

A general purpose formulation for nonsmooth dynamics with finite rotations: Application to the woodpecker toy

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Abstract

The aim of this work is to extend the finite element multibody dynamics approach to problems involving frictional contacts and impacts. The nonsmooth generalized- α (NSGA) scheme is adopted, which imposes bilateral and unilateral constraints both at position and velocity levels avoiding drift phenomena. This scheme can be implemented in a general purpose simulation code with limited modifications of pre-existing elements. The study of the woodpecker toy dynamics sets up a good example to show the capabilities of the NSGA scheme within the context of a general finite element framework. This example has already been studied by many authors who generally adopted a model with a minimal set of coordinates and small rotations. It is shown that good results are obtained using a general purpose finite element code for multibody dynamics, in which the equations of motion are assembled automatically and large rotations are easily taken into account. In addition, comparing results between different models of the woodpecker toy, the importance of modelling large rotations and the horizontal displacement of the woodpecker's sleeve is emphasized.

1 Introduction

The general aim of this work is to extend the nonlinear finite element approach for flexible multibody systems [14] to contact problems involving friction and impacts. Several authors studied finite element strategies for the solution of flexible impact problems from an implicit point of view, within the framework of the Newmark, Hilber-Hughes-Taylor (HHT) and energy-momentum integrators [20]. However, we are interested to solve not only flexible contact but also rigid contact between bodies and rigid bilateral constraints, as usually found when modeling multibody systems such as the woodpecker example. Nonsmooth contact dynamic methodologies, *e.g.* as those proposed in [24, 33, 23, 34, 1, 15], pose as interesting frameworks which solve efficiently problems involving strong discontinuities as the ones characterizing the phenomena just described. In this work, the nonsmooth generalized- α (NSGA) method extended to frictional contacts is adopted, based on the fact that it is specifically tailored for problems involving rigid and flexible components undergoing impacts with friction. Another nice property of the NSGA is that its implementation into a general finite element framework does not require a substantial modification of the library of elements. Once the integrator is implemented within the code, pre-existing elements can be readily used in the nonsmooth modeling of contact problems with impact. The adoption of a nonsmooth time integrator brings advantages even for contact problems involving only flexible elements, due to the fact that artificial impacts of numerical origin can be present. Additionally, the nonlinear finite element approach offers an efficient framework to deal with finite rotations in mechanical systems and to assemble the equations of motion automatically. The combination of the finite element approach with the NSGA solver thus leads to a powerful modeling and simulation tool for systems composed by rigid and/or flexible bodies including joints and contacts with friction and impacts. In this paper, the capabilities of this tool are illustrated based on the woodpecker toy case study.

The woodpecker toy is an example that is usually found in papers dealing with nonsmooth contact dynamics methodologies. It involves several rigid bodies which are subject to frictional contact with impacts with a rich dynamic transition between the gap, stick and slip states. The toy consists of a pole,



Figure 1: Woodpecker toy

a sleeve with an internal diameter slightly larger than the pole diameter, a woodpecker body with the beak and a spring which attaches the bird's body with the sleeve, see Fig. 1. It is a non conservative system where the gravity acts as the only energy source. The potential energy is not only transformed into kinetic energy but also dissipated through the frictional impacts.

The example was first introduced by Pfeiffer [27], where the loss of energy due to the impacts was represented by an equivalent heuristic model. Glocker and Pfeiffer [16] studied the problem including a contact model for the beak. The equations of motion were based on a Coulomb friction model and a Poisson impact law, and their solution was computed using the Moreau-Jean scheme. In an experiment, these authors measured a falling distance per cycle of 5.3 mm and a limit cycle frequency of $f = 9.2$ Hz. The problem was also studied in [29, 21, 22], where a minimal set of 3 coordinates was used to describe the woodpecker toy: the vertical position of the sleeve y , the sleeve rotation ϕ_M , and the bird rotation ϕ_S . In order to simplify the model, these authors additionally assumed that the bird undergoes small rotations and that the displacement of the sleeve in the horizontal direction can be neglected. Glocker and Studer [17] studied the problem with the same assumptions but using the Newton impact law. A bifurcation analysis of the system and an exhaustive explanation of all the phases of the motion of the woodpecker was done by Leine and Van Campen [22]. Slavic and Boltezar [31] extended the methodology of the previous works to account for contact constraints between bodies with arbitrary shapes, with a specific description of the beak shape. Recently, Charles *et al.* [8] presented an algorithm for problems with impacts and friction, where the woodpecker toy model from Glocker and Studer [16] is used as a benchmark test. Also recently, Peng *et al.* [25, 26] analyzed this problem using a symplectic method for time integration.

All the previous works dealing with the nonsmooth dynamics of the woodpecker toy specifically deduced the equations of motion for the considered problem resulting in a model with a minimal set of coordinates. They assumed small rotations and neglected the horizontal displacement of the sleeve, except for Slavic and Boltezar [31]. For all of them, the numerical solution was also obtained using first-order time integrators, such as the Moreau-Jean scheme.

The use of a model with a minimal set of coordinates has the advantage that the equations of motion are compact and only unilateral constraints are involved. However, this approach can hardly be automated in a general purpose software and the scalability to more complex mechanisms is quite difficult to achieve. Using a finite element approach, the equations of motion can be assembled automatically. It should be remarked that with the adoption of this paradigm, in addition to unilateral constraints, bilateral constraints dealing with kinematic joints and rigid bodies have to be managed. Using the nonsmooth generalized- α method both types of constraints can be exactly satisfied without any drift both at position and at velocity levels.

This paper extends the finite element multibody dynamics approach [14] to problems involving contacts. Currently, several nonsmooth contact dynamic algorithms are implemented in many codes, for example in LMGC90 [11], SICONOS [2], Chrono [18], Algoryx¹ (its bases are presented in [19]) and Oofelie [7]. Specifically, our nonsmooth contact algorithm was implemented in Oofelie which is a nonlinear modular code well-suited to model complex articulated systems including rigid and flexible components. This software is used for academic studies as well as for industrial applications. The equations of motion of mechanical systems exhibiting nonsmooth phenomena and finite rotations are assembled and solved in an automatic manner, as will be shown with the woodpecker toy example. The Newton impact model was chosen instead of the Poisson model. This selection is based on the fact that the Newton's law can

¹<https://www.algoryx.se>

be easily combined with the Coulomb's friction law. Furthermore, the Newton impact model was already implemented in the two versions of the nonsmooth generalized- α time integration scheme [10, 5].

The work is organized as follows. First, a brief explanation of the equations of motion and of the nonsmooth generalized- α time integrator is presented. Then, the woodpecker toy problem is defined and the strategy used to solve it is introduced. Finally, the numerical results are presented and discussed, and the conclusions are stated.

2 Equations of motion

According to the nonlinear finite element method, a mechanical system is represented using a set q of nodal position and orientation variables. For systems with finite rotations, q does not belong to a vector space but to a Lie group [6]. In Oofelie, the position of a node with respect to the inertial frame is treated as a vector $\mathbf{x} \in \mathbb{R}^3$ and the translation velocity is represented by a vector $\mathbf{u} = \dot{\mathbf{x}}$. The orientation of a node with respect to the inertial frame is represented by a rotation matrix $\mathbf{R}$ which belongs to the rotation group $SO(3)$ and the rotational velocity is represented by the angular velocity vector in body-axes $\boldsymbol{\Omega} \in \mathbb{R}^3$. We have $\dot{\mathbf{R}} = \mathbf{R}\mathbf{S}(\boldsymbol{\Omega})$, where the operator $\mathbf{S}(\cdot)$ maps a 3×1 vector into a 3×3 skew-symmetric matrix. Thus, the configuration variable q collects the position vectors and rotation matrices of all the nodes. The translation and angular velocity vectors are assembled in a global velocity vector $\mathbf{v}$, which is related to $\dot{q}$ by a kinematic compatibility condition of the general form $\dot{q} = h(q, \mathbf{v})$.

The equations of motion for a system with unilateral and bilateral constraints, like the woodpecker toy, are written in the following form with the constraints at velocity level:

$$\dot{q}^+ = h(q, \mathbf{v}^+) \quad (1a)$$

$$\mathbf{M}(q)d\mathbf{v} - \mathbf{g}_q^T d\mathbf{i} - \mathbf{f}(q, \mathbf{v}, t)dt = \mathbf{0} \quad (1b)$$

$$-\mathbf{g}_q^{\overline{\mathcal{U}}} \mathbf{v} = \mathbf{0} \quad (1c)$$

$$-\left(g_{Nq}^j \mathbf{v}^{j,+} + e_N^j g_{Nq}^j \mathbf{v}^{j,-}\right) \in \partial\psi_{\mathbb{R}^+}(di_N^j),$$

$$\text{if } g_N^j(q) \leq 0, \quad \forall j \in \mathcal{U} \quad (1d)$$

$$-\left(\mathbf{g}_{Tq}^j \mathbf{v}^{j,+} + e_T^j \mathbf{g}_{Tq}^j \mathbf{v}^{j,-}\right) \in \partial\psi_{C(di_N^j)}(di_T^j),$$

$$\text{if } g_N^j(q) \leq 0, \quad \forall j \in \mathcal{U} \quad (1e)$$

where

- • $\mathcal{U}$ denotes the set of indices of the unilateral constraints, $\overline{\mathcal{U}}$ is its complementarity set, i.e., the set of bilateral constraints and $\mathcal{C} = \mathcal{U} \cup \overline{\mathcal{U}}$ is the full set of constraints.
- • $\dot{q}^+(t) = \lim_{\gamma \rightarrow t, \gamma > t} \dot{q}(\gamma)$ and $\mathbf{v}^+(t) = \lim_{\gamma \rightarrow t, \gamma > t} \mathbf{v}(\gamma)$ are the right limits of the velocities, and $\mathbf{v}^-(t) = \lim_{\gamma \rightarrow t, \gamma < t} \mathbf{v}(\gamma)$ is the left limit.
- • The convention $\mathbf{v}(t) = \mathbf{v}^+(t)$ and $\dot{q}(t) = \dot{q}^+(t)$ will be used in the remaining part of this work for conciseness of notations.
- • $d\mathbf{i}$ is the impulse measure of the contact reaction and the bilateral forces.
- • di_N and di_T are the impulse measures of the contact reaction forces in the normal and tangential directions.
- • $\mathbf{g}$ is the combined set of bilateral and unilateral constraints, and $\mathbf{g}_q(q)$ is the corresponding gradient.
- • g_{Nq} and $\mathbf{g}_{Tq}$ are the gradient matrices in the normal and tangential directions at contact point of the contacting bodies, respectively.
- • At impact, the velocities $\mathbf{v}$ and $\mathbf{v}^-$ are related by the Newton impact law $g_{Nq}^j \mathbf{v}^j + e_N^j g_{Nq}^j \mathbf{v}^{j,-} = 0$ in the normal direction, and $\mathbf{g}_{Tq}^j \mathbf{v}^j + e_T^j \mathbf{g}_{Tq}^j \mathbf{v}^{j,-} = 0$ in the tangential direction.
- • $e_N^j \in [0, 1]$ and $e_T^j \in [-1, 1]$ are the normal and the tangential coefficients of restitution at the contact point $j \in \mathcal{U}$, respectively.

- • $\psi_{\mathbb{R}^+}$ is the indicator function of the real half line $\mathbb{R}^+$ and $\partial\psi_{\mathbb{R}^+}$ is the sub-differential of $\psi_{\mathbb{R}^+}$. Then, $\psi_{C(\mathbf{di}_N)}$ is the indicator function of a section of the Coulomb's friction cone. Equation (1e) represents the frictional unilateral conditions. We refer the reader to [32] in order to have a deeper understanding of how to write the impact law as measure differential inclusions.
- • $\mathbf{f}(q, \mathbf{v}, t) = \mathbf{f}^{ext}(t) - \mathbf{f}^{cin}(q, \mathbf{v}) - \mathbf{f}^{damp}(q, \mathbf{v}) - \mathbf{f}^{int}(q)$ collects the external, complementary inertia, damping and internal forces.
- • $\mathbf{M}(q)$ is the mass matrix which may, in general, depend on the configuration.
- • $d\mathbf{v}$ is the differential measure associated with the velocity $\mathbf{v}$, assumed to be of bounded variation.
- • t is time, and dt is the corresponding standard Lebesgue measure.

Remark: Neglecting singular measures, the impulse measure can be decomposed as $d\mathbf{i} = \boldsymbol{\lambda}dt + \sum_i \mathbf{p}_i \delta_{t_i}$ where $\boldsymbol{\lambda}$ are the smooth Lagrange multipliers, δ_{t_i} the Dirac delta supported at t_i , and $\mathbf{p}_i$ is the impulse at the instant t_i . The inclusions in Eqs. (1c) and (1d) should be understood as compact notations for inclusions: (i) in $\partial\psi_{\mathbb{R}^+}(\lambda_N^j)$ and $\partial\psi_{C(\lambda_N^j)}(\boldsymbol{\lambda}_T^j)$ during the smooth parts of the motion and (ii) in $\partial\psi_{\mathbb{R}^+}(p_{iN}^j)$ and $\partial\psi_{C(p_{iN}^j)}(\mathbf{p}_{iT}^j)$ at the impact times t_i .

3 Time integration method

The nonsmooth generalized- α scheme was introduced by Brüls *et al.* [5] for the solution of contact and impact problems in multibody dynamics. This algorithm artificially splits the motion into smooth and nonsmooth (impulsive) components. In order to correctly capture vibration effects of problems with flexible components, the smooth part is integrated using the second-order accurate generalized- α method, whereas the nonsmooth part is integrated using a first-order scheme. One distinctive feature of this method is that, through the adoption of a similar procedure to that presented by Gear, Gupta and Leimkuhler [13], the unilateral and bilateral constraints are simultaneously satisfied both at position and velocity levels in order to avoid drift phenomena. The resulting scheme is characterized by a set of three coupled sub-problems: one for the smooth prediction of the motion and two others for the corrections at position and at velocity levels.

In this work, the decoupled version of the nonsmooth generalized- α method presented in [10] is used to integrate the equations of motion. Two noticeable differences with respect to its predecessor [5] should be highlighted. On the one hand, the definition of the splitting is modified in order to ensure a decoupling of the three different subsets of equations that need to be solved at every time step. This feature improves considerably the robustness of the integrator for problems involving nonlinear bilateral constraints and flexible elements. On the other hand, the equations are formulated by using an augmented Lagrangian approach, which combines Lagrange multipliers and penalty terms. Additionally, the method has been extended to deal with frictional contacts. A detailed explanation can be found in [10] for frictionless problems and in [12] for the frictional case. In what follows, a brief description is given by presenting the final set of time discretized equations for the three sub-problems, and highlighting the specific treatment of finite rotations.

The key for the treatment of finite rotation variables is to represent the current configuration q_{n+1} based on an increment vector $\Delta\mathbf{q}_{n+1}$ with respect to the previous configuration q_n according to

$$q_{n+1} = q_n \circ \exp(\mathbf{S}(\Delta\mathbf{q}_{n+1})) \quad (2)$$

where $\mathbf{S}$ is an operator which maps a vector to an element of the Lie algebra, $\exp$ is the exponential map of the Lie group and $\circ$ is the composition rule [6].

The splitting strategy decomposes the physical position increment and velocity variable at time t_{n+1} as

$$\Delta\mathbf{q}_{n+1} = \Delta\tilde{\mathbf{q}}_{n+1} + \mathbf{U}_{n+1} \quad (3a)$$

$$\mathbf{v}_{n+1} = \tilde{\mathbf{v}}_{n+1} + \mathbf{W}_{n+1} \quad (3b)$$

where $\Delta\tilde{\mathbf{q}}_{n+1}$ and $\tilde{\mathbf{v}}_{n+1}$ are the smooth contributions to the position and velocity fields, and the position correction $\mathbf{U}_{n+1}$ and the velocity jump $\mathbf{W}_{n+1}$ are the nonsmooth contributions.

The smooth motion is defined by a modified form of the equations of motion at time step t_{n+1} where the contributions of the contact and associated reaction forces are ignored, *i.e.*,

$$\mathbf{M}(\tilde{q}_{n+1})\dot{\tilde{\mathbf{v}}}_{n+1} - \mathbf{f}(\tilde{q}_{n+1}, \tilde{\mathbf{v}}_{n+1}, t_{n+1}) - \mathbf{g}_{\tilde{q},n+1}^{\bar{\mathcal{U}},T} \tilde{\lambda}_{n+1}^{\bar{\mathcal{U}}} = \mathbf{0} \quad (4a)$$

$$-\mathbf{g}_{\tilde{q},n+1}^{\bar{\mathcal{U}}} \tilde{\mathbf{v}}_{n+1} = \mathbf{0} \quad (4b)$$

with the notation $\mathbf{g}_{\tilde{q},n+1}^T = \mathbf{g}_q^T(\tilde{q}_{n+1})$; $\tilde{\lambda}^{\bar{\mathcal{U}}}$ is a Lagrange multiplier associated with the bilateral constraint, $\dot{\tilde{\mathbf{v}}}$ is the smooth acceleration, $\tilde{\mathbf{v}}$ the smooth velocity, *i.e.*, the velocity of the system without considering impacts, and $\tilde{q}_{n+1}$ is the smooth configuration computed from $\Delta\tilde{\mathbf{q}}_{n+1}$ using $\tilde{q}_{n+1} = q_n \circ \exp(\mathbf{S}(\Delta\tilde{\mathbf{q}}_{n+1}))$.

These equations are completed with the difference equations of the generalized- α scheme,

$$\Delta\tilde{\mathbf{q}}_{n+1} = h\mathbf{v}_n + h^2(0.5 - \beta)\mathbf{a}_n + h^2\beta\mathbf{a}_{n+1} \quad (5a)$$

$$\tilde{\mathbf{v}}_{n+1} = \mathbf{v}_n + h(1 - \gamma)\mathbf{a}_n + h\gamma\mathbf{a}_{n+1} \quad (5b)$$

$$(1 - \alpha_m)\mathbf{a}_{n+1} + \alpha_m\mathbf{a}_n = (1 - \alpha_f)\dot{\tilde{\mathbf{v}}}_{n+1} + \alpha_f\dot{\tilde{\mathbf{v}}}_n \quad (5c)$$

where $h = t_{n+1} - t_n$ is the step size and $\mathbf{a}_{n+1}$ is a pseudo acceleration term that arises in the generalized- α integrator scheme [4]. The numerical coefficients γ , β , α_m , and α_f can be chosen from the desired value of the spectral radius at infinite frequencies ρ_∞ according to [9].

After the evaluation of the smooth motion, the position correction $\mathbf{U}_{n+1}$ is computed in order to obtain a position field q_{n+1} which satisfies the bilateral constraints and the unilateral constraints in the normal and tangential directions. At position level the Lagrange multiplier of the constraints is denoted by $\boldsymbol{\nu}$ which is split into a normal $\nu_N^j \in \mathcal{R}$ and tangential $\boldsymbol{\nu}_T^j := [\nu_{T1} \ \nu_{T2}]^T \in \mathcal{R}^2$ components at contact point j . The set of equations for the position correction are given by

$$\mathbf{M}(\tilde{q}_{n+1})\mathbf{U}_{n+1} - h^2\mathbf{f}_{n+1}^p - \mathbf{g}_{q,n+1}^T \boldsymbol{\nu}_{n+1} = \mathbf{0} \quad (6a)$$

$$-\mathbf{g}_{n+1}^{\bar{\mathcal{U}}} = \mathbf{0} \quad (6b)$$

$$-g_N^j \in \partial\psi_{\mathbb{R}^+}(\nu_{N,n+1}^j), \quad \forall j \in \mathcal{U} \quad (6c)$$

$$\begin{aligned} -\mathbf{g}_{T,n+1}^j &\in \partial\psi_{C(\nu_{N,n+1}^j)}(\boldsymbol{\nu}_{T,n+1}^j), \\ &\text{if } g_N^j(q) \leq 0, \quad \forall j \in \mathcal{U} \end{aligned} \quad (6d)$$

where $\mathbf{f}_{n+1}^p = \mathbf{f}(q_{n+1}, \tilde{\mathbf{v}}_{n+1}, t_{n+1}) - \mathbf{f}(\tilde{q}_{n+1}, \tilde{\mathbf{v}}_{n+1}, t_{n+1}) + (\mathbf{g}_{q,n+1}^T - \mathbf{g}_{\tilde{q},n+1}^T)\tilde{\lambda}_{n+1}$, and where the gap $\mathbf{g}$ is split into a normal $g_N^j \in \mathcal{R}$ and tangential $\mathbf{g}_T^j := [g_{T1} \ g_{T2}]^T \in \mathcal{R}^2$ components at contact point j . After the computation of the position field, the velocity jump $\mathbf{W}_{n+1}$ is computed such that the velocity $\mathbf{v}_{n+1}$ satisfies the bilateral constraints at velocity level, and the unilateral constraints in the normal and tangential directions according to the impact law and sticking conditions. The equations for the velocity jump are given by

$$\mathbf{M}(q_{n+1})\mathbf{W}_{n+1} - h\mathbf{f}_{n+1}^* - \mathbf{g}_{q,n+1}^T \boldsymbol{\Lambda}_{n+1} = \mathbf{0} \quad (7a)$$

$$-\mathbf{g}_{q,n+1}^{\bar{\mathcal{U}}} \mathbf{v}_{n+1} = \mathbf{0} \quad (7b)$$

$$\begin{aligned} -\left(g_{Nq,n+1}^j \mathbf{v}_{n+1}^j + e_N^j g_{Nq,n+1}^j \mathbf{v}_n^j\right) &\in \partial\psi_{\mathbb{R}^+}(\Lambda_{N,n+1}^j) \\ &\text{if } g_N^j(q) \leq 0, \end{aligned} \quad (7c)$$

$$\begin{aligned} -\left(\mathbf{g}_{Tq,n+1}^j \mathbf{v}_{n+1}^j + e_T^j \mathbf{g}_{Tq,n}^j \mathbf{v}_n^j\right) &\in \partial\psi_{C(\Lambda_{N,n+1}^j)} \\ &\text{if } g_N^j(q) \leq 0, \end{aligned} \quad (7d)$$

where $\mathbf{f}_{n+1}^* = \mathbf{f}(q_{n+1}, \mathbf{v}_{n+1}, t_{n+1}) - \mathbf{f}(\tilde{q}_{n+1}, \tilde{\mathbf{v}}_{n+1}, t_{n+1}) + (\mathbf{g}_{q,n+1}^T - \mathbf{g}_{\tilde{q},n+1}^T)\tilde{\lambda}_{n+1} - (\mathbf{M}(q_{n+1}) - \mathbf{M}(\tilde{q}_{n+1}))\dot{\tilde{\mathbf{v}}}_{n+1}$, and $\boldsymbol{\Lambda}$ is a Lagrange multiplier for the unilateral and bilateral constraints that represents an impulse obtained as the integral of the reaction forces in the time interval $(t_n, t_{n+1}]$. Analogously to $\boldsymbol{\nu}$, $\boldsymbol{\Lambda}$ which is split into a normal $\Lambda_N^j \in \mathcal{R}$ and tangential $\boldsymbol{\Lambda}_T^j := [\Lambda_{T1} \ \Lambda_{T2}]^T \in \mathcal{R}^2$ components at contact point j .

It should be noted that the three sub-problems (4, 6, 7) can be solved sequentially in a decoupled manner, as the smooth sub-problem does not depend on the corrections at position and at velocity level. Then, each sub-problem can be solved using a semi-smooth Newton method, where an augmented

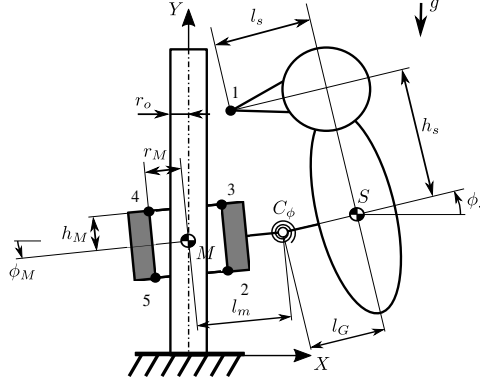


Figure 2: Woodpecker model (not to scale)

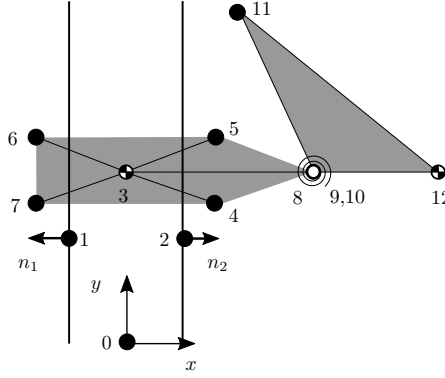


Figure 3: Woodpecker toy discretization: nodes' IDs

Lagrangian approach is adopted for the problems at position and at velocity levels, as similarly done by Alart and Curnier [3] for quasi-static problems with unilateral constraints.

Remark: in the proposed framework, it is mandatory to impose the Coulomb friction constraint at position level because the position correction is blind to the correction done at velocity level (recall that the position sub-problem is decoupled from the velocity one). Another important aspect is that friction constraints are nonholonomic and, therefore, we formulate those constraints on position increments and not on the global position field.

4 Problem definition for the woodpecker toy

The woodpecker toy is simulated using a general purpose software which implements the NSGA algorithm of the previous section. All the translations in the z direction and rotations around the x and y axes are imposed to zero, leading to a planar system (Fig. 2). We remark that other works analyzing the woodpecker toy were usually formulated in a 2D configuration. For this reason, we decided to simulate the woodpecker example in a planar configuration. However, our code is also able to handle three-dimensional configurations as well, and this mechanism could have been analyzed in a spatial configuration as proposed in Zander [36]. Table 1 gives the parameters of the model used in the numerical simulation according to [27], where COM and COR refer to the center of mass and to the coefficient of restitution, respectively. In Fig. 2, the letters M and S make reference to the sleeve and the woodpecker body respectively, and the numbers 1-5 are the IDs of the contact elements accordingly specified in Tab. 2: pole-beak (ID: 1), down right sleeve-pole (ID: 2), upper right sleeve-pole (ID: 3), upper left sleeve-pole (ID: 4) and down left sleeve-pole (ID: 5).

Table 1: Woodpecker model parameters and initial conditions

Geometrical characteristics	
pole radius	$r_O = 0.0025$ m
sleeve radius	$r_M = 0.0031$ m
half sleeve height	$h_M = 0.0058$ m
distance: spring - COM sleeve	$l_M = 0.0100$ m
distance: spring - COM woodpecker	$l_G = 0.0150$ m
distance: beak - COM woodpecker	$h_S = 0.0200$ m
distance: beak - COM woodpecker	$l_S = 0.0201$ m
Inertial properties	
sleeve mass	$m_M = 0.0003$ kg
woodpecker mass	$m_S = 0.0045$ kg
sleeve moment of inertia	$J_M = 5.0 \cdot 10^{-9}$ kg·m ²
woodpecker moment of inertia	$J_S = 7.0 \cdot 10^{-7}$ kg·m ²
Force elements	
angular stiffness	$c_\phi = 0.0056$ Nm/rad
gravity	$g = 9.81$ m/s ²
Contact parameters	
COR in normal direction	$\varepsilon_{N1} = 0.5$
COR in normal direction	$\varepsilon_{N2} = \varepsilon_{N3} = 0$
	$\varepsilon_{N4} = \varepsilon_{N5} = 0$
COR in tangential direction	$\varepsilon_{T1} = \varepsilon_{T2} = \varepsilon_{T3} = 0$
	$\varepsilon_{T4} = \varepsilon_{T5} = 0$
frictional coefficient	$\mu_1 = \mu_2 = \mu_3 = 0.3$
	$\mu_4 = \mu_5 = 0.3$
Initial conditions	
sleeve angular position	$\phi_{MO} = -0.1036$ rad
woodpecker angular position	$\phi_{SO} = -0.2788$ rad
woodpecker vertical velocity	$\mathbf{v}_o = -0.3411$ m/s
sleeve angular velocity	$\omega_{MO} = 0.0$ rad/s
woodpecker angular velocity	$\omega_{SO} = -7.4583$ rad/s
sleeve position X	$X_M = 0$ m

Table 2: Finite elements description

Element type	Description	Nodes
Friction Contact	Pole - Beak	2 11
Friction Contact	Pole - Sleeve	2 4
Friction Contact	Pole - Sleeve	2 5
Friction Contact	Pole - Sleeve	1 6
Friction Contact	Pole - Sleeve	1 7
Rigid Body	Sleeve	3 4 5 6 7 8
Rigid Body	Body bird	12 10 11
Hinge	Sleeve-Body	8 10 9
Spring	Torsional spring	0 9

4.1 Decomposition in elements

The system is described using nodes and elements, in a finite element approach [14]. Each node defines a position and/or an orientation in the 3D space whilst the elements represent the interplay between them. In general, the components of the system to analyze are modeled by elements implemented in the finite element library, such as, for example, rigid bodies, flexible beams, springs and hinges. Then, the contributions of the different elements are assembled numerically to obtain the discretized equations of motion of the system, which afterwards are integrated in time (in our case, using the nonsmooth generalized- α method).

Flexible elements, like springs or beams, do not have any constraint associated. These elements only contribute to the equations of motion with a force term that depends on the position, orientation and/or velocity of the nodes defining the specific element. On the other hand, rigid bodies (through fixed-length constraints) and kinematic joints contribute with bilateral constraints that connect the element's nodes. Because of that, it is important to use an algorithm that imposes the constraints not only at velocity level but also at position level, otherwise a drift at position level can occur, as shown in [5]. Frictional contacts are modeled using a node to face frictional contact element.

In Fig. 3, the set of elements used to simulate the woodpecker toy are shown, where the numbers refer to the ID of the node. Two rigid body elements (with the associated fixed-length constraints), one hinge element, one torsional spring element and five frictional contact elements are used. In Tab. 2 the list of elements and the nodes belonging to each one are shown. In our philosophy a rigid body is described by a node at the center of mass, and a set of additional nodes that are attached to the center of mass by means of rigid links, *i.e.* fixed-length constraints. Three bilateral constraints are introduced per rigid link in order to model the rigidity of the body. Each contact element involves only two nodes, the first one is the node attached to the contact surface. In this example the position of the contact surface node is fixed in time, but the element is capable to deal in an implicit manner with the translation and rotation of the contact surface. The hinge element is composed of three nodes and introduces six bilateral constraints. The first two nodes are the position of the material points to be joined by the hinge, whilst the third node is used to store the relative angle of the hinge, which can be used to model a torsional spring or impose an external torque, for instance, to simulate the action of a motor. Three of the five bilateral constraints guarantee that the two nodes remain together. The next two fix the rotations about the two directions that are perpendicular to the rotation axis of the hinge. The last constraint defines the value of the relative angle. A more extended explanation of the formulation of the rigid body, the hinge and the spring elements can be found in [14].

4.2 Elements' contribution

This subsection presents the different contributions of the elements and shows that many of them can be evaluated from standard operators available in nonlinear finite element codes.

It should be observed that the structure of the equations of the NSGA method, *i.e.* Eqs. (4,6,7), is quite similar to the structure of the equations for problems without contacts and impacts. This allows to reuse many of the implemented capabilities in the finite element code in problems involving nonsmooth phenomena. In order to assemble Eqs. (4, 6, 7), the functions computing the elements' contributions are called several times with different input variables. The standard functions for the elements are:

- • Bilateral Internal Forces: computes the reaction forces of the general form $\mathbf{g}_q^T \boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ is a vector with the same dimensions as the Lagrange multipliers. The input variables are the vector $\boldsymbol{\lambda}$ and the configuration variable q .
- • Flexible Internal Forces $\mathbf{f}(q, \mathbf{v})$: computes internal forces of the elements that are not related with constraints. The inputs are the configuration variable and the velocity vector.
- • Inertial Forces: computes $\mathbf{M}\mathbf{a}$, where $\mathbf{a}$ is a vector with the same dimensions as the acceleration. The input variables are $\mathbf{a}$ and the configuration variable.
- • Complementarity Inertial Forces: computes forces complementing the term $\mathbf{M}\mathbf{a}$ (computed by the Inertial Forces), such as it could be the case of gyroscopic forces.
- • Constraints: computes the contribution of the position constraints. The input variable is the configuration variable.
- • Velocity Constraints: computes the contribution of the velocity constraints. The input variables are the configuration variable and the velocity vectors at time t_n and t_{n+1} .

For the woodpecker toy model introduced before, the following evaluation of functions are involved. In the case of the spring element, the function of Flexible Internal Forces is called only three times, once for each sub-problem, with the inputs $\tilde{q}_{n+1}, \tilde{\mathbf{v}}_{n+1}$ at smooth level, $q_{n+1}, \tilde{\mathbf{v}}_{n+1}$ at position level and $q_{n+1}, \mathbf{v}_{n+1}$ at velocity level. For the hinge and rigid body elements, the function of Bilateral Internal Forces is called with the input variables $\tilde{\boldsymbol{\lambda}}, \boldsymbol{\nu}, \mathbf{\Lambda}$ for the smooth, the position and the velocity sub-problems, respectively. In addition, in order to impose the constraints at position level, the function Constraints is called with the configuration variable q_{n+1} as input. In order to impose the constraints at velocity level, the function of Velocity Constraints is called for the smooth sub-problem with $\tilde{q}_{n+1}, \tilde{\mathbf{v}}_{n+1}$ as inputs, and it is called again for the velocity sub-problem with the inputs $q_{n+1}, \mathbf{v}_{n+1}$. For the rigid body element the function of Inertial Forces and Complementarity Inertial Forces are called with the input values $\tilde{q}_{n+1}, \dot{\tilde{\mathbf{v}}}_{n+1}$ for the smooth sub-problem, with $q_{n+1}, \mathbf{U}_{n+1}$ for the position sub-problem and with $q_{n+1}, \mathbf{W}_{n+1}$ for the velocity sub-problem.

The contact element has been explicitly developed for the NSGA algorithm. Due to the fact that this element does not contribute to the smooth sub-problem and does neither to the inertia-like term (the nodes defining the element are mass-less nodes), the contact element involves only two functions:

- • Nonsmooth Residue Position: computes all the contributions of the contact element to the position sub-problem, see Eqs. (6). The input variables are the configuration variable at time t_{n+1} and the Lagrange multiplier $\boldsymbol{\nu}$.
- • Nonsmooth Residue Velocity: computes all the contribution of the contact element to the velocity sub-problem, see Eqs. (7). The input variables are the velocity vector at time t_n and t_{n+1} , the configuration variable at time t_{n+1} and the Lagrange multipliers $\mathbf{\Lambda}$.

5 Numerical results

In this section, the dynamics of the woodpecker toy is numerically analyzed by studying three cases which correspond to different models of the toy. In case I, the model consists in assuming small rotations and in neglecting the horizontal displacement of the sleeve, which results in a total of 3 coordinates as done, for instance, in [17]. The main distinction to other works is that, in this case, the equations of the bird

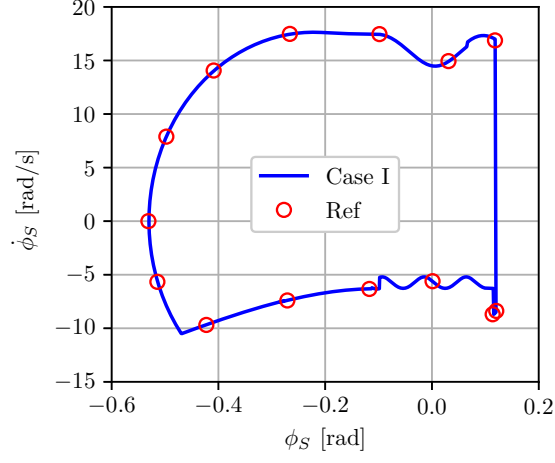


Figure 4: Case I: phase diagram for the bird

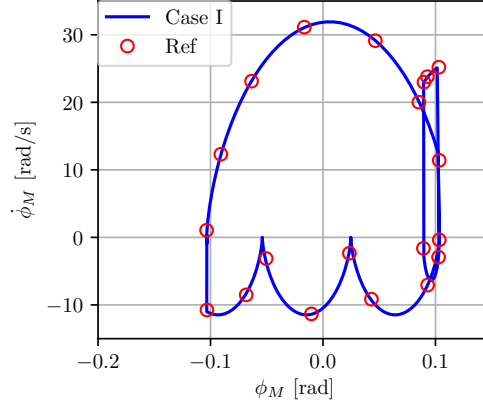


Figure 5: Case I: phase diagram for the sleeve

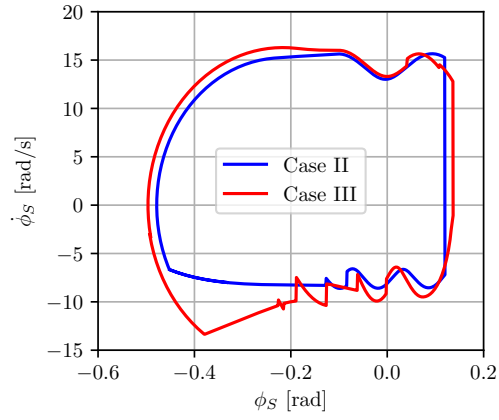


Figure 6: Cases II-III: phase diagram for the bird

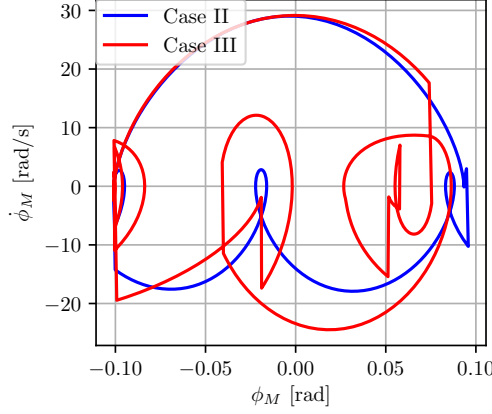


Figure 7: Cases II-III: phase diagram for the sleeve

and the sleeve are implemented as a finite element, a kind of super-element, which provides a way of “connecting” the degrees of freedom (DoFs) of the model to other classical finite elements. This allows us to keep modeling the contact using the element presented in [12] and to integrate the equations of motion by using the NSGA presented in [10]. In cases II and III, the finite element model with large rotations of the woodpecker toy introduced in the previous section is studied. In case II, the horizontal displacement of the sleeve center of mass (x_m) is fixed, whereas in case III this displacement is free. In case II, since the displacement of the center of mass is fixed, it is possible to apply a symmetry condition and the contacts 4 and 5 can be removed [16, 17, 21, 31]. The Coulomb friction model is adopted in this work. For the simulation of the woodpecker toy, dynamic friction models could give better results. However, in the Coulomb friction model the friction force is independent of the velocity magnitude and, therefore, it has a simpler structure and fewer parameters to be calibrated [30]. In order to compare the numerical solutions with the references from the literature, identical static and kinetic friction coefficients are assumed. In what follows, the results obtained using a time step of $h = 10^{-4}$ s and a spectral radius $\rho_\infty = 0.8$ are analyzed.

In Figs. 4 and 5 the phase diagrams for the center of mass of the bird and the sleeve are respectively shown for case I. It can be observed that the solution obtained with our framework agrees with the solution found in other works, such as in [17] or in [21], which use the same model for woodpecker (denoted in the figures by “Ref”). This test case mainly states that the NSGA time integrator and the adopted element for modeling the contact do not introduce any numerical artifact. In Figs. 6 and 7 the phase diagrams for the center of mass of the bird and the sleeve are respectively shown for cases II and III. The first direct observation is the disparity between these phase plots and those for case I. Clearly, the assumption of small rotations does not apply to this problem, as the solutions for cases II and III, in which large rotations are modeled, deviate considerably from the one of case I.

In order to understand the dynamics of the problem, Fig. 8 shows the correlation of the contact status at velocity level to the evolution of the rotation of the bird ϕ_S and the vertical displacement y_M and the rotation of the sleeve ϕ_M for case II. We can identify the following evolution:

1. Beak impact. ϕ_S shows a large velocity jump, and this jump is also evidenced in the evolution of y_M and ϕ_M .
2. Sleeve impact with the pole, followed by a rebound. Velocity jumps appear in every DoF.
3. Second sleeve impact, followed by permanent contact with sliding. Again, we can see velocity jumps in every DoF.
4. Transition from sliding to sticking. The y_M coordinate is now constant.
5. Transition from sticking to sliding. This phase has no associated jumps.
6. Transition from sliding to free motion.

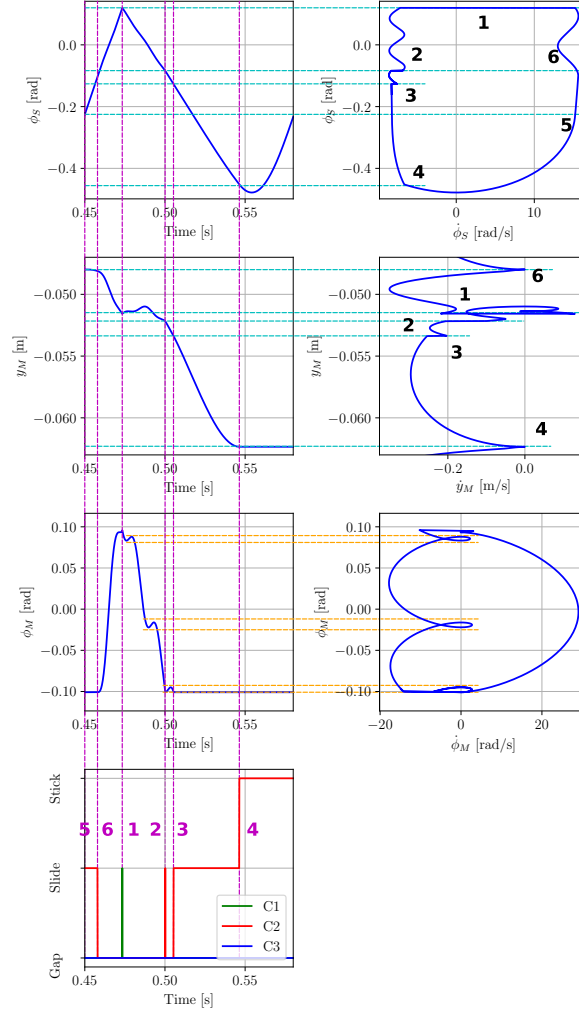


Figure 8: Case II: contact status correlated to evolution of the the rotation of the bird ϕ_S , the sleeve's vertical displacement y_M and the rotation of the sleeve ϕ_M . In the last figure of the sequence, the contact status of the contacts of the beak (C1) and the sleeve (C2 and C3) are shown. These contacts are numbered following the order of appearance in Table 2.

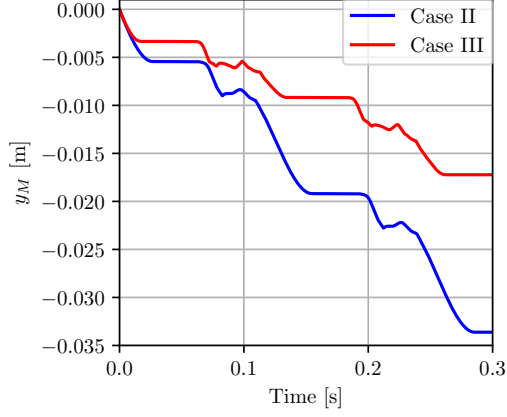


Figure 9: Sleeve vertical displacement comparison

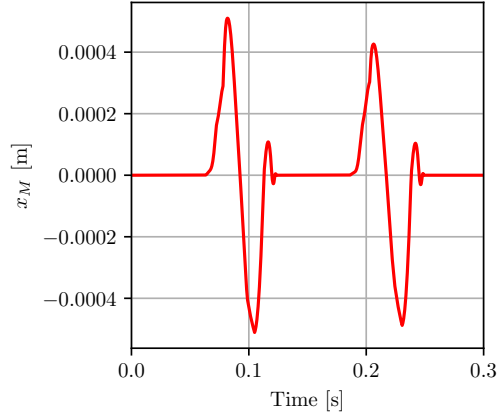


Figure 10: Case III: sleeve horizontal displacement

It is interesting to mention that we observe that in cases II and III, the contact associated to the upper-right point of the sleeve (C3 in Fig. 8) is never active, whilst this is not the case for case I.

The second observation emanating from the comparison of the phase diagrams is the disparity between cases II and III. These results strongly imply that neglecting the horizontal displacement of the sleeve center of mass has a strong influence on the dynamics of the woodpecker toy. The cycle frequency for case II is $f_{II} = 7.66$ Hz and $f_{III} = 7.96$ Hz for case III. The vertical displacements of the sleeve in each cycle is $h_{II} = 14.3$ mm and $h_{III} = 7.6$ mm for cases II and III respectively. This fact was also recognized in reference [31], although their predicted falling distance per cycle was much larger than our predictions, and well apart from the experimental results reported in [16]. Besides, it is shown that accounting for the horizontal displacement of the sleeve has a large influence on the predicted falling distance per cycle. Despite their similar behaviour and close frequencies, fixing the horizontal displacement of the sleeve extends the sticking face, which in turns slows down the fall of the sleeve, a phenomenon that is clearly observed in Fig. 9. In case III, after tuning the friction coefficient, we observe that for $\mu = 0.35$ the frequency is $f_{III}^* = 8.3$ Hz, 9.78% lower than the experimental value reported in [16] which is 9.2 Hz, while the sleeve falling distance is $h_{III}^* = 5.3$ mm which is identical to the experimental measurement. Comparing with the numerical solutions given by Glocker and Pfeiffer [16], it can be found that they get a cycle frequency of 8.98 Hz which is 2.4% lower than the experimental value, whilst the sleeve falling distance is 5.7 mm, 7.5% greater than in the experiments.

In case III, in contrast to case II, the sleeve is completely free to move. The position of the sleeve

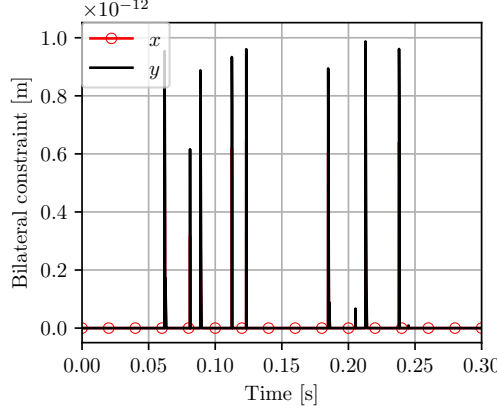


Figure 11: Case III: bilateral constraints for node 4

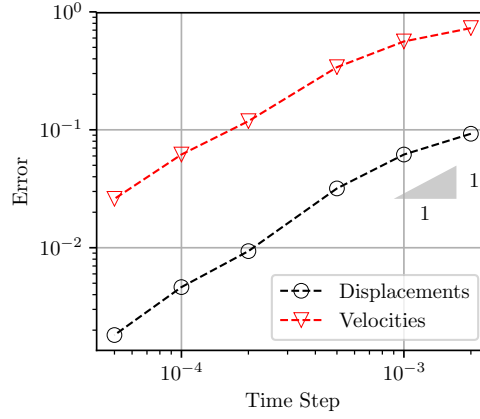


Figure 12: Case II: convergence study

center of mass in the horizontal direction is shown in Figs. 10. The bilateral constraints are satisfied at position level without any drift as highlighted in Fig. 11, where the bilateral constraints introduced by the rigid body modeling of the sleeve (node 4) are shown. The error obtained for this bilateral constraint is below the adopted tolerance.

In order to provide more confidence in the numerical solutions obtained for cases II and III, the numerical convergence of the involved models and methods is analyzed next. For that purpose, the error is evaluated using the L^1 norm, which is defined as $\text{Error}(h) = \sum_{n=0}^N |f_n - f(t_n)| / \sum_{n=0}^N |f(t_n)|$, where N is the number of time steps, that depends on the time increment h , f_n is the numerical solution and $f(t_n)$ is the reference solution. Since no analytical solution is available for the current problem, the numerical solution corresponding to a time increment $h = 10^{-5}$ s is taken as reference solution. In Figs. 12 and 13, the obtained results for the y component of the displacement and velocity fields of node 3 can be observed. As it can be appreciated, a global convergence of order 1 is achieved, which is the expected order when adopting the NSGA as time integrator.

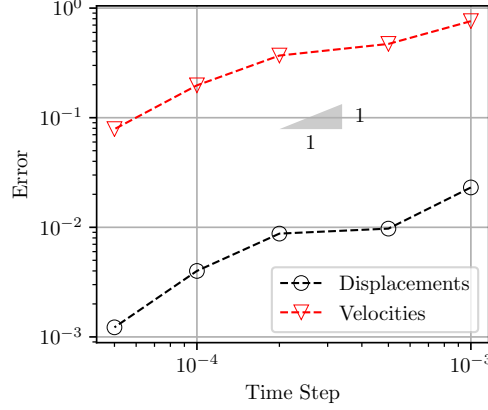


Figure 13: Case III: convergence study

6 Conclusions

The finite element multibody dynamics approach extended to problems involving frictional contacts and impacts is applied to obtain the solution of the woodpecker toy test example. The solution procedure is based on the decoupled version of the NSGA method [10], which is characterized by three sub-problems that can be solved in a decoupled sequential manner. Friction is introduced at the element level adopting the formulation presented in detail in [3]. The woodpecker example is a complex and interesting test-case, because its dynamics involves nonsmooth phenomena with impacts and stick-slip transitions at the different contacts between the pole, the beak and the sleeve, which separate smooth phases of motion. This dynamics is very sensitive to the modelling assumptions, in particular regarding the sleeve motion and the geometric nonlinearities caused by finite rotations. Simplifying assumptions often found in the literature, which linearize the rotations and/or neglect the horizontal motion of the sleeve, affect the behaviour and should therefore be avoided. Without such simplifying assumptions, the complexity of the problem increases significantly both at the level of the formulation of the equations of motion and of their numerical solution. This paper shows that the equations of motion of the system can be automatically generated based on the library of elements available in the proposed simulation code, which relieves the user from a heavy and error-prone derivation. The equations are expressed in terms of absolute translations and rotations of the nodes, which are not independent variables but have to satisfy a set of bilateral constraints. The nonsmooth equations of motion are thus formulated in terms of differential measures and involve both bilateral constraints and unilateral constraints. In addition, the finite rotation variables do not belong to a linear space and cannot be treated as vector quantities. The decoupled NSGA method is then able to tackle the problem in its general form, with the advantages that the bilateral and unilateral constraints are strictly satisfied at position and velocity levels, regardless of the time step size. Compared to other time-stepping schemes, this integration scheme is also known to better capture the nonlinear phenomena in the smooth phases of motion. From the results of the woodpecker simulation, it is shown that by considering large rotations and the horizontal displacement of the sleeve, a richer dynamic behaviour is obtained as well as a good match to experimental results with an error of 9.78 % and 0% in the frequency and in the falling height, respectively. The woodpecker test-case thus illustrates the generality and relevance of the proposed simulation framework, which allows the user to eliminate doubtful assumptions, and still permits an efficient formulation and numerical solution of the problem.

In this work, the Newton's impact law is adopted to model the nonsmooth restitution of the velocity in the normal and tangential directions of the contact. As part of future work the implementation of Poisson's impact law [8, 35, 28] in the context of the NSGA will be investigated. Zander [36] studied the woodpecker toy dynamics where the pole is modeled with a flexible rod. This is an interesting example that deserves a detailed analysis which will also be done in a future work.

Acknowledgments

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