# Generalized VANDERMONDE determinants

MOTIVATION:

The problem of *polynomial interpolation* is that of finding a polynomial function $p : \mathbf{R} \longrightarrow \mathbf{R}$ with degree $\leqslant n$ whose values coincide with those of a function $f$ at distinct abscissae $x_0, x_1, \ldots, x_n$. The well-known solution, rather ingenious, is given by:

$$p(x) = \sum_{0 \leqslant k \leqslant n} f(x_k) \prod_{\substack{0 \leqslant j \leqslant n \\ j \neq k}} \frac{x - x_j}{x_k - x_j}$$

(this is LAGRANGE *interpolation polynomial*). But if we don't know this tricky solution, we look for the polynomial under the form $p(x) = a_0 + a_1 x + \cdots + a_n x^n$; the coefficients have to fulfill the system

$$\begin{cases} a_0 + a_1 x_0 + \cdots + a_n x_0^n = f(x_0) \\ a_0 + a_1 x_1 + \cdots + a_n x_1^n = f(x_1) \\ \cdots \\ a_0 + a_1 x_n + \cdots + a_n x_n^n = f(y_n) \end{cases}$$

(with unknowns $a_k$). This is a linear system with as many equations as unknowns; it has a unique solution if and only if its coefficient matrix has non-zero determinant. This matrix is a VANDERMONDE *matrix*:

$$\begin{pmatrix} 1 & x_0 & \ldots & x_0^n \\ 1 & x_1 & \ldots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \ldots & x_n^n \end{pmatrix}. \tag{1}$$

Classically, its determinant can be factored as

$$\prod_{0 \leqslant j < k \leqslant n} (x_k - x_j),$$

and we observe that it is non-zero since the given abscissae are assumed to be distinct: the problem of polynomial interpolation has a unique solution.

Another problem about determining a polynomial function $p : \mathbf{R} \longrightarrow \mathbf{R}$ with degree $\leqslant n$ is that of the TAYLOR polynomial: given a function $f$, $n$ times differentiable at some point $x_0$, we want that $p(x_0)$ coincide with $f(x_0)$, that $p'(x_0)$ coincide with $f'(x_0)$, ..., that $p^{(n)}(x_0)$ coincide with $f^{(n)}(x_0)$; there is only one abscissa, but "higher order" conditions. The coefficients of $p$ are now subject to:

$$\begin{cases} a_0 + a_1 x_0 + \cdots + a_n x_0^n = f(x_0) \\ \qquad a_1 + \cdots + n a_n x_0^n = f'(x_0) \\ \cdots \\ \qquad\qquad a_n = \frac{1}{n!} f^{(n)}(x_0). \end{cases}$$

The situation is simpler than in the first problem, since the matrix of this system is an upper triangular matrix with only ones on its diagonal: hence the determinant is 1 and this problem has always a unique solution.

We also might have to find a polynomial function subject to mixed conditions. For instance, if $f$ is a real function and $x_1, x_2, x_3$ three abscissae, we want to find a polynomial $p$ (as simple as possible, i.e. of lowest degree), such that

$$\begin{cases} p(x_1) = f(x_1) \\ p'(x_1) = f'(x_1) \\ p(x_2) = f(x_2) \\ p(x_3) = f(x_3) \\ p'(x_3) = f'(x_3) \\ p''(x_3) = f''(x_3); \end{cases} \tag{2}$$

in other words, we ask that $p$ coincide with $f$ at all three abscissae, but also that the derivatives coincide at $x_1$ and $x_3$ and that the second derivatives coincide at $x_3$. Since there are six conditions, it is reasonable to search for $p$ among polynomials with six coefficients, i.e. whose degree is $\leqslant 5$. If we note $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$, the coefficients have to fullfill the conditions

$$\begin{cases} a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + a_4x_1^4 + a_5x_1^5 = f(x_1) \\ a_1 + 2a_2x_1 + 3a_3x_1^2 + 4a_4x_1^3 + 5a_5x_1^4 = f'(x_1) \\ a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3 + a_4x_2^4 + a_5x_2^5 = f(x_2) \\ a_0 + a_1x_3 + a_2x_3^2 + a_3x_3^3 + a_4x_3^4 + a_5x_3^5 = f(x_3) \\ a_1 + 2a_2x_3 + 3a_3x_3^2 + 4a_4x_3^3 + 5a_5x_3^4 = f'(x_3) \\ a_2 + 3a_3x_3 + 6a_4x_3^2 + 10a_5x_3^3 = \frac{1}{2}f''(x_3). \end{cases}$$

Again, with respect to the unknowns $a_0, a_1, \ldots, a_5$, this is a square linear system; its matrix is now

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \\ 0 & 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 \\ 0 & 0 & 1 & 3x_3 & 6x_3^2 & 10x_3^3 \end{pmatrix}. \tag{3}$$

This time, the computation of its determinant is neither classical nor immediate. But we would like to be sure that it is non-zero, in such a way that the function $p$ exist and be unique.

Let us now formulate the problem in full generality and abstraction.

Let $\overline{k}$ be a *multi-index*, i.e. a list $(k_1, k_2, \ldots, k_n)$ of natural numbers; $n$ will be its *length* and $k = k_1 + k_2 + \cdots + k_n$ its *weight*. Let furthermore $A$ be a commutative ring; we shall work with matrices with entries in the ring $A[X_1, X_2, \ldots, X_n]$ of polynomials with $n$ indeterminates. Our goal is to prove the following

***Theorem.*** *Let*

$$L_k(X_i) = \begin{pmatrix} 1 & X_i & X_i^2 & X_i^3 & \ldots & X_i^{k-1} \end{pmatrix}, \quad V_i^{\overline{k}} = \begin{pmatrix} L_k(X_i) \\ \partial_i L_k(X_i) \\ \frac{1}{2}\partial_i^2 L_k(X_i) \\ \vdots \\ \frac{1}{(k_i-1)!}\partial_i^{k_i-1} L_k(X_i) \end{pmatrix} \quad \text{and} \quad V^{\overline{k}} = \begin{pmatrix} V_1^{\overline{k}} \\ V_2^{\overline{k}} \\ \vdots \\ V_n^{\overline{k}} \end{pmatrix}$$

*($\partial_i$ being the derivation with respect to the $i$th indeterminate). In these conditions,*

$$\text{dét}\, V^{\overline{k}} = \prod_{1 \leqslant i < j \leqslant n} (X_j - X_i)^{k_i k_j}. \tag{4}$$

For instance (using, for simplicity, $X = X_1$, $Y = X_2$, $Z = X_3$),

$$V^{(2,1,3)} = \begin{pmatrix} 1 & X & X^2 & X^3 & X^4 & X^5 \\ 0 & 1 & 2X & 3X^2 & 4X^3 & 5X^4 \\ 1 & Y & Y^2 & Y^3 & Y^4 & Y^5 \\ 1 & Z & Z^2 & Z^3 & Z^4 & Z^5 \\ 0 & 1 & 2Z & 3Z^2 & 4Z^3 & 5Z^4 \\ 0 & 0 & 1 & 3Z & 6Z^2 & 10Z^3 \end{pmatrix} \quad \text{and} \quad \text{dét}\, V^{(2,1,3)} = (Y-X)^2(Z-X)^6(Z-Y)^3.$$

This is the matrix (3) that arises in the resolution of problem (2). When $\overline{k} = (1, 1, \ldots, 1)$,

$$V^{(1,1,\ldots,1)} = \begin{pmatrix} 1 & X_1 & \ldots & X_1^n \\ 1 & X_2 & \ldots & X_2^n \\ \vdots & \vdots & & \vdots \\ 1 & X_n & \ldots & X_n^n \end{pmatrix}: \tag{5}$$

this matrix is merely the classical VANDERMONDE matrix, as it appears in the polynomial interpolation problem. Furthermore, when $\overline{k} = (n)$,

$$V^{(n)} = \begin{pmatrix} 1 & X_1 & X_1^2 & \dots & X_1^n \\ 0 & 1 & 2X_1 & \dots & nX_1^{n-1} \\ 0 & 0 & 1 & \dots & \frac{n(n-1)}{2}X_1^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{6}$$

is the matrix that appears in the problem of the TAYLOR polynomial. In this case, the right hand side of (4) is 1, since the product is empty when the length of $\overline{k}$ is 1.

Let us note that one of the $k_i$'s may be equal to zero; however, in such a case, the corresponding group of lines, in $V^{\overline{k}}$, is empty: the only effect is to shift the numbering of the following indeterminates. For instance,

$$V^{(2,1)} = \begin{pmatrix} 1 & X_1 & X_1^2 \\ 0 & 1 & 2X_1 \\ 1 & X_2 & X_2^2 \end{pmatrix}, \quad \text{while} \quad V^{(2,0,1)} = \begin{pmatrix} 1 & X_1 & X_1^2 \\ 0 & 1 & 2X_1 \\ 1 & X_3 & X_3^2 \end{pmatrix}.$$

Considering multi-indices with null components is thus pointless, although it is not forbidden.

Before proving the theorem, we have to state and justify two lemmas.

**Lemma 1.** If $P \in A[X]$, then

$$P' = \left.\frac{P(X+H)-P}{H}\right|_{H=0}.$$

The right-hand side of this equality must be understood as follows: $P(X+H)$ is the two-indeterminate ($X$ and $H$) polynomial obtained by the substitution of $X+H$ to $X$ in $P$; the difference $P(X+H)-P =$ $= P(X+H)-P(X)$ vanishes when $H=0$; hence, after the polynomial remainder theorem, it is divisible by $H$; it follows that $(P(X+H)-P)/H$ is a polynomial (rather than a fraction), again with two indeterminates $X$ and $H$); finally, $(P(X+H)-P)/H|_{H=0}$ is the result of the substitution of 0 to $H$ in this polynomial; it is a polynomial in $X$.

PROOF: Since both maps from $A[X]$ into itself

$$P \mapsto P' \quad \text{and} \quad P \mapsto (P(X+H)-P)/H|_{H=0}$$

are linear, it is enough to prove the equality when $P$ is one of the elements of the basis $(1, X, X^2, \dots)$ of $A[X]$. But, for any natural number $n$,

$$\left.\frac{(X+H)^n - X^n}{H}\right|_{H=0} = \left.\frac{\sum\limits_{0\leqslant i\leqslant n}\binom{n}{i}X^{n-i}H^i - X^n}{H}\right|_{H=0} = \left.\frac{\sum\limits_{0<i\leqslant n}\binom{n}{i}X^{n-i}H^i}{H}\right|_{H=0} = \left.\sum_{0<i\leqslant n}\binom{n}{i}X^{n-i}H^{i-1}\right|_{H=0} =$$

$$= \binom{n}{1}X^{n-1} = nX^{n-1} = (X^n)'. \quad \blacksquare$$

**Lemma 2.** If $P \in A[X]$, then

$$P^{(d)} = \left.\frac{1}{H^d}\sum_{0\leqslant j\leqslant d}(-1)^j\binom{d}{j}P(X+(d-j)H)\right|_{H=0}.$$

This result generalizes lemma 1, which is the particular case $d=1$.

FIRST PROOF: We can work by induction, just as when we prove the binomial formula (see [2]) or the general LEIBNIZ' rule for the $n$th derivative of a product (see [3]). We have here to cope with questions of sign, but everything is ok. $\blacksquare$

SECOND PROOF: As for the first lemma, it is sufficient to prove the result when $P = X^n$ ($n \in \mathbf{N}$). In this case, the right-hand side becomes

$$\left.\frac{1}{H^d}\sum_{0\leqslant j\leqslant d}(-1)^j\binom{d}{j}(X+(d-j)H)^n\right|_{H=0} = \left.\frac{1}{H^d}\sum_{0\leqslant j\leqslant d}(-1)^j\binom{d}{j}\sum_{0\leqslant i\leqslant n}\binom{n}{i}X^{n-i}((d-j)H)^i\right|_{H=0}$$

$$= \left.\sum_{0\leqslant i\leqslant n}\binom{n}{i}X^{n-i}\Big(\sum_{0\leqslant j\leqslant d}(-1)^j\binom{d}{j}(d-j)^i\Big)H^{i-d}\right|_{H=0}.$$

The sum in the parentheses is something that we recognise: $\sum\limits_{0\leqslant j\leqslant d}(-1)^j\binom{d}{j}(d-j)^i$ is the number of onto maps from a $i$ element set to a $d$ element set (see for instance [1]); this number is also $d!$ times the Stirling number of the second kind $S(i,d)$. This number is obviously zero when $i<d$ (hopefully!): so in the last sum, the negative powers of $H$ disappear and we have, as expected, a polynomial in $H$ — not a rational fraction. It follows that the principal sum is null when $n<d$, since in this case all the terms disappear. Next, if $n\geqslant d$, the value at $H=0$ of this polynomial is its constant term, obtained when $i=d$; the sum $\sum\limits_{0\leqslant j\leqslant d}(-1)^j\binom{d}{j}(d-j)^d$, in this case, is the number of onto maps from a $d$ element set to a $d$ element set, i.e. $d!$. And

$$\frac{1}{H^d}\sum_{0\leqslant j\leqslant d}(-1)^j\binom{d}{j}(X+(d-j)H)^n\Bigg|_{H=0}=\binom{n}{d}X^{n-d}\Big(\sum_{0\leqslant j\leqslant d}(-1)^j\binom{d}{j}(d-j)^d\Big)=\binom{n}{d}X^{n-d}d!=$$
$$=\frac{n!}{(n-d)!}X^{n-d}=n(n-1)\cdots(n-d+1)X^{n-d}=(X^n)^{(d)}.\quad\blacksquare$$

Proof of the theorem: In matrix $V^{\overline{k}}$, the $i$th group of lines can be rewritten, thanks to the lemmas:

$$V_i^{\overline{k}}=\left(\begin{array}{c}L_k(X_i)\\ \frac{1}{H}(L_k(X_i+H)-L_k(X_i))\\ \frac{1}{2H^2}(L_k(X_i+2H)-2L_k(X_i+H)+L_k(X_i))\\ \vdots\\ \frac{1}{(k_i-1)!H^{k_i-1}}\Big(\sum\limits_{0\leqslant j\leqslant k_i-1}(-1)^j\binom{k_i-1}{j}L_k(X_i+(k_i-1-j)H)\Big)\end{array}\right)\Bigg|_{H=0}.$$

In the computation of dét $V^{\overline{k}}$, we shall factor out $1/H$ in the first line of each such group, $1/(2H^2)$ in the second one, …, $1/((k_i-1)!H^{k_i-1}$ in the last one; so we have divided the determinant by

$$\Delta:=(1!\cdot 2!\cdots(k_1-1)!)\cdots(1!\cdot 2!\cdots(k_n-1)!)H^{(k_1-1)k_1/2+\cdots+(k_n-1)k_n/2}. \tag{7}$$

The $i$th group of lines becomes:

$$\left(\begin{array}{c}L_k(X_i)\\ L_k(X_i+H)-L_k(X_i)\\ L_k(X_i+2H)-2L_k(X_i+H)+L_k(X_i)\\ \vdots\\ \sum\limits_{0\leqslant j\leqslant k_i-1}(-1)^j\binom{k_i-1}{j}L_k(X+(k_i-1-j)H)\end{array}\right)\Bigg|_{H=0}.$$

In this new group of lines, we add the first line to the second one, we substract it from the third one, and so on, to get

$$\left(\begin{array}{c}L_k(X_i)\\ L_k(X_i+H)\\ L_k(X_i+2H)-2L_k(X_i+H)\\ \vdots\\ \sum\limits_{0\leqslant j\leqslant k_i-2}(-1)^j\binom{k_i-1}{j}L_k(X+(k_i-1-j)H)\end{array}\right)\Bigg|_{H=0};$$

these operations left the determinant unchanged. We add now twice the second line to the third one, we substract it three times to the fourth line, etc., whigh gives

$$\left(\begin{array}{c}L_k(X_i)\\ L_k(X_i+H)\\ L_k(X_i+2H)\\ \vdots\\ \sum\limits_{0\leqslant j\leqslant k_i-3}(-1)^j\binom{k_i-1}{j}L_k(X+(k_i-1-j)H)\end{array}\right)\Bigg|_{H=0};$$

again, the determinant remains unchanged. Going on the same way, we obtain at last

$$\left.\begin{pmatrix} L_k(X_i) \\ L_k(X_i+H) \\ L_k(X_i+2H) \\ \vdots \\ L_k(X+(k_i-1)H) \end{pmatrix}\right|_{H=0},$$

and the matrix

$$W^{\overline{k}} = \left.\begin{pmatrix} L_k(X_1) \\ L_k(X_1+H) \\ \vdots \\ L_k(X_1+(k_1-1)H) \\ \vdots \\ L_k(X_n) \\ L_k(X_n+H) \\ \vdots \\ L_k(X_n+(k_n-1)H) \end{pmatrix}\right|_{H=0}$$

formed by these $n$ groups of lines still has the same determinant as the former. In other words,

$$\text{dét}\, V^{\overline{k}} = \frac{1}{\Delta}\, \text{dét}\, W^{\overline{k}}\Big|_{H=0}, \tag{8}$$

where $\Delta$ is defined by (7). But the determinant of $W^{\overline{k}}$ is a classical VANDERMONDE determinant; its value is

$$((X_1+H)-X_1)((X_1+2H)-X_1)\cdots((X_1+(k_1-1)H)-X_1)(X_2-X_1)\cdots((X_n+(k_n-1)H)-(X_n+(k_n-2)H)),$$

i. e. the product of all differences

$$(X_j+j'H)-(X_i+i'H)$$

where $1 \leqslant i \leqslant j \leqslant n$, $0 \leqslant i' < k_i$, $0 \leqslant j' < k_j$ and, when $i = j$, $i' < j'$. Formally:

$$\text{dét}\, W^{\overline{k}} = \prod_{\substack{1\leqslant i\leqslant n \\ 0\leqslant i'<j'\leqslant k_i}} \big((X_i-j'H)-(X_i-i'H)\big) \cdot \prod_{\substack{1\leqslant i<j\leqslant n \\ 0\leqslant i'<k_i \\ 0\leqslant j'<k_j}} \big((X_j-j'H)-(X_i-i'H)\big) \tag{9}$$

For a fixed $i$,

$$\prod_{0\leqslant i'<j'<k_i} \big((X_i-j'H)-(X_i-i'H)\big) = \prod_{0\leqslant i'<j'<k_i} (i'-j')H = 1!\cdots(k_i-1)!H^{(k_i-1)+\cdots+1} = 1!\cdots(k_i-1)!H^{(k_i-1)k_i/2}$$

The first product in the right-hand side of (9) is thus:

$$\begin{aligned}\prod_{\substack{1\leqslant i\leqslant n \\ 0\leqslant i'<j'<k_i}} \big((X_i-j'H)-(X_i-i'H)\big) &= \prod_{1\leqslant i\leqslant n} 1!\cdots(k_i-1)!H^{(k_i-1)k_i/2} \\ &= (1!\cdot 2!\cdots(k_1-1)!)\cdots(1!\cdot 2!\cdots(k_n-1)!)H^{(k_1-1)k_1/2+\cdots+(k_n-1)k_n/2} \\ &= \Delta.\end{aligned}$$

Hence,

$$\begin{aligned}\text{dét}\, V^{\overline{k}} = \prod_{\substack{1\leqslant i<j\leqslant n \\ 0\leqslant i'<k_i \\ 0\leqslant j'<k_j}} \big((X_j-j'H)-(X_i-i'H)\big)\Big|_{H=0} = \prod_{\substack{1\leqslant i<j\leqslant n \\ 0\leqslant i'<k_i \\ 0\leqslant j'<k_j}} \big(X_j-X_i+(i'-j')H\big)\Big|_{H=0} = \\ = \prod_{\substack{1\leqslant i<j\leqslant n \\ 0\leqslant i'<k_i \\ 0\leqslant j'<k_j}} (X_j-X_i) = \prod_{1\leqslant i<j\leqslant n} (X_j-X_i)^{k_ik_j},\end{aligned}$$

and the proof is complete. ∎

## References

[1] Pascal DUPONT, *Surjections*, Losanges **15** (2011), 29–36.

[2] https://en.wikipedia.org/wiki/Binomial_theorem (viewed on November 5, 2021)

[3] https://en.wikipedia.org/wiki/General_Leibniz_rule (viewed on November 5, 2021)