\beta}} \colon (\mathbb{R}_{\geq 0})^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{val}_{\boldsymbol{\beta}}(a) = \sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{i=0}^{n} \beta_i} \tag{2.1}$$

for any infinite word a over $\mathbb{R}_{\geq 0}$, provided that the series converges. A *$\boldsymbol{\beta}$-representation* of a non-negative real number x is an infinite word $a \in \mathbb{N}^{\mathbb{N}}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$. A $\boldsymbol{\beta}$-representation is said to be *finite* if it ends with infinitely many zeros, and *infinite* otherwise. The *length* of a finite $\boldsymbol{\beta}$-representation is the length of the longest prefix ending in a non-zero digit. When a $\boldsymbol{\beta}$-representation is finite, we sometimes omit to write the tail of zeros.

2.1. **Greedy algorithm on $[0, 1)$.** For $x \in [0, 1)$, a distinguished $\boldsymbol{\beta}$-representation $\varepsilon_{\boldsymbol{\beta},0}(x)\varepsilon_{\boldsymbol{\beta},1}(x)\varepsilon_{\boldsymbol{\beta},2}(x)\cdots$, called the *greedy $\boldsymbol{\beta}$-expansion* is obtained from the *greedy algorithm.* If the first N digits of the greedy $\boldsymbol{\beta}$-expansion of x are given by $\varepsilon_{\boldsymbol{\beta},0}(x), \ldots, \varepsilon_{\boldsymbol{\beta},N-1}(x)$, then the next digit $\varepsilon_{\boldsymbol{\beta},N}(x)$ is the greatest integer such that

$$\sum_{n=0}^{N} \frac{\varepsilon_{\boldsymbol{\beta},n}(x)}{\prod_{k=0}^{n} \beta_k} \leq x.$$

In particular, for all $n \in \mathbb{N}$, the digit $\varepsilon_{\boldsymbol{\beta},n}(x)$ belongs to the alphabet $[\![0, \lceil \beta_n \rceil - 1]\!]$. The greedy algorithm can be equivalently defined as follows:

- $\varepsilon_{\boldsymbol{\beta},0}(x) = \lfloor \beta_0 x \rfloor$ and $r_{\boldsymbol{\beta},0}(x) = \beta_0 x - \varepsilon_{\boldsymbol{\beta},0}(x)$
- $\varepsilon_{\boldsymbol{\beta},n}(x) = \lfloor \beta_n r_{\boldsymbol{\beta},n-1}(x) \rfloor$ and $r_{\boldsymbol{\beta},n}(x) = \beta_n r_{\boldsymbol{\beta},n-1}(x) - \varepsilon_{\boldsymbol{\beta},n}(x)$ for $n \in \mathbb{N}_{\geq 1}$.

The obtained $\boldsymbol{\beta}$-representation is denoted by $d_{\boldsymbol{\beta}}(x)$ and is called the *greedy $\boldsymbol{\beta}$-expansion* of x. When the context is clear, we simply denote $\varepsilon_{\boldsymbol{\beta},n}(x)$ by $\varepsilon_n(x)$ and $r_{\boldsymbol{\beta},n}(x)$ by $r_n(x)$.

Example 1. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$ already studied in [4] and [5]. We have $d_{\boldsymbol{\beta}}(\frac{-5+2\sqrt{13}}{3}) = 11$ and $d_{\boldsymbol{\beta}}(\frac{2+\sqrt{13}}{9}) = (10)^{\omega}$ where the ω notation means an infinite repetition.

Example 2. Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$ and consider the Cantor base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ from [4] defined by

$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases} \tag{2.2}$$

[1]If $i > j$, we take the convention that $[\![i, j]\!]$ is the empty set.

for all $n \in \mathbb{N}$, where rep_2 is the function mapping any non-negative integer to its 2-expansion and $|u|_1$ is the number of occurrences of the letter 1 in the word u. We get $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ where the infinite word $\beta_0\beta_1\beta_2\cdots$ is the Thue-Morse word over the alphabet $\{\alpha, \beta\}$. The greedy $\boldsymbol{\beta}$-expansion of $\frac{1}{2}$ has 10001 as a prefix and $d_{\boldsymbol{\beta}}(\frac{65-18\sqrt{13}}{6}) = 1002$.

2.2. **Lazy algorithm on $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$.** Considering a Cantor base $\boldsymbol{\beta}$, define

$$x_{\boldsymbol{\beta}} = \sum_{n \in \mathbb{N}} \frac{\lceil \beta_n \rceil - 1}{\prod_{k=0}^{n} \beta_k}.$$

Either this series converges or $x_{\boldsymbol{\beta}} = +\infty$. In both cases, this corresponds to the greatest real number that has a $\boldsymbol{\beta}$-representation $a_0a_1a_2\cdots$ such that for all $n \in \mathbb{N}$ the letter a_n belongs to the alphabet $[\![0, \lceil \beta_n \rceil - 1]\!]$.

Since the greedy algorithm converges on $[0, 1)$, it can be easily seen that $x_{\boldsymbol{\beta}} \geq 1$. Moreover, for all $n \in \mathbb{N}$,

$$x_{\boldsymbol{\beta}^{(n)}} = \frac{x_{\boldsymbol{\beta}^{(n+1)}} + \lceil \beta_n \rceil - 1}{\beta_n}. \tag{2.3}$$

Hence, it can be easily proved that $x_{\boldsymbol{\beta}} = 1$ if and only if $\beta_n \in \mathbb{N}_{\geq 2}$ for all $n \in \mathbb{N}$.

Example 3. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$ from Example 1. We get $x_{\boldsymbol{\beta}} = \frac{5+7\sqrt{13}}{18} \simeq 1.67$ and $x_{\boldsymbol{\beta}^{(1)}} = \frac{2+\sqrt{13}}{3} \simeq 1.86$.

Example 4. Let $\alpha, \beta > 1$ and let $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ be the Thue-Morse Cantor base on $\{\alpha, \beta\}$ defined as (2.2). For all $n \geq 1$, let

$$x_n = \sum_{m=0}^{2^n - 1} \frac{\lceil \beta_m \rceil - 1}{\prod_{k=0}^{m} \beta_k}.$$

We get $x_{\boldsymbol{\beta}} = \lim_{n \to +\infty} x_n$. Similarly, let $\overline{\boldsymbol{\beta}}$ denote the Cantor base $\overline{\boldsymbol{\beta}} = (\overline{\beta_n})_{n \in \mathbb{N}}$ where $\overline{\alpha} = \beta$ and $\overline{\beta} = \alpha$. We get $\overline{\boldsymbol{\beta}} = (\beta, \alpha, \alpha, \beta, \alpha, \beta, \beta, \alpha, \ldots)$. For all $n \geq 1$, denote

$$y_n = \sum_{m=0}^{2^n - 1} \frac{\lceil \overline{\beta_m} \rceil - 1}{\prod_{k=0}^{m} \overline{\beta_k}}.$$

By definition of the Thue-Morse sequence, for all $n \in \mathbb{N}$ we have

$$(\beta_{2^n}, \beta_{2^n+1}, \ldots, \beta_{2^{n+1}-1}) = (\overline{\beta_0}, \overline{\beta_1}, \ldots, \overline{\beta_{2^n-1}}).$$

Moreover, for all $n \geq 1$ the sequence $(\beta_0, \ldots, \beta_{2^n-1})$ has the same number of α and β. We get $\prod_{k=0}^{2^n-1} \beta_k = (\alpha\beta)^{2^{n-1}}$. Hence, we have

$$\begin{cases} x_1 = \frac{\lceil \alpha \rceil - 1}{\alpha} + \frac{\lceil \beta \rceil - 1}{\alpha\beta}, \\ y_1 = \frac{\lceil \beta \rceil - 1}{\beta} + \frac{\lceil \alpha \rceil - 1}{\beta\alpha}, \\ x_{n+1} = x_n + \frac{1}{(\alpha\beta)^{2^{n-1}}} y_n, \quad \forall n \geq 1 \\ y_{n+1} = y_n + \frac{1}{(\alpha\beta)^{2^{n-1}}} x_n, \quad \forall n \geq 1. \end{cases}$$

That is, for all $n \geq 1$, we have

$$v_{n+1} = A_n v_n$$

where,

$$v_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} 1 & \frac{1}{(\alpha\beta)^{2^{n-1}}} \\ \frac{1}{(\alpha\beta)^{2^{n-1}}} & 1 \end{pmatrix}.$$

For all $n \geq 1$, the eigenvalues of the matrix A_n are $1 + \frac{1}{(\alpha\beta)^{2^{n-1}}}$ and $1 - \frac{1}{(\alpha\beta)^{2^{n-1}}}$ of eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ respectively. Moreover, we have

$$v_1 = \tfrac{x_1+y_1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{x_1-y_1}{2}\begin{pmatrix}1\\-1\end{pmatrix}.$$

We obtain

$$\begin{aligned} v_{n+1} &= A_nA_{n-1}\cdots A_1 v_1 \\ &= \tfrac{x_1+y_1}{2} A_nA_{n-1}\cdots A_1\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{x_1-y_1}{2} A_nA_{n-1}\cdots A_1\begin{pmatrix}1\\-1\end{pmatrix} \\ &= \tfrac{x_1+y_1}{2}\prod_{k=1}^{n}\big(1+\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{x_1-y_1}{2}\prod_{k=1}^{n}\big(1-\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\begin{pmatrix}1\\-1\end{pmatrix}. \end{aligned}$$

Then, the value of $x_{\boldsymbol{\beta}}$ can be computed by

$$x_{\boldsymbol{\beta}} = \lim_{n\to+\infty} x_n = \tfrac{x_1+y_1}{2}\prod_{k\in\mathbb{N}_{\geq 1}}\big(1+\tfrac{1}{(\alpha\beta)^{2^{n-1}}}\big) + \tfrac{x_1-y_1}{2}\prod_{k\in\mathbb{N}_{\geq 1}}\big(1-\tfrac{1}{(\alpha\beta)^{2^{n-1}}}\big).$$

We now study the two infinite products in the above formula. We have

$$\begin{aligned} &\Big(\prod_{k\in\mathbb{N}_{\geq 1}}\big(1+\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\Big)\Big(\prod_{k\in\mathbb{N}_{\geq 1}}\big(1-\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\Big) \\ &= \prod_{k\in\mathbb{N}_{\geq 1}}\Big(\big(1+\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\big(1-\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\Big) \\ &= \prod_{k=2}^{\infty}\big(1-\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big). \end{aligned}$$

Hence, we get

$$\prod_{k\in\mathbb{N}_{\geq 1}}\big(1+\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big) = \frac{1}{1-\frac{1}{\alpha\beta}}.$$

Moreover, consider the function f defined by $f(z) = \sum_{m\in\mathbb{N}}(-1)^{t_m}z^m$ where $t_0t_1t_2\cdots$ is the Thue-Morse sequence over the alphabet $\{0,1\}$. By the infinite product definition of the Thue-Morse sequence, we get

$$\prod_{k\in\mathbb{N}_{\geq 1}}\big(1-\tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big) = f(\tfrac{1}{\alpha\beta}).$$

Then, the value of $x_{\boldsymbol{\beta}}$ can be computed by

$$x_{\boldsymbol{\beta}} = \tfrac{x_1+y_1}{2}\Big(\frac{1}{1-\frac{1}{\alpha\beta}}\Big) + \tfrac{x_1-y_1}{2} f(\tfrac{1}{\alpha\beta}).$$

In particular, by considering the Cantor base from Example 2, a computer approximation of $f(\frac{1}{\alpha\beta})$ gives 0.627941. Hence, we get $x_{\boldsymbol{\beta}} \simeq 1.73295$.

Example 5. Consider the Cantor base $\boldsymbol{\beta} = (1+\frac{1}{n+1})_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, we have $\beta_n = \frac{n+2}{n+1}$ so we get

$$x_{\boldsymbol{\beta}} = \sum_{n\in\mathbb{N}}\frac{1}{\prod_{k=0}^{n}\frac{k+2}{k+1}} = \sum_{n\in\mathbb{N}}\frac{1}{n+2} = +\infty.$$

As said in [5, Section 3], if $x_{\boldsymbol{\beta}} < +\infty$, the other extreme $\boldsymbol{\beta}$-expansions of real number, namely the *lazy* $\boldsymbol{\beta}$*-expansions*, is defined. Hence, from now on, consider a Cantor base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ such that $x_{\boldsymbol{\beta}} < +\infty$. For instance, any Cantor base $\boldsymbol{\beta}$ that takes only finitely many values has finite corresponding $x_{\boldsymbol{\beta}}$.

In the greedy algorithm, each digit is chosen as the largest possible at the considered position. On the contrary, in the lazy algorithm, each digit is chosen as the least possible at each step. The *lazy algorithm* is defined as follows: for $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$, if the first N digits of the lazy $\boldsymbol{\beta}$-expansion of x are given by $\xi_{\boldsymbol{\beta},0}(x), \ldots, \xi_{\boldsymbol{\beta},N-1}(x)$, then the next digit $\xi_{\boldsymbol{\beta},N}(x)$ is the least element in $[\![0, \lceil\beta_N\rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{\xi_{\boldsymbol{\beta},n}(x)}{\prod_{k=0}^{n}\beta_k} + \sum_{n=N+1}^{+\infty} \frac{\lceil\beta_n\rceil - 1}{\prod_{k=0}^{n}\beta_k} \geq x.$$

The lazy algorithm can be equivalently defined as follows:

- $\xi_{\boldsymbol{\beta},0}(x) = \lceil\beta_0 x - x_{\boldsymbol{\beta}^{(1)}}\rceil$ and $s_{\boldsymbol{\beta},0}(x) = \beta_0 x - \xi_{\boldsymbol{\beta},0}(x)$
- $\xi_{\boldsymbol{\beta},n}(x) = \lceil\beta_n s_{\boldsymbol{\beta},n-1}(x) - x_{\boldsymbol{\beta}^{(n+1)}}\rceil$ and $s_{\boldsymbol{\beta},n}(x) = \beta_n s_{\boldsymbol{\beta},n-1}(x) - \xi_{\boldsymbol{\beta},n}(x)$ for $n \in \mathbb{N}_{\geq 1}$.

The obtained $\boldsymbol{\beta}$-representation of $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ is denoted by $\ell_{\boldsymbol{\beta}}(x)$ and is called the *lazy* $\boldsymbol{\beta}$*-expansion* of x. As before, if the context is clear, the indexes $\boldsymbol{\beta}$ in the writings $\xi_{\boldsymbol{\beta},n}(x)$ and $s_{\boldsymbol{\beta},n}(x)$ are omitted.

Example 6. We continue Examples 1 and 3. The first 5 digits of $\ell_{\boldsymbol{\beta}}(\frac{35-5\sqrt{13}}{18})$ are 10212.

3. Flip greedy and get lazy

In [5, Section 5], in the alternate base framework, both greedy and lazy expansions were compared. The following result generalizes this comparison to the Cantor base expansions.

For a Cantor base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$, we let $A_{\boldsymbol{\beta}}$ denote the (possibly infinite) alphabet $[\![0, \sup_{n\in\mathbb{N}}(\lceil\beta_n\rceil - 1)]\!]$. Note that, if the supremum is infinite, the alphabet $A_{\boldsymbol{\beta}}$ is made of all non-negative integers. Any greedy and lazy $\boldsymbol{\beta}$-expansion belongs to $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ and more precisely to the set of infinite words $a \in A_{\boldsymbol{\beta}}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, the letter a_n belongs to $[\![0, \lceil\beta_n\rceil - 1]\!]$. From now on, let $\prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$ denote this set of infinite words.

Let $\theta_{\boldsymbol{\beta}}$ be the map defined by

$$\begin{aligned} \theta_{\boldsymbol{\beta}} \colon \prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!] &\to \prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!], \\ a_0a_1\cdots &\mapsto (\lceil\beta_0\rceil - 1 - a_0)(\lceil\beta_1\rceil - 1 - a_1)\cdots . \end{aligned}$$

The map $\theta_{\boldsymbol{\beta}}$ is continuous with respect to the topology induced by the prefix distance, bijective and the inverse map $\theta_{\boldsymbol{\beta}}^{-1}$ is the map $\theta_{\boldsymbol{\beta}}$ itself. For any infinite word $a \in \prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$, we get

$$\operatorname{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) = x_{\boldsymbol{\beta}} - \operatorname{val}_{\boldsymbol{\beta}}(a). \tag{3.1}$$

Moreover, the map $\theta_{\boldsymbol{\beta}}$ is decreasing with respect to the lexicographic order, that is, for all infinite words a and b in $\prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$, we get

$$a <_{\text{lex}} b \iff \theta_{\boldsymbol{\beta}}(a) >_{\text{lex}} \theta_{\boldsymbol{\beta}}(b). \tag{3.2}$$

The map $\theta_{\boldsymbol{\beta}}$ is the key of the reasoning of this paper. In fact, as shown in the following result, it will allow us to "flip" the greedy expansions in order to get the lazy ones.

Proposition 7. *For all $x \in [0, 1)$ and all $n \in \mathbb{N}$, we have $\xi_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) = \lceil\beta_n\rceil - 1 - \varepsilon_{\boldsymbol{\beta},n}(x)$ and $s_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) = x_{\boldsymbol{\beta}^{(n+1)}} - r_{\boldsymbol{\beta},n}(x)$. In particular, we get $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x) = \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x))$.*

Proof. Consider $x \in [0,1)$. We proceed by induction on n. By (2.3), we have

$$\begin{aligned}\xi_{\boldsymbol{\beta},0}(x_{\boldsymbol{\beta}} - x) &= \lceil \beta_0(x_{\boldsymbol{\beta}} - x) - x_{\boldsymbol{\beta}^{(1)}} \rceil \\ &= \lceil \lceil \beta_0 \rceil - 1 - \beta_0 x \rceil \\ &= \lceil \beta_0 \rceil - 1 + \lceil -\beta_0 x \rceil \\ &= \lceil \beta_0 \rceil - 1 - \lfloor \beta_0 x \rfloor \\ &= \lceil \beta_0 \rceil - 1 - \varepsilon_{\boldsymbol{\beta},0}(x).\end{aligned}$$

Moreover, we get

$$\begin{aligned}s_{\boldsymbol{\beta},0}(x_{\boldsymbol{\beta}} - x) &= \beta_0(x_{\boldsymbol{\beta}} - x) - (\lceil \beta_0 \rceil - 1 - \varepsilon_{\boldsymbol{\beta},0}(x)) \\ &= \beta_0 x_{\boldsymbol{\beta}} - (\lceil \beta_0 \rceil - 1) - (\beta_0 x - \varepsilon_{\boldsymbol{\beta},0}(x)) \\ &= x_{\boldsymbol{\beta}^{(1)}} - r_{\boldsymbol{\beta},0}(x)\end{aligned}$$

where (2.3) is used again in the last equality. By induction, for all $n \in \mathbb{N}_{\geq 1}$, we have

$$\begin{aligned}\xi_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) &= \lceil \beta_n s_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x) - x_{\boldsymbol{\beta}^{(n+1)}} \rceil \\ &= \lceil \beta_n(x_{\boldsymbol{\beta}^{(n)}} - r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x)) - x_{\boldsymbol{\beta}^{(n+1)}} \rceil \\ &= \lceil \lceil \beta_n \rceil - 1 - \beta_n r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x) \rceil \\ &= \lceil \beta_n \rceil - 1 - \lfloor \beta_n r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x) \rfloor \\ &= \lceil \beta_n \rceil - 1 - \varepsilon_{\boldsymbol{\beta},n}(x)\end{aligned}$$

and

$$\begin{aligned}s_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) &= \beta_n s_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x) - \xi_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) \\ &= \beta_n(x_{\boldsymbol{\beta}^{(n)}} - r_{\boldsymbol{\beta},n-1}(x)) - (\lceil \beta_n \rceil - 1 - \varepsilon_{\boldsymbol{\beta},n}(x)) \\ &= x_{\boldsymbol{\beta}^{(n+1)}} - r_{\boldsymbol{\beta},n}(x).\end{aligned}$$

In particular, we can conclude that $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x) = \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x))$. □

Example 8. Let $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$ be the alternate base considered in Example 1. By Proposition 7, the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - \frac{-5+2\sqrt{13}}{3} = \frac{25-5\sqrt{13}}{18}$ equals $10(21)^\omega$ since $d_{\boldsymbol{\beta}}(\frac{-5+2\sqrt{13}}{3}) = 11$. This coincides with Example 6.

Example 9. We continue Example 2. The lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - \frac{1}{2} \simeq 1.23$ has 11120 as a prefix.

Thanks to Proposition 7, in the sequel, results from [4] on greedy $\boldsymbol{\beta}$-expansions will be translated in terms of lazy $\boldsymbol{\beta}$-expansions. The differences between the greedy and lazy $\boldsymbol{\beta}$-expansions will be highlighted in the text.

4. First properties of lazy expansions

For any alphabet A, the *shift operator over* A, denoted by σ_A, is defined by

$$\sigma_A \colon A^{\mathbb{N}} \to A^{\mathbb{N}},\ a_0a_1a_2\cdots \mapsto a_1a_2a_3\cdots .$$

Throughout the text, whenever there is no ambiguity on the alphabet, we simply write σ instead of $\sigma_{A_{\boldsymbol{\beta}}}$.

Lemma 10. *For all* $n \in \mathbb{N}$, *we have* $\sigma^n \circ \theta_{\boldsymbol{\beta}} = \theta_{\boldsymbol{\beta}^{(n)}} \circ \sigma^n$ *on* $\prod_{n\in\mathbb{N}}[\![0, \lceil \beta_n \rceil - 1]\!]$.

Proof. This is a straightforward verification. □

Proposition 11. *For all $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and all $n \in \mathbb{N}$, we have*

$$\sigma^n(\ell_{\boldsymbol{\beta}}(x)) = \ell_{\boldsymbol{\beta}^{(n)}}(s_{\boldsymbol{\beta},n-1}(x)).$$

Proof. This is a consequence of Proposition 7, Lemma 10 and [4, Proposition 8] since for all $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ we have

$$\begin{aligned}\sigma^n(\ell_{\boldsymbol{\beta}}(x)) &= \sigma^n \circ \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)) \\ &= \theta_{\boldsymbol{\beta}^{(n)}} \circ \sigma^n(d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)) \\ &= \theta_{\boldsymbol{\beta}^{(n)}}(d_{\boldsymbol{\beta}^{(n)}}(r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x))) \\ &= \ell_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x)) \\ &= \ell_{\boldsymbol{\beta}^{(n)}}(s_{\boldsymbol{\beta},n-1}(x)).\end{aligned}$$

□

Proposition 12. *Let a be an infinite word over $\mathbb{N}$ and $x \in (x_{\boldsymbol{\beta}}-1, x_{\boldsymbol{\beta}}]$. We have $a = \ell_{\boldsymbol{\beta}}(x)$ if and only if $a \in \prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$, $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$ and for all $\ell \in \mathbb{N}$,*

$$\sum_{n=\ell+1}^{+\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k} > \frac{x_{\boldsymbol{\beta}^{(\ell+1)}} - 1}{\prod_{k=0}^{\ell} \beta_k}.$$

Proof. Consider $a \in \mathbb{N}^{\mathbb{N}}$ and $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$. By Proposition 7, we have $a = \ell_{\boldsymbol{\beta}}(x)$ if and only if $a \in \prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$ and $\theta_{\boldsymbol{\beta}}(a) = d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)$. By [4, Lemma 9], we get $a = \ell_{\boldsymbol{\beta}}(x)$ if and only if $a \in \prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$, $\mathrm{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) = x_{\boldsymbol{\beta}} - x$ and for all $N \in \mathbb{N}$,

$$\sum_{n=N+1}^{+\infty} \frac{\lceil\beta_n\rceil - 1 - a_n}{\prod_{k=0}^{n} \beta_k} < \frac{1}{\prod_{k=0}^{N} \beta_k}.$$

We conclude the proof by (3.1) and by definition of $x_{\boldsymbol{\beta}^{(N+1)}}$. □

Proposition 13. *The lazy $\boldsymbol{\beta}$-expansion of a real number $x \in (x_{\boldsymbol{\beta}}-1, x_{\boldsymbol{\beta}}]$ is lexicographically minimal among all $\boldsymbol{\beta}$-representations of x in $\prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$.*

Proof. Let $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and let $a \in \prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$ be a $\boldsymbol{\beta}$-representation of x. Suppose that $a <_{\mathrm{lex}} \ell_{\boldsymbol{\beta}}(x)$. By (3.2), we get $\theta_{\boldsymbol{\beta}}(a) >_{\mathrm{lex}} \theta_{\boldsymbol{\beta}}(\ell_{\boldsymbol{\beta}}(x))$. By (3.1), $\theta_{\boldsymbol{\beta}}(a)$ is a $\boldsymbol{\beta}$-representation of $x_{\boldsymbol{\beta}} - x$. Moreover, by Proposition 7 and since the inverse map $\theta_{\boldsymbol{\beta}}^{-1}$ is the map $\theta_{\boldsymbol{\beta}}$ itself, we have $\theta_{\boldsymbol{\beta}}(\ell_{\boldsymbol{\beta}}(x)) = d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)$. This is absurd since, by [4, Proposition 12], $d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)$ is lexicographically maximal among all $\boldsymbol{\beta}$-representations of $x_{\boldsymbol{\beta}} - x$. □

Note that, contrary to [4, Proposition 12], it cannot be stated that "the lazy $\boldsymbol{\beta}$-expansion of a real number $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ is lexicographically minimal among all $\boldsymbol{\beta}$-representations of x". In fact, the alphabet of the $\boldsymbol{\beta}$-representations of x must be fixed as shown in the following example.

Example 14. Let $\boldsymbol{\beta}$ be the alternate base from Example 1 and consider $x = 8-2\sqrt{13}$. We have $x \in (x_{\boldsymbol{\beta}}-1, x_{\boldsymbol{\beta}}]$ and the lazy $\boldsymbol{\beta}$-expansion of x has 01 as a prefix. However, the infinite word 003330^ω is a $\boldsymbol{\beta}$-representation of x and $003330^\omega <_{\mathrm{lex}} \ell_{\boldsymbol{\beta}}(x)$. This does not contradict Proposition 13 since the infinite word 003330^ω does not belong to $\prod_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$.

Proposition 15. *The function $\ell_{\boldsymbol{\beta}} \colon (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}] \to A_{\boldsymbol{\beta}}^{\mathbb{N}}$ is increasing:*

$$\forall x, y \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}], \quad x < y \iff \ell_{\boldsymbol{\beta}}(x) <_{\mathrm{lex}} \ell_{\boldsymbol{\beta}}(y).$$

Proof. Consider $x, y \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$. By [4, Proposition 13], Proposition 7 and (3.2), we have

$$x < y \iff x_{\boldsymbol{\beta}} - x > x_{\boldsymbol{\beta}} - y$$

β	2	$\frac{11}{5}$	$\frac{5}{2}$
x_β	$\frac{\lceil 2\rceil-1}{2-1}=1$	$\frac{\lceil\frac{11}{5}\rceil-1}{\frac{11}{5}-1}=\frac{5}{3}$	$\frac{\lceil\frac{5}{2}\rceil-1}{\frac{5}{2}-1}=\frac{4}{3}$
$\ell_\beta(x_\beta-\frac{1}{2})$	01^ω	$1221\cdots$	$1211\cdots$

FIGURE 1. Some lazy $\boldsymbol{\beta}$-expansions when $\boldsymbol{\beta}=(\beta,\beta,\ldots)$ with $\beta>1$.

$$\begin{aligned}
&\iff d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-x) >_{\text{lex}} d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-y)\\
&\iff \theta_{\boldsymbol{\beta}}(\ell_{\boldsymbol{\beta}}(x)) >_{\text{lex}} \theta_{\boldsymbol{\beta}}(\ell_{\boldsymbol{\beta}}(y))\\
&\iff \ell_{\boldsymbol{\beta}}(x) <_{\text{lex}} \ell_{\boldsymbol{\beta}}(y). \qquad \square
\end{aligned}$$

Remark 16. Considering two Cantor bases $\boldsymbol{\alpha}=(\alpha_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\beta}=(\beta_n)_{n\in\mathbb{N}}$ such that for all $n\in\mathbb{N}$, $\prod_{i=0}^n\alpha_i\le\prod_{i=0}^n\beta_i$, by [4, Proposition 15], we have $d_{\boldsymbol{\alpha}}(x)\le_{\text{lex}} d_{\boldsymbol{\beta}}(x)$ for all $x\in[0,1)$. However, an analogous result cannot be obtained for the lazy expansions. In fact, since the interval of definition of the lazy expansions depends on the considered Cantor base, it is not possible to state a result of the form "for all $x\in I$, we have $\ell_{\boldsymbol{\alpha}}(x)\le_{\text{lex}}\ell_{\boldsymbol{\beta}}(x)$ (or $\ell_{\boldsymbol{\alpha}}(x)\ge_{\text{lex}}\ell_{\boldsymbol{\beta}}(x)$)" where I is a fixed interval. Moreover, it is neither correct to say "for all $x\in[0,1)$, we have $\ell_{\boldsymbol{\alpha}}(x_{\boldsymbol{\alpha}}-x)\le_{\text{lex}}\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-x)$ (or $\ell_{\boldsymbol{\alpha}}(x_{\boldsymbol{\alpha}}-x)\ge_{\text{lex}}\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\alpha}}-x)$)". Indeed, this can already be seen while considering real bases, that is $\boldsymbol{\beta}=(\beta,\beta,\ldots)$ with $\beta>1$, as illustrated in Figure 1 (where the notation β, x_β and $\ell_\beta(\cdot)$ are used instead of $\boldsymbol{\beta}$, $x_{\boldsymbol{\beta}}$ and $\ell_{\boldsymbol{\beta}}(\cdot)$).

Remark 17. Note that, some results as Propositions 11 and 15 could also have been proved easily without any prerequisite from [4]. In this paper, a choice has been made, that is to use as much as possible Proposition 7 and results from [4].

5. Quasi-lazy expansions

In this section, we define the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}}-1$ in order to obtain an analogous of Parry's theorem [10] characterizing the lazy expansions of real numbers in $(x_{\boldsymbol{\beta}}-1,x_{\boldsymbol{\beta}}]$.

First, let us define the *quasi-greedy $\boldsymbol{\beta}$-expansion of* 1 by

$$d^*_{\boldsymbol{\beta}}(1)=\lim_{x\to1^-}d_{\boldsymbol{\beta}}(x) \tag{5.1}$$

where the limit is taken with respect to the prefix distance of infinite words. Note that this limit exists by left continuity of $d_{\boldsymbol{\beta}}$ in the neighborhood of 1.

Remark 18. The quasi-greedy $\boldsymbol{\beta}$-expansion of 1 obtained in (5.1) coincides with the one defined in [4]. In fact, let $t_0t_1\cdots$ denote the quasi-greedy $d^*_{\boldsymbol{\beta}}(1)$ from [4]. By [4, Theorem 26 and Corollary 36], for all $n\in\mathbb{N}$, the word $t_0\cdots t_n0^\omega$ is the greedy $\boldsymbol{\beta}$-expansion of a real number in $x_n\in[0,1)$. We have $\lim_{n\to+\infty}x_n=1$ and $\lim_{n\to+\infty}d_{\boldsymbol{\beta}}(x_n)=d^*_{\boldsymbol{\beta}}(1)$. Hence, in what follows, the results from [4] in terms of $d^*_{\boldsymbol{\beta}}(1)$ can be used.

Note that, in [4], we made a choice of definition for the greedy $\boldsymbol{\beta}$-expansion of 1 and defined $d^*_{\boldsymbol{\beta}}(1)$ accordingly. However, in this paper I decided not to define the greedy $\boldsymbol{\beta}$-expansion of 1. In fact, if this were the case, one would have expected to define the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}}-1$ analogously. This would have been done by extending the lazy algorithm over $x_{\boldsymbol{\beta}}-1$ as in [5]. However, in that case, $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1)$ would have not been the image of $d_{\boldsymbol{\beta}}(1)$, chosen as in [4], by the map $\theta_{\boldsymbol{\beta}}$ when β_0 is an integer since if $\beta_0\in\mathbb{N}_{\ge2}$, we have $d_{\boldsymbol{\beta}}(1)=\beta_00^\omega$ whereas the first letter of $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1)$ is 0.

In order to get similar results from [4] for lazy expansions, we define the *quasi-lazy $\boldsymbol{\beta}$-expansion* of $x_{\boldsymbol{\beta}} - 1$ as follows:

$$\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \lim_{x \to (x_{\boldsymbol{\beta}}-1)^+} \ell_{\boldsymbol{\beta}}(x). \tag{5.2}$$

Again, this limit exists by right continuity of $\ell_{\boldsymbol{\beta}}$ in the neighborhood of $x_{\boldsymbol{\beta}} - 1$. Let us first prove that, similarly to Proposition 7, the "flip" of the quasi-greedy $\boldsymbol{\beta}$-expansions of 1 is the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$.

Proposition 19. *We have $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \theta_{\boldsymbol{\beta}}(d^*_{\boldsymbol{\beta}}(1))$.*

Proof. Consider a sequence of real numbers $(x_n)_{n\in\mathbb{N}} \in [0,1)^{\mathbb{N}}$ such that $\lim_{n\to+\infty} x_n = 1$. We have $(x_{\boldsymbol{\beta}} - x_n)_{n\in\mathbb{N}} \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]^{\mathbb{N}}$ and $\lim_{n\to+\infty}(x_{\boldsymbol{\beta}} - x_n) = x_{\boldsymbol{\beta}} - 1$. Hence, by continuity of $\theta_{\boldsymbol{\beta}}$ and by Proposition 7, we get

$$\begin{aligned} \theta_{\boldsymbol{\beta}}(d^*_{\boldsymbol{\beta}}(1)) &= \lim_{n\to+\infty} \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x_n)) \\ &= \lim_{n\to+\infty} \ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x_n) \\ &= \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1). \end{aligned}$$ □

Example 20. Consider the alternate base from Example 1. We have $d^*_{\boldsymbol{\beta}}(1) = 200(10)^\omega$, $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$, $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = 012(02)^\omega$ and $\ell_{\boldsymbol{\beta}^{(1)}}(x_{\boldsymbol{\beta}^{(1)}} - 1) = (02)^\omega$.

Proposition 21. *The quasi-lazy expansion $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ is a $\boldsymbol{\beta}$-representation of $x_{\boldsymbol{\beta}} - 1$.*

Proof. This is direct by Proposition 19, [4, Proposition 22] and (3.1). □

Note that, in comparison with the quasi-greedy $\boldsymbol{\beta}$-expansion of 1 which is always infinite, the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ can be finite.

Example 22. Consider an alternate base $\boldsymbol{\beta} = \overline{(\beta_0, \ldots, \beta_{p-1})}$ such that for all $i \in [\![0, p-1]\!]$, $\beta_i \in \mathbb{N}_{\geq 2}$. We get $d^*_{\boldsymbol{\beta}^{(i)}}(1) = ((\beta_i - 1)\cdots(\beta_{p-1}-1)(\beta_0-1)\cdots(\beta_{i-1}-1))^\omega$ and since $x_{\boldsymbol{\beta}^{(i)}} = 1$ for all $i \in [\![0, p-1]\!]$, we have $\ell^*_{\boldsymbol{\beta}^{(i)}}(0) = 0^\omega$.

The following result gives a necessary condition on the Cantor base $\boldsymbol{\beta}$ to have a finite quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$.

Proposition 23. *If the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ is finite of length $n \in \mathbb{N}$, then $x_{\boldsymbol{\beta}^{(n)}} = 1$.*

Proof. Suppose that $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \ell_0 \cdots \ell_{n-1} 0^\omega$ with $n \in \mathbb{N}$ and $\ell_{n-1} \neq 0$ (if it exists, that is if $n \neq 0$). By Proposition 19, we get that

$$d^*_{\boldsymbol{\beta}}(1) = (\lceil \beta_0 \rceil - 1 - \ell_0)\cdots(\lceil \beta_{n-1} \rceil - 1 - \ell_{n-1})(\lceil \beta_n \rceil - 1)(\lceil \beta_{n+1} \rceil - 1)\cdots.$$

However, by [4, Proposition 30], we know that

$$\sigma^n(d^*_{\boldsymbol{\beta}}(1)) = (\lceil \beta_n \rceil - 1)(\lceil \beta_{n+1} \rceil - 1)\cdots \leq_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1).$$

Hence, we obtain that $\sigma^n(d^*_{\boldsymbol{\beta}}(1)) = d^*_{\boldsymbol{\beta}^{(n)}}(1)$. We conclude that

$$\begin{aligned} x_{\boldsymbol{\beta}^{(n)}} &= \mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(d^*_{\boldsymbol{\beta}}(1))) \\ &= \mathrm{val}_{\boldsymbol{\beta}^{(n)}}(d^*_{\boldsymbol{\beta}^{(n)}}(1)) \\ &= 1. \end{aligned}$$ □

Corollary 24. *If the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ is finite of length $n \in \mathbb{N}$, then $\beta_k \in \mathbb{N}_{\geq 2}$ for all $k \geq n$.*

As the following example shows, the necessary conditions given by the previous proposition and corollary are not sufficient.

Example 25. Consider the Cantor base $\boldsymbol{\beta} = (\frac{4}{3}, 2, 2, 2, 2, 2 \cdots)$. We have $x_{\boldsymbol{\beta}} = \frac{3}{2}$ and $x_{\boldsymbol{\beta}^{(n)}} = 1$ for all $n \geq 1$. However, we have $d^*_{\boldsymbol{\beta}}(1) = (10)^\omega$ and $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = (01)^\omega$.

An infinite word in $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ is said *ultimately maximal* if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, the $(n+1)^{\text{st}}$ letter of $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ is $\lceil\beta_n\rceil - 1$.

Lemma 26. *The infinite word $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ cannot be ultimately maximal.*

Proof. This is a direct consequence of Proposition 19 since $d^*_{\boldsymbol{\beta}}(1)$ is infinite. □

We now prove that $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ is lexicographically smaller than all $\boldsymbol{\beta}$-representations of real numbers in $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ belonging to $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$.

Proposition 27. *If a is an infinite word in $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$, then $a >_{\text{lex}} \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$.*

Proof. Let a be an infinite word in $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$. Then $\theta_{\boldsymbol{\beta}}(a)$ is an infinite word over $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ and by (3.1), we have $\mathrm{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) = x_{\boldsymbol{\beta}} - \mathrm{val}_{\boldsymbol{\beta}}(a) \in [0, 1)$. By [4, Proposition 23], we get that $\theta_{\boldsymbol{\beta}}(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$. Moreover, by Proposition 19, we have $d^*_{\boldsymbol{\beta}}(1) = \theta_{\boldsymbol{\beta}}(\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1))$. Hence, by (3.2), we conclude that $a >_{\text{lex}} \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$. □

Note that, similarly to Proposition 13, Proposition 27 is weaker than its analogous greedy one [4, Proposition 23] since we fix the alphabet of the $\boldsymbol{\beta}$-representations. A stronger result cannot be stated as illustrated in the next example.

Example 28. Continuing Examples 14 and 20, the infinite word 003330^ω is a $\boldsymbol{\beta}$-representation of $8 - 2\sqrt{13}$. However $003330^\omega <_{\text{lex}} 012(02)^\omega = \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$.

By [4, Proposition 23], the word $d^*_{\boldsymbol{\beta}}(1)$ is lexicographically maximal among all infinite $\boldsymbol{\beta}$-representations of all real numbers in $[0, 1]$. The following result gives the translation of this property in terms of the lazy representations.

Proposition 29. *The quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ is the lexicographically least $\boldsymbol{\beta}$-representation of $x_{\boldsymbol{\beta}} - 1$ in $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ that is not ultimately maximal.*

Proof. By Proposition 21 and Lemma 26, the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ is a $\boldsymbol{\beta}$-representation of $x_{\boldsymbol{\beta}} - 1$ in $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ which is not ultimately maximal. Moreover, let a be an infinite word in $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = x_{\boldsymbol{\beta}} - 1$ and suppose that $a <_{\text{lex}} \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$. As above, we get $\theta_{\boldsymbol{\beta}}(a) >_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$ with $\mathrm{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) = 1$. By [4, Proposition 23], the word $\theta_{\boldsymbol{\beta}}(a)$ must be a finite $\boldsymbol{\beta}$-representation of 1. By setting N to the length of the longest prefix of $\theta_{\boldsymbol{\beta}}(a)$ not ending with 0, we get $a_n = \lceil\beta_n\rceil - 1$ for all $n \geq N$, that is a is ultimately maximal in $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$. □

6. Admissible sequences

We let $D'_{\boldsymbol{\beta}}$ denote the subset of $A^{\mathbb{N}}_{\boldsymbol{\beta}}$ of all lazy $\boldsymbol{\beta}$-expansions of real numbers in the interval $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and let $S'_{\boldsymbol{\beta}}$ denote the topological closure of $D'_{\boldsymbol{\beta}}$ with respect to the prefix distance of infinite words:

$$D'_{\boldsymbol{\beta}} = \{\ell_{\boldsymbol{\beta}}(x) \colon x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]\} \quad \text{and} \quad S'_{\boldsymbol{\beta}} = \overline{D'_{\boldsymbol{\beta}}}.$$

The following result links the sets $D'_{\boldsymbol{\beta}}$ and $S'_{\boldsymbol{\beta}}$ with their analogous greedy ones $D_{\boldsymbol{\beta}} = \{d_{\boldsymbol{\beta}}(x) \colon x \in [0, 1)\}$ and $S_{\boldsymbol{\beta}} = \overline{D_{\boldsymbol{\beta}}}$ from [4].

Proposition 30. *The maps* $\theta_{\boldsymbol{\beta}}\big|_{D_{\boldsymbol{\beta}}}\colon D_{\boldsymbol{\beta}} \to D'_{\boldsymbol{\beta}}$ *and* $\theta_{\boldsymbol{\beta}}\big|_{S_{\boldsymbol{\beta}}}\colon S_{\boldsymbol{\beta}} \to S'_{\boldsymbol{\beta}}$ *are both bijective.*

Proof. By Proposition 7, the map $\theta_{\boldsymbol{\beta}}\big|_{D_{\boldsymbol{\beta}}}$ is well defined and surjective. Hence, by continuity of the map $\theta_{\boldsymbol{\beta}}$, the map $\theta_{\boldsymbol{\beta}}\big|_{S_{\boldsymbol{\beta}}}$ is also well defined and surjective. Moreover, since the map $\theta_{\boldsymbol{\beta}}$ is injective, so are the maps $\theta_{\boldsymbol{\beta}}\big|_{D_{\boldsymbol{\beta}}}$ and $\theta_{\boldsymbol{\beta}}\big|_{S_{\boldsymbol{\beta}}}$. □

Note that, in the particular case of alternate bases, Proposition 30 can be deduced from [5, Remark 6.3].

Proposition 31. *Let* $a, b \in S'_{\boldsymbol{\beta}}$.

(1) *If* $a <_{\text{lex}} b$ *then* $\text{val}_{\boldsymbol{\beta}}(a) \le \text{val}_{\boldsymbol{\beta}}(b)$.
(2) *If* $\text{val}_{\boldsymbol{\beta}}(a) < \text{val}_{\boldsymbol{\beta}}(b)$ *then* $a <_{\text{lex}} b$.

Proof. Suppose that $a, b \in S'_{\boldsymbol{\beta}}$ are such that $a <_{\text{lex}} b$. By Proposition 30 and (3.2), we have $\theta_{\boldsymbol{\beta}}(a), \theta_{\boldsymbol{\beta}}(b) \in S_{\boldsymbol{\beta}}$ and $\theta_{\boldsymbol{\beta}}(a) >_{\text{lex}} \theta_{\boldsymbol{\beta}}(b)$. By [4, Proposition 31], we $\text{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) \ge \text{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(b))$. We conclude the proof of the first item by (3.1). The second item immediately follows. □

We are now able to state a Parry-like theorem for Cantor real bases in the lazy framework.

Theorem 32. *Let* a *be an infinite word over* $\mathbb{N}$.

(1) *The word* a *belongs to* $D'_{\boldsymbol{\beta}}$ *if and only if* $a \in \prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ *and for all* $n \in \mathbb{N}$,
$$\sigma^n(a) >_{\text{lex}} \ell^*_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - 1).$$
(2) *The word* a *belongs to* $S'_{\boldsymbol{\beta}}$ *if and only if* $a \in \prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ *and for all* $n \in \mathbb{N}$,
$$\sigma^n(a) \ge_{\text{lex}} \ell^*_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - 1).$$

Proof. Let a be an infinite word. We have $a \in D'_{\boldsymbol{\beta}}$ if and only if $a \in \prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ and $\theta_{\boldsymbol{\beta}}(a) \in D_{\boldsymbol{\beta}}$. Moreover, by [4, Theorem 26], we have $\theta_{\boldsymbol{\beta}}(a) \in D_{\boldsymbol{\beta}}$ if and only if $\sigma^n(\theta_{\boldsymbol{\beta}}(a)) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$ for all $n \in \mathbb{N}$. However, for all $n \in \mathbb{N}$, by Lemma 10, we have $\sigma^n(\theta_{\boldsymbol{\beta}}(a)) = \theta_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a))$ and by Proposition 19, we have $d^*_{\boldsymbol{\beta}^{(n)}}(1) = \theta_{\boldsymbol{\beta}^{(n)}}(\ell^*_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - 1))$. Hence, the first item follows from (3.2). The second item can be proved in a similar fashion by using [4, Proposition 30]. □

Example 33. Consider $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$ from Example 1. In view of Example 20, the sequence $a = (2120)^\omega$ belongs to $D'_{\boldsymbol{\beta}}$.

Note that in Theorem 32, the hypothesis that a belongs to $\prod_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ is required. For otherwise, any sequence a such that $a_n > \lceil\beta_n\rceil - 1$ for all $n \in \mathbb{N}$ would belong to $D'_{\boldsymbol{\beta}}$.

As a consequence of Theorem 32, we can characterize the set $D'_{\boldsymbol{\beta}}$ by translating [4, Proposition 34 and Corollaries 35 and 36] to the lazy framework. To do so, we define sets of finite words $X'_{\boldsymbol{\beta},n}$ for $n \in \mathbb{N}_{\ge 1}$ as follows. If $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \ell_0\ell_1\cdots$ then, for all $n \in \mathbb{N}_{\ge 1}$, we let
$$X'_{\boldsymbol{\beta},n} = \{\ell_0 \cdots \ell_{n-2}s \colon s \in [\![\ell_{n-1} + 1, \lceil\beta_{n-1}\rceil - 1]\!]\}.$$
Note that $X'_{\boldsymbol{\beta},n}$ is empty if and only if $\ell_{n-1} = \lceil\beta_{n-1}\rceil - 1$.

Proposition 34. *We have*
$$D'_{\boldsymbol{\beta}} = \bigcup_{n_0\in\mathbb{N}_{\ge 1}} X'_{\boldsymbol{\beta},n_0}\Bigg(\bigcup_{n_1\in\mathbb{N}_{\ge 1}} X'_{\boldsymbol{\beta}^{(n_0)},n_1}\Bigg(\bigcup_{n_2\in\mathbb{N}_{\ge 1}} X'_{\boldsymbol{\beta}^{(n_0+n_1)},n_2}\Bigg(\quad\cdots\quad\Bigg)\Bigg)\Bigg).$$

Therefore, we have $D'_{\boldsymbol{\beta}} = \bigcup_{n\in\mathbb{N}_{\ge 1}} X'_{\boldsymbol{\beta},n} D'_{\boldsymbol{\beta}^{(n)}}$ *and any prefix of* $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1)$ *belongs to* $\mathrm{Pref}(D'_{\boldsymbol{\beta}})$.

Proof. This follows from Propositions 19, 30 and [4, Proposition 34] since $w_0w_1\cdots w_{n-1} \in X'_{\boldsymbol{\beta},n}$ if and only if $(\lceil\beta_0\rceil - 1 - w_0)(\lceil\beta_1\rceil - 1 - w_1)\cdots(\lceil\beta_{n-1}\rceil - 1 - w_{n-1}) \in X_{\boldsymbol{\beta},n}$. □

As in the greedy case, for lazy expansions in alternate bases, Proposition 34 can be straightened as follows. Consider an alternate base $\boldsymbol{\beta}$ of length p. We define sets of finite words $Y'_{\boldsymbol{\beta},h}$ for $h \in [\![0, p-1]\!]$ as follows. If $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \ell_0\ell_1\cdots$ then, for all $h \in [\![0, p-1]\!]$, we let

$$Y'_{\boldsymbol{\beta},h} = \{\ell_0\cdots\ell_{n-2}s \colon n \in \mathbb{N}_{\ge 1},\ n \bmod p = h,\ s \in [\![\ell_{n-1}+1, \lceil\beta_{n-1}\rceil - 1]\!]\}.$$

Note that $Y'_{\boldsymbol{\beta},h}$ is empty if and only if for all $n \in \mathbb{N}_{\ge 1}$ such that $n \bmod p = h$, $\ell_{n-1} = \lceil\beta_{n-1}\rceil - 1$. Moreover, unlike the sets $X'_{\boldsymbol{\beta},n}$ defined above, the sets $Y'_{\boldsymbol{\beta},h}$ can be infinite.

Proposition 35. *Let* $\boldsymbol{\beta}$ *be an alternate base of length* p. *We have*

$$D'_{\boldsymbol{\beta}} = \bigcup_{h_0=0}^{p-1} Y'_{\boldsymbol{\beta},h_0}\left(\bigcup_{h_1=0}^{p-1} Y'_{\boldsymbol{\beta}^{(h_0)},h_1}\left(\bigcup_{h_2=0}^{p-1} Y'_{\boldsymbol{\beta}^{(h_0+h_1)},h_2}\left(\quad\cdots\quad\right)\right)\right).$$

Therefore, we have $D'_{\boldsymbol{\beta}} = \bigcup_{h=0}^{p-1} Y'_{\boldsymbol{\beta},h} D'_{\boldsymbol{\beta}^{(h)}}$.

7. The lazy $\boldsymbol{\beta}$-shift

This section is concerned with the study of the lazy $\boldsymbol{\beta}$-shift. First, let us define

$$\Delta'_{\boldsymbol{\beta}} = \bigcup_{n\in\mathbb{N}} D'_{\boldsymbol{\beta}^{(n)}} \quad\text{and}\quad \Sigma'_{\boldsymbol{\beta}} = \overline{\Delta'_{\boldsymbol{\beta}}}.$$

By Proposition 30, we get

$$\Delta'_{\boldsymbol{\beta}} = \bigcup_{n\in\mathbb{N}} \theta_{\boldsymbol{\beta}^{(n)}}(D_{\boldsymbol{\beta}^{(n)}}). \tag{7.1}$$

Proposition 36. *The sets* $\Delta'_{\boldsymbol{\beta}}$ *and* $\Sigma'_{\boldsymbol{\beta}}$ *are both shift-invariant.*

Proof. Let a be an infinite word over $\mathbb{N}$. By (7.1), if a belongs to $\Delta'_{\boldsymbol{\beta}}$, then there exists $n \in \mathbb{N}$ and an infinite word $b \in D_{\boldsymbol{\beta}^{(n)}}$ such that $a = \theta_{\boldsymbol{\beta}^{(n)}}(b)$. We obtain that $\sigma(a) = \sigma(\theta_{\boldsymbol{\beta}^{(n)}}(b)) = \theta_{\boldsymbol{\beta}^{(n+1)}}(\sigma(b))$ by Lemma 10. By [4, Theorem 26], $\sigma(b) \in D_{\boldsymbol{\beta}^{(n+1)}}$ so $\sigma(a) \in D'_{\boldsymbol{\beta}^{(n+1)}}$. Then, it is easily seen that if $a \in S'_{\boldsymbol{\beta}^{(n)}}$ then $\sigma(a) \in S'_{\boldsymbol{\beta}^{(n+1)}}$. □

Since the set $\Sigma'_{\boldsymbol{\beta}}$ is shift-invariant and closed with respect to the topology induced by the prefix distance on infinite words, we conclude that the subset $\Sigma'_{\boldsymbol{\beta}}$ of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ is a subshift, which we call the *lazy* $\boldsymbol{\beta}$*-shift*.

Remark 37. It is important to remark that the lazy $\boldsymbol{\beta}$-shift is not the lazy $\boldsymbol{\beta}$-shift defined in [5]. In fact, as said in [5, Remark 36], there is two ways to extend the notion of β-shift from the real base case to the alternate bases or more generally to the Cantor base framework.

Recall that the set of finite factors and the set of prefixes of all elements in a language L are respectively denoted $\mathrm{Fac}(L)$ and $\mathrm{Pref}(L)$. Let us now study the factors of the lazy $\boldsymbol{\beta}$-shift.

Proposition 38. *We have* $\mathrm{Fac}(D'_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Delta'_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}})$.

Proof. By definition, we have $\mathrm{Fac}(D'_{\boldsymbol\beta}) \subseteq \mathrm{Fac}(\Delta'_{\boldsymbol\beta}) = \mathrm{Fac}(\Sigma'_{\boldsymbol\beta})$. It remains to show that $\mathrm{Fac}(D'_{\boldsymbol\beta}) \supseteq \mathrm{Fac}(\Delta'_{\boldsymbol\beta})$. Let $f \in \mathrm{Fac}(\Delta'_{\boldsymbol\beta})$. By (7.1), there exist $n \in \mathbb{N}$ and $b \in D_{\boldsymbol\beta^{(n)}}$ such that $f \in \mathrm{Fac}(\theta_{\boldsymbol\beta^{(n)}}(b))$. In particular, $f \in \mathrm{Fac}(\theta_{\boldsymbol\beta}(0^n b))$ where, by [4, Theorem 26], $0^n b \in D_{\boldsymbol\beta}$. We obtain that $f \in \mathrm{Fac}(\theta_{\boldsymbol\beta}(D_{\boldsymbol\beta})) = \mathrm{Fac}(D'_{\boldsymbol\beta})$ by Proposition 30. □

Corollary 39. *We have*

$$\mathrm{Fac}(\Sigma'_{\boldsymbol\beta}) = \bigcup_{n\in\mathbb{N}} \theta_{\boldsymbol\beta^{(n)}}\big(\mathrm{Pref}(D_{\boldsymbol\beta^{(n)}})\big).$$

Proof. By Propositions 36 and 38, we have $\mathrm{Fac}(\Sigma'_{\boldsymbol\beta}) = \mathrm{Pref}(\Delta'_{\boldsymbol\beta}) = \bigcup_{n\in\mathbb{N}} \mathrm{Pref}(D'_{\boldsymbol\beta^{(n)}})$. The conclusion follows from Proposition 30. □

In the alternate base framework, an analogue of Bertrand-Mathis' theorem [1] can be stated for the lazy $\boldsymbol\beta$-shift. To do so, recall that a subshift S of $A^{\mathbb{N}}$ is called *sofic* if the language $\mathrm{Fac}(S) \subseteq A^*$ is accepted by a finite automaton.

Theorem 40. *Let $\boldsymbol\beta$ be an alternate base of length p. The lazy $\boldsymbol\beta$-shift $\Sigma'_{\boldsymbol\beta}$ is sofic if and only if for all $i \in [\![0, p-1]\!]$, $\ell^*_{\boldsymbol\beta^{(i)}}(x_{\boldsymbol\beta^{(i)}} - 1)$ is ultimately periodic.*

In order to prove this result, let us construct an automaton $\mathcal{A}'_{\boldsymbol\beta}$ in the case where all quasi-lazy expansions are ultimately periodic and state some results in order to link this automaton with the one used in the greedy case (see [4, Theorem 48]) called $\mathcal{A}_{\boldsymbol\beta}$. Roughly, if all the quasi-lazy expansions are ultimately periodic, then so are the quasi-greedy expansions and the "image" of the automaton $\mathcal{A}_{\boldsymbol\beta}$ under the maps $\theta_{\boldsymbol\beta^{(i)}}$ with $i \in [\![0, p-1]\!]$ is an automaton accepting $\mathrm{Fac}(\Sigma'_{\boldsymbol\beta})$. This notion of "image" of the automaton under the maps $\theta_{\boldsymbol\beta^{(i)}}$ will be clearer in what follows, more precisely in Lemmas 42 and 44.

Henceforth, let $\boldsymbol\beta$ be an alternate base of length p and suppose that for all $i \in [\![0, p-1]\!]$, $\ell^*_{\boldsymbol\beta^{(i)}}(x_{\boldsymbol\beta^{(i)}} - 1)$ is ultimately periodic and write[2]

$$\ell^*_{\boldsymbol\beta^{(i)}}(x_{\boldsymbol\beta^{(i)}} - 1) = \ell_0^{(i)} \cdots \ell_{m_i-1}^{(i)} \big(\ell_{m_i}^{(i)} \cdots \ell_{m_i+n_i-1}^{(i)}\big)^\omega.$$

Without loss of generality, from now on, suppose that n_i is a multiple of p (it suffices to take the least common multiple of p and the length of the period). For all $i \in [\![0, p-1]\!]$, by Proposition 19, we get[3]

$$d^*_{\boldsymbol\beta^{(i)}}(1) = t_0^{(i)} \cdots t_{m_i-1}^{(i)} \big(t_{m_i}^{(i)} \cdots t_{m_i+n_i-1}^{(i)}\big)^\omega$$

with $t_n^{(i)} = \lceil \beta_{i+n} \rceil - 1 - \ell_n^{(i)}$ for all $n \in [\![0, m_i+n_i-1]\!]$. Hence, all quasi-greedy expansions of 1 are ultimately periodic. Let $\mathcal{A}_{\boldsymbol\beta}$ be the automaton over the alphabet $A_{\boldsymbol\beta}$ from [4, Section 7.3] which accepts $\mathrm{Fac}(\Sigma_{\boldsymbol\beta})$ (see [4, Theorem 48]). Recall that $\mathcal{A}_{\boldsymbol\beta} = (Q, I, F, A_{\boldsymbol\beta}, \delta)$ where

$$\begin{aligned} Q &= \big\{q_{i,j,k} \colon i,j \in [\![0,p-1]\!],\ k \in [\![0, m_i+n_i-1]\!]\big\},\\ I &= \big\{q_{i,i,0} \colon i \in [\![0,p-1]\!]\big\},\\ F &= Q \end{aligned}$$

and, for each $i, j \in [\![0, p-1]\!]$ and each $k \in [\![0, m_i+n_i-1]\!]$, we have

$$\delta(q_{i,j,k}, t_k^{(i)}) = \begin{cases} q_{i,(j+1) \bmod p, k+1} & \text{if } k \neq m_i+n_i-1 \\ q_{i,(j+1) \bmod p, m_i} & \text{else} \end{cases}$$

[2]Recall that $\ell^*_{\boldsymbol\beta^{(i)}}(x_{\boldsymbol\beta^{(i)}} - 1)$ can be finite, hence, n_i can be equal to 1 and $\ell_{m_i}^{(i)} = 0$.

[3]Note that the preperiod and period m_i and n_i may be not minimal.

and for all $s \in [\![0, t_k^{(i)} - 1]\!]$, we have

$$\delta(q_{i,j,k}, s) = q_{(j+1) \bmod p,(j+1) \bmod p,0}.$$

Define the automaton $\mathcal{A}'_{\boldsymbol{\beta}} = (Q, I, F, A_{\boldsymbol{\beta}}, \delta')$ where for each $i, j \in [\![0, p-1]\!]$ and each $k \in [\![0, m_i + n_i - 1]\!]$, we have

$$\delta'(q_{i,j,k}, \ell_k^{(i)}) = \begin{cases} q_{i,(j+1) \bmod p,k+1} & \text{if } k \neq m_i + n_i - 1 \\ q_{i,(j+1) \bmod p,m_i} & \text{else} \end{cases} \tag{7.2}$$

and for all $s \in [\![\ell_k^{(i)} + 1, \lceil \beta_j \rceil - 1]\!]$, we have

$$\delta'(q_{i,j,k}, s) = q_{(j+1) \bmod p,(j+1) \bmod p,0}. \tag{7.3}$$

Since we supposed that the parameters n_i, with $i \in [\![0, p-1]\!]$, were multiples of p, we get the following result.

Lemma 41. *In the automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$, for all $i, j \in [\![0, p-1]\!]$ and $k \in [\![0, m_i + n_i - 1]\!]$, the state $q_{i,j,k}$ is accessible if and only if $i + k \equiv j \pmod p$.*

Proof. Let us prove the result for the automaton $\mathcal{A}'_{\boldsymbol{\beta}}$. The reasoning for the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ is similar. Suppose that $i + k \equiv j \pmod p$. There exists a path from $q_{i,i,0}$ to $q_{i,j,k}$ labeled by $\ell_0^{(i)} \cdots \ell_k^{(i)}$. In fact, for all $k' \in [\![0, k-1]\!]$, we have

$$\delta'(q_{i,(i+k') \bmod p,k'}, \ell_{k'}^{(i)}) = q_{i,(i+k'+1) \bmod p,k'+1}. \tag{7.4}$$

Conversely, let $i, j \in [\![0, p-1]\!]$ and $k \in [\![0, m_i + n_i - 1]\!]$. Suppose that the state $q_{i,j,k}$ is accessible. Let c be an initial path ending in $q_{i,j,k}$. By definition of the transitions, if a path starts in $q_{i',i',0}$ with $i' \in [\![0, p-1]\!] \setminus \{i\}$ and ends in $q_{i,j,k}$ then it necessarily goes through $q_{i,i,0}$ by using a transition of the form (7.3). Hence, we may suppose that the path c only uses transitions of the form (7.2). The conclusion follows since for all $k' \in [\![0, k-1]\!]$, we have (7.4) and

$$\delta'(q_{i,(i+m_i+n_i-1) \bmod p,m_i+n_i-1}, \ell_{m_i+n_i-1}^{(i)}) = q_{i,(i+m_i+n_i) \bmod p,m_i}$$

where $n_i \equiv 0 \pmod p$ by assumption. □

By the previous lemma, from now on, we consider the automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$ by preserving only the set

$$\{q_{i,(i+k) \bmod p,k} \colon i \in [\![0, p-1]\!],\ k \in [\![0, m_i + n_i - 1]\!]\}$$

of accessible states and we keep the same notation.

Lemma 42. *Let $a \in A_{\boldsymbol{\beta}}$, $i_1, i_2 \in [\![0, p-1]\!]$ and $k_1 \in [\![0, m_{i_1} + n_{i_1} - 1]\!]$, $k_2 \in [\![0, m_{i_2} + n_{i_2} - 1]\!]$. We have*

$$\delta(q_{i_1,(i_1+k_1) \bmod p,\, k_1}, a) = q_{i_2,(i_2+k_2) \bmod p,\, k_2}$$

if and only if

$$\delta'(q_{i_1,(i_1+k_1) \bmod p,\, k_1}, \lceil \beta_{i_1+k_1} \rceil - 1 - a) = q_{i_2,(i_2+k_2) \bmod p,\, k_2}.$$

Proof. Fix $a \in A_{\boldsymbol{\beta}}$, $i \in [\![0, p-1]\!]$ and $k \in [\![0, m_i + n_i - 1]\!]$. By definition of the automaton $\mathcal{A}_{\boldsymbol{\beta}}$, from $q_{i,(i+k) \bmod p,\, k}$ we have the following transitions

$$\delta(q_{i,(i+k) \bmod p,\, k}, a) = \begin{cases} q_{i,(i+k+1) \bmod p,\, k+1} & \text{if } a = t_k^{(i)} \text{ and } k \neq m_i + n_i - 1 \\ q_{i,(i+m_i) \bmod p,\, m_i} & \text{if } a = t_k^{(i)} \text{ and } k = m_i + n_i - 1 \\ q_{(i+k+1) \bmod p,(i+k+1) \bmod p,\, 0} & \text{if } a \in [\![0, t_k^{(i)} - 1]\!]. \end{cases}$$

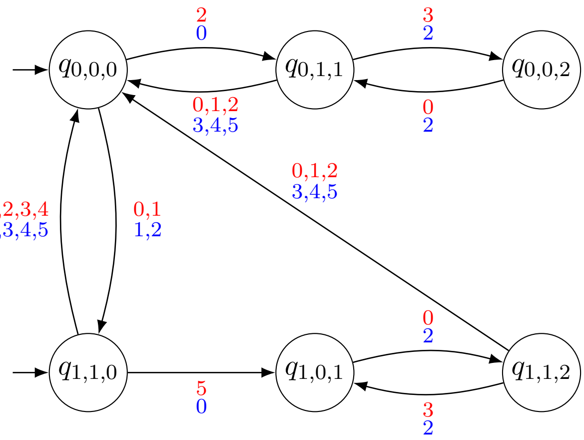

FIGURE 2. An accessible automaton accepting $\mathrm{Fac}(\Sigma_{\overline{(\varphi^2,3+\sqrt{5})}})$ (labels above and red) and $\mathrm{Fac}(\Sigma'_{\overline{(\varphi^2,3+\sqrt{5})}})$ (labels below and blue).

Similarly, by definition of $\mathcal{A}'_{\boldsymbol{\beta}}$, we have

$$\delta'(q_{i,(i+k) \bmod p,k}, a) = \begin{cases} q_{i,(i+k+1) \bmod p, k+1} & \text{if } a = \ell_k^{(i)} \text{ and } k \neq m_i + n_i - 1 \\ q_{i,(i+m_i) \bmod p, m_i} & \text{if } a = \ell_k^{(i)} \text{ and } k = m_i + n_i - 1 \\ q_{(i+k+1) \bmod p,(i+k+1) \bmod p, 0} & \text{if } a \in [\![\ell_k^{(i)}+1, \lceil \beta_{i+k} \rceil - 1]\!]. \end{cases}$$

We get the conclusion since $\ell_k^{(i)} = \lceil \beta_{i+k} \rceil - 1 - t_k^{(i)}$, and hence $a \in [\![0, t_k^{(i)} - 1]\!]$ if and only if $\lceil \beta_{i+k} \rceil - 1 - a \in [\![\ell_k^{(i)} + 1, \lceil \beta_{i+k} \rceil - 1]\!]$. □

Example 43. Let $\boldsymbol{\beta} = \overline{(\varphi^2, 3+\sqrt{5})}$. We have $d_{\boldsymbol{\beta}}(1) = 2(30)^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 5(03)^\omega$ and $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = 02^\omega$, $\ell_{\boldsymbol{\beta}^{(1)}}(x_{\boldsymbol{\beta}^{(1)}} - 1) = 02^\omega$. The corresponding accessible automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$ are depicted in Figure 2 with red and blue labels respectively.

Lemma 44. *Let $i \in [\![0, p-1]\!]$ and consider $w \in A_{\boldsymbol{\beta}}^{\mathbb{N}}$. The word w is accepted in $\mathcal{A}_{\boldsymbol{\beta}}$ from $q_{i,i,0}$ if and only if $\theta_{\boldsymbol{\beta}^{(i)}}(w)$ is accepted in $\mathcal{A}'_{\boldsymbol{\beta}}$ from $q_{i,i,0}$.*

Proof. This immediately follows from Lemma 42. □

We are now ready to prove Theorem 40.

Proof of Theorem 40. Suppose that, for all $i \in [\![0, p-1]\!]$, $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1)$ is ultimately periodic. For all $i \in [\![0, p-1]\!]$, let

$$\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) = \ell_0^{(i)} \cdots \ell_{m_i-1}^{(i)} \big(\ell_{m_i}^{(i)} \cdots \ell_{m_i+n_i-1}^{(i)}\big)^\omega$$

with n_i multiple of p. By Proposition 19, for all $i \in [\![0, p-1]\!]$, we obtain

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} \cdots t_{m_i-1}^{(i)} \big(t_{m_i}^{(i)} \cdots t_{m_i+n_i-1}^{(i)}\big)^\omega$$

with $t_n^{(i)} = \lceil \beta_{i+n} \rceil - 1 - \ell_n^{(i)}$ for all $n \in [\![0, m_i + n_i - 1]\!]$. Let $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$ be the automata associated with the greedy and lazy expansions respectively. By [4, Theorem 48], for each $i \in [\![0, p-1]\!]$, the language accepted in $\mathcal{A}_{\boldsymbol{\beta}}$ from the initial state $q_{i,i,0}$ is precisely $\mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. Hence, by Lemma 44, in $\mathcal{A}'_{\boldsymbol{\beta}}$ the language accepted from the initial state $q_{i,i,0}$ is precisely $\theta_{\boldsymbol{\beta}^{(i)}}(\mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}}))$. We get the conclusion by Corollary 39.

Conversely, suppose that there exists $j \in [\![0, p-1]\!]$ such that $\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}} - 1)$ is not ultimately periodic, then we prove that $\Sigma'_{\boldsymbol{\beta}}$ is not sofic. This follows the same lines as in the greedy case (see [4, Theorem 48]). Hence, in the subsequent, the main ideas of the proof are given. Let

$$\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) = \ell_0^{(i)} \ell_1^{(i)} \cdots \quad \text{for every } i \in [\![0, p-1]\!].$$

We define a partition $(G_1, \ldots, G_q)$ of $[\![0, p-1]\!]$ as follows. Let $r = \mathrm{Card}\{\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) \colon i \in [\![0, p-1]\!]\}$ and let $i_1, \ldots, i_r \in [\![0, p-1]\!]$ be such that $\ell^*_{\boldsymbol{\beta}^{(i_1)}}(x_{\boldsymbol{\beta}^{(i_1)}} - 1), \ldots, \ell^*_{\boldsymbol{\beta}^{(i_r)}}(x_{\boldsymbol{\beta}^{(i_r)}} - 1)$ are pairwise distinct and $\ell^*_{\boldsymbol{\beta}^{(i_1)}}(x_{\boldsymbol{\beta}^{(i_1)}} - 1) <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} \ell^*_{\boldsymbol{\beta}^{(i_r)}}(x_{\boldsymbol{\beta}^{(i_r)}} - 1)$. Let $q \in [\![1, r]\!]$ be the unique index such that $\ell^*_{\boldsymbol{\beta}^{(i_q)}}(x_{\boldsymbol{\beta}^{(i_q)}} - 1) = \ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}} - 1)$ where $\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}} - 1)$ is not ultimately periodic by assumption. We set

$$G_s = \{i \in [\![0, p-1]\!] \colon \ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) = \ell^*_{\boldsymbol{\beta}^{(i_s)}}(x_{\boldsymbol{\beta}^{(i_s)}} - 1)\} \quad \text{for } s \in [\![1, q-1]\!]$$

and

$$G_q = \{i \in [\![0, p-1]\!] \colon \ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) \geq_{\mathrm{lex}} \ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}} - 1)\}.$$

For each $s \in [\![1, q-1]\!]$, we write $G_s = \{i_{s,1}, \ldots, i_{s,\alpha_s}\}$ where $i_{s,1} < \ldots < i_{s,\alpha_s}$ and we use the convention that $i_{s,\alpha_s+1} = i_{s+1,1}$ for $s \leq q-2$ and $i_{q-1,\alpha_{q-1}+1} = j$. Moreover, we let $g \in \mathbb{N}_{\geq 1}$ be such that for all $i, i' \in [\![0, p-1]\!]$ such that $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) \neq \ell^*_{\boldsymbol{\beta}^{(i')}}(x_{\boldsymbol{\beta}^{(i')}} - 1)$, the length-$g$ prefixes of $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1)$ and $\ell^*_{\boldsymbol{\beta}^{(i')}}(x_{\boldsymbol{\beta}^{(i')}} - 1)$ are distinct. Then, for $s \in [\![1, q-1]\!]$, we define C_s to be the least $c \in \mathbb{N}_{\geq 1}$ such that $\ell^{(i_s)}_{g-1+c} < \lceil \beta_{i_s+g-1+c} \rceil - 1$. Finally, let $N \in \mathbb{N}_{\geq 1}$ be such that $pN \geq \max\{g, C_1, \ldots, C_{q-1}\}$.

For all $m \in \mathbb{N}$, consider

$$w^{(m)} = \left(\prod_{s=1}^{q-1} \prod_{k=1}^{\alpha_s} \ell_0^{(i_s)} \cdots \ell_{g-1}^{(i_s)} (\lceil \beta_{i_{s,k}+g} \rceil - 1) \cdots (\lceil \beta_{i_{s,k+1}+p(2N+1)-1} \rceil - 1) \right) \ell_0^{(j)} \cdots \ell_{m-1}^{(j)}.$$

Now, let $m, n \in \mathbb{N}$ be distinct. Since $\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}} - 1)$ is not ultimately periodic, $\sigma^m\big(\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}} - 1)\big) \neq \sigma^n\big(\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}} - 1)\big)$. Thus, there exists $k \in \mathbb{N}_{\geq 1}$ such that $\ell_m^{(j)} \cdots \ell_{m+k-2}^{(j)} = \ell_n^{(j)} \cdots \ell_{n+k-2}^{(j)}$ and $\ell_{m+k-1}^{(j)} \neq \ell_{n+k-1}^{(j)}$. Without loss of generality, we suppose that $\ell_{m+k-1}^{(j)} < \ell_{n+k-1}^{(j)}$. Let $z = \ell_m^{(j)} \cdots \ell_{m+k-1}^{(j)}$. Similarly to proof of [4, Theorem 48], it can be shown that $w^{(m)} z \in \mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}}) \cap \mathrm{Pref}(D'_{\boldsymbol{\beta}^{(i_{1,1})}})$ and $w^{(n)} z \notin \mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}})$. □

Remark 45. In the proof of the necessary condition of Theorem 40, the parameters $\{r, i_1, \ldots, i_r, q, G_1, \ldots, G_q, \ldots\}$ may not coincide with the ones in the necessary condition of [4, Theorem 48]. In fact, it may happen for example that there exist $i, j \in [\![0, p-1]\!]$, such that $d^*_{\boldsymbol{\beta}^{(i)}}(1) >_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(j)}}(1)$ whereas $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) \leq_{\mathrm{lex}} \ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}} - 1)$. For instance, this is illustrated by Examples 22 and 43.

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