## $S$-adic characterizaion of dendric shift spaces

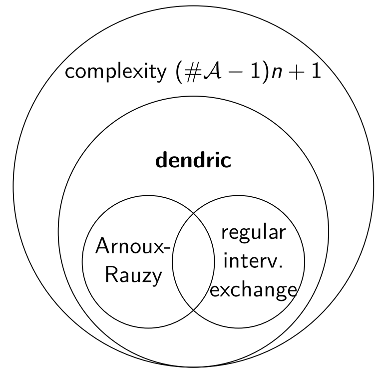
France Gheeraert

joint work with Marie Lejeune et Julien Leroy

8-9 July 2021

## Dendric shift spaces

Introduced in 2013 by V. Berthé, C. De Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer and G. Rindone.




## Definition

Given a (recurrent) language $\mathcal{L}$ (from an infinite word or a shift) and a word $w \in \mathcal{L}$

$$E^-_{\mathcal{L}}(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}\}$$
$$E^+_{\mathcal{L}}(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}\}$$
$$E_{\mathcal{L}}(w) = \{(a,b) \in E^-_{\mathcal{L}}(w) \times E^+_{\mathcal{L}}(w) \mid awb \in \mathcal{L}\}$$

**Definition**

The *extension graph* of $w \in \mathcal{L}$ is the bipartite graph $\mathcal{E}_{\mathcal{L}}(w)$ with vertices $E^-_{\mathcal{L}}(w) \sqcup E^+_{\mathcal{L}}(w)$ and such that, for all $a \in E^-_{\mathcal{L}}(w)$, $b \in E^+_{\mathcal{L}}(w)$, there is an edge $(a,b)$ if and only if $(a,b) \in E_{\mathcal{L}}(w)$.

A word is *dendric* it its extension graph is a tree.
A language is *dendric* if all its element are.

## Question

Can we find a condition $\mathcal{C}$ such that

A language $\mathcal{L}$ is (recurrent) dendric if and only if there exists a sequence $(\sigma_n : \mathcal{A}^*_{n+1} \to \mathcal{A}^*_n)$ of morphisms satisfying the condition $\mathcal{C}$ and such that

$$w \in \mathcal{L} \Leftrightarrow \exists N \in \mathbb{N}, a \in \mathcal{A}_{N+1} \text{ st. } w \in \mathrm{Fac}(\sigma_0 \dots \sigma_N(a))?$$

For example, there is a well-known condition for Sturmian languages, namely $(\sigma_n) \in \{L_0, L_1\}^{\mathbb{N}}$ and it is non ultimately constant, where

$$L_0 : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases} \qquad L_1 : \begin{cases} 0 \mapsto 10 \\ 1 \mapsto 1 \end{cases}$$

## Ternary case

In the ternary case ($\#\mathcal{A} = 3$), we have found such a condition:

the sequence $(\sigma_n)_{n \geq 0}$ is primitive and labels a path in a specific graph (with two vertices).

The construction is based on $S$-adic representations using return words (Berthé *et al.*)

Details can be found in the following article:
*$S$-adic characterization of minimal ternary dendric shifts*, to appear in *Ergodic Theory and Dynamical Systems*

## Open questions

Finding a condition $\mathcal{C}$ to characterize

1. dendric languages on larger alphabets
   - work in progress
2. languages of sub-linear complexity
   - $\mathcal{S}$-adic conjecture
3. or any other family of languages