Bouligand Influence Function and Robustness of Support Vector Machines

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Notation

Assumptions:

- $X \subseteq \mathbb{R}^d$, $Y \subseteq \mathbb{R}$, $X \neq \varnothing$, $Y \neq \varnothing$
- $\mathcal{D} = \mathcal{D}_n = ((x_1, y_1), \ldots, (x_n, y_n))$, $1 \leq i \leq n$
- (X_i, Y_i) i.i.d. $\sim \mathrm{P} \in \mathcal{M}_1$, P (totally) unknown

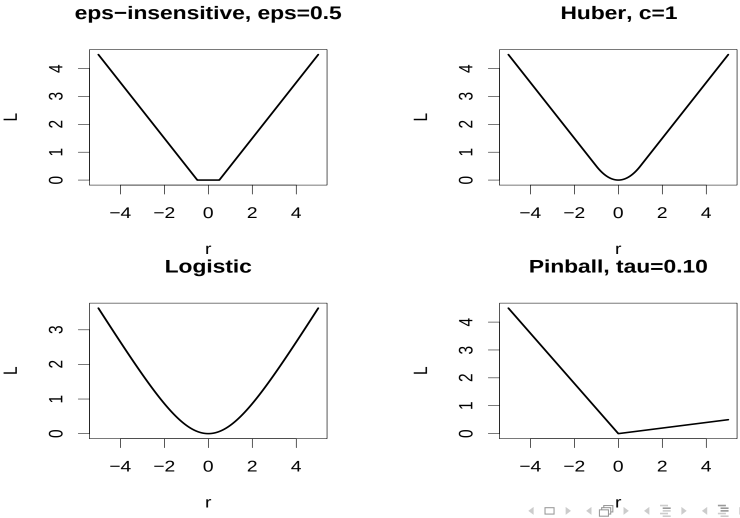
Aim:

- $f(x_i)$ = quantity of interest of $\mathrm{P}_{Y_i | X_i = x_i}$

Assumption:

- Loss function: $L : Y \times \mathbb{R} \to [0, \infty)$, $L(y_i, f(x_i))$, convex

Loss functions for regression



Kernel methods

- **Kernel**: $k : X \times X \to \mathbb{R}$, if $\exists$ Hilbert space $\mathcal{H}$ and $\Phi : X \to \mathcal{H}$ such that

$$k(x,x') = \langle \Phi(x), \Phi(x') \rangle, \quad \forall x, x' \in X$$

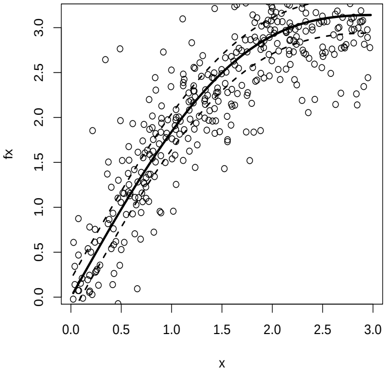
Reproducing Kernel Hilbert Space (RKHS)

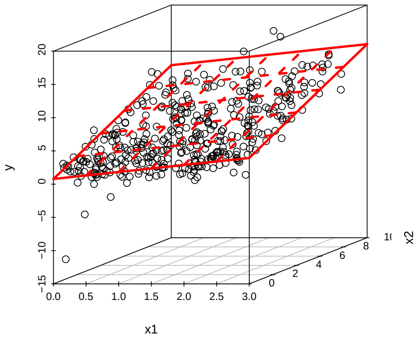
$\mathcal{H}$ a Hilbert space of functions $f : X \to \mathbb{R}$. A reproducing kernel for $\mathcal{H}$ is a kernel k with

$$f(x) = \langle f, k(x,\cdot) \rangle \quad \forall\, f \in \mathcal{H}, \forall\, x \in X.$$

- **Canonical feature map**: $\Phi(x) = k(x,\cdot),\ x \in X$
- $k \rightleftarrows$ RKHS unique
- **Bounded**: $||k||_\infty := \sqrt{\sup_{x \in \mathcal{X}} k(x,x)} < \infty$
- **GRBF**: $k(x,x') = e^{-\gamma ||x - x'||_2^2}$, $\gamma > 0$

Example for feature map $\Phi(\mathbf{x}) = \mathbf{k}(\mathbf{x}, \cdot)$





Support Vector Machines (SVMs)

Definition

Kernel Based Regression (KBR) operator

$$S(\mathrm{P}) = f_{\mathrm{P},\lambda} = \arg\min_{f \in \mathcal{H}} \mathbb{E}_{\mathrm{P}} L\big(Y_i, f(X_i)\big) + \lambda \, \|f\|_{\mathcal{H}}^2 \ ,$$

where $\mathrm{P} \in \mathcal{M}_1$, $\mathcal{H}$ is a RKHS and $\lambda > 0$.

Kernel Based Regression estimator

$$S(\mathrm{P}_n) = \arg\min_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} L\big(Y_i, f(X_i)\big) + \lambda \, \|f\|_{\mathcal{H}}^2 \ ,$$

where $\mathrm{P}_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, y_i)}$.

Learnability of SVMs

- Universal (weak) consistency:
  $\mathcal{R}_{L,\mathrm{P}}(f_{\mathrm{P}_n}) \xrightarrow{\mathrm{P}} \inf_{f \in \mathcal{H}} \mathcal{R}_{L,\mathrm{P}}(f) := \mathcal{R}^*_{L,\mathrm{P},\mathcal{H}}$
- $L-$risk consistency:
  $\mathcal{R}_{L,\mathrm{P}}(f_{\mathrm{P}_n,\lambda_n}) \xrightarrow{\mathrm{P}} \mathcal{R}^*_{L,\mathrm{P}}$, where
  $\mathcal{R}^*_{L,\mathrm{P}} := \inf_{f:X \to \mathbb{R} \text{ measurable}} \mathcal{R}_{L,\mathrm{P}}(f)$ for suitable $\lambda_n \downarrow 0$

Christmann & Steinwart (2007)

Question

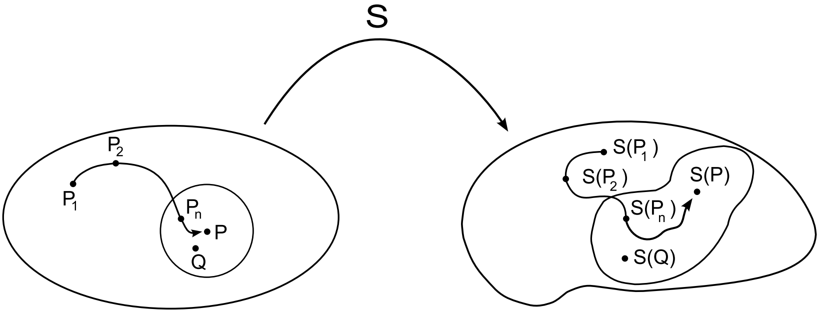
"Which properties must

- $S(\mathrm{P}) = f_{\mathrm{P},\lambda}$,
- k,
- and L

have for good robustness properties of SVMs?"

Robustness

What is the impact on $S(\mathrm{P}) = f_{\mathrm{P},\lambda}$ due to violations from
(X_i, Y_i) i.i.d. $\sim \mathrm{P}$, $\mathrm{P} \in \mathcal{M}_1$ unknown ?



Bouligand differentiability

Bouligand-derivative

$f : X \to Z$ is **Bouligand-differentiable** at $x_0 \in X$, if $\exists$ a positive homogeneous function $\nabla^B f(x_0) : X \to Z$ such that

$$f(x_0 + h) = f(x_0) + \nabla^B f(x_0)(h) + o(h)\,,$$

i.e.

$$\lim_{h \downarrow 0} \left\| f(x_0 + h) - f(x_0) - \nabla^B f(x_0)(h) \right\|_Z / \, \|h\|_X = 0.$$

Strong approximation

$f : X \to Z$ **strongly approximates** $F : X \times Y \to Z$ in x at (x_0, y_0) (notation: $f \approx_x F$) if $\forall \varepsilon > 0\ \exists$ neighborhoods $\mathcal{N}(x_0)$ of x_0 and $\mathcal{N}(y_0)$ of y_0 such that $\forall x, x' \in \mathcal{N}(x_0), \forall y \in \mathcal{N}(y_0)$

$$\left\| \big(F(x,y) - f(x)\big) - \big(F(x',y) - f(x')\big) \right\|_Z \leq \varepsilon \left\| x - x' \right\|_X .$$

Strong Bouligand-derivative

$F : X \times Y \to Z$ has partial B-derivative $\nabla_1^B F(x_0, y_0)$ w.r.t. x at (x_0, y_0). Then $\nabla_1^B F(x_0, y_0)$ is **strong** if

$$F(x_0, y_0) + \nabla_1^B F(x_0, y_0)(x - x_0) \approx_x F$$

at (x_0, y_0).

Robinson (1991)

Bouligand influence function

BIF (C&VM '07)

The **Bouligand influence function (BIF)** of a function $S : \mathcal{M}_1 \to \mathcal{H}$ for a distribution P in the direction of a distribution $\mathrm{Q} \neq \mathrm{P}$ is the special B-derivative (if it exists)

$$\lim_{\varepsilon \downarrow 0} \frac{\left\| S\big((1-\varepsilon)\mathrm{P} + \varepsilon \mathrm{Q}\big) - S(\mathrm{P}) - \mathrm{BIF}(\mathrm{Q}; S, \mathrm{P}) \right\|_{\mathcal{H}}}{\varepsilon} = 0.$$

If BIF exists, then Hampel's IF exists and $\mathrm{BIF} = \mathrm{IF}$

Goal: Bounded BIF

Main result

Assumptions

- $X \subset \mathbb{R}^d$, $Y \subset \mathbb{R}$ closed sets,
- $\mathcal{H}$ is RKHS with **bounded**, measurable kernel k,
- $f_{\mathrm{P},\lambda} \in \mathcal{H}$,
- $L : Y \times \mathbb{R} \to [0, \infty)$ **convex** and **Lipschitz continuous** w.r.t. the 2^{nd} argument with uniform Lipschitz constant $|L|_1 := \sup_{y \in Y} |L(y, \cdot)|_1 \in (0, \infty)$,
- L has measurable partial B-derivatives w.r.t. to the 2^{nd} argument with $\kappa_1 := \sup_{y \in Y} \left\| \nabla_2^B L(y, \cdot) \right\|_\infty \in (0, \infty)$, $\kappa_2 := \sup_{y \in Y} \left\| \nabla_{2,2}^B L(y, \cdot) \right\|_\infty < \infty$,

Assumptions

- $\delta_1 > 0$, $\delta_2 > 0$,
- $\mathcal{N}_{\delta_1}(f_{\mathrm{P},\lambda}) := \{f \in \mathcal{H}; \|f - f_{\mathrm{P},\lambda}\|_{\mathcal{H}} < \delta_1\}$,
- $\lambda > \frac{1}{2}\kappa_2 \|\Phi\|_{\mathcal{H}}^3$,
- P, Q probability measures on $\big(X \times Y, \mathcal{B}(X \times Y)\big)$ with $\mathbb{E}_{\mathrm{P}}|Y| < \infty$ and $\mathbb{E}_{\mathrm{Q}}|Y| < \infty$.
- Define $G : (-\delta_2, \delta_2) \times \mathcal{N}_{\delta_1}(f(\mathrm{P}, \lambda)) \to \mathcal{H}$,

$$G(\varepsilon, f) := 2\lambda f + \mathbb{E}_{(1-\varepsilon)\mathrm{P}+\varepsilon\mathrm{Q}} \nabla_2^B L(Y, f(X))\Phi(X)\,,$$

- $G(0, f_{\mathrm{P},\lambda}) = 0$ and $\nabla_2^B G(0, f_{\mathrm{P},\lambda})$ is **strong**.

Theorem (C&VM '07)

Then $\mathrm{BIF}(\mathrm{Q}; S, \mathrm{P})$ with $S(\mathrm{P}) := f_{\mathrm{P},\lambda}$

1. exists,
2. equals

$$T^{-1}\Big(\mathbb{E}_{\mathrm{P}} \nabla_2^B L(Y, f_{\mathrm{P},\lambda}(X))\Phi(X) \\ -\mathbb{E}_{\mathrm{Q}} \nabla_2^B L(Y, f_{\mathrm{P},\lambda}(X))\Phi(X)\Big),$$

where $T : \mathcal{H} \to \mathcal{H}$ with
$T = 2\lambda\, \mathrm{id}_{\mathcal{H}} + \mathbb{E}_{\mathrm{P}} \nabla_{2,2}^B L(Y, f_{\mathrm{P},\lambda}(X)) \langle \Phi(X), \cdot \rangle_{\mathcal{H}} \Phi(X)$, and

3. is bounded.

Examples

The assumptions of the theorem are valid and thus $\mathrm{BIF}(\mathrm{Q}; S, \mathrm{P})$ exists and is bounded, if

ϵ-insensitive loss L_ϵ, pinball loss L_τ

$\forall \delta > 0\, \exists$ positive constants ξ_{P}, ξ_{Q}, c_{P}, and c_{Q} such that $\forall t \in \mathbb{R}$ with $|t - f_{\mathrm{P},\lambda}(x)| \le \delta \|k\|_\infty$ the following inequalities hold $\forall a \in [0, 2\delta\|k\|_\infty]$ and $\forall x \in X$:

$$\mathrm{P}\big(Y \in [t, t+a] \,\big|\, x\big) \le c_{\mathrm{P}} a^{1+\xi_{\mathrm{P}}}$$
$$\mathrm{Q}\big(Y \in [t, t+a] \,\big|\, x\big) \le c_{\mathrm{Q}} a^{1+\xi_{\mathrm{Q}}} .$$

The assumptions of the theorem are valid and thus $\mathrm{BIF}(\mathrm{Q}; S, \mathrm{P})$ exists and is bounded, if

Huber loss L_{Huber}

$$\begin{aligned} &\forall x \in X: \\ &\quad \mathrm{P}\big(Y \in \{f_{\mathrm{P},\lambda}(x) - c, f_{\mathrm{P},\lambda}(x) + c\} \,\big|\, x\big) \\ &= \mathrm{Q}\big(Y \in \{f_{\mathrm{P},\lambda}(x) - c, f_{\mathrm{P},\lambda}(x) + c\} \,\big|\, x\big) \\ &= 0\,. \end{aligned}$$

Logistic loss L_{log}

No special assumptions on the probabilities needed.

Summary

Support vector machines

- Non-parametric and flexible
- Robust:
  - $\mathrm{BIF}(\mathrm{Q}; T, \mathrm{P})$ is bounded for regression if $\nabla_2^B L$ and k bounded
- Applications: insurance tariffs, credit scoring in banks, fraud detection, data mining, genomics, ...

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- Robinson (1991). An implicit-function theorem for a class of non-smooth functions. *Mathematics of Operations Research*, **16**, 292-309.
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More on the theorem

For the proof of the theorem we showed:

i. For some χ and each $f \in \mathcal{N}_{\delta_1}(f_{\mathrm{P},\lambda})$, $G(\cdot\,, f)$ is Lipschitz continuous on $(-\delta_2, \delta_2)$ with Lipschitz constant χ.

ii. G has partial B-derivatives with respect to ε and f at $(0, f_{\mathrm{P},\lambda})$.

iii. $\nabla_2^B G(0, f_{\mathrm{P},\lambda})\big(\mathcal{N}_{\delta_1}(f_{\mathrm{P},\lambda}) - f_{\mathrm{P},\lambda}\big)$ is a neighborhood of $0 \in \mathcal{H}$.

iv. $\delta\big(\nabla_2^B G(0, f_{\mathrm{P},\lambda}), \mathcal{N}_{\delta_1}(f_{\mathrm{P},\lambda}) - f_{\mathrm{P},\lambda}\big) \;=:\; d_0 > 0.$

v. For each $\xi > d_0^{-1}\chi$ there exist $\delta_3, \delta_4 > 0$, a neighborhood $\mathcal{N}_{\delta_3}(f_{\mathrm{P},\lambda}) := \{f \in \mathcal{H}; \|f - f_{\mathrm{P},\lambda}\|_{\mathcal{H}} < \delta_3\}$, and a function $f^* : (-\delta_4, \delta_4) \to \mathcal{N}_{\delta_3}(f_{\mathrm{P},\lambda})$ satisfying

- **v.1)** $f^*(0) = f_{\mathrm{P},\lambda}$.
- **v.2)** $f^*(\cdot)$ is Lipschitz continuous on $(-\delta_4, \delta_4)$ with Lipschitz constant $|f^*|_1 = \xi$.
- **v.3)** For each $\varepsilon \in (-\delta_4, \delta_4)$ is $f^*(\varepsilon)$ the unique solution of $G(\varepsilon, f) = 0$ in $(-\delta_4, \delta_4)$.
- **v.4)** It holds
  $$\nabla^B f^*(0)(u) = \left(\nabla_2^B G(0, f_{\mathrm{P},\lambda})\right)^{-1} \left(-\nabla_1^B G(0, f_{\mathrm{P},\lambda})(u)\right).$$