Portfolio choice and mental accounts: A comparison with traditional approaches

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ABSTRACT

This paper analyses the ability of a realistic mental accounting model for portfolio choice (presented in Hübner and Lejeune (2021)) to compete with traditional utility alternatives. Das, Markowitz, Scheid, and Statman (2010) and Hübner and Lejeune (2021) have shown that the mental accounting framework embeds optimal allocations derived from quadratic utility optimization and/or the Gaussian distribution assumption. We complement their work by demonstrating here the flexibility and numerical superiority of the optimization outputs that a non-Gaussian version of HAMA generates with respect to two traditional expected utility maximization rules that produce a wide and relevant spectrum of portfolio allocation rules for a variety of realistic investor types: the decay rate approach proposed by Stutzer (2003), which is analogous to maximizing expected utility, and the class of flexible three-parameter utility functions (FTP) introduced by Conniffe (2007) that encompasses a wide set of popular utility functions.

Key words: mental accounts, portfolio choice, horizon, upside potential, risk aversion

JEL classification: G02, G11, D14.

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1 Introduction

What portfolio choice criterion should guide investors towards an asset allocation that is optimal at satisfying their goals? Which probability distribution characterizes the riskiness of financial asset returns? What is a sensible function representing investors' attitude towards this riskiness? These are relevant questions that financial advisers have to deal with in order to formulate appropriate asset allocation advice. Markowitz's (1952) seminal work offers one of the early answers to these questions, through the mean-variance paradigm. If investors are characterized by quadratic preferences and/or if asset returns display spherically generated distributions (Chamberlain, 1983), portfolios can be ranked according to the Sharpe ratio (Sharpe, 1994), computed ex-post on historical data. However, the first condition leads to a decreasing marginal utility, an implausible characterization of the preferences of most investors. The second one contradicts substantial empirical evidence on the unconditional distribution of asset returns (see, for instance, Badrinath and Chatterjee, 1988; Corrado and Su, 1997; Fong, 1997; Harvey and Siddique, 2000; Aparicio and Estrada, 2001). Even when one only fits the unconditional distribution of stock returns, departures from normality due to fat tails (high kurtosis) and to left or right asymmetry (skewness) are pervasive, precluding the straight application of the mean-variance framework.

Another traditional approach to portfolio decision problems relies on expected utility maximization. Restrictions on higher-order derivatives of utility functions allow for the consideration of asymmetric risks (Kraus and Litzenberger, 1976) or fat-tail risks (Fang and Lai, 1997). However, no utility-based framework has managed to circumvent the fundamental argument put forward by Brockett and Kahane (1992): for any well-behaved utility function, i.e. one whose derivatives alternate in signs, it is possible to find returns distributions such that expected utility maximization leads to a solution with a low expectation, a high variance and a low skewness. A comprehensive theoretical framework with limited assumptions over returns distributions should therefore not posit moment preference, although parametric models of the expected utility approach necessarily do so. For this reason, the fundamental question: "which utility function of the rational investor, universal enough to encompass each profile, but sufficiently well-behaved to get a tractable treatment, should be used in practice?" remains unanswered.

Pure parametric approaches to utility maximization are doomed to expose themselves to a similar criticism. Furthermore, imposing a parametric utility function results in a "one-size-fits-it-all" characterization of investors' preferences. This is incompatible with the mental accounting behavior of individuals, as stressed by Thaler (1985, 1999) in his behavioral exposition of the portfolio choice problem. In our paper, we move away from the traditional utility theory stream, and follow the behavioral mental accounting approach to propose an alternative framework that (i) starts from an inductive and positive treatment of investors' portfolio choices rather than a deductive and normative set of predictions derived from a pre-specified utility function; (ii) acknowledges that individual investors are likely to agree on a loose information set, like some characteristics of the distribution of financial returns, rather than imposing a homogenous and sophisticated probabilistic knowledge across all potential states of the world; and – as the proof of the pudding is in the eating – (iii) it ensures that portfolio choices obtained with reasonably parameterized utility functions are attainable with a reasonable calibration in our behavioral framework as well, even though the opposite may not be true. We believe that the first two differentiators represent interesting improvements over utility theory. Our purpose is not to *impose* them as criteria of dominance over traditional utility functions; rather, we *propose* an alternative, hopefully intuitive way of framing the investor's behavior. The last aspect, which is output-oriented, imposes itself as an effectiveness criterion.

Our approach is based on the following intuitive observation: in practice, investors do not know their utility function and have to rely on incomplete but readily usable returns characteristics, such as some unconditional moments of the distribution. The associated parameters of their portfolio choice problem, including parameters characterizing risk aversion, should reflect this incomplete information set. In their inspiring work, Das et al. (2010) show that the risk aversion parameter of quadratic utility maximizers can be inferred in the behavioral mental accounting approach without specifying any utility function. In their mental accounting model (hereafter MA), investors divide their global investment portfolio into distinct goal-based accounts. Each of these accounts is associated with a threshold return and a tolerance probability of failing to reach this return. Investors, with the help of their financial advisers, are better at stating their threshold returns and a tolerance probabilities rather than a value for their degree of risk aversion. However, the

MA model has limits in two relevant dimensions that have implications for capturing investors' attitude towards portfolio risk. First, when goals are associated with a specific investment horizon, it does not provide any guidance on how the threshold return should be set in function of horizon nor on the model implications for the horizon dimension. Second, it focuses on downside risk, i.e. the probability of falling short the threshold return. While this safety-first approach characterizes some investors (or some mental accounts for a same investor), it misses investors whose risk-taking is balanced between safety and upside potential, a behavior depicted in the SP/A theory of Lopes (1987). Statman (2008) also argue that top layers of mental accounts can be designed for upside potential, that is a goal that reflects a chance of getting rich. Consequently, investors may differ in the degree of gain-loss asymmetry they assign to their portfolios, and this dimension is missing in the baseline MA framework.

In their contribution, Hübner and Lejeune (2021) propose a Horizon-Asymmetry Mental Accounting model (hereafter HAMA) that addresses these missing dimensions. In their parsimonious extension of the MA model each individual investment goal is represented by a set of four parameters: one associated with the threshold return, a tolerance probability of shortfall, a gain-loss asymmetry parameter and an investment horizon. The HAMA framework complies with the desired elements (i) and (ii) underlined above. It inherits the intuitive non-parametric approach from the behavioral MA model. Moreover, the HAMA portfolio choice program considers that the distributional properties of financial portfolios returns are summarized by their first four moments, reflecting the limited information set at the disposal of investors or their allocation advisers.[1] Moreover, in relation with (iii), Hübner and Lejeune (2021) show that HAMA encompasses the optimal quadratic utility allocations. However it remains a research question whether HAMA competes with more flexible parameterized utility functions. The goal of the present paper is to test the flexibility and numerical superiority of the optimization outputs that HAMA generates with respect to two parametric expected utility maximization rules that produce, in our view, a wide and relevant spectrum of portfolio allocation rules for a variety of realistic investor types. The first class of utility functions is the expected power utility one. We analyze this comparison through a mapping

[1] Although affirming the existence of these moments represents a strong assumption regarding some evidence from many series of returns (see Jondeau and Rockinger, 1999), the assumption is commonly used for the derivation of asset-pricing models.

into the decay rate approach proposed by Stutzer (2003), which is analogous to maximizing expected utility when one refers to the large deviations theory. As HAMA, the decay rate approach is a behavioral portfolio optimization program that integrates a notion of horizon, and therefore is a relevant competitor to which the added value of HAMA has to be demonstrated. The second expected utility maximization rule is the so-called class of flexible three-parameter utility functions (FTP) introduced by Conniffe (2007) that encompasses a wide set of popular utility functions. The FTP class is fully characterized by a triplet of investor-specific parameters, which is the same number of parameters used in the HAMA characterization of preferences (once the horizon dimension is fixed). In that sense, the FTP model can be seen as the parametric counterpart of HAMA, and is thus an interesting challenger. The objective of the quantitative analysis of this paper is to identify whether some HAMA-generated allocations are inaccessible with traditional utility-based options portfolios. This test helps us to identify which gaps in the traditional individual portfolio choice literature can be fulfilled with a flexible mental account specification, and what are the sets of parameters that make these unmatched allocations possible. For both utility functions, we leave aside the issue of the estimation of the multivariate distribution of asset returns by fixing the same information set in all portfolio optimization cases, namely the first four moments of the sample distribution.

Using numerical techniques, we show that while HAMA portfolio program can easily recover decay rate optimal allocations and FTP optimal portfolios, a reverse matching is not necessarily true. Contrary to the decay rate approach, the HAMA is able to decouple horizon and tolerance probability dimensions, while enabling the investor to explicitly state objectives in terms of investment horizon. A numerical optimization underlines that HAMA optimal portfolios encompass FTP ones, while the opposite cannot be achieved. Interestingly, the FTP model does not map easily into optimal positions held by investors who associate a high value with upside potential.

The paper is organized as follows. Section 1 presents the HAMA approach to portfolio choice optimization. The second section compares HAMA with Stutzer (2003)'s decay rate approach. Section 3 proposes a horse race between HAMA and the FTP class of utility functions, using numerical optimization on data on Fama-French portfolios. Section 4 concludes the article.

2 The Horizon-Asymmetry Mental Accounting model (HAMA)

2.1 Theoretical foundations

Following a goal-based mental accounting approach (Das et al., 2010; Hübner and Lejeune, 2021), investors maximize the expected return of their portfolio associated with a mental account m, subject to a shortfall probability:

$$\max_{p} E\left[R_p\right] \tag{1}$$

$$\text{s.t.} \quad \mathbb{P}\left(\bar{R}_p^{H_m} \leq E\left[R_p\right] - \lambda_m\right) - \gamma_m \mathbb{P}\left(\bar{R}_p^{H_m} \geq E\left[R_p\right] + \lambda_m\right) \leq \Omega_m \tag{2}$$

where R_p is the 1-period the log gross return of the portfolio, and $\bar{R}_p^{H_m} = H_m^{-1} \sum_{t=1}^{H_m} R_{p,t}$ is the mean of the log gross returns over an horizon H_m. The HAMA framework implies that investors assess the riskiness of investment alternatives with respect to their probability of falling short the expected return by an amount λ_m over their investment horizon. The threshold return in the original MA model Das et al. (2010) is thus here expressed in terms of deviation from expected returns $(E\left[R_p\right] - \lambda_m)$. It can be interpreted as a relative Value-at-Risk threshold defined by λ_m. This formulation helps us to define a symmetric Value-at-Potential level $(E\left[R_p\right] + \lambda_m)$ without using any additional parameter. When $\gamma > 0$, investors value the upside potential of portfolios and it relaxes their optimization constraints. In other words, they balance downside risk with upside potential benefits, and γ_m is a parameter that governs the degree of gain-loss asymmetry. If $\gamma_m = 0$ investors behave as of the safety-first kind (as in Das et al., 2010), and only focus on the downside risk of financial assets. When γ_m is equal to 1, a symmetric treatment is achieved between downside and upside dimensions.

2.2 Practical implementation

The implementation of HAMA portfolio program (1)-(2) requires appropriate estimates for the shortfall and upside probabilities. Rather than assuming a probability distribution for $\mathbb{P}$, like the Gaussian distribution, we use the following intuition. In practice, while investors agree that asset returns are not normally distributed, they don't have any consensus on which functional form should applied instead. In contrast, investors (or their financial advisers) have to rely on less complete but readily usable returns characteristics, that is the empirical estimates of volatility, asymmetry and fat-tailedness. For i.i.d. period log returns, it is easy to derive the moments of the mean of a sequence of n drawings of these returns, $\overline{R}_p^n = \frac{1}{n}\sum_{t=1}^{n} R_{p,t}(1)$. These moments are equal to: $\mu_{2,p}^n = \frac{\mu_{2,p}}{n}$, $\mu_{3,p}^n = \frac{\mu_{3,p}}{n^2}$, $\mu_{4,p}^n = \frac{\mu_{4,p}}{n^3} + \frac{3(n-1)\mu_{2,p}^2}{n^3}$.

A generalization of Chebyshev's inequality found in the work by Mallows (1956) enables us to link this limited set of available information to estimates of the probabilities of downside risk and upside potential in (2). If $\mu_{3,p}$ and $\mu_{4,p}$ exist and are known, the shortfall constraint of a portfolio p for a mental account m is given by:

$$\pi_p(-\lambda_m, H_m) - \gamma_m \pi_p(\lambda_m, H_m) \leq \Omega_m \tag{3}$$

$$\text{where } \pi_p(x,H) := \pi_p\left(x, H; \mu_{2,p}, \mu_{3,p}, \mu_{4,p}\right) = \frac{\Delta_p}{Q_p^2(x) + \Delta_p(1 + \frac{Hx^2}{\mu_{2,p}})} \tag{4}$$

$$Q_p(x) = -\frac{Hx^2}{\mu_{2,p}} + \frac{\mu_{3,p}x}{\mu_{2,p}^2} + 1 \tag{5}$$

$$\Delta_p = \frac{1}{H}\left(\frac{\mu_{4,p}}{\mu_{2,p}^2} - \frac{\mu_{3,p}^2}{\mu_{2,p}^3} - 3\right) + 2 \tag{6}$$

$$\text{under the constraint } \Delta_p > 0$$

and

$$\frac{\partial \pi_p(x,H)}{\partial \mid x \mid} < 0 \ \forall x \tag{7}$$

$$\frac{\partial \pi_p(x,H)}{\partial H} < 0 \ \forall \varphi_p \leq H \leq \Phi_p \tag{8}$$

where

$$\varphi_p = \min\left[\frac{(-x^2\mu_{3,p}^2 + \mu_{3,p}^2 - \mu_{2,p}\mu_{4,p} - 3\mu_{2,p}^3 x^2 + 3\mu_{2,p}^3 + \mu_{2,p}\mu_{4,p}x^2)^2}{4\mu_{2,p}^3 x^2 \mu_{3,p}^2}, \frac{x^2\mu_{3,p}^2}{\mu_{2,p}^3(1-x^2)^2}\right] \tag{9}$$

$$\Phi_p = \max\left[\frac{(-x^2\mu_{3,p}^2 + \mu_{3,p}^2 - \mu_{2,p}\mu_{4,p} - 3\mu_{2,p}^3 x^2 + 3\mu_{2,p}^3 + \mu_{2,p}\mu_{4,p}x^2)^2}{4\mu_{2,p}^3 x^2 \mu_{3,p}^2}, \frac{x^2\mu_{3,p}^2}{\mu_{2,p}^3(1-x^2)^2}\right] \tag{10}$$

The proof of this proposition follows directly from Mallows (1956) by using the moments of $\overline{R}_p^H$. The proof for the sign of the derivative with respect to $\mid x \mid$ is straightforward. The sign of the second derivative can be obtained by using the variable transformation $y = \frac{\sqrt{H}x}{\sqrt{\mu_{2,p}}}$ and noting that $\frac{d\pi_p(y,H)}{dH} = \frac{\partial \pi_p(y,H)}{\partial y}\frac{dy}{dH} + \frac{\partial \pi_p(y,H)}{\partial H}$. The first term is always strictly negative and the second term is non-positive for $\varphi_p \leq H \leq \Phi_p$, which completes the proof. $\square$

Probability estimate $\pi_p(x, H)$ is an upper bound, function of unconditional moments. For $x > 0$, it is an estimate of $\mathbb{P}\left(\overline{R}_p^H - E[R_p] > x\right)$, for $x < 0$ it approximates $\mathbb{P}\left(\overline{R}_p^H - E[R_p] < x\right)$. Moreover, for a given horizon, the larger the interval x around expected returns, the more important the probability mass inside $[E[R_p - x; E[R_p + x]$, and the lower the estimate $\pi_p(x, H)$. For a given x, the longer the horizon H, the lower $\pi_p(x, H)$.

Substituting $\mathbb{P}$ by its estimate π_p, the HAMA portfolio selection program is thus formulated as follows:

$$\max_{p} E\left[R_p\right] \tag{11}$$

$$\text{s.t.} \quad \pi_p(-\lambda_m, H_m) - \gamma_m \pi_p(\lambda_m, H_m) \leq \Omega_m \tag{12}$$

Hence, the preference of HAMA investors are characterized by 4 parameters: the investment horizon H_m, an interval over expected returns λ_m, a gain-loss asymmetry parameter γ_m and a tolerance probability Ω_m.

3 Mapping into the expected power utility criterion

Hübner and Lejeune (2021) show that the HAMA model covers the standard mean-variance approach as a special case. The risk aversion coefficient in a quadratic utility setup is pinned down by an intuitive set of parameters that capture threshold return, tolerance probability and horizon dimensions. In this section, we compare the HAMA model with a behavioral approach to portfolio selection, based on a decay rate criterion, that generalizes the use of power utility in the objective function of portfolio choice. The approach is proposed by Stutzer (2003) and, like the Das et al. (2010) model, enables the investor to substitute the specification of its risk aversion coefficient (here in a power utility setup) with the choice, more intuitive, of a target return. As in the HAMA framework, the decay rate approach refers to a threshold return and a shortfall probability. Moreover, it also relies on the convergence properties of the shortfall probability in order to derive optimal portfolios, and an implicit notion of horizon emanates from the approach. On account of these similarities with our model, it is interesting to compare the two approaches, and to present the contribution of ours. Furthermore, this comparison enables us to hit two targets with one shot. Using the large deviations theory, Stutzer (2003) interestingly shows the analogy of the decay rate approach with expected power utility. Specifically, decay rate investors are equivalent to expected power utility investors, with a risk aversion coefficient dependent on target returns and on the investment opportunity set. Hence, a mapping into the expected utility approach derives from a mapping into the decay rate approach.

The foundation of the decay rate approach is a shortfall probability problem. Investors select their optimal portfolio in order to ensure the minimization of the probability that the growth rate of invested wealth (i.e. the portfolio log return) will not exceed an investor-selected target growth rate (r). As mentioned above, the convergence properties of the shortfall probability make sure that the shortfall probability decays to zero asymptotically as the time horizon goes to infinity, when one selects a portfolio with expected returns higher than the target rate. For a finite time horizon, however, the shortfall probability is likely to vary from portfolio to portfolio. Instead of choosing a specific value for the horizon, Stutzer (2003) argues that a sensible strategy is to choose a portfolio that makes the shortfall probability decay to zero as fast as possible as time tends towards infinity. Consequently, under this approach, investors select portfolios that maximize the decay rate of the shortfall probability. Stutzer (2003) further shows that, for i.i.d. returns, the maximization of the decay rate ensures that the shortfall probability is minimized for all horizons. This characteristic of the decay rate approach differs from our approach. In the HAMA framework, investors select portfolios such that the shortfall probability is negligible for a finite investment horizon, which does not necessarily imply that the decay rate of the optimal portfolio is maximized. In the Gaussian version of the approach, the decay rate optimization is equivalent to the following criterion (see

Stutzer, 2003, for a detailed derivation):[2]

$$\arg\max_p D_p(\log(r)) \equiv \arg\max_p \frac{1}{2}\left(\frac{E[R_p]-\log(r)}{\sqrt{Var[R_p]}}\right)^2 \tag{13}$$

which corresponds to equation (7) in Stutzer (2003), and where r is the target return. Furthermore, Stutzer (2003) demonstrates an analogy between decay rate optimization and portfolio selection under the power utility criterion. Using the large deviations theory, the author shows that decay rate optimization is equivalent to using power utility with a risk aversion parameter that depends on the target return. When log returns are normally distributed and i.i.d., the risk aversion parameter (here below $1-\theta_{max}(p)$) can be inferred from the optimal portfolio as follows (we refer to Stutzer, 2003, for a detailed derivation):

$$\text{Risk aversion} \equiv 1-\theta_{max}(p) = 1-\frac{\log(r)-E\left[R_p\right]}{Var\left[R_p\right]} \tag{14}$$

which corresponds to equation (18) in Stutzer (2003), and where $\theta_{max}(p)$ depends on the target rate r and the portfolio p, and $\theta_{max}(p) < 0$. As discussed in Stutzer (2003), the latter condition implies that the third derivative of the power utility function is positive, which in turn involves preference for positive skewness. Therefore, in contrast with the HAMA framework, the decay rate approach posits moment preferences and does not comply with Brockett and Kahane's (1992) critique.

Example

Let us consider a simple example in order to illustrate the mechanics of the decay rate approach and compare it to the HMA model. The investment opportunity set consists in a risky return series with yearly mean and standard deviation of respectively 15% and 25%, and a riskless asset delivering a return of 3% per year. The assets are thus similar to the risky and money-like assets used for the mapping into the mean-variance setup in the previous example. Consider that the investor has a target return of 5%. The decay rate maximization (13) leads to an optimal allocation of 85.27%

[2]Stutzer (2003) also presents a portfolio implementation of the decay rate approach for a non-Gaussian (and i.i.d.) case. A mapping of HAMA into this non-Gaussian i.i.d decay rate approach can be implemented using the tools described in the appendix of Hübner and Lejeune (2021). The arguments and findings made under the Gaussian version presented here remain valid under the non-Gaussian mapping. Consequently, we choose to only present a mapping into the Gaussian version of the decay rate approach, which we think will be more intuitive to the reader.

in the risky asset, and 14.73% in the riskless asset. Figure 1 reports the evolution of the shortfall probability depending on the time horizon for the optimal portfolio. Two other portfolios are also represented. The first one consists of an investment of solely 50% in the risky asset. The second is more aggressive with a risky share of 125%, meaning that it holds a short position in the riskless asset. It is clear from the figure that the decay rate of the shortfall probability is maximized for the optimal portfolio. The slope of the relationship between shortfall probabilities and horizon is more pronounced for this portfolio. Moreover, the shortfall probability is the lowest at each point of time.

When a tolerance probability is specified, it is possible to infer an investment horizon for which the optimal decay rate portfolio respects the shortfall probability constraint. For instance, it takes around 19 years for the optimal portfolio to respect a tolerance probability of 10%. This observation suggests that an HAMA type of investor with an investment horizon of 19 years, a tolerance probability of 10% and a target rate of 5% will select the decay rate optimal portfolio for his/her investment. Indeed, at the 19-year horizon, the decay rate optimal portfolio is the only portfolio that respects the shortfall probability threshold of 10% for a target rate of 5%. Note that by its nature, the HAMA parameter λ can directly be inferred from the target return r. In our example, a target return of 5% implies that $\lambda = E[R_p] - \log(r) = 5.78\%$, where $E[R_p]$ is the expected log return of the optimal decay rate portfolio. Assuming Gaussian distribution of returns and implementing the maximization under constraint given by the HAMA model in (1)-(2) with $\lambda = 5.78\%$, $\Omega = 10\%$ and H=19, we obtain the same optimal portfolio as the one obtained by decay rate optimization. Other pairs of horizon and tolerance probabilities also lead an HAMA investor to select the decay rate portfolio as an optimal allocation. For instance, an investor with a horizon of around 30 years and a tolerance probability of 5% will also choose the share of 85.27% in the risky asset. Therefore, there is a direct mapping of the decay rate parameter, i.e. the target return r, into HAMA parameters (λ, H, and Ω). For a given target return r, there is one unique parameter λ^* and a multitude of pairs (H^*, Ω^*) that reconcile the solution of the HAMA and the decay rate optimization problems. As discussed above, it is possible to infer the risk aversion coefficient of power utility investors from the decay rate optimal portfolio. For our example with a target rate of 5% and a portfolio with an optimal risky share of 85.27%, the corresponding risk aversion coefficient amounts to 1.89. Hence, if one considers investors with power utility functions, all HAMA investors characterized by λ^* and with horizon and tolerance probability falling in the range of pairs (H^*, Ω^*) share a risk aversion coefficient of 1.89. This result suggests that decay rate investors trade off horizon and tolerance for a given threshold return. If they want to target a shorter horizon, they need to be more tolerant in order to obtain the given target return.

Note that a fair matching of the one-parameter decay rate model would require to use only one HAMA parameter to replicate its optimal allocations.[3] Table 1 reports results of such a matching. For several pairs of the tolerance probability (Omega) and the investment horizon (H), the interval over expected returns (lambda) is used in order to replicate the decay rate allocation obtained in our example. The distance between HAMA and decay rate optimal weights, given by Equation (18) and described later in this paper, is extremely small, indicating an accurate matching.

Table 1: Results of a numerical matching of decay rate optimal portfolio with HAMA optimization where the interval over expected returns (lambda) is the only degree of freedom used.

H	Omega	lambda	distance in weights	H	Omega	lambda	distance in weights
5	0.01	20.4%	0.00	20	0.01	10.3%	0.01
5	0.05	14.4%	0.00	20	0.05	7.2%	0.00
5	0.1	11.2%	0.00	20	0.1	5.6%	0.00
5	0.15	9.1%	0.00	20	0.15	4.5%	0.00
5	0.2	7.4%	0.00	20	0.2	3.7%	0.00
10	0.01	14.4%	0.00	25	0.01	9.1%	0.00
10	0.05	10.2%	0.00	25	0.05	6.5%	0.00
10	0.1	7.9%	0.00	25	0.1	5.0%	0.00
10	0.15	6.4%	0.00	25	0.15	4.1%	0.00
10	0.2	5.2%	0.00	25	0.2	3.3%	0.00
15	0.01	11.8%	0.00	30	0.01	8.3%	0.00
15	0.05	8.3%	0.00	30	0.05	5.9%	0.00
15	0.1	6.5%	0.00	30	0.1	4.6%	0.00
15	0.15	5.2%	0.00	30	0.15	3.7%	0.00
15	0.2	4.3%	0.00	30	0.2	3.0%	0.00

We have just seen that a decay rate allocation can be recovered using appropriate HAMA parameters. Let us now illustrate that the reverse is not necessarily true. Consider an HAMA investor who selects the following inputs for one of his/her accounts: $\Omega = 10\%$, $\lambda = 4\%$ and an investment horizon $H = 15$ years. This investor finds it optimal to hold a share of 52.56% in the risky asset in this account. This optimal portfolio yields an expected log return of 8.16%, which implies a

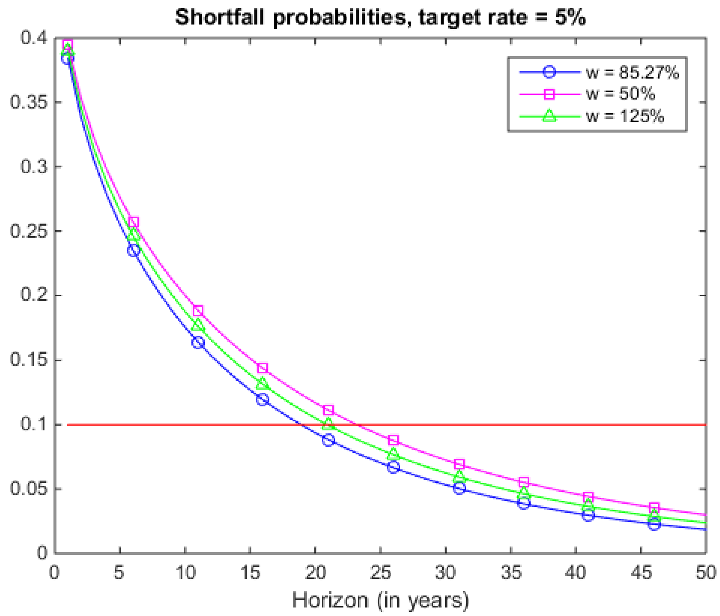
[3]We thank an anonymous referee who ask us for digging deeper into this direction.

target return r of 7.42% ($= \exp(E[R_p] - \lambda) - 1$). However, when optimizing the decay rate for this implied target return, the optimal share on the risky asset amounts to 126%, far above the optimal share of the HAMA portfolio. The decay rate portfolio generates an expected return of 12.98% and implies a value for λ of 5.82%, which is inconsistent with the parameter initially selected by the HAMA investor. The optimal share of 52.56% can be matched with a decay rate optimization with a target return of 3.75% as input. However, this allocation delivers an implied λ of 4.47% (i.e., 8.16% - log(3.75%)), which is again inconsistent with the preference of the HAMA investor. The reconciliation is not possible because the HAMA investor has preferences for horizon and tolerance probability that do not match any of the pairs (H^*, Ω^*) associated with a target return that is consistent with $\lambda = 4\%$. This finding simply emerges from the fact that the HAMA framework enables the investor to decouple preferences for horizon and tolerance probability.

There is another limit to the matching of HAMA allocation with decay rate optimization. Let us now assume the following combination of HAMA parameters: $\Omega = 5\%$, $\lambda = 1\%$ and an investment horizon $H = 10$ years. This combination generates an optimal share of 8.36% in the risky asset and an expected log return of 3.88%. These statistics imply a target return r of 2.92%, which is lower than the riskless rate of 3%. In this case, decay rate optimization would lead to an optimal portfolio that consists solely of the riskless asset, as this allocation implies an instantaneous decay towards a shortfall probability of 0. The efficient frontier of the decay rate approach therefore has a lower bound equal to the riskless rate of 3%. It is thus too narrow to cover a whole range of relevant HAMA optimal allocations, as the one underlined above.

Our example shows that the decay rate parameter, r, can be matched by appropriate HAMA parameters. However, the reverse mapping shows that the decay rate optimization does not necessarily lead to an optimal allocation consistent with the one implied by HAMA parameters. All in all, these examples point out that the HAMA model encompasses, and is more general than, the decay rate approach. Moreover, the HAMA framework enables the investor to explicitly state objectives in terms of investment horizon and to disentangle horizon and tolerance probability dimensions. Another advantage of the HAMA approach lies in the absence of any utility assumption, while decay rate allocations reflect optimal allocation of power utility investors. Finally, the formulation

Figure 1: Evolution of the probability to fall short of a target rate of 5% with respect to the investment horizon



of the HAMA portfolio problem directly in terms of a shortfall probability constraint paves the way to a straightforward introduction of the consideration for upside potential. This extension, proposed in Section 2, is not obvious to address in the decay rate approach.

4 Mapping into a flexible three-parameter utility framework

In this section, we include the class of flexible three-parameter utility functions in the analysis (hereafter FTP, see Conniffe, 2007; Meyer, 2010). Though the approach is not popular in the portfolio selection literature (at least in part due to the complexity of its structural form), it is the most general class of utility function suggested so far and is remarkably flexible in its capacity to represent varying preferences for risk. The FTP approach is a generalization of Xie's (2000) power risk aversion (PRA) utility functions, but also encompasses other types of utility functions including the hyperbolic absolute risk aversion (HARA) family (Merton, 1971). In particular, the FTP form can replicate increasing, constant and decreasing absolute as well as relative risk aversion characterizations (i.e., IARA, CARA, DARA, IRRA, CRRA and DRRA representations). Moreover, the FTP function is characterized by three parameters, making the approach even with the HAMA model as regards the available degrees of freedom once horizon, absent from the FTP specification, is kept fixed. The fact that the FTP approach covers a wide range of risk preferences, and that it is characterized by the same number of parameters makes it a serious candidate for a horse race with the HAMA model. The FTP utility function is as follows:

$$U(x) = \frac{1}{\xi}\left\{1 - \left[1 - k\xi\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right)\right]^{\frac{1}{k}}\right\} \tag{15}$$

with

$$1 - k\xi\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right) > 0 \tag{16}$$

Parameters k, ξ and σ characterize the magnitude and shape of the risk aversion measure. We refer to Conniffe (2007) for the different restrictions that can be made on parameters and the corresponding types of risk aversion that these restrictions characterize. The FTP portfolio selection problem consists in the maximization of the expected returns adjusted for "FTP risk", computed using a fourth-order Taylor series expansion of $U(x)$:

$$\max_p E\left[R_p\right] - \frac{U''\left(E\left[R_p\right]\right)}{2}\mu_p^2 + \frac{U'''\left(E\left[R_p\right]\right)}{6}\mu_p^3 - \frac{U''''\left(E\left[R_p\right]\right)}{24}\mu_p^4. \tag{17}$$

We implement a two-way comparison between the two approaches. First, we map the HAMA parameters into the FTP framework by finding HAMA parameters that deliver optimal allocations close to the ones derived with the FTP utility maximization. This mapping into the FTP class of utility functions provides us with an assessment of the capacity of the HAMA model to replicate the range of risk preferences covered by the FTP framework. Second, a reverse mapping is implemented where we map FTP parameters into the HAMA framework. This reverse mapping helps us to investigate whether the HAMA model covers optimal allocations that cannot be easily replicated with the FTP utility optimization.

The HAMA (first mapping) or FTP (second, reverse mapping) parameters are found using numerical techniques in order to minimize the relative point-wise distance in weights normalized by the sum of the absolute values of weights:

$$d = \frac{\sqrt{\sum_{i=1}^{N} (w_i - w_i^*)^2}}{\sum_{i=1}^{N} |w_i^*|} \tag{18}$$

where w_i is the weight assigned to asset i in the targeted optimal portfolio, and w_i^* its correspondent in the HAMA or FTP portfolio. Nevertheless, portfolios can differ in the absolute values of their exposures. Therefore, the same relative error in weights produces a greater distance for portfolios with larger absolute exposures. For instance, if there is a relative error of 10% for a weight of 0.5, this translates into an increment $(w_i - w_i^*)^2$ of 0.0025, while the same relative error for a weight of 1.5 corresponds to a squared difference of 0.0225. By normalizing by the sum of the absolute values of optimal weights, we obtain a comparable metric across portfolios with small and large exposures.

Data

For the practical portfolio implementation, we use the 10 domestic (US) industry portfolios provided by Fama and French. The database can easily be downloaded from Kenneth French's website and is widely used in academic research in finance.[4] The Fama-French industry portfolios contain value-weighted assets and are regularly re-balanced. Each stock from NYSE, AMEX, and NASDAQ exchanges is assigned to an industry portfolio at the end of June of each year in the sample. This association is based on Compustat SIC codes for the previous fiscal year (or CRSP SIC codes whenever Compustat SIC codes are not available). We use an updated version of the database that consists of annual returns running from 1927 through 2013. Descriptive statistics and cross-correlations of the industry portfolios are reported in Table 2. Consumer durables, denoted *Durbl*, have the highest mean return, but they also make up the leading portfolio in terms of volatility and kurtosis. This asset is thus expected to obtain relatively more weights in more aggressive strategies, such as the growth-optimal strategy. Note that *Durbl* also displays the highest positive skewness, meaning that a high excess kurtosis implies important upside potential. The Consumer Non-Durables (*NoDur*) and Telecommunication (*Telcm*) portfolios are the least volatile portfolios, with returns close to a normal distribution (i.e. with skewness and excess kurtosis at near zero, and

[4] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

with Chi-square statistics that fail to reject the null of normally distributed data), and might then be favored by more defensive strategies. Under the *Other* portfolio, assets assigned to the following sectors can be found: Mines, Construction, Building Maintenance, Transport, Hotels, Bus Services, Entertainment, and Finance. This portfolio has the worst skewness of all the industry portfolios, with relatively important volatility. Cross-correlations among Fama-French's domestic industry portfolios vary between 0.39 and 0.88, indicating diversification potential in the asset universe.

Table 2: Summary statistics and cross-correlations of the asset universe. Assets are the 10 domestic industry portfolios provided by Fama and French and available for download from Kenneth French's website. Value-weighted linear annual returns are reported for a period from 1927 to 2013. The last column of the upper panel reports the p-values of a Jarque-Bera test for normal distribution.

Statistics	mean	std	skewness	kurtosis	JB pvalue
NoDur	0.127	0.192	-0.130	2.769	>0.50
Durbl	0.152	0.349	0.853	6.055	0.00
Manuf	0.130	0.242	0.399	5.561	0.00
Enrgy	0.134	0.218	-0.083	3.454	>0.50
HiTec	0.139	0.277	0.092	2.886	>0.50
Telcm	0.113	0.196	-0.050	3.207	>0.50
Shops	0.134	0.244	-0.091	2.712	>0.50
Hlth	0.138	0.217	0.152	2.666	>0.50
Utils	0.113	0.219	0.167	3.869	0.13
Other	0.116	0.237	-0.402	3.243	0.18

Correlations	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
NoDur	100%									
Durbl	69%	100%								
Manuf	77%	87%	100%							
Enrgy	52%	58%	78%	100%						
HiTec	59%	71%	79%	54%	100%					
Telcm	59%	53%	56%	39%	63%	100%				
Shops	87%	78%	79%	50%	72%	60%	100%			
Hlth	74%	47%	61%	39%	59%	49%	72%	100%		
Utils	64%	45%	52%	51%	50%	61%	59%	59%	100%	
Other	83%	78%	88%	76%	72%	64%	81%	65%	64%	100%

Matching of FTP allocations with the HAMA model

The mapping is implemented in two steps, as follows. First, we derive the optimal allocation of the FTP model for selected ranges of values for its parameters. Second, for each optimal allocation from the FTP model, we find the combination of the parameters of the HAMA model that replicate the optimal allocation. This is implemented using a numerical routine that gives the set of HAMA parameters that minimize the distance in portfolio weights. The first step requires the specification of reasonable and representative combinations of the three FTP parameters, k, ξ and σ. Except for some combinations of parameter values that lead to well-known utility class, such as the PRA and HARA family, described in Conniffe (2007), the literature is silent on the selection of plausible values for the FTP parameters. We use values described in Conniffe (2007) as a starting point for generating FTP allocations. In order to be as exhaustive as possible, we extend this range of parameter values and consider the following values for each parameter: $k \in [-2, -1, -0.5, 0, 0.5, 1, 2]$, $\xi \in [0.25, 0.5, 1, 1.5, 5, 10]$ and $\sigma \in [-0.05, -0.01, 0, 0.25, 0.5, 1, 1.25, 1.5]$. This selected ranges of values bring about 336 possible combinations of the three FTP parameters. We further test whether these combinations of the FTP parameters are plausible. Indeed, some combinations might lead to complex values for the expected utility. For instance, if $k \neq 1$ and $1 - k\xi \left(x^{1-\sigma} - 1\right) / (1 - \sigma) \leq 0$, the utility function (15) is not well defined. Moreover, as explained in Conniffe (2007), some combinations of the parameters might lead to a negative Arrow-Pratt coefficient of absolute risk aversion. We disregard these combinations in our analysis. Finally, several combinations of the parameters lead to a risk aversion coefficient of 0, and the growth optimal portfolio is selected. As seen in the previous section, the growth optimal is a special case of the HAMA model. Therefore we also filter out any redundant allocations associated with the growth optimal portfolio. Due to these filters, 151 out of the initial 336 possible combinations of the FTP parameters are used in the mapping analysis. For each parameter, the frequency of values are reported in Figure 4.

Table 3 reports the results of the matching analysis on the total sample of 151 allocations. In the Panel A, the first four columns report statistics on the FTP parameters used to generate optimal allocations to be matched. The next four columns report minimum, median, mean and maximum values of portfolio statistics, and of the Arrow-Pratt coefficient of absolute risk aversion associated with the optimal FTP allocations. The risk aversion coefficient display values from 0 to 19.7, with an average at 6.1, which indicates that the selected FTP parameters cover a large range of portfolio profiles, from very defensive to very aggressive profiles. The next three panels of Table 3 are related to the optimized HAMA parameters that match all 151 allocations of the FTP framework. The horizon dimension is fixed at $H = 5, 10$ and 30 years. For a given horizon, the corresponding panel reports the statistics for the HAMA parameters associated with all optimal FTP allocations. Distance metrics indicate that the HAMA model is able to closely replicate the FTP allocations.

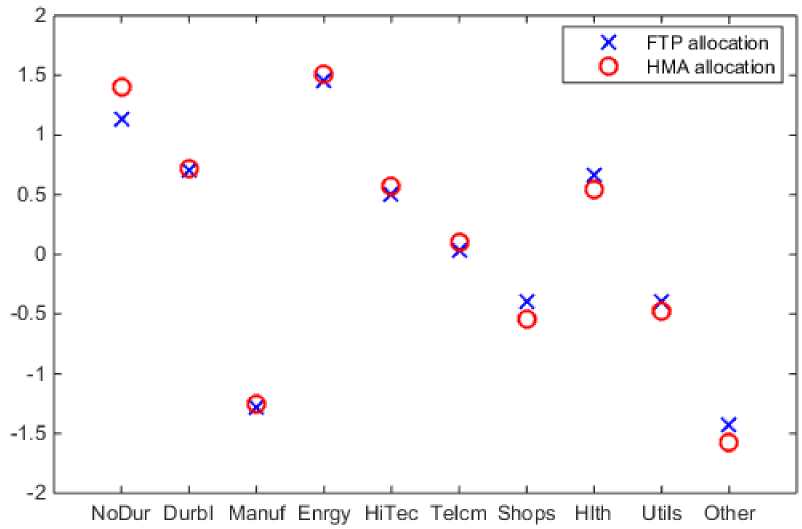
Median and average distance values (not reported in the table) are around 0.02, and a maximum value of 0.0514 is observed. The worst matching associated with the maximum distance value is reported in Figure 2. The high distance value is mostly due to differences in the weights of the Non-durable, Shops and Other assets, though the signs of the allocation are respected. Portfolio weights of the two optimal allocations are nevertheless very close to each other, indicating that the mapping is accurate. The results of this mapping analysis confirm the good performance of the HAMA approach in the replication of FTP optimal allocations and its associated wide range of risk aversion profiles.

Table 3: Matching of FTP allocations with the HAMA model. The upper panel reports statistics on the FTP parameters and associated optimal portfolios. The next three panels report statistics on the HAMA parameters that match FTP allocations for three different horizon: 5, 10 and 30 years. All 151 FTP allocations are matched. The maximum distance metric is equal to 0.0514.

Panel A	**FTP parameters**			**Optimal portfolio statistics**				
	k	***g***	***s***	***mean***	***std***	***skewn.***	***kurt.***	***risk av.***
min	-2.00	0.25	-0.05	0.12	0.12	-0.54	2.27	0.00
median	0.50	1.00	0.50	0.13	0.13	-0.27	2.33	4.99
mean	0.55	2.43	0.66	0.13	0.14	-0.30	2.54	6.09
max	2.00	10.00	1.50	0.16	0.30	-0.14	4.92	19.68

Panel B	***H = 5 years***			***H = 10 years***			***H = 30 years***		
	λ	γ	Ω	λ	γ	Ω	λ	γ	Ω
min	0.06	0.00	0.01	0.04	0.00	0.02	0.03	0.00	0.01
median	0.11	0.31	0.10	0.08	0.40	0.08	0.05	0.53	0.05
mean	0.12	0.34	0.11	0.08	0.42	0.09	0.05	0.57	0.07
max	0.53	3.04	0.32	0.19	0.99	0.35	0.50	3.75	0.26

Figure 2:
Worst match of FTP allocations with the HAMA parameters. Distance metric $D_{worst} = 0.0514$.



Matching of HAMA allocations with the FTP framework

The reverse mapping proceeds as follows. We select a range of values for the HAMA parameters and calculate the associated optimal portfolio allocations: $H \in [1, 5, 10, 20]$, $\lambda \in [0.01, 0.05, 0.1, 0.15, 0.2]$, $\gamma \in [0, 0.25, 0.5, 0.75, 1]$ and $\Omega \in [0.01, 0.05, 0.1, 0.15, 0.2]$. It gives a total of 500 allocations. However, due to the risky aspect of Fama-French assets, some combinations of HAMA parameters are too conservative, a no portfolio satisfies the shortfall probability constraint. Moreover, as in the previous section, we also remove any redundant growth optimal allocation to avoid any bias in the results. After the application of these two filters we end up with 153 HAMA allocations to be matched. Next, numerical optimization techniques are used in order to obtain the FTP parameters that minimize the distance between FTP optimal allocations and HAMA optimal portfolios. Results are presented in Table 4.

Panel A reports statistics on HAMA parameters and their associated optimal portfolios. The skewness and kurtosis of HAMA portfolios cover a large range of values, going from -1.55 to 0.89 for skewness and 2.39 to 9.46 for kurtosis. Panel B of Table 4 shows statistics on the optimized FTP parameters, the distance metric and the optimal HAMA portfolios when the distance metric is lower than (or equal to) 0.04 (i.e., for the best matches). Finally, Panel C reports the same statistics for the worst matches, i.e. those associated with a distance higher than 0.04. Extreme mean and maximum values of the worst-match category underline the difficulties of the FTP model to replicate some of the HAMA optimal portfolios.[5] A comparison of the statistics of the HAMA portfolios of the two categories shed light on these difficulties. Portfolios in the worst-match category are associated with a larger range of skewness and kurtosis values than the values observed for the best-match category. This suggests that the HAMA portfolios of the worst-match category have relatively unusually skewed and leptokurtic distributions. Figure 3 confirms this observation. It reports the combination of values of skewness and volatility (left panel) and of skewness and kurtosis (right panel) of the optimal HAMA portfolios. The red circles indicate portfolios within the worst-match category, for which the distance metric is higher than 0.04. The blue squares represent portfolios within the best-match category. The left panel underlines that although FTP allocations fare quite well in spanning the volatility dimension, they struggle to cover extremely positive and negative values of portfolios' skewness. The right panel indicates that FTP allocations have also hard time to mimic portfolios associated with an important kurtosis. The optimal HAMA portfolios associated with large values of skewness and kurtosis tend to correspond to investors with an important consideration for the upside potential of assets. Hence, the results of the reverse mapping presented here suggest that the FTP approach is not able to replicate the HAMA optimal allocations of these high-γ investors.

[5] In the matching of HAMA allocations with FTP parameters implemented in this section, we do not impose boundaries on FTP parameters, which explains that the optimizer might visit regions associated with extreme values. Imposing bounds on FTP parameters does not improve the matching.

Table 4: Results of the reverse matching. HAMA optimal portfolios, associated with the parameters and statistics reported in Panel A, are matched with FTP parameters. Panel B shows statistics for the best matches, i.e. those with a distance metric lower than or equal to 0.04. The worst matches are reported in Panel C.

	HAMA parameters				**Optimal HAMA portfolio statistics**			
Panel A	$\boldsymbol{H}$	$\boldsymbol{\lambda}$	$\boldsymbol{\gamma}$	$\boldsymbol{\Omega}$	***mean***	***std***	***skewn.***	***kurt.***
min	1.00	0.01	0.00	0.01	0.10	0.12	-1.55	2.39
median	5.00	0.15	0.50	0.10	0.15	0.22	-0.35	3.95
mean	8.79	0.13	0.43	0.09	0.14	0.22	-0.37	4.04
max	20.00	0.20	1.00	0.20	0.16	0.34	0.89	9.46
Panel B					**Optimal HAMA portfolio statistics**			
distance ≤ 0.04	k	ξ	σ	***distance***	***mean***	***std***	***skewn.***	***kurt.***
min	-1.00	-0.54	-0.02	0.00	0.12	0.12	-0.71	2.39
median	0.79	1.00	0.01	0.01	0.15	0.21	-0.34	3.68
mean	0.74	2.19	0.04	0.01	0.15	0.21	-0.34	3.69
max	10.58	15.00	0.32	0.04	0.16	0.31	-0.05	5.86
nb of allocations	104							
Panel C					**Optimal HAMA portfolio statistics**			
distance > 0.04	k	ξ	σ	***distance***	***mean***	***std***	***skewn.***	***kurt.***
min	-1.53	-0.39	-0.02	0.04	0.10	0.13	-1.55	2.67
median	0.76	7.67	0.00	0.09	0.14	0.23	-0.67	4.34
mean	46.66	7070.40	0.25	0.12	0.14	0.23	-0.43	4.77
max	1000.00	58990.01	5.19	0.55	0.15	0.34	0.89	9.46
nb of allocations	49							

Figure 3: Values of volatility, skewness and kurtosis of HAMA portfolios used in the reverse mapping. Blue squares indicate HAMA portfolios that are reasonably matched by the FTP model. Red circles represents HAMA portfolios that cannot be replicated by the FTP model.

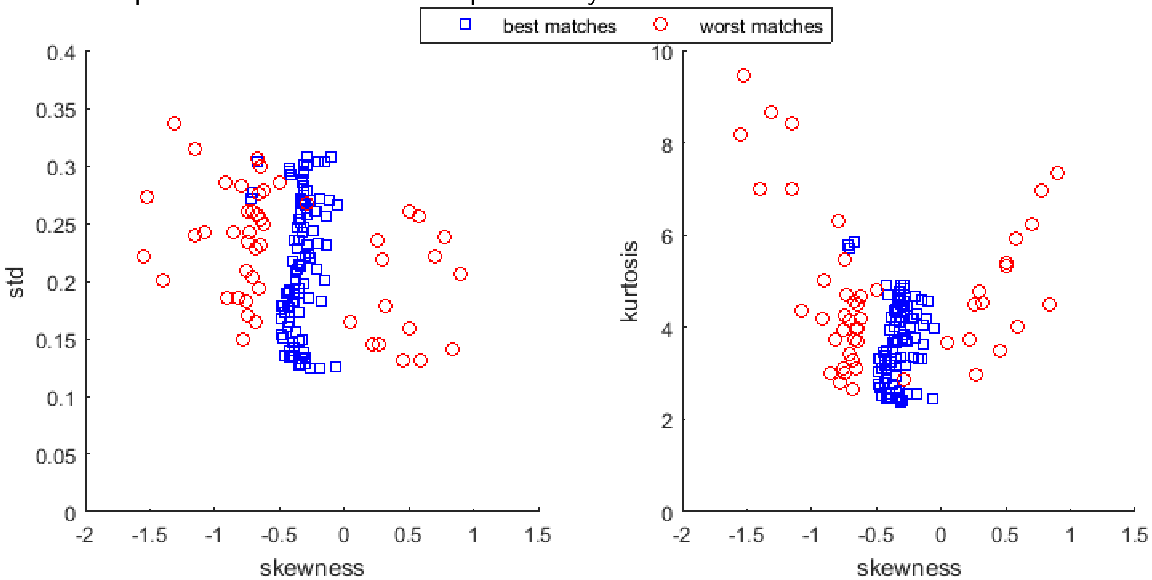
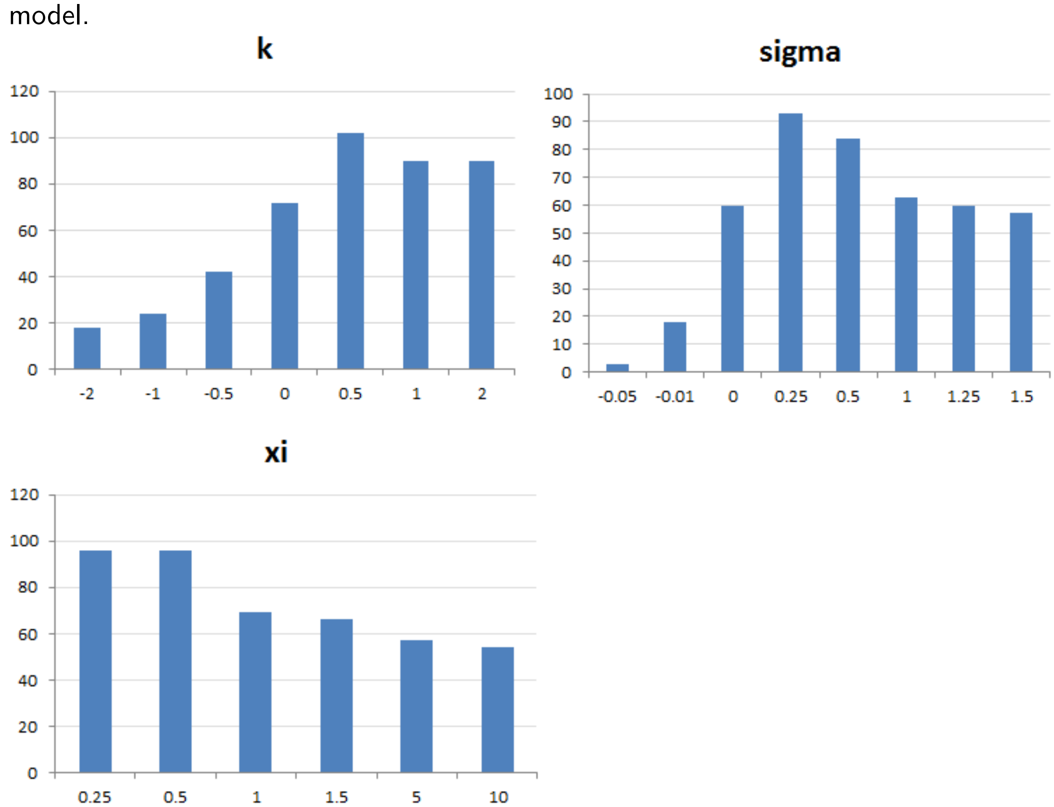


Figure 4: Frequencies of values used for FTP parameters in the matching of FTP allocations with the HAMA model.



5 Conclusion

The paper compares a behavioral mental accounting framework to portfolio choice, HAMA, to traditional utility approaches. HAMA takes the realistic view that, in practice, investors do not know their own utility function, and have hard time to translate their risk attitudes into the coefficients (like the risk aversion coefficient) used in these parametric setups. Besides, while investors agree that the Gaussian assumption is not necessarily appropriate in practice, they have incomplete knowledge about asset returns distributions. Instead of assuming any specific probability function, investors (or their financial advisers) use empirical estimates for the first four unconditional moments in order to assess their probabilities of reaching or exceeding a financial objective. The set of parameters required for solving their portfolio problem reduces to a threshold return (or a VaR threshold), a tolerance probability, an investment horizon and a weight assigned to the upside potential of portfolios.

We show that the HAMA model is not solely an intuitive tool for characterizing investor's risk-return attitudes, but that it also competes with traditional utility approaches, even very flexible ones. While the HAMA approach covers optimal allocations obtained in Stutzer's (2003) decay rate optimization, which is analogue to the expected power utility criterion, and a general class of flexible three-parameter (FTP) utility functions presented in Conniffe (2007); Meyer (2010). However a reverse-mapping exercise indicate that the reverse is not possible. Though also based on a notion of horizon, the decay rate approach does not dissociate preferences for investment horizon and tolerance probability, and cannot reproduce optimal HAMA portfolios associated with a target return smaller than the riskless rate. The FTP approach, for its part, has hard time to replicate HAMA allocations associated with extreme asymmetry and fat-tailness held by investors with a large consideration for the upside potential of portfolios. These results indicate that HAMA covers a larger set of heterogeneous risk profiles compared to those competitors.

The application of the HAMA framework is not limited to portfolio choices and has interesting implications for future research. The framework has the potential to contribute to the literature on multi-moment asset-pricing models, with interesting implications for the derivation of CAPM-type of equilibria (capital market line, security market line) under less restrictive assumptions than traditional models (see, for instance, Kraus and Litzenberger, 1976; Simaan, 1993; Jurczenko and Maillet, 2001). Moreover, as the HAMA risk measure (probability of shortfall) refers to a notion of time, it can be matched with the one on Treasury securities (i.e. maturity or duration), which is estimated with the information contained in multiple points on the yield curve. Information on interest rates ensures the consistency of the framework with a non-constant term structure. Such matching is not easily transportable to other risk measures. Through an arbitrage argument, a link can be made between the risk and return of treasuries and a market portfolio, and market-wide attitudes towards risk can be extracted. In particular, such a link can be used to endogenously characterize the dynamics of expected market returns and its components. If market-wide attitudes towards risk are a leading indicator of market sentiment, the estimation of these dynamics with stock market and term structure data should enable us to test new predictors of future excess stock returns and financial stress.

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