Computational & Multiscale
Mechanics of Materials

CM3

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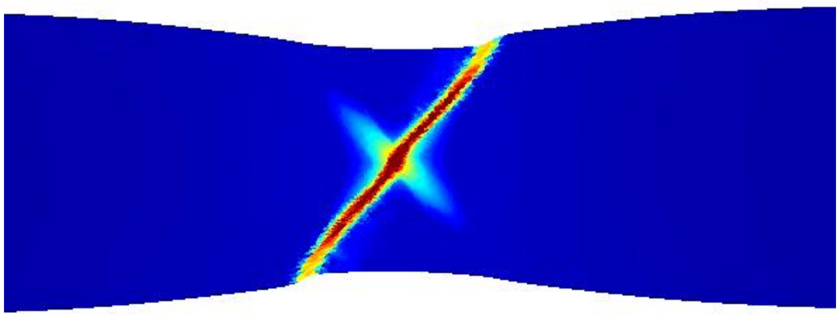
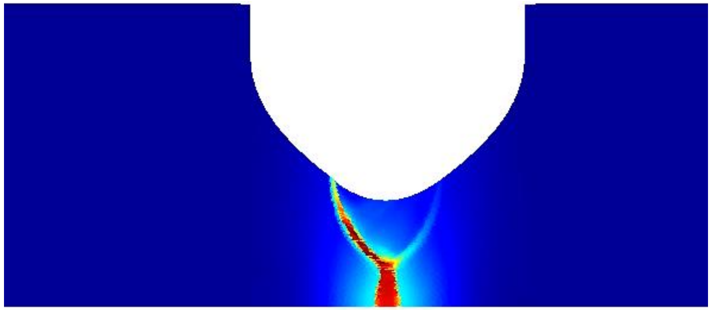
A Non-Local Ductile Failure Model Accounting for Void Growth and Coalescence at Low and High Stress Triaxiality





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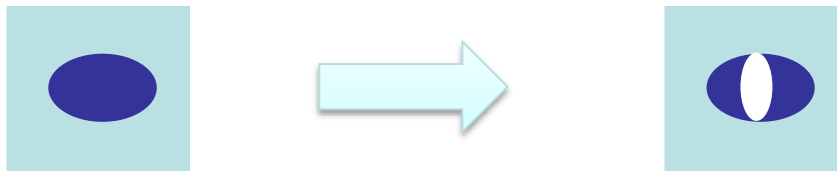
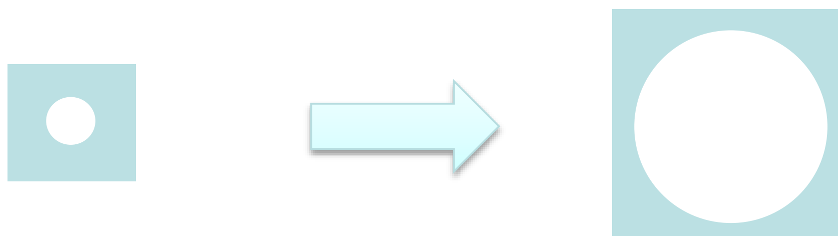
Introduction

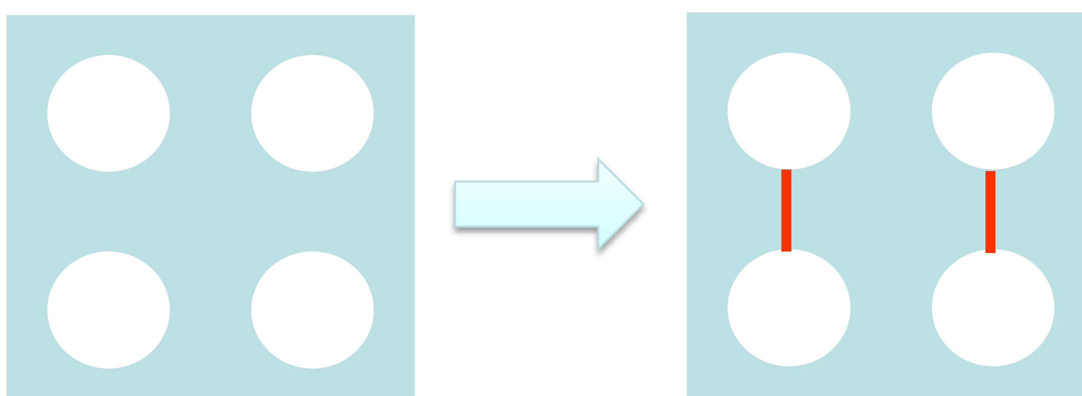
- Ductile failure: failure mechanism
  - Void nucleation (dislocation motion, particle/matrix decohesion, particle cracking, …)



  - Void growth of existing voids (because of plastic incompressibility)

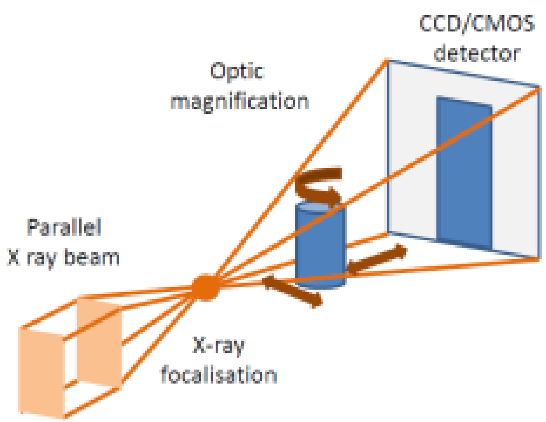


  - Void coalescence (crack growth by shrinking of ligaments between voids)

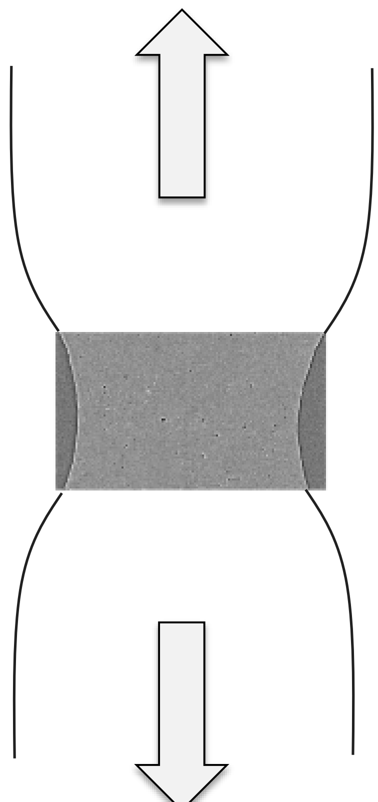


- Ductile failure: complex coalescence scenarios
  - What does happen inside a « ductile » material under large strain ?

X-ray tomography of in-situ tensile tests
= scanner for materials



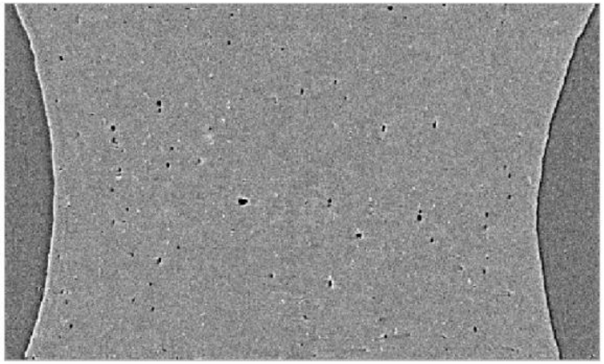
Tests performed at the ESRF synchrotron in Grenoble by F. *Hannard (Ph. D. UCL) collaboration with Dr. E. maire INSA Lyon*



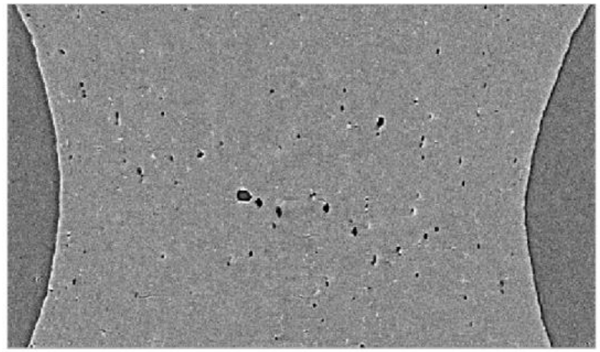
(Hannard et al. 2016)

- Ductile failure: complex coalescence scenarios
  - What does happen inside a « ductile » material under large strain ?

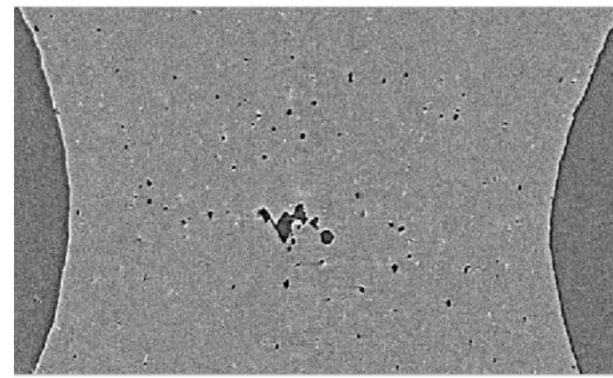
X-ray tomography of in-situ tensile tests on Al 6056



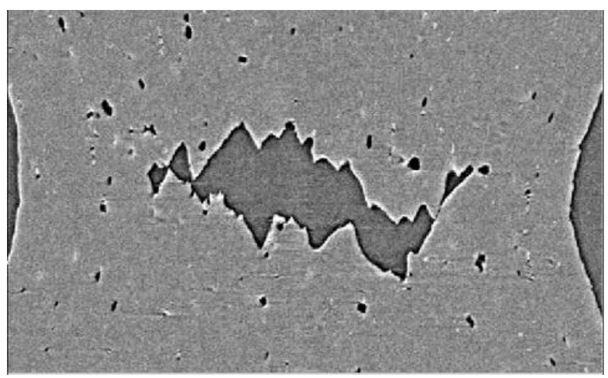
$\varepsilon = 23\%$



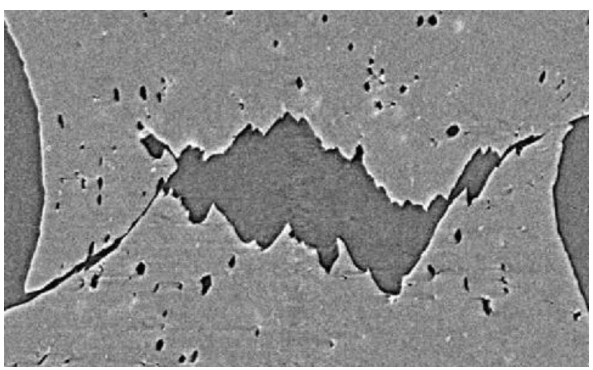
$\varepsilon = 30\%$



$\varepsilon = 38\%$



$\varepsilon = 50\%$

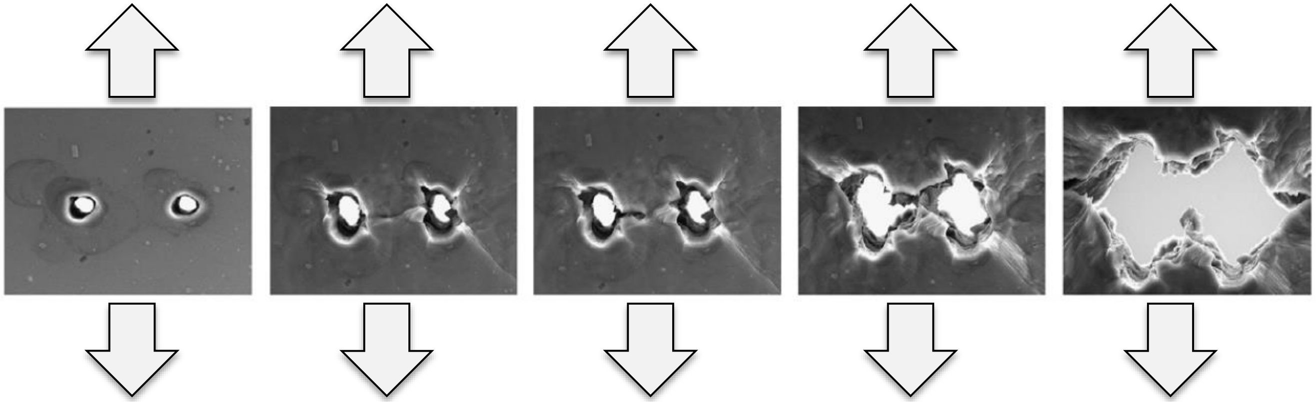


$\varepsilon = 60\%$

(Hannard et al. 2016)

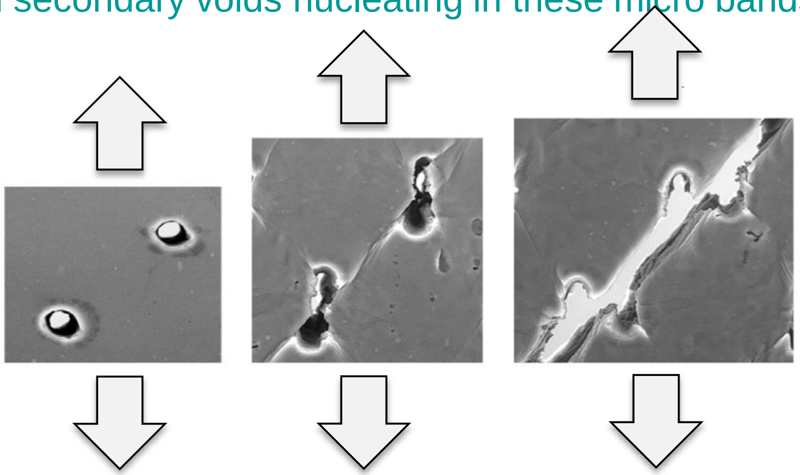
- Ductile failure: complex coalescence scenarios
  - Localization band perpendicular to the main loading direction
    - Shrinking of ligaments between voids

Internal necking coalescence



  - Micro shear bands inclined to the main loading direction
    - Joining primary voids
    - Possibly with secondary voids nucleating in these micro bands

Shear driven coalescence



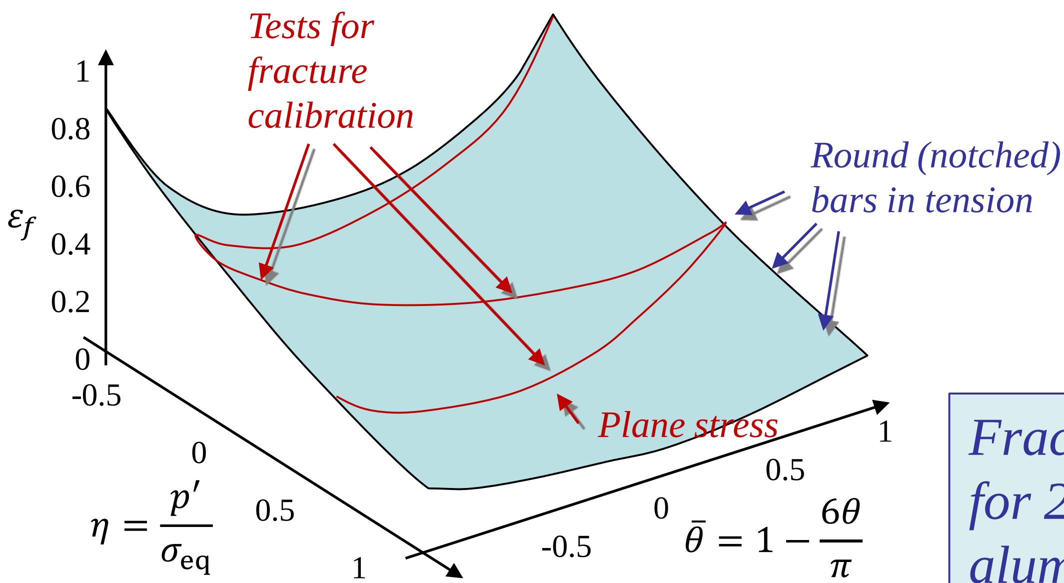
(Weck & Wilkinson 2008)

- Ductile failure: stress-state dependent fracture strain
  - Stress triaxiality dependent

$$\eta = \frac{p'}{\sigma_{\mathrm{eq}}} \in [-\infty \ \infty] \qquad p' = \frac{\mathrm{tr}\left(\boldsymbol{\sigma}\right)}{3} \qquad \sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\mathrm{dev}\left(\boldsymbol{\sigma}\right) : \mathrm{dev}\left(\boldsymbol{\sigma}\right)}$$

  - Lode dependent

$$\theta = \frac{1}{3}\arccos\left(\frac{27J_3}{2\sigma_{\mathrm{eq}}^3}\right) \qquad J_3 = \det\left(\mathrm{dev}\left(\boldsymbol{\sigma}\right)\right)$$

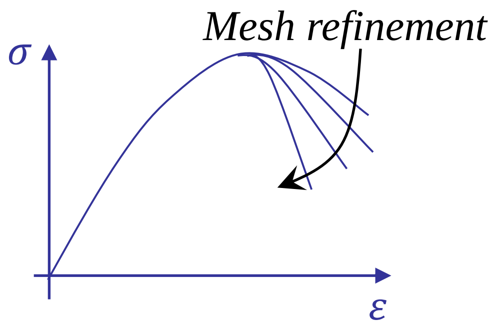


Fracture locus for 2024-T351 aluminum alloy

(Bai & Wierzbicki 2010)

- Objective & Methodology
  - Develop a multi-surface model incorporating
    - Void growth phase
    - Internal necking coalescence phase
      - Driven by maximum principal stress
    - Shear driven coalescence phase
      - Driven by maximum shear stress
  - In a nonlocal formalism
    - Why?
      - Local forms suffer from mesh-dependency

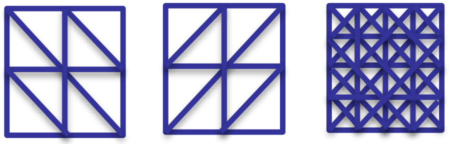
The maximum principal stress & maximum shear stress are Lode-dependent!



    - Implicit formulation *(Peerlings et al. 1998)*
      - Introduction of a characteristic length l_c
      - New non-local degrees of freedom $\bar{Z}_k$
      - New Helmholtz-type equation to be solved

$$\bar{Z}_k - Z_k - l_{ck}^2 \Delta_0 \bar{Z}_k = 0\,,\ \text{with } k = 1, \ldots, N$$

      - Damage indicators depend on the nonlocal variable



The numerical results change without convergence

    - Multiple nonlocal variables can be considered
      - Damage indicators depend on N different sources

Multi-surface nonlocal porous model

- Porous plasticity corotational approach
  - Yield condition

$$\Phi_{\mathrm{nl}} = \Phi_{\mathrm{nl}}\left(\boldsymbol{\sigma}; \sigma_{\mathrm{Y}}, \mathbf{Y}\right) = 0$$

  - Plastic flow

$$\mathbf{D}^{\mathrm{p}} = \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1} = \dot{\mu}\frac{\partial \Phi_{\mathrm{nl}}}{\partial \boldsymbol{\sigma}}$$

  - Evolution laws
    - Equivalent matrix plastic strain rate:

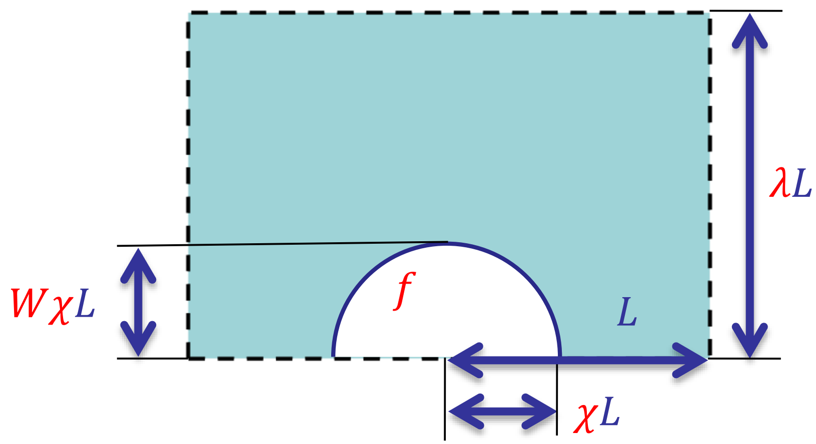
$$\dot{\varepsilon}_{\mathrm{m}} = \frac{\boldsymbol{\sigma} : \mathbf{D}^{\mathrm{p}}}{\left(1-f\right)\sigma_{\mathrm{Y}}}$$

    - Isotropic hardening law:

$$\sigma_{\mathrm{Y}} = \sigma_{\mathrm{Y}}^{0} + R\left(\varepsilon_{\mathrm{m}}\right)$$

    - Evolution laws for void characteristics

$$\mathbf{Y} = \begin{bmatrix} f & \chi & W & \lambda \end{bmatrix}^{T}$$



Void characteristics Y:

Porosity : f
Void ligament ratio: χ
Void aspect ratio: W
Void spacing ratio: λ

Yield surface Φ_{nl} and evolution laws for Y depend on the void expansion solution:
- *Void growth;*
- *Internal necking coalescence; or*
- *Shear driven coalescence*

- Nonlocal void characteristics
  - Nonlocal plastic state by implicit formulation
    - Nonlocality of the volumetric plastic deformation

$$\begin{cases} \varepsilon_{\mathrm{v}} = \mathrm{tr}\left(\mathbf{D}^{\mathrm{p}}\right) \\ \varepsilon_{\mathrm{v}} = \bar{\varepsilon}_{\mathrm{v}} - l_{\varepsilon_{\mathrm{v}}}^{2} \Delta_0 \bar{\varepsilon}_{\mathrm{v}} \end{cases}$$

    - Nonlocality of the deviatoric plastic deformation

$$\begin{cases} \varepsilon_{\mathrm{d}} = \sqrt{\frac{2}{3} \mathrm{dev}\left(\mathbf{D}^{\mathrm{p}}\right) : \mathrm{dev}\left(\mathbf{D}^{\mathrm{p}}\right)} \\ \varepsilon_{\mathrm{d}} = \bar{\varepsilon}_{\mathrm{d}} - l_{\varepsilon_{\mathrm{d}}}^{2} \Delta_0 \bar{\varepsilon}_{\mathrm{d}} \end{cases}$$

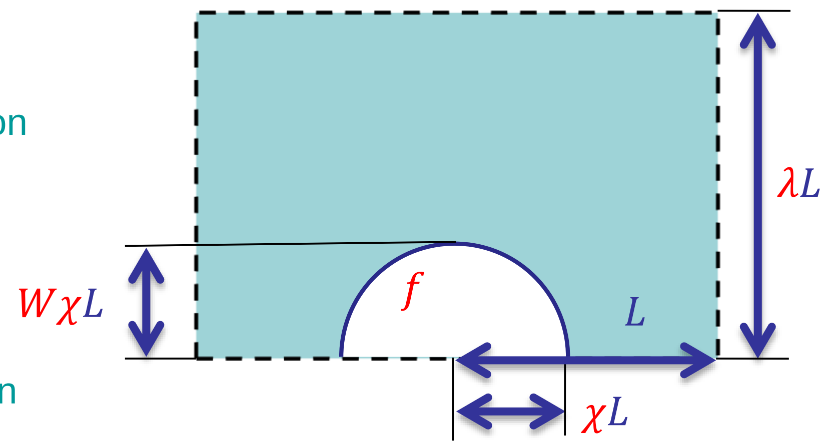
    - Nonlocality of the matrix plastic deformation

$$\varepsilon_{\mathrm{m}} = \bar{\varepsilon}_{\mathrm{m}} - l_{\varepsilon_{\mathrm{m}}}^{2} \Delta_0 \bar{\varepsilon}_{\mathrm{m}}$$

  - Nonlocal evolution laws for void characteristics

$$\mathbf{Y} = \mathbf{Y}(\varepsilon_{\mathrm{m}}, \varepsilon_{\mathrm{v}}, \varepsilon_{\mathrm{d}}, \boldsymbol{\sigma})$$

$$\mathbf{Y} = \mathbf{Y}_{\mathrm{nl}}(\varepsilon_{\mathrm{m}}, \varepsilon_{\mathrm{v}}, \varepsilon_{\mathrm{d}}, \bar{\varepsilon}_{\mathrm{m}}, \bar{\varepsilon}_{\mathrm{v}}, \bar{\varepsilon}_{\mathrm{d}}, \boldsymbol{\sigma})$$



Void characteristics Y:

Porosity : f
Void ligament ratio: χ
Void aspect ratio: W
Void spacing ratio: λ

Multi-surface nonlocal porous model

- Void growth phase – GTN model
  - Yield condition

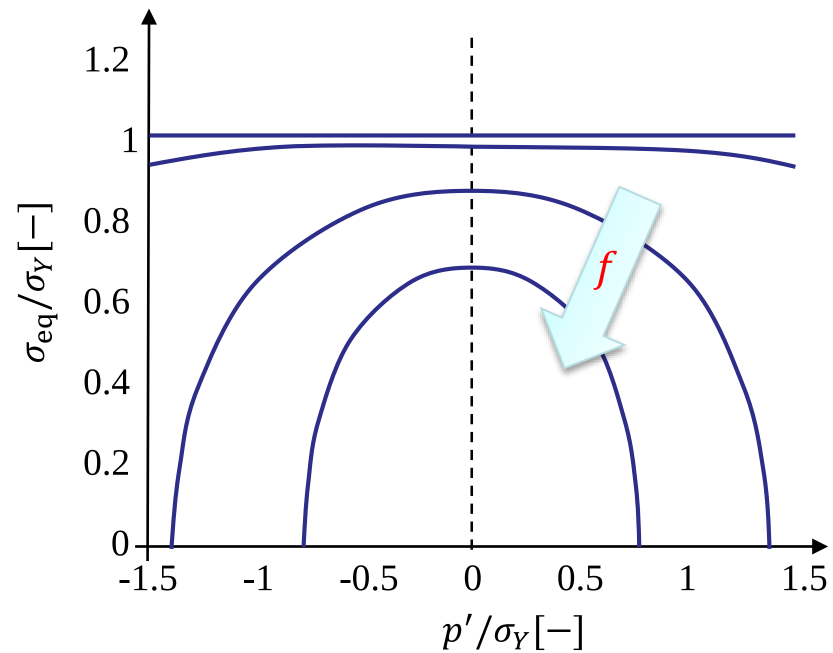
$$\begin{cases} \Phi_{\mathrm{nl}} = \Phi_{\mathrm{G}} = \dfrac{\hat{\sigma}_{\mathrm{G}}}{\sigma_{\mathrm{Y}}} - 1 = 0 \\ \hat{\sigma}_{\mathrm{G}}\left(\sigma_{\mathrm{eq}}, p', \sigma_{\mathrm{Y}}, f\right) = \dfrac{\sqrt{\sigma_{\mathrm{eq}}^2 + 2\sigma_{\mathrm{Y}}^2 f q_1 \left[\cosh\left(\frac{3}{2} q_2 \frac{p'}{\sigma_{\mathrm{Y}}}\right) - 1\right]}}{1 - q_1 f} \end{cases}$$

$$p' = \frac{\mathrm{tr}\left(\boldsymbol{\sigma}\right)}{3}$$

$$\sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\mathrm{dev}\left(\boldsymbol{\sigma}\right) : \mathrm{dev}\left(\boldsymbol{\sigma}\right)}$$

  - Parameters:
    - q_1 and q_2
  - Hardening law
    - $\sigma_{\mathrm{Y}}(\varepsilon_m)$
    - Matrix plastic deformation ε_m
  - Nonlocal evolution laws for void characteristics

$$\mathbf{Y}_{\mathrm{nl}} = \mathbf{Y}_{\mathrm{G}}(\varepsilon_{\mathrm{m}}, \varepsilon_{\mathrm{v}}, \varepsilon_{\mathrm{d}}, \bar{\varepsilon}_{\mathrm{m}}, \bar{\varepsilon}_{\mathrm{v}}, \bar{\varepsilon}_{\mathrm{d}}, \boldsymbol{\sigma})$$



Multi-surface nonlocal porous model

- Void growth phase - $\mathbf{Y}_{\mathrm{nl}} = \mathbf{Y}_{\mathrm{G}}(\varepsilon_{\mathrm{m}}, \varepsilon_{\mathrm{v}}, \varepsilon_{\mathrm{d}}, \bar{\varepsilon}_{\mathrm{m}}, \bar{\varepsilon}_{\mathrm{v}}, \bar{\varepsilon}_{\mathrm{d}}, \boldsymbol{\sigma})$
  - Nonlocal porosity evolution

$$\dot{f} = \dot{f}_{\mathrm{gr}} + \dot{f}_{\mathrm{nu}} + \dot{f}_{\mathrm{sh}}$$

    - Growth part

$$\dot{f}_{\mathrm{gr}} = (1-f)\,\mathrm{tr}\,(\mathbf{D}^{\mathrm{p}}) \Rightarrow \dot{f}_{\mathrm{gr}} = (1-f)\,\dot{\bar{\varepsilon}}_{\mathrm{v}}$$

    - Nucleation part

$$\dot{f}_{\mathrm{nu}} = A_n\,(\varepsilon_{\mathrm{m}})\,\dot{\varepsilon}_{\mathrm{m}} \Rightarrow \dot{f}_{\mathrm{nu}} = A_n\,(\bar{\varepsilon}_{\mathrm{m}})\,\dot{\bar{\varepsilon}}_{\mathrm{m}}$$

    - Shear part

$$\dot{f}_{\mathrm{sh}} = k_w \phi_\eta \phi_\omega f \frac{\mathrm{dev}(\boldsymbol{\sigma}) : \mathbf{D}^{\mathrm{p}}}{\sigma_{\mathrm{eq}}} \Rightarrow \dot{f}_{\mathrm{sh}} = k_w \phi_\eta \phi_\omega f\,\dot{\bar{\varepsilon}}_d$$

(Nahshon and Hutchinson 2008)

    - Parameter: η_s, k_w

$$\phi_\eta = \exp\left(-\frac{1}{2}\left(\frac{\eta}{\eta_s}\right)^2\right) \quad \& \quad \phi_\omega = 1 - \omega^2$$

$$\mathbf{D}^{\mathrm{p}} = \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1} = \dot{\mu}\frac{\partial \Phi_{\mathrm{nl}}}{\partial \boldsymbol{\sigma}}$$

$$\varepsilon_{\mathrm{v}} = \mathrm{tr}\,(\mathbf{D}^{\mathrm{p}})$$

$$\frac{\mathrm{dev}\,(\boldsymbol{\sigma}) : \mathbf{D}^{\mathrm{p}}}{\sigma_{\mathrm{eq}}} = \dot{\varepsilon}_{\mathrm{d}}$$

$$\omega = \frac{27 J_3}{2{\sigma_{\mathrm{eq}}}^3}\,.$$

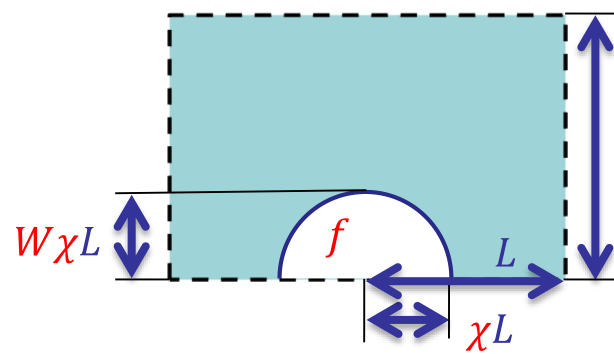
$$\eta = \frac{p'}{\sigma_{\mathrm{eq}}}$$

  - Voids shape evolution

$$\begin{cases} \dot{\lambda} = \kappa \lambda \dot{\bar{\varepsilon}}_{\mathrm{d}} \\ \chi = \left(\dfrac{3 f \lambda}{2W}\right)^{\frac{1}{3}} \\ \dot{W} = 0 \end{cases}$$

Periodic distribution $\kappa = 1.5$,
Random distribution $\kappa = 0$,
Clustered distribution $0 < \kappa < 1.5$

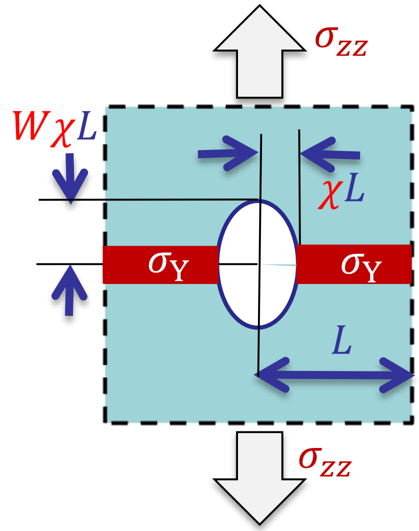
(Benzerga et al. 2016)



- Internal necking – Coalescence
  - Thomason coalescence onset
    - Localized plastic flow in ligament
    - Limit load factor for uniaxial tension

(Thomason 1985)

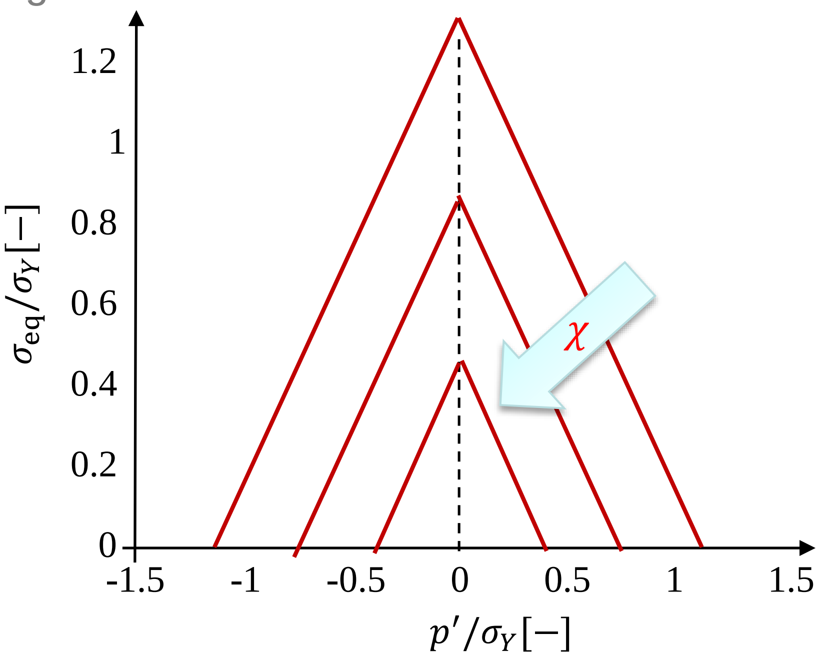
$$C_{\mathrm{Tf}}\left(\mathbf{Y}\right)=\frac{\sigma_{zz}}{\sigma_{\mathrm{Y}}}=\left(1-\chi^{2}\right)\left[h\left(\frac{1-\chi}{W\chi}\right)^{2}+g\sqrt{\frac{1}{\chi}}\right]$$



    - Parameters:
      - $g=0.1,\ h=1.24$ are generally adopted
  - New yield surface accounting for general loading
    - Driven by maximum principal stress (MPS)

$$\begin{cases}\Phi_{\mathrm{nl}}=\Phi_{\mathrm{T}}=\dfrac{\hat{\sigma}_{\mathrm{T}}}{\sigma_{\mathrm{Y}}}-1=0\\[2ex]\hat{\sigma}_{\mathrm{T}}=\dfrac{1}{C_{\mathrm{Tf}}}\underbrace{\left(\dfrac{2}{3}\sigma_{\mathrm{eq}}\cos\theta+|p'|\right)}_{\text{MPS}}\end{cases}$$

$$\boxed{\theta\left(\sigma_{\mathrm{eq}},J_{3}\right)=\frac{1}{3}\arccos\frac{27J_{3}}{2{\sigma_{\mathrm{eq}}}^{3}}\,.\qquad p'=\frac{\mathrm{tr}\left(\boldsymbol{\sigma}\right)}{3}}$$



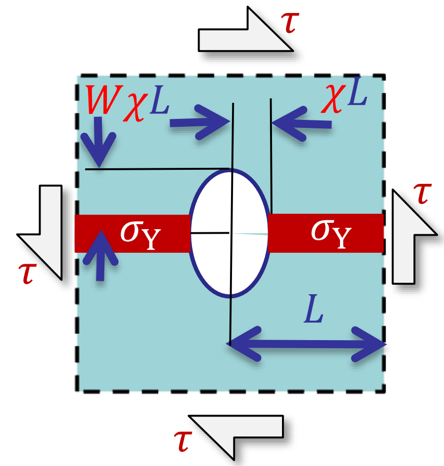
  - Evolution laws for void characteristics

$$\mathbf{Y}_{\mathrm{nl}}=\mathbf{Y}_{\mathrm{T}}(\varepsilon_{\mathrm{m}},\varepsilon_{\mathrm{v}},\varepsilon_{\mathrm{d}},\bar{\varepsilon}_{\mathrm{m}},\bar{\varepsilon}_{\mathrm{v}},\bar{\varepsilon}_{\mathrm{d}},\boldsymbol{\sigma})$$

Multi-surface nonlocal porous model

- Shear driven – Coalescence
  - Thomason-like coalescence onset
    - Limit load factor

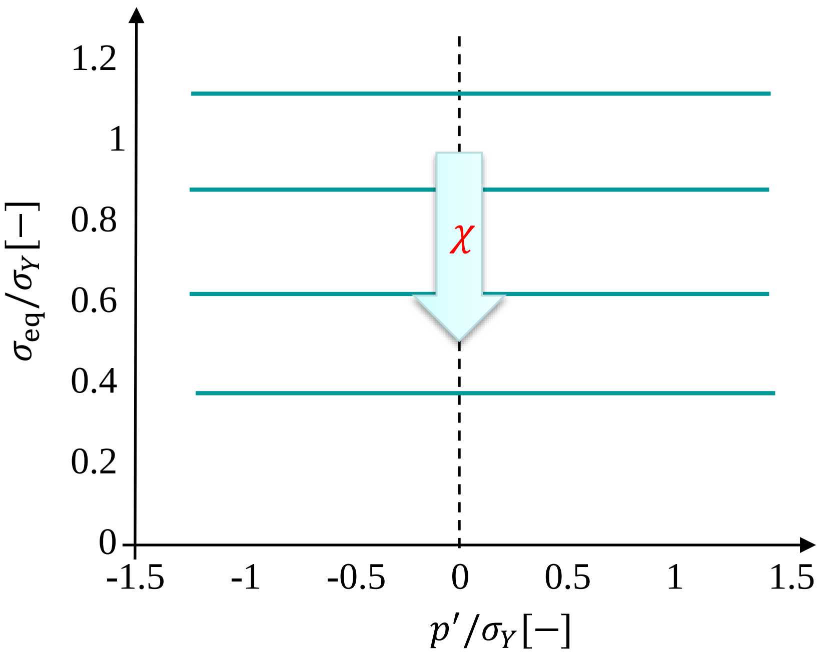
$$C_{\mathrm{Sf}}\left(\mathbf{Y}\right)=\frac{\sqrt{3}\tau}{\sigma_{\mathrm{Y}}}=\xi\left(1-\chi^{2}\right)$$



    - Parameter ξ
      - $\xi=1$ for σ_Y uniform inside localization band
      - $\xi>1$ is used for real distribution
  - New yield surface
    - Driven by maximum shear stress (MSS)

$$\begin{cases}\Phi_{\mathrm{nl}}=\Phi_{\mathrm{S}}=\dfrac{\hat{\sigma}_{\mathrm{S}}}{\sigma_{\mathrm{Y}}}-1=0\\ \hat{\sigma}_{\mathrm{S}}=\dfrac{\sqrt{3}\tau}{C_{\mathrm{Sf}}}=\underbrace{\dfrac{\sigma_{\mathrm{eq}}}{C_{\mathrm{Sf}}}\left(\dfrac{\sin\theta}{2}+\dfrac{\sqrt{3}\cos\theta}{2}\right)}_{\text{MSS}}\end{cases}$$

$$\boxed{\theta\left(\sigma_{\mathrm{eq}},J_{3}\right)=\frac{1}{3}\arccos\frac{27J_{3}}{2{\sigma_{\mathrm{eq}}}^{3}}}$$



  - Evolution laws for void characteristics

$$\mathbf{Y}_{\mathrm{nl}}=\mathbf{Y}_{\mathrm{S}}(\varepsilon_{\mathrm{m}},\varepsilon_{\mathrm{v}},\varepsilon_{\mathrm{d}},\bar{\varepsilon}_{\mathrm{m}},\bar{\varepsilon}_{\mathrm{v}},\bar{\varepsilon}_{\mathrm{d}},\boldsymbol{\sigma})$$

- Competition between different modes
  - Yield surface

$$\Phi_{\mathrm{e}} = \frac{\hat{\sigma}}{\sigma_{\mathrm{Y}}} - 1 = 0$$

    - Effective stress

$$\hat{\sigma} = \max\left(\hat{\sigma}_{\mathrm{G}}, \hat{\sigma}_{\mathrm{T}}, \hat{\sigma}_{\mathrm{S}}\right)$$

    - Approximated form

$$\hat{\sigma} = \left(\hat{\sigma}_{\mathrm{G}}^{m} + \hat{\sigma}_{\mathrm{T}}^{m} + \hat{\sigma}_{\mathrm{S}}^{m}\right)^{\frac{1}{m}}$$

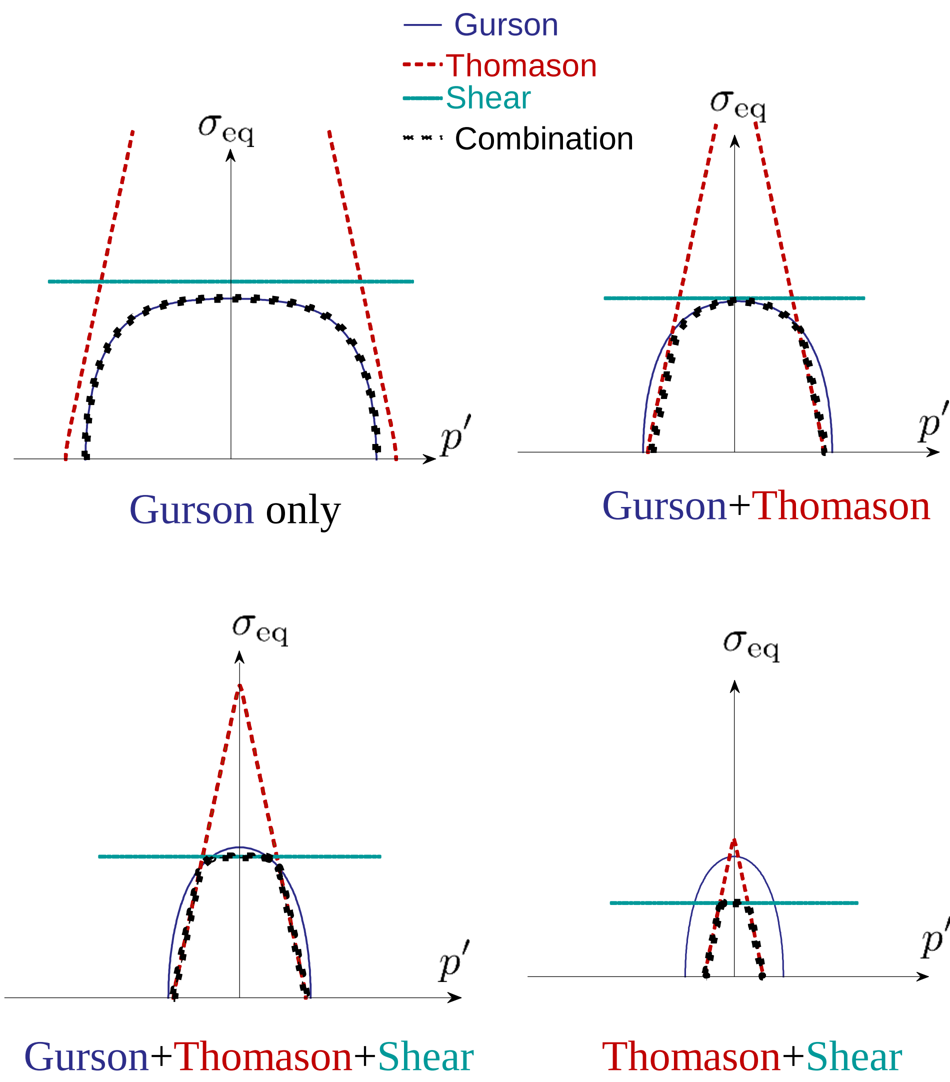
    - Smoothing using $m \gg 1$

  - Onset of void necking coalescence

$$\dot{\varepsilon}_{\mathrm{m}} > 0\,, \text{ and } \hat{\sigma}_T > \max\left(\hat{\sigma}_G, \hat{\sigma}_S\right)$$

  - Onset of void shear coalescence

$$\dot{\varepsilon}_{\mathrm{m}} > 0\,, \text{ and } \hat{\sigma}_S > \max\left(\hat{\sigma}_G, \hat{\sigma}_T\right)\,.$$



Multi-surface nonlocal porous model

- Solution under proportional loadings
  - Constant
    - Stress triaxiality (η); and
    - Normalized Lode angle ($\bar{\theta}$)
  - $\varepsilon_{\mathrm{dc}}$- ductility = plastic deformation at coalescence onset

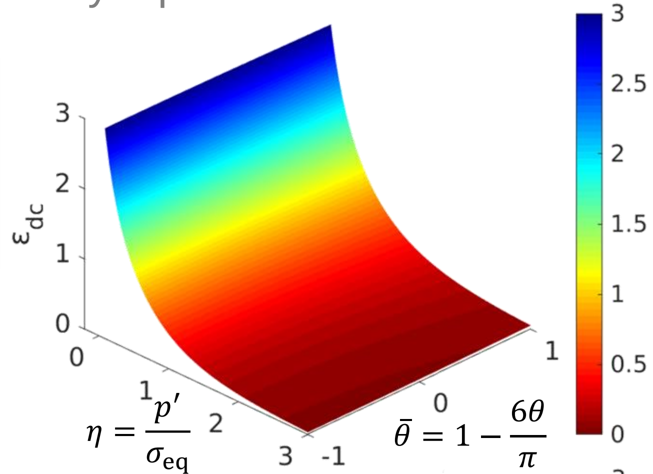
$$\eta = \frac{p'}{\sigma_{\mathrm{eq}}} \qquad p' = \frac{\mathrm{tr}\,(\boldsymbol{\sigma})}{3}$$

$$\sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\mathrm{dev}\,(\boldsymbol{\sigma}) : \mathrm{dev}\,(\boldsymbol{\sigma})}$$

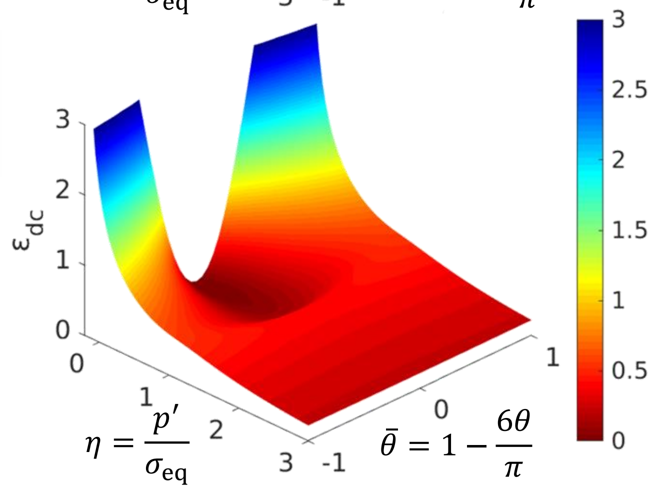
$$\bar{\theta} = 1 - \frac{6\theta}{\pi}$$

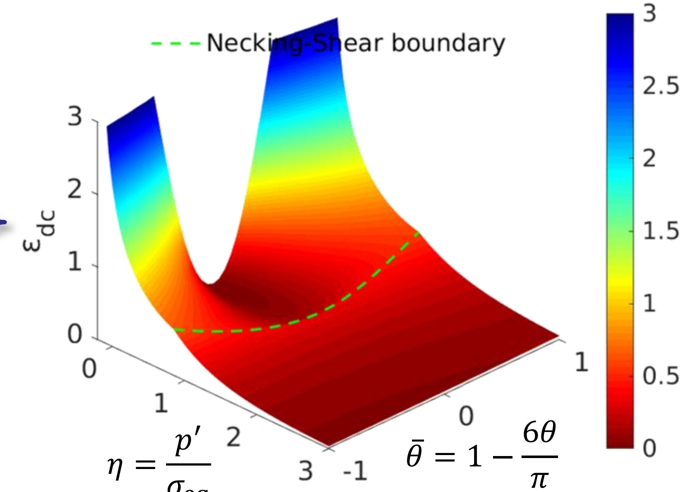
$$\theta\,(\sigma_{\mathrm{eq}}, J_3) = \frac{1}{3}\arccos\frac{27 J_3}{2{\sigma_{\mathrm{eq}}}^3}$$

Internal necking coalescence



Shear driven coalescence





Multisurface coalescence

- Solution under proportional loadings
  - Constant
    - Stress triaxiality (η); and
    - Normalized Lode angle ($\bar{\theta}$)
  - $\varepsilon_{\mathrm{dc}}$- ductility = plastic deformation at coalescence onset
  - Three parameters to calibrate
    - Nucleation under shearing: : η_s, k_w
    - Failure onset under a pure shearing $\varepsilon_{\mathrm{ds}}$

$$\xi = \frac{1 - q_1 f_0 \exp\left(k_\omega \varepsilon_{\mathrm{d}s}\right)}{1 - \chi_0^2 \exp\left[\frac{2}{3}\left(\kappa + k_\omega\right)\varepsilon_{\mathrm{d}s}\right]}$$

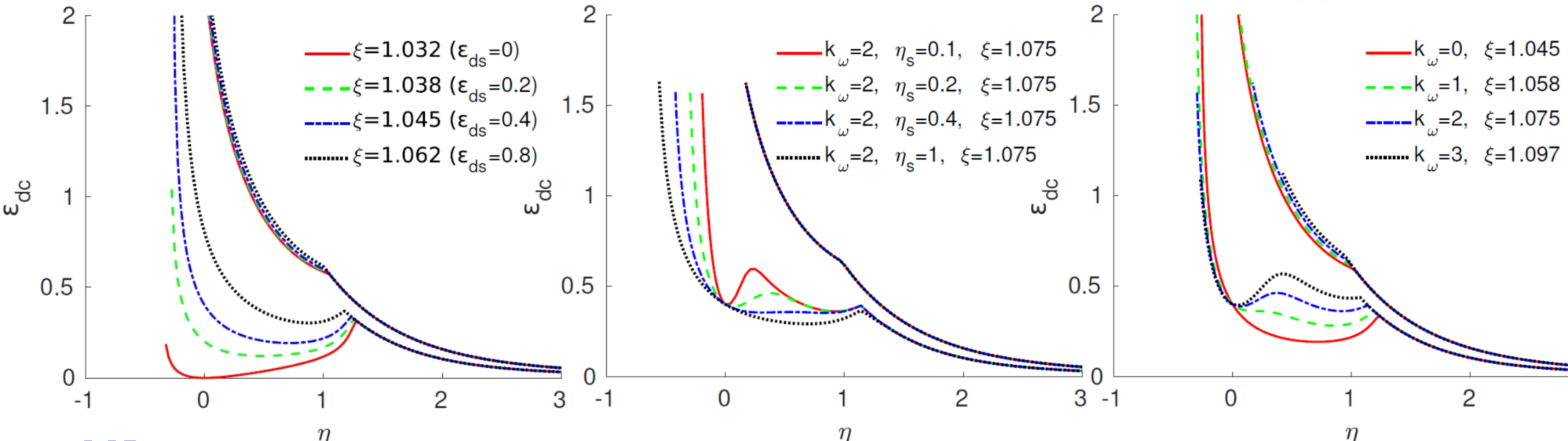
$$\eta = \frac{p'}{\sigma_{\mathrm{eq}}} \qquad p' = \frac{\mathrm{tr}\,(\boldsymbol{\sigma})}{3}$$

$$\omega = \frac{27 J_3}{2{\sigma_{\mathrm{eq}}}^3}\,.$$

$$\dot{f}_{\mathrm{sh}} = k_w \phi_\eta \phi_\omega f \dot{\bar{\varepsilon}}_d$$

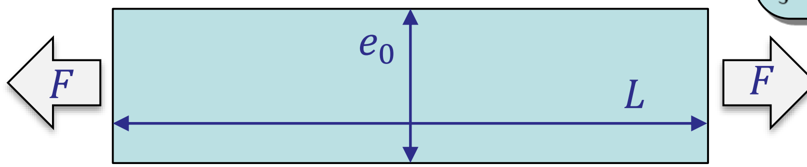
$$\phi_\eta = \exp\left(-\frac{1}{2}\left(\frac{\eta}{\eta_s}\right)^2\right)$$

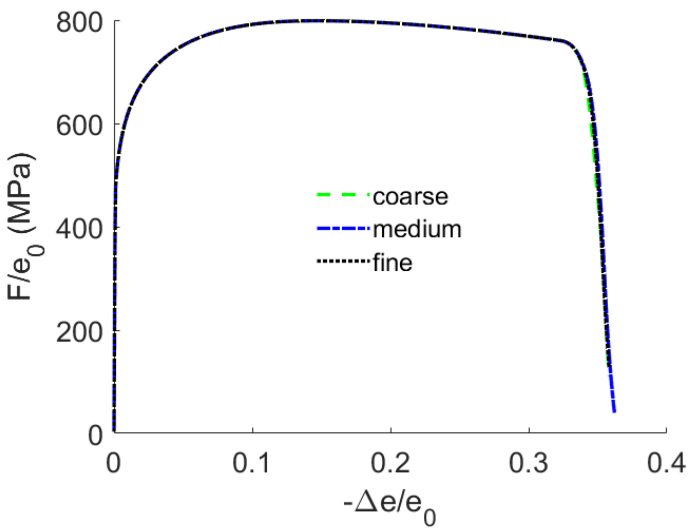
$$\phi_\omega = 1 - \omega^2$$



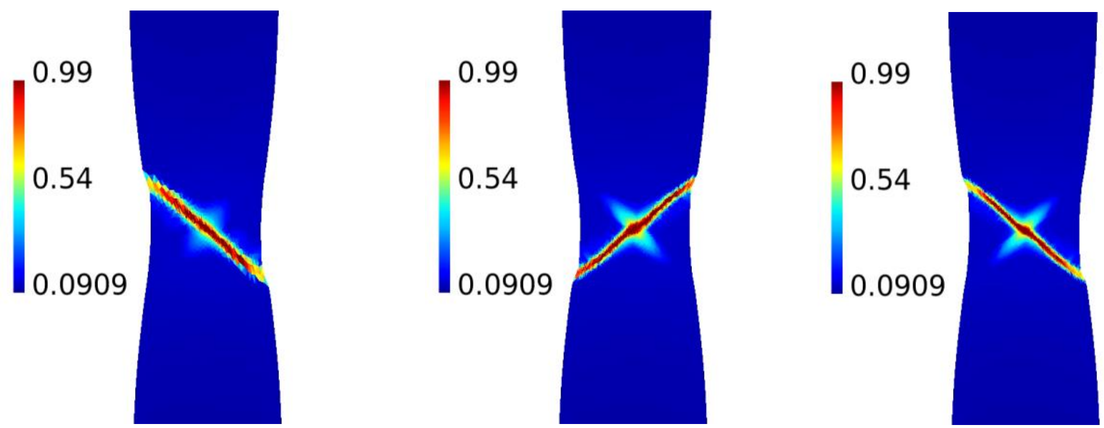
- Plane strain smooth specimen under tensile loading
  - Verification of the nonlocal model: mesh convergence

$$L = 12.5\ \text{mm}$$
$$e_0 = 3\ \text{mm}$$
$$\xi = 1.015\ \ (\varepsilon_{ds} = 0.95)$$





Distribution of void ligament ratio χ

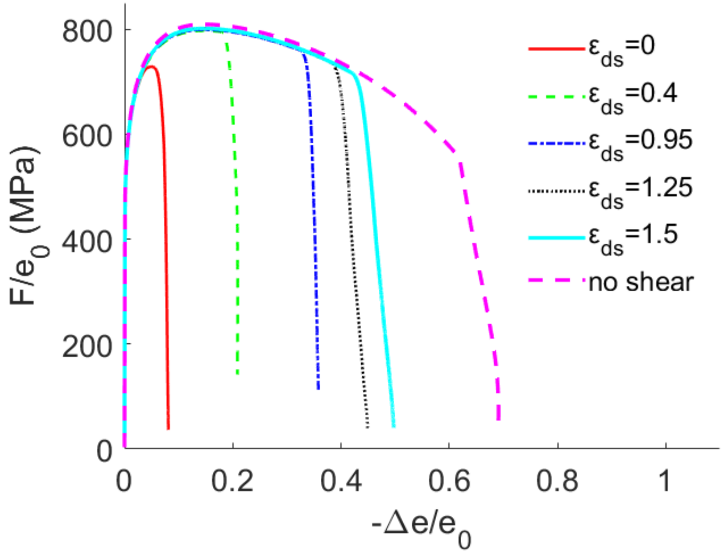


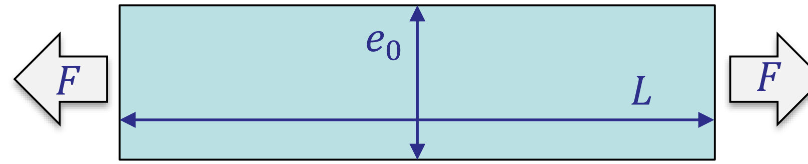
Coarse *Medium* *Fine*

Capture slant fracture

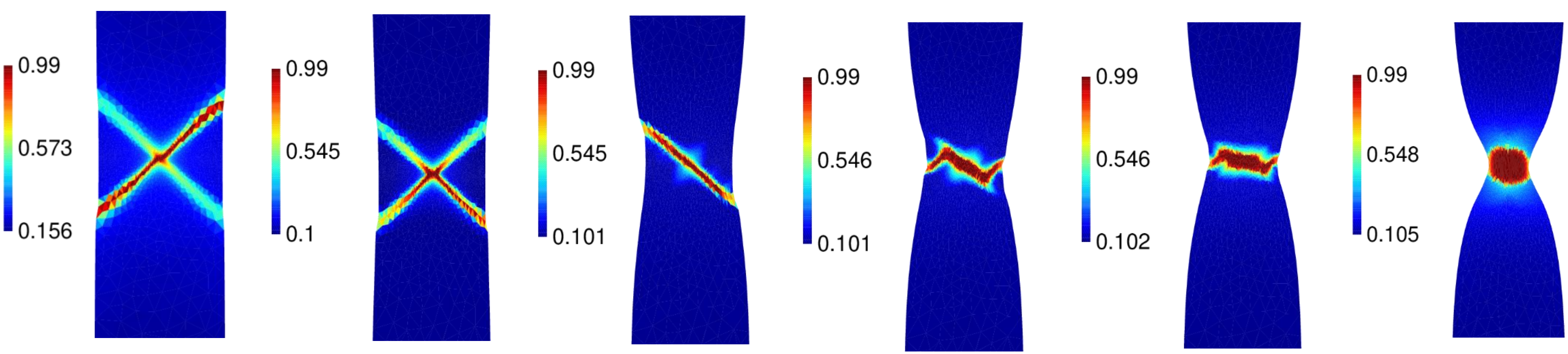
- Plane strain smooth specimen under tensile loading
  - Effect of ξ

$L = 12.5$ mm
$e_0 = 3$ mm



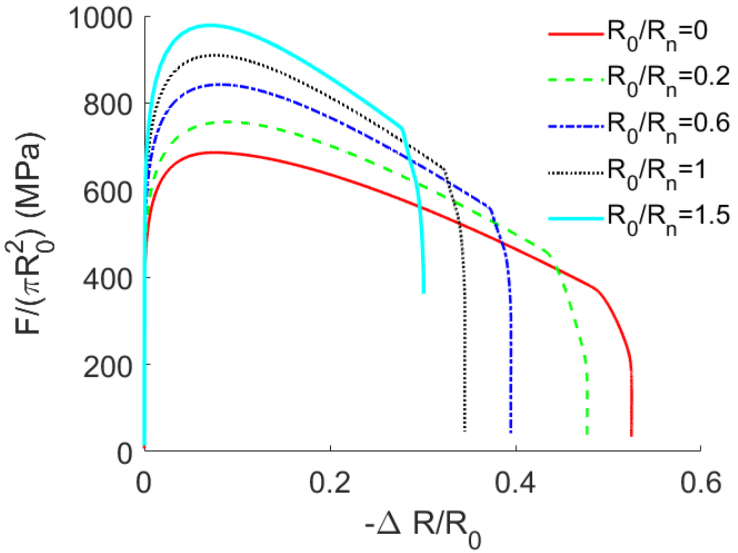


Distribution of void ligament ratio χ

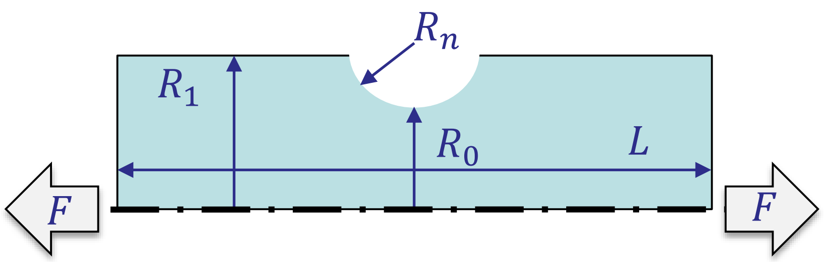


$\varepsilon_{ds} = 0$ | $\varepsilon_{ds} = 0.4$ | $\varepsilon_{ds} = 0.95$ | $\varepsilon_{ds} = 1.25$ | $\varepsilon_{ds} = 1.5$ | no shear

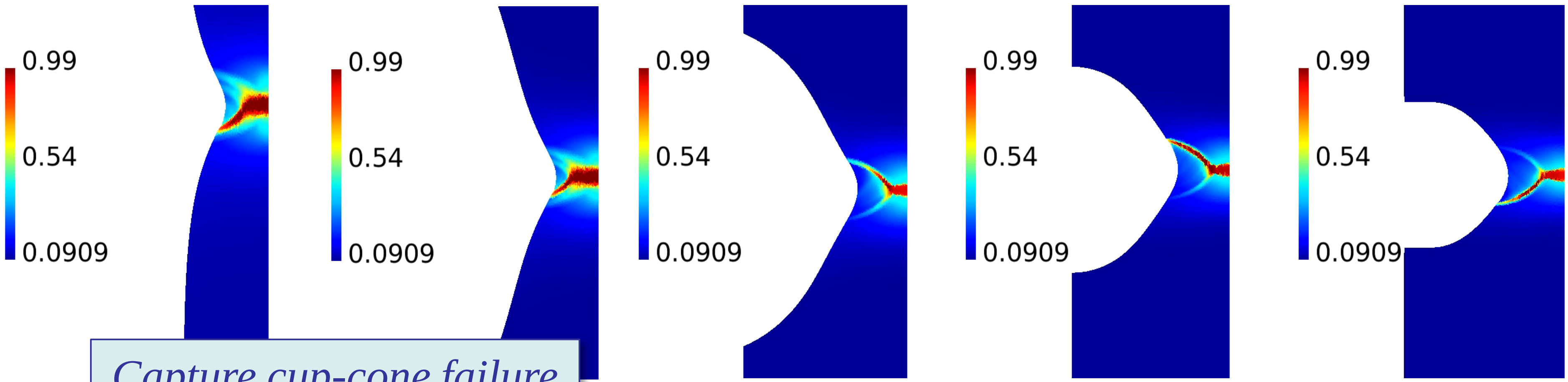
- Axisymmetric (notched) specimens under tensile loading
  - Different notch radii: $R_0/R_n = 0, 0.2, 0.6, 1, 1.5$



$R_0 = 3$ mm
$R_1 = 6$ mm
$L = 25$ mm
$\xi = 1.015 \ (\varepsilon_{ds} = 0.95)$



Distribution of void ligament ratio χ



Capture cup-cone failure

Conclusion

- Objective
  - Simulation of ductile failure incorporating void growth & coalescence deformation modes
- Methodology
  - Nonlocal porous plasticity
  - Multi-surface model incorporating
    - Void growth;
    - Internal necking coalescence; and
    - Shear driven coalescence
- Results
  - The proposed framework is able to model
    - The slant fracture mode in plane strain smooth specimens
    - The cup-cone fracture mode in axisymmetric smooth & notched specimens

Thank you for your attention

- Computational & Multiscale Mechanics of Materials – CM3
- http://www.ltas-cm3.ulg.ac.be/
- B52 - Quartier Polytech 1
- Allée de la découverte 9, B4000 Liège
- L.Noels@ulg.ac.be

Predictor-corrector scheme

- Elastic predictor

$$\mathbf{F}^{\mathrm{ppr}} = \mathbf{F}_n^{\mathrm{p}} \qquad \mathbf{F}^{\mathrm{epr}} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{ppr}-1}$$

- Plastic corrector (fully implicit radial return)

$$\begin{cases} \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{pr}} - \mathbb{C} : \Delta \mathbf{E}^p \,, \\ \boldsymbol{\sigma} = J^{-1} \boldsymbol{\tau} \,, \\ \sigma_{\mathrm{Y}} = \sigma_{\mathrm{Y}} \left(\varepsilon_{\mathrm{m}n} + \Delta \varepsilon_{\mathrm{m}} \right) \,, \\ \mathbf{Y} = \mathbf{Y}_n + \Delta \mathbf{Y} \left(\Delta \bar{\mathbf{Z}}, \boldsymbol{\sigma} \right) \,, \\ \Phi_{\mathrm{nl}} \left(\boldsymbol{\sigma}; \sigma_{\mathrm{Y}}, \mathbf{Y} \right) = 0 \,, \\ \Delta \mathbf{E}^p - \Delta \mu \mathbf{N}^{\mathrm{p}} \left(\boldsymbol{\sigma}; \sigma_{\mathrm{Y}}, \mathbf{Y} \right) = \mathbf{0} \,, \text{ and} \\ \boldsymbol{\sigma} : \Delta \mathbf{E}^p - (1 - f) \, \sigma_{\mathrm{Y}} \Delta \varepsilon_{\mathrm{m}} = 0 \,. \end{cases}$$

Unknowns: $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$, σ_{Y}, $\Delta\varepsilon_{\mathrm{m}}$, $\mathbf{Y}$, $\Delta\mathbf{E}^p$, and $\Delta\mu$