Cycle selections

Marie Baratto

QuantOM, HEC Management School of the University of Liege, Liège, Belgium.
email: `marie.baratto@uliege.be`.

Yves Crama

QuantOM, HEC Management School of the University of Liege, Liège, Belgium.
email: `yves.crama@uliege.be`

Abstract

We introduce the following *cycle selection problem* which is motivated by an application to kidney exchange problems. Given a directed graph $G = (V, A)$, a *cycle selection* is a subset of arcs $B \subseteq A$ forming a union of directed cycles. A related optimization problem, the *Maximum Weighted Cycle Selection problem* can be defined as follows: given a weight $w_{i,j} \in \mathbb{R}$ for all arcs $(i, j) \in A$, find a cycle selection B which maximizes $w(B)$. We prove that this problem is strongly NP-hard. Next, we focus on cycle selections in complete directed graphs. We provide several ILP formulations of the problem: an arc formulation featuring an exponential number of constraints which can be separated in polynomial time, four extended compact formulations, and an extended non compact formulation. We investigate the relative strength of these formulations. We concentrate on the arc formulation and on the description of the associated cycle selection polytope. We prove that this polytope is full-dimensional, and that all the inequalities used in the arc formulation are facet-defining. Furthermore, we describe three new classes of facet-defining inequalities and a class of valid inequalities. We also consider the consequences of including additional constraints on the cardinality of a selection or on the length of the associated cycles.

Keywords: Digraph, Cycle, Kidney exchange, Extended formulation, Polytope, Facet.

1 Introduction

1.1 Problem definition

This paper introduces and investigates a combinatorial optimization problem originally motivated by an application to kidney exchange programs. The motivation will be further developed in Section 1.2 hereunder but for now, we start with a mathematical definition of the problem.

Our graph-theoretic terminology is standard and follows [Bang-Jensen and Gutin, 2009]. All directed graphs (or digraphs) we consider in this paper are loopless and have no parallel arcs. For a digraph $G = (V, A)$, we let $|V| = n$ and $|A| = m$. The digraph $G = (V, A)$ is *complete* if A contains all pairs of distinct vertices (i, j), for $i, j \in V$. A *(directed) cycle* of a digraph G is a sequence of the form $C = (v_1, a_1, v_2, a_2, \ldots, v_k, a_k, v_{k+1})$ where $k \geq 2$, $v_1, v_2, \ldots, v_k$ are distinct vertices of G, $v_1 = v_{k+1}$, $a_1, a_2, \ldots, a_k$ are distinct arcs, v_i is the tail of a_i and v_{i+1} is its head for $i = 1, 2, \ldots, k$. The *length* of C is k, and we say that C is a k-cycle. When no confusion can arise, we often identify a cycle with its set of arcs, so that we can speak of a union of cycles, for example.

Let us now introduce a new definition. Given a directed graph $G = (V, A)$, where V is the set of vertices and A is the set of arcs of G, we say that a subset of arcs $B \subseteq A$ is a *cycle selection* in G if the arcs of B form a union (possibly empty) of directed cycles. Equivalently, B is a cycle selection if and only if each arc of B is contained in a directed cycle of $G_B = (V, B)$. And equivalently again, B is a cycle selection of G if and only if each arc of B is contained in a strong (or strongly connected) component of $G_B = (V, B)$: the equivalence follows from the observation that an arc of B, say (i, j), is in a cycle of G_B if and only if i and j are in a same strong component of G_B.

As an illustration, Figure 2 displays the collection of nonempty selections of the digraph represented in Figure 1.

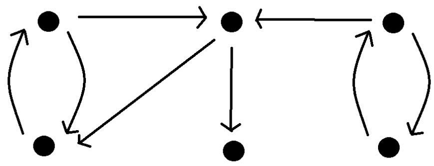

Figure 1: A directed graph.

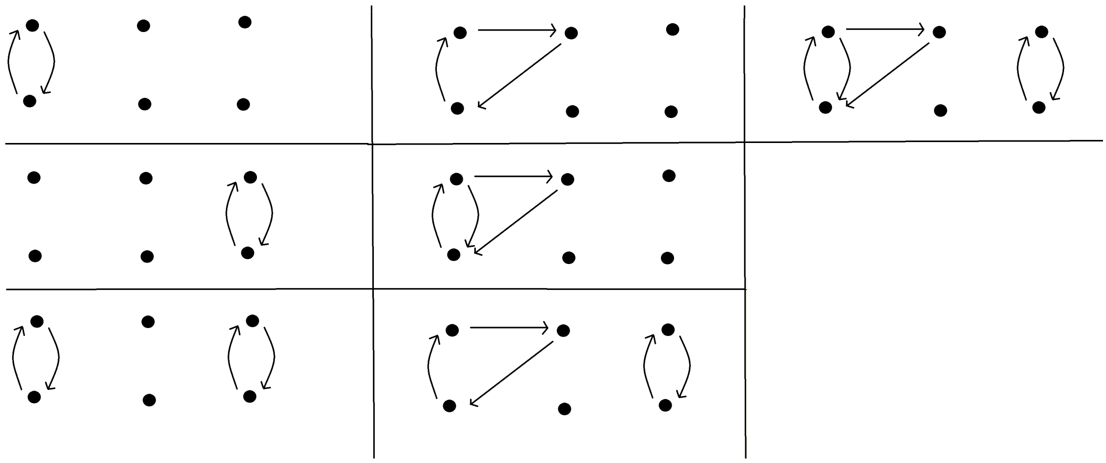

Figure 2: All nonempty selections of the directed graph in Figure 1.

In view of the above definitions, the time complexity to verify that a subset $B \subseteq A$ is a cycle selection is $O(n+m)$, using for example Tarjan's algorithm to identify all strong components of G_B ([Tarjan, 1972]).

The *maximum weighted cycle selection* (**MWCS**) problem, or cycle selection problem for short, is the following optimization problem: given a digraph $G = (V, A)$ and a weight $w_{i,j} \in \mathbb{R}$ for each arc $(i, j) \in A$, find a cycle selection B which maximizes $w(B) = \sum_{(i,j)\in B} w_{i,j}$.

This article investigates several properties of the cycle selection problem. Section 1.2 lays out the motivation for studying it. Section 1.3 provides a literature review of previous related work in order to contextualize the cycle selection problem and to position our contributions. Section 2 discusses the complexity of the problem. Next, various integer linear programming formulations of the cycle selection problem are proposed in Section 3, namely, an arc-based formulation (Section 3.1), several extended compact formulations (Section 3.2), and an extended non compact one (Section 3.3). We establish the relative strength of the linear relaxations of these formulations. Section 4 investigates the facial structure of the cycle selection polytope associated with the arc formulation for a complete digraph. We prove that the polytope is full dimensional and that all the inequalities used in the ILP formulation are facet-defining. Furthermore, we describe three additional classes of facet-defining inequalities and one class of valid inequalities. Section 5 considers the extension of the cycle selection problem which arises when a constraint is placed on the cardinality of the selection and of the cycles that it includes. Finally, Section 6 presents some conclusions and perspectives for future research.

1.2 Motivation

Our motivation to study cycle selections originally stems from optimization problems arising in the context of kidney exchange programs (KEPs). Let us briefly explain how KEPs work. Nowadays, the preferred treatment option offered to patients with an end-stage renal disease is kidney transplant from a living donor. This option exists primarily when a patient has a relative willing to donate one of its healthy kidneys. However, in many situations, patients are unable to receive a kidney from their associated healthy donor because of ABO blood type incompatibility or tissue type incompatibility. Kidney exchange programs try to alleviate this difficulty by enlisting a large number of incompatible patient-donor pairs, say, pairs (P_i, D_i) made up of patient P_i and donor D_i, for $i = 1, \ldots, n$. Considering such a pool makes it potentially feasible to transplant kidneys in cyclic fashion with, for example, D_1 donating a kidney to P_2, D_2 donating one to P_3, and D_3 donating one to P_1 ([Roth et al., 2004]).

Given a pool of patients, one can build a *compatibility digraph* $G = (V, A)$ whose vertices are the pairs $v_i = (P_i, D_i)$, and A contains the arc (v_i, v_j) if D_i appears to be compatible with P_j, based on blood and tissue type. Maximizing the number of feasible cyclic transplants amounts now to finding in G a collection of *vertex-disjoint* cycles whose union contains as many arcs as possible. (Beside cycles, some KEPs may also take non-closed directed paths in consideration, but we disregard this option here.) There is a large amount of literature documenting various formulations and matching algorithms to solve this optimization problem; see, e.g., [Constantino et al., 2013], [Dickerson et al., 2016], [Biró et al., 2021] and the literature review in Section 1.3.

One of the remaining issues with this approach, is that in reality, blood type and tissue type are not the only determinants of the feasibility of a transplant. The decision to perform a transplant is based on more complex, so-called crossmatch tests of compatibility between donor and patient. In practice, for cost- and time-efficiency reasons, crossmatch tests are only performed *after* an intended transplant has been identified.

As a result, incompatibilities may be revealed *after* the identification of potential exchange cycles, which, as as consequence, may completely fail to be implemented.

A way to tackle this issue is to *first* select a restricted, but promising subset of arcs, to crossmatch them in order to test their compatibility, and only then to run the matching algorithm in order to find an optimal set of exchange cycles. [Smeulders et al., 2021] have proposed a stochastic integer programming formulation of this approach. Namely, they introduce a *two-stage selection problem* which, given a *testing budget* $\mathbf{B}$, identifies (in stage 1) a subset of arcs $B \subseteq A$ with $|B| \leq \mathbf{B}$ such that the expected number of transplants in the graph (V, B) (in stage 2) is maximized. Solving this optimization problem is numerically challenging.

[Smeulders et al., 2021] tightened the formulation of the two-stage selection problem by adding constraints which enforce that the set B must be a cycle selection: indeed, arcs that are not contained in any cycle cannot be used in transplants and hence, should not be selected in the first stage. Their work motivates our attempt to develop a better understanding of the cycle selection problem and of its ILP formulations.

1.3 Literature review

Rather surprisingly in view of their natural definition, cycle selections have apparently not been previously investigated in the literature, except for the paper of [Smeulders et al., 2021] mentioned above and to which we return in Section 3.2.2. Our review, therefore, is limited to a number of related, but different combinatorial problems.

The weighted *girth* problem asks for a simple cycle of minimum total weight in a weighted *undirected* graph G. The *cycle cone* and *cycle polytope* are, respectively the cone generated by the incidence vectors of cycles of G and the convex hull of these vectors. Thus, the weighted girth problem is the optimization problem over the cycle polytope. It is NP-hard in general, but is polynomially solvable when certain restrictions are placed on the cost vectors. On the other hand, the optimization problem over the cycle cone is polynomially solvable. A linear system describing the cycle cone is given in [Seymour, 1979]. An alternative proof of this result, as well as additional properties of the cycle cone and the cycle polytope, are established in [Coullard and Pulleyblank, 1989].

[Bauer, 1997] studies the facial structure of the cycle polytope associated with a complete undirected graph on n vertices. She proves that this polytope is full dimensional for $n \geq 4$, she provides an ILP formulation for it, and she proves that all inequalities in the ILP formulation are facet-defining when $n \geq 6$. She also presents additional classes of facet-defining valid inequalities, as well as a complete linear description of the cycle polytope when $n \leq 6$. [Bauer et al., 2002] extend the previous results to the case where the cycles are restricted to have length at most $\mathbf{K}$, where $0 \leq \mathbf{K} \leq n$. They also experiment with a branch-and-cut algorithm for the solution of the corresponding optimization problem.

[Balas and Oosten, 2000] investigate the minimum girth problem and the cycle polytope associated with complete *directed* graphs. The optimization problem is again NP-hard, but it is polynomially solvable when all cycles have a positive weight. [Balas and Oosten, 2000] propose an arc-based ILP formulation of the problem. They prove that the cycle polytope on a complete graph with n vertices is a face of the related polytope on a complete graph with $n+1$ vertices. This leads them to an efficient general lifting procedure. They also give a partial description of the facets of the cycle polytope. The article [Balas and Rüdiger, 2009] is a continuation of the previous one. It further studies the cycle polytope, the cycle cone, the upper cycle polyhedron, the dominant of the cycle polytope and their relationships.

[Hartmann and Özlük, 2001] carry out a polyhedral analysis of the $\mathbf{K}$-cycle polytope, which is the convex hull of the incidence vectors of all simple directed cycles with length exactly $\mathbf{K}$. They determine the dimension of the $\mathbf{K}$-cycle polytope. They describe several sets of valid inequalities and discuss the complexity of the associated separation problems. They also investigate the relationship between the $\mathbf{K}$-cycle polytopes of directed and undirected graphs.

In a separate stream of research, the *cardinality constrained multi-cycle* (**CCMC**) problem has been recently studied by several researchers. Given a weighted digraph $G = (V, A)$ and an integer $\mathbf{K}$, the problem is here to find a set of arcs with maximum weight forming a union of *vertex-disjoint* cycles of length at most $\mathbf{K}$. **CCMC** is the underlying combinatorial optimization problem to be solved by kidney exchange programs, as explained in Section 1.2. It is NP-hard for each fixed $\mathbf{K} \geq 3$, and polynomially solvable when $\mathbf{K} = 2$ or when $\mathbf{K} = n$ (see [Abraham et al., 2007], [Roth et al., 2007]). Several IP formulations have been proposed for this problem and are reviewed in [Mak-Hau, 2017] and [Biró et al., 2021]. In particular, [Abraham et al., 2007] and [Roth et al., 2007] give two formulations of exponential size, one where the variables are associated with the arcs of G, and another one where the variables are associated with cycles. Later, [Constantino et al., 2013] and [Dickerson et al., 2016], among others, proposed more complex but compact (polynomial-size) formulations of **CCMC**, including an *extended edge* formulation and a *position-indexed* formulation. [Dickerson et al., 2016] also study the relative strength of the linear relaxation of different formulations.

[Mak-Hau, 2018] focuses on the polyhedral structure of the arc-based formulation proposed by [Roth et al., 2007] when G is a complete digraph. The author proves that three classes of constraints in the initial formulation of the problem are facet-defining for the **CCMC** polytope. Furthermore, she identifies four new classes of valid inequalities. [Lam and Mak-Hau, 2020] extend the theoretical results of [Mak-Hau, 2018] and build on them to develop an efficient branch-and-bound-and-cut algorithm for the **CCMC** problem.

Obviously, cycles and unions of vertex-disjoint cycles of a digraph G are cycle selections of G, so that the following inclusions hold:

$$\text{cycle polytope} \subseteq \text{cycle selection polytope}$$

and

$$\textbf{CCMC} \text{ polytope} \subseteq \text{cycle selection polytope.}$$

The cycle selection problem and the associated polytope have apparently not been investigated until now, but we will be able to draw some inspiration from previous work on related problems in the remainder of the paper.

2 Complexity

The *maximum weighted cycle selection* (**MWCS**) problem has been introduced in Section 1. When all arc weights are nonnegative, a maximum cycle selection of $G = (V, A)$ consists of all the arcs contained in strong components of G. Therefore, in this case, **MWCS** is solvable in linear, $O(n + m)$ time by a simple application of Tarjan's strong component algorithm ([Tarjan, 1972]).

For arbitrary weights, however, **MWCS** is NP-hard. To see this, consider the corresponding decision problem: given a digraph $G = (V, A)$, a weight $w_{i,j} \in \mathbb{N}$ for each arc $(i, j) \in A$, and a number $w_0 \in \mathbb{N}$, is there a cycle selection B such that $w(B) \geq w_0$?

Theorem 1. *The decision version of the maximum weighted cycle selection problem is strongly NP-complete, even when G is a complete digraph.*

Proof. The problem **MWCS** is clearly in NP. We will prove that **MWCS** is strongly NP-complete by reducing the *hitting set* problem **HS** to it. Recall the definition of the hitting set problem: given a finite set X, a collection $T = \{T_1, \dots, T_r\}$ of subsets of X, and $t \in \mathbb{N}$, is there a subset $H \subseteq X$ such that $|H| \leq t$ and $T_j \cap H \neq \emptyset$ for all $j \in \{1, \dots, r\}$? Note that for any instance of **HS**, we can assume without loss of generality that each element of X is in one of the subsets $T_1, \dots, T_r$, and that $t < r$. **HS** is known to be strongly NP-complete ([Karp, 1972]).

With an instance (X, T, t) of **HS**, we associate an instance (G, w, w_0) of **MWCS** where G is the complete digraph on the set of vertices $V = \{0\} \cup X \cup T$, the weights on the arcs are:

- for all $j = 1, \ldots, r$, $w(T_j, 0) = r$,
- for all $i \in X$, $w(0, i) = -1$,
- for all $i \in X$ and for all $j = 1, \ldots, r$, if $i \in T_j$, then $w(i, T_j) = 0$,
- all the other arcs have weight $-r$,

and $w_0 = r^2 - t$.

We claim that this instance of **MWCS** has a YES answer if and only if the original instance of **HS** has a YES answer.

First, suppose that the original instance of **HS** has a YES answer, i.e., suppose there exists $H \subseteq X$ where $|H| \leq t$ and $H \cap T_j \neq \emptyset$ for all $j \in \{1, \ldots, r\}$. Then, let us define a cycle selection B in the following way:

$$\begin{aligned} B = &\{(T_j, 0) : j \in \{1, \ldots, r\}\} \cup \{(0, i) : i \in H\} \\ &\cup \{(i, T_j) : j \in \{1, \ldots, r\}, i \in H \cap T_j\}. \end{aligned}$$

Since each element of H is in one of $T_1, \ldots, T_r$, and since H is a hitting set, B is the union of the 3-cycles $(T_j, 0, i)$, for $j = 1, 2, \ldots, r$ and $i \in H \cap T_j$. Thus, B is indeed a cycle selection and its weight is $w(B) = r^2 - |H| \geq r^2 - t = w_0$, so that the instance of **MWCS** has a YES answer.

Next, suppose conversely that the instance of **MWCS** has a YES answer, in other words that there is a cycle selection B with weight at least $w_0 = r^2 - t$. First, note that each arc of B should be in one of the three sets below:

- $\{(T_j, 0) : j \in \{1, \ldots, r\}\}$,
- $\{(0, i) : i \in X\}$,
- $\{(i, T_j) : j \in \{1, \ldots, r\}, i \in T_j\}$.

Indeed, all arcs not in these three sets have a negative weight $(-r)$ and their inclusion in B would result in a total weight at most equal to $r^2 - r < r^2 - t$, which contradicts our assumption. Moreover, B must contain all arcs $(T_j, 0)$ for all $j \in \{1, \ldots, r\}$, because otherwise $w(B) \leq (r-1)r < r^2 - t$, again a contradiction.

Let now $H = \{i \in X : (0, i) \in B\}$. Then,

$$w(B) = r^2 + \sum_{i \in H} w(0, i) = r^2 - |H|.$$

Since $w(B) \geq r^2 - t$, it follows that $|H| \leq t$.

Finally, we claim that H is a hitting set of T. Indeed, for each $j = 1, \ldots, r$, the arc $(T_j, 0) \in B$ must lie in a cycle of $G_B = (V, B)$. Hence, B must also contain at least one arc of the form (i, T_j) for some i in T_j. Then, $(0, i)$ also is in B, so that i is in H. In conclusion, H is a hitting set with size $|H| \leq t$, meaning that **HS** has a YES answer. □

Since all cycles considered in the proof have length exactly 3, it follows that **MWCS** is NP-complete even when the cycle selection is restricted to contain cycles of length at most 3. On the other hand, **MWCS** is trivially solved when restricted to cycles of length 2: indeed, in this case, the arc $(i, j) \in A$ is in an optimal cycle selection if and only if $(j, i) \in A$ and $w_{i,j} + w_{j,i} \geq 0$.

3 Formulations

3.1 Arc formulation

Let $G = (V, A)$ be an arbitrary directed graph, with $|V| = n$ and $|A| = m$. In order to obtain a first IP formulation for cycle selections, we introduce the "natural" arc variables $\beta_{i,j} \in \{0, 1\}$, where $\beta_{i,j} = 1$ if arc (i, j) is selected and 0 otherwise, for all $(i, j) \in A$.

The set of vectors of $\{0, 1\}^m$ associated with cycle selections is denoted by P_G, or simply P (we usually omit the reference to G, which will be clear from the context). The convex hull of P (or P_G) is denoted by P^* (or P^*_G) and is called the *cycle selection polytope* associated with G. Since **MWCS** is the linear optimization problem over P^* and is NP-hard, it is probably hopeless to obtain a complete linear description of P^*. One of our main goals in this paper will be to produce a (partial) description of P^* for complete digraphs.

For now, consider the following set of constraints:

$$\beta_{i,j} \in \{0, 1\} \qquad \forall (i, j) \in A \tag{1}$$

$$\beta_{i,j} \leq \sum_{(l,k) \in A: l \in V \setminus S, k \in S} \beta_{l,k} \qquad \forall (i, j) \in A, \forall S \subseteq V : i \in S, j \in V \setminus S. \tag{2}$$

We call (2) the *return inequalities* for the set P. (The return inequalities are formally similar to the inequalities defining the cycle cone of an undirected graph; see [Seymour, 1979], [Bauer, 1997].)

Theorem 2. *The constraints* (1)-(2) *provide a correct formulation of the cycle selection problem, that is,*

$$P = \left\{ \beta \in \{0, 1\}^m : \beta_{i,j} \leq \textstyle\sum_{(l,k) \in A: l \in V \setminus S, k \in S} \beta_{l,k} \quad \forall (i, j) \in A, \forall S \subseteq V : i \in S, j \in V \setminus S \right\}.$$

Proof. To show that (2) is valid for P, suppose that β describes a cycle selection B which contains the arc (i, j), so that $\beta_{i,j} = 1$. Let $S \subset V$ be a subset of vertices containing i, but not j. Since B is a cycle selection, there is a directed cycle C such that $(i, j) \in C \subseteq B$, i.e., $\beta_{l,k} = 1$ for all $(l, k) \in C$. At least one arc (l, k) of C must have $l \notin S$ and $k \in S$, and hence (2) is satisfied.

Conversely, suppose that the point $\beta \in \{0,1\}^m$ satisfies the return inequalities (2). Let $B = \{(i,j) : \beta_{i,j} = 1\}$ and $G_B = (V, B)$. For any fixed arc (i,j) such that $\beta_{i,j} = 1$, we must show that (i,j) is contained in at least one directed cycle of G_B. Let $S \subseteq V$ be the set of those vertices k such that there is a directed path $\pi_{k,i}$ from k to i in G_B. Note that $i \in S$. If $j \in S$, then (i,j) is indeed in a directed cycle which is the union of the path $\pi_{j,i}$ and the arc (i,j). Otherwise, $j \in V \setminus S$ and $i \in S$. Because β satisfies the inequality (2), there exists $(l,k) \in A$ such that $l \in V \setminus S$, $k \in S$ and $\beta_{l,k} = 1$. But then (l,k) and $\pi_{k,i}$ together form a path from l to i and thus l should be in S, which brings a contradiction. $\square$

We refer to (1)-(2) as the *arc formulation* of the cycle selection problem, and we define the associated relaxed polytope

$$PL = \left\{ \beta \in [0,1]^m : \beta_{i,j} \leq \textstyle\sum_{(l,k)\in A: l \in V\setminus S, k \in S} \beta_{l,k} \quad \forall (i,j) \in A, \forall S \subseteq V : i \in S, j \in V \setminus S \right\}. \tag{3}$$

There holds

$$P \subseteq P^* \subseteq PL.$$

Because of the exponential number of return inequalities (2), even optimizing a linear function over PL may not be easy. But our next result implies that cutting plane methods can be used efficiently (and that linear optimization over PL is polynomial, by virtue of the equivalence of optimization and separation; see [Grötschel et al., 1981],[Conforti et al., 2014]).

Theorem 3. *The separation problem for the relaxed polytope PL is solvable in polynomial time.*

Proof. The separation problem is the following: given a vector $\beta \in [0,1]^m$, is there $(i,j) \in A$ and $S \subset V$ such that $i \in S, j \in V \setminus S$, and $\beta_{i,j} > \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k}$? There are m arcs (i,j) to check, so we can ask the question for each such arc successively.

Since $\beta_{i,j}$ is fixed, we just need to solve $\min_{S\subset V} \sum_{(l,k)\in A: l \in V\setminus S, k\in S} \beta_{l,k}$ which is the min-cut problem with source j, sink i, and capacity $\beta_{l,k}$ on each arc (l,k). This (j,i)-min-cut problem is solvable in polynomial time. $\square$

3.2 Compact extended formulations

The arc formulation presented in the previous section contains an exponential number of return inequalities (2). The aim of this section is to present several compact, polynomial-size formulations of the cycle selection problem and to compare them with the arc formulation.

3.2.1 Extended arc formulations

We start with three formulations based on the relation between cycle selections and circulations. Recall that a *circulation* in a directed graph $G = (V, A)$ is a flow-vector $x \in \mathbb{R}_+^{|A|}$ which is balanced at every vertex, that is, such that

$$\sum_{h:(h,k)\in A} x_{h,k} = \sum_{h:(k,h)\in A} x_{k,h} \qquad \forall k \in V.$$

The *support* of a circulation x is the set $C(x) = \{(i,j) \in A : x_{i,j} > 0\}$. It can be viewed as a cycle selection consisting of m cycles or less (see, e.g., Corollary 4.3.3 in [Bang-Jensen and Gutin, 2009]). Conversely, every cycle selection B gives rise to a circulation x^B whose support is exactly B, as follows. For each arc $(u,v) \in B$, let $C^{(u,v)}$ be a cycle of G_B containing (u,v), and put a flow of one unit on $C^{(u,v)}$, that is define $x_{i,j}^{(u,v)} = 1$ if $(i,j) \in C^{(u,v)}$, $x_{i,j}^{(u,v)} = 0$ otherwise. Finally, define a circulation x^B as the sum of the cycle flows $x^{(u,v)}$, that is, let $x^B = \sum_{(u,v)\in B} x^{(u,v)}$. Note that this construction does not univocally define x^B, because the choice of the cycles $C^{(u,v)}$ is not unique, but this will be irrelevant for our purpose.

In particular, when x is a binary circulation, then $C(x) = \{(i,j) : x_{i,j} = 1\}$ is an *arc-disjoint* union of cycles, i.e., a special type of cycle selection. If moreover

$$\sum_{h:(h,k)\in A} x_{h,k} \leq 1 \qquad \forall k \in V,$$

then the support of a binary circulation is a *vertex-disjoint* union of cycles.

These observations lead to different formulations for cycle selections. A first *simple extended arc formulation* is as follows: vector $\beta \in \mathbb{R}^{|A|}$ defines a selection if and only there exists $x \in \mathbb{R}_+^{|A|}$ such that

$$x_{i,j} \leq m\,\beta_{i,j} \qquad \forall (i,j) \in A \tag{4}$$

$$\beta_{i,j} \leq x_{i,j} \qquad \forall (i,j) \in A \tag{5}$$

$$\sum_{h:(h,k)\in A} x_{h,k} = \sum_{h:(k,h)\in A} x_{k,h} \qquad \forall k \in V \tag{6}$$

$$0 \leq \beta_{i,j} \leq 1 \qquad \forall (i,j) \in A \tag{7}$$

$$\beta_{i,j} \text{ integer} \qquad \forall (i,j) \in A. \tag{8}$$

Indeed, any feasible solution of (4)-(8) defines a subset of arcs B (associated with β) and a circulation x such that B is exactly the support of x. Therefore, B is a cycle selection. Conversely, every cycle selection B gives rise to a feasible solution (β, x^B) as explained above.

A second, more complex but as we will see, tighter formulation relies on expressing that each arc (u,v) of a cycle selection must be contained in the support $C^{(u,v)}$ of a representative *binary* circulation $x^{(u,v)}$ (note that $C^{(u,v)}$ and $x^{(u,v)}$ may differ for each arc (u,v)). We define $x_{i,j}^{(u,v)} = 1$ if $(i,j) \in C^{(u,v)}$, and we interpret $x_{i,j}^{(i,j)} = 1$ to mean that arc (i,j) is in the cycle selection, that is, we identify $x_{i,j}^{(i,j)}$ with $\beta_{i,j}$.

The cycle selection problem can now be formulated as follows:

$$x_{i,j}^{(u,v)} \leq x_{i,j}^{(i,j)} \qquad \forall (i,j) \in A, \forall (u,v) \in A \tag{9}$$

$$\sum_{h:(h,k)\in A} x_{h,k}^{(u,v)} = \sum_{h:(k,h)\in A} x_{k,h}^{(u,v)} \qquad \forall k \in V, \forall (u,v) \in A \tag{10}$$

$$0 \leq x_{i,j}^{(u,v)} \leq 1 \qquad \forall (i,j) \in A, \forall (u,v) \in A \tag{11}$$

$$x_{i,j}^{(u,v)} \text{ integer} \qquad \forall (i,j) \in A, \forall (u,v) \in A \tag{12}$$

Constraints (10)-(12) enforce that each vector $x^{(u,v)}$ is indeed a binary circulation, and constraints (9) enforce that arc (i,j) can be in the support $C^{(u,v)}$ only if it is selected at all (if $x_{i,j}^{(u,v)} = 1$, then $x_{i,j}^{(i,j)} \equiv \beta_{i,j}$ must be 1 as well).

The constraints (9)-(12) provide a correct extended formulation of the cycle selection problem. We refer to it as the *extended arc formulation* of the cycle selection problem, and we note that it is formally similar to the *extended edge formulation* of [Constantino et al., 2013] for the cardinality-constrained multi-cycle problem (**CCMC**, see Section 1.3). It contains a polynomial number of variables ($O(n^4)$) and a polynomial number of constraints ($O(n^4)$).

Finally, the extended arc formulation can be further adapted by insisting that the support of each binary circulation $x^{(u,v)}$ should consist of vertex-disjoint cycles. This can be achieved by adding the following constraints to the extended arc formulation:

$$\sum_{h:(k,h)\in A} x_{k,h}^{(u,v)} \leq 1 \qquad \forall k \in V, \forall (u,v) \in A. \tag{13}$$

We refer to (9)-(13) as the *modified extended arc formulation* for cycle selections.

Remark 1. One may want to further strengthen these extended formulations so that $C^{(u,v)}$ is a single cycle for each (u,v). This would require to include additional exponential families of inequalities describing the cycle polytope, see [Balas and Oosten, 2000], [Balas and Rüdiger, 2009]. □

We now aim to establish the relation between the arc formulation of Section 3 and the different extended arc formulations introduced above. Let us first denote as P_{EA} the polytope defined by the linear relaxation (9)-(11) of the extended arc formulation, and recall that PL is the solution set of the relaxation (3) of the arc formulation.

Theorem 4. *The polytope PL is the projection of the polytope P_{EA} on the space $\mathbb{R}^A$ of the variables $\beta_{i,j} \equiv x_{i,j}^{(i,j)}$, $(i,j) \in A$.*

Proof. To prove first that the projection of P_{EA} is contained in PL, let us consider a point $x \in P_{EA}$ and let us set $\beta_{i,j} = x_{i,j}^{(i,j)}$ for all $(i,j) \in A$. We must show that $\beta \in PL$.

The bounding constraints $0 \leq \beta_{i,j} \leq 1$ are satisfied. So, we only need to show that, for each fixed arc $(i,j) \in A$ and each fixed subset $S \subseteq V$ with $i \in S$, $j \notin S$, the return inequality

$$\beta_{i,j} \leq \sum_{(l,k)\in A: l \in V\setminus S, k\in S} \beta_{l,k} \tag{14}$$

can be deduced from the inequalities defining P_{EA}. Let us add up the following inequalities:

$$x_{l,k}^{(i,j)} \leq x_{l,k}^{(l,k)} \qquad \forall (l,k) \in (V \setminus S, S), \tag{15}$$

$$\sum_{h:(k,h)\in A} x_{k,h}^{(i,j)} - \sum_{h:(h,k)\in A} x_{h,k}^{(i,j)} = 0 \qquad \forall k \in S. \tag{16}$$

This yields a new inequality with right-hand side equal to:

$$\sum_{(l,k)\in A: l\in V\setminus S, k\in S} x_{l,k}^{(l,k)} = \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k}.$$

In the left-hand side of the summation, each variable $x_{l,k}^{(i,j)}$, $(l,k) \in (V \setminus S, S)$, appears once with coefficient $+1$ in (15) and once with coefficient -1 in (16). Also, each variable $x_{k,h}^{(i,j)}$ with $k, h \in S$ appears once with coefficient $+1$ and once with coefficient -1 in (16). As a result, the left-hand side of the summation boils down to $\sum_{(k,h)\in A: k\in S, h\in V\setminus S} x_{k,h}^{(i,j)}$. Since $x_{k,h}^{(i,j)} \geq 0$ for all $(k,h) \in A$, the left-hand side of the inequality is at least $x_{i,j}^{(i,j)} = \beta_{i,j}$, and hence we obtain the return inequality (14).

Next, to prove that PL is contained in the projection of P_{EA}, we must show that for each feasible solution $\beta^* \in PL$ of the relaxed arc formulation, there exists a solution $x \in P_{EA}$ with $x_{i,j}^{(i,j)} = \beta_{i,j}^*$ for all $(i,j) \in A$.

For every fixed arc $(i,j) \in A$, denote as $G^{(i,j)} = (V, A)$ the digraph (V, A) equipped with the following lower bound $\ell_{h,k}^{(i,j)}$ and upper bound $c_{h,k}^{(i,j)}$ on each arc (h,k) in A:

- if $h \neq i$ and $k \neq j$, then $\ell_{h,k}^{(i,j)} = 0$ and $c_{h,k}^{(i,j)} = \beta_{h,k}^*$;
- if $h = i$ and $k \neq j$, then $\ell_{i,k}^{(i,j)} = c_{i,k}^{(i,j)} = 0$;
- if $h \neq i$ and $k = j$, then $\ell_{h,j}^{(i,j)} = c_{h,j}^{(i,j)} = 0$;
- if $h = i$ and $k = j$, then $\ell_{i,j}^{(i,j)} = c_{i,j}^{(i,j)} = \beta_{i,j}^*$.

We say that a circulation x is *feasible* in $G^{(i,j)}$ if $\ell_{h,k}^{(i,j)} \leq x_{h,k} \leq c_{h,k}^{(i,j)}$ for each arc (h,k) in A. In view of Hoffman's circulation theorem ([Hoffman, 1960], [Bang-Jensen and Gutin, 2009]), there exists a feasible circulation in $G^{(i,j)}$ if and only if

$$\sum_{(h,k)\in A: h\in S, k\notin S} \ell_{h,k}^{(i,j)} \leq \sum_{(h,k)\in A: h\notin S, k\in S} c_{h,k}^{(i,j)} \quad \text{for all } S \subseteq V. \tag{17}$$

Let us verify that this is indeed the case. Fix the set $S \subseteq V$. With our definition of the lower and upper bounds, the left-hand side of the inequality (17) is zero (and hence, the inequality is trivially satisfied) unless $i \in S$ and $j \notin S$, in which case it is equal to $\ell_{i,j}^{(i,j)} = \beta_{i,j}^*$. But then, the right-hand side of (17) is equal to $\sum_{(h,k)\in A: h\notin S, k\in S} \beta_{h,k}^*$. So, (17) boils down to a return inequality, and it is satisfied in view of the feasibility of β^* for the relaxed arc formulation.

So, we conclude from Hoffman's theorem that for each $(i,j) \in A$, there exists a feasible circulation $x^{(i,j)}$ in $G^{(i,j)}$. Note that due to the bounds on the arcs of $G^{(i,j)}$, $x^{(i,j)}_{i,j} = \beta^*_{i,j}$. Moreover, the collection of circulations $x^{(i,j)}$, for all $(i,j) \in A$, satisfies the constraints (9)–(11) of the extended arc formulation. Indeed, the constraints (10) are satisfied by definition of circulations, and the constraints (11) are satisfied because of the lower and upper bounds on the arcs of $G^{(i,j)}$. As for constraints (9), consider $(u,v) \in A$ and $(i,j) \in A$.

- If $(u,v) = (i,j)$, then (9) is trivial.
- If $u = i$ and $v \neq j$, or if $u \neq i$ and $v = j$, then $x^{(u,v)}_{i,j} \leq c^{(u,v)}_{i,j} = 0$ (upper bound on $x^{(u,v)}_{i,j}$ in the graph $G^{(u,v)}$).
- If $u \neq i$ and $v \neq j$, then $x^{(u,v)}_{i,j} \leq c^{(u,v)}_{i,j} = \beta^*_{i,j}$.

Hence, constraints (9) are indeed satisfied. This concludes the proof. □

In view of Theorem 4, the linear relaxation P_{EA} of the extended arc formulation is equivalent to the linear relaxation PL of the arc formulation when it comes to solving the maximum weighted cycle selection problem. We are now going to show that the same conclusion applies when we consider the modified extended arc formulation. We denote by P_{MEA} the polytope defined by inequalities (9)-(11) and (13).

Theorem 5. *The polytope PL is the projection of the polytope P_{MEA} on the space $\mathbb{R}^A$ of the variables $\beta_{i,j} \equiv x^{(i,j)}_{i,j}$, $(i,j) \in A$.*

Proof. Since $P_{MEA} \subseteq P_{EA}$, Theorem 4 immediately implies that the projection of P_{MEA} is contained in PL.

For the reverse inclusion, consider the collection of circulations $x^{(i,j)}$ obtained in the proof of Theorem 4. Each circulation $x^{(i,j)}$ can be written as a positive linear combination of the form

$$x^{(i,j)} = \sum_{C \in \mathcal{C}^{(i,j)}} \lambda^{(i,j)}_C \xi^C, \tag{18}$$

where $\mathcal{C}^{(i,j)}$ is a collection of directed cycles forming the support of $x^{(i,j)}$, ξ^C is the incidence vector of cycle C, and $\lambda^{(i,j)}_C > 0$ for all $C \in \mathcal{C}^{(i,j)}$ (see, e.g., [Bang-Jensen and Gutin, 2009]). If some cycle $C \in \mathcal{C}^{(i,j)}$ does not contain the arc (i,j), then we can remove this cycle from the collection $\mathcal{C}^{(i,j)}$, and the right-hand side of (18) still defines a feasible circulation as required in the proof of Theorem 4. So, we can assume without loss of generality that $(i,j) \in C$, or equivalently, $\xi^C_{i,j} = 1$, for all $C \in \mathcal{C}^{(i,j)}$. It then follows from (18) that

$$x^{(i,j)}_{i,j} = \sum_{C \in \mathcal{C}^{(i,j)}} \lambda^{(i,j)}_C. \tag{19}$$

We want to show now that inequality (13) is satisfied, for an arbitrary $k \in V$ and for $(u, v) = (i, j)$. The left-hand side of (13) is

$$\begin{aligned}\sum_{h:(k,h)\in A} x_{k,h}^{(i,j)} &= \sum_{h:(k,h)\in A} \sum_{C\in\mathcal{C}^{(i,j)}} \lambda_C^{(i,j)} \xi_{k,h}^C \\ &= \sum_{C\in\mathcal{C}^{(i,j)}} \lambda_C^{(i,j)} \Big(\sum_{h:(k,h)\in A} \xi_{k,h}^C \Big).\end{aligned}$$

For each cycle C, $\sum_{h:(k,h)\in A} \xi_{k,h}^C$ is either 1 (if vertex k is on the cycle) or 0 (otherwise). So, we get:

$$\sum_{h:(k,h)\in A} x_{k,h}^{(i,j)} \leq \sum_{C\in\mathcal{C}^{(i,j)}} \lambda_C^{(i,j)},$$

and from (19),

$$\sum_{h:(k,h)\in A} x_{k,h}^{(i,j)} \leq x_{i,j}^{(i,j)} \leq 1,$$

so that the constraint (13) is satisfied. This implies that the collection of circulations $x^{(i,j)}$, for all $(i, j) \in A$, satisfies the constraints (9)–(11) and (13) of the modified extended arc formulation, which concludes the proof. $\square$

Finally, we return to the simple extended arc formulation. Let us denote as P_{SEA} the set of solutions of the relaxed formulation (4)-(7). The proof of the following result suggests that this formulation is quite loose.

Theorem 6. *The polytope PL is included in the projection of the polytope P_{SEA} on the space $\mathbb{R}^A$ of the variables $\beta_{i,j}$, $(i, j) \in A$, and the inclusion is strict for complete digraphs on $n \geq 2$ vertices.*

Proof. Given $\beta^* \in PL$, consider again the collection of circulations $x^{(u,v)}$, $(u, v) \in A$ obtained in the proof of Theorem 4. Let us define $x^* = \sum_{(u,v)\in A} x^{(u,v)}$, and let us show that $(\beta^*, x^*) \in P_{SEA}$. First, it is clear that x^* is a circulation, i.e., it satisfies the equations (6). Since $\beta^*_{i,j} = x_{i,j}^{(i,j)}$, it follows that $\beta^*_{i,j} \leq \sum_{(u,v)\in A} x_{i,j}^{(u,v)} = x^*_{i,j}$, meaning that equation (5) is satisfied. Finally, in view of equation (9), $x^*_{i,j} = \sum_{(u,v)\in A} x_{i,j}^{(u,v)} \leq \sum_{(u,v)\in A} x_{i,j}^{(i,j)} = m\beta^*_{i,j}$, hence equation (4) is satisfied and $(\beta^*, x^*) \in P_{SEA}$, as required.

To prove that the inclusion is strict when $n \geq 2$, consider the following assignment (only the nonzero values are displayed):

- $\beta_{1,2} = 1.0$, $\beta_{2,1} = 0.5$,
- $x_{1,2} = x_{2,1} = 1$.

Then, $(\beta, x) \in P_{SEA}$, but $\beta \notin PL$ since β does not satisfy the return inequality

$$\beta_{1,2} \leq \sum_{k\in V\setminus\{2\}} \beta_{2,k}. \qquad \square$$

3.2.2 Position-indexed formulation

Another extended formulation has been proposed by [Smeulders et al., 2021]; it is inspired by the position-indexed edge formulation of the **CCMC** problem ([Dickerson et al., 2016]).

Assuming (without loss of generality) that the vertex-set of the digraph $G = (V, A)$ is $V = \{1, \ldots, n\}$, let us denote by V^l the subset of vertices $\{l, \ldots, n\}$, for each l in V. Given binary values for the arc variables $\beta_{i,j}$, define $B^l = \{(i, j) \in A : i \in V^l, j \in V^l, \beta_{i,j} = 1\}$. Let us then introduce a new set of position-indexed binary variables:

$$\phi^l_{i,j,k} \text{ for all } (i,j) \in A, l \in V, k \in \kappa(i,j,l) \text{ where } \kappa(i,j,l) = \begin{cases} \{1\} \text{ if } i = l \\ \{2, ..., n\} \text{ if } j = l \\ \{2, ..., n-1\} \text{ if } i, j > l \end{cases}$$

with the interpretation that $\phi^l_{i,j,k}$ is equal to 1 if arc (i, j) is in position k in a cycle of the digraph (V^l, B^l) containing vertex l, and 0 otherwise.

[Smeulders et al., 2021] propose the following formulation:

$$\beta_{i,j} \le \sum_{l \in V} \sum_{k \in \kappa(i,j,l)} \phi^l_{i,j,k} \qquad \forall (i,j) \in A \tag{20}$$

$$\phi^l_{i,j,k} \le \beta_{i,j} \qquad \forall l \in V, (i,j) \in A^l, k \in \kappa(i,j,l) \tag{21}$$

$$\phi^l_{i,j,k} \le \sum_{h:(h,i) \in A^l \wedge k-1 \in \kappa(h,i,l)} \phi^l_{h,i,k-1} \qquad \forall l \in V, (i,j) \in A^l, k \in \kappa(i,j,l), k > 1 \tag{22}$$

$$\phi^l_{i,j,k} \le \sum_{h:(j,h) \in A^l \wedge k+1 \in \kappa(j,h,l)} \phi^l_{j,h,k+1} \qquad \forall l \in V, (i,j) \in A^l, j \neq l, k \in \kappa(i,j,l) \tag{23}$$

$$0 \le \phi^l_{i,j,k} \le 1 \qquad \forall l \in V, (i,j) \in A^l, k \in \kappa(i,j,l) \tag{24}$$

$$0 \le \beta_{i,j} \le 1 \qquad \forall (i,j) \in A \tag{25}$$

$$\phi^l_{i,j,k} \text{ integer} \qquad \forall l \in V, (i,j) \in A^l, k \in \kappa(i,j,l) \tag{26}$$

$$\beta_{i,j} \text{ integer} \qquad \forall (i,j) \in A \tag{27}$$

Constraints (20) express that if an arc is selected, then it is part of at least one cycle, and constraints (21) ensure that if an arc is in a cycle, then it has to be selected. Constraints (22) enforce that if arc (i, j) is in position k in some cycle of (V^l, B^l), then there must be a preceding arc in position $k - 1$, unless $k = 1$ (when $k = 1$, then there is no preceding arc, but because of the definition of $\kappa(i, j, l)$, i is necessarily equal to l for the variables $\phi^l_{i,j,1}$). Similarly, constraints (23) enforce that arc (i, j) must have a succeeding arc unless $j = l$ which means that a cycle is completed.

The inequalities (20)-(27) provide a compact *position-indexed formulation* for cycle selections, with $O(n^4)$ variables and constraints. Their linear relaxation (20)-(25) describes a polytope P_{PI}. We next show that this relaxation is weaker than the relaxation PL of the arc formulation.

Theorem 7. *The polytope PL is included in the projection of the polytope P_{PI} on the space $\mathbb{R}^A$ of the variables $\beta_{i,j}$, $(i,j) \in A$, and the inclusion is strict for complete digraphs on $n \geq 4$ vertices.*

Proof. We must prove that for any feasible solution $\beta \in PL$, there exists a solution (β, ϕ) in P_{PI}.

As in the proof of Theorem 5, consider a collection of circulations $x^{(i,j)}$, $(i,j) \in A$, and write each $x^{(i,j)}$ as a positive linear combination of the form

$$x^{(i,j)} = \sum_{C \in \mathcal{C}^{(i,j)}} \lambda_C^{(i,j)} \xi^C, \tag{28}$$

where $\mathcal{C}^{(i,j)}$ is a collection of directed cycles containing the arc (i,j), ξ^C is the incidence vector of cycle c, $\lambda_C^{(i,j)} > 0$ for all $C \in \mathcal{C}^{(i,j)}$, and

$$\beta_{i,j} = \sum_{C \in \mathcal{C}^{(i,j)}} \lambda_C^{(i,j)}. \tag{29}$$

For any two arcs (i,j), $(u,v) \in A$, and for all k, l, define $\mathcal{C}^{(u,v)}(i,j,k,l)$ as the set of cycles $C \in \mathcal{C}^{(u,v)}$ such that arc (i,j) is in position k in C, and the lowest-indexed vertex of C is l.

For all $(i,j) \in A, l \in V, k \in \kappa(i,j,l)$, set now

$$\phi_{i,j,k}^l = \max_{(u,v) \in A} \sum_{C \in \mathcal{C}^{(u,v)}(i,j,k,l)} \lambda_C^{(u,v)}. \tag{30}$$

We claim that (β, ϕ) satisfies inequalities (20)-(25). First, for each $(i,j) \in A$, in view of (29), of $\mathcal{C}^{(i,j)} = \bigcup_{k,l} \mathcal{C}^{(i,j)}(i,j,k,l)$, and of (30), we get

$$\beta_{i,j} = \sum_{C \in \mathcal{C}^{(i,j)}} \lambda_C^{(i,j)} = \sum_{k,l} \sum_{C \in \mathcal{C}^{(i,j)}(i,j,k,l)} \lambda_C^{(i,j)} \leq \sum_{k,l} \phi_{i,j,k}^l,$$

which is exactly inequality (20).

Consider next the inequalities (21). For given i,j,k,l, the maximum in the right-hand side of (30) is achieved for some arc (u,v). With this value of (u,v),

$$\phi_{i,j,k}^l = \sum_{C \in \mathcal{C}^{(u,v)}(i,j,k,l)} \lambda_C^{(u,v)} \leq \sum_{C \in \mathcal{C}^{(u,v)} : (i,j) \in C} \lambda_C^{(u,v)} \leq \beta_{i,j}.$$

The last inequality holds by construction of the circulation $x^{(u,v)}$ in Theorem 4: the sum of the weights $\lambda_c^{(u,v)}$ of the cycles involved in $\mathcal{C}^{(u,v)}$ cannot exceed the upper bound $\beta_{s,t}$ on any arc (s,t). In particular, it cannot exceed $\beta_{i,j}$.

For the inequality (22) associated with i, j, k, l, where $k > 1$, assume again that the maximum in equation (30) is achieved for arc (u, v). For each cycle $C \in \mathcal{C}^{(u,v)}(i, j, k, l)$, there is an arc $(h(C), i)$ in position $k - 1$ in c. So,

$$\begin{aligned}
\phi^l_{i,j,k} &= \sum_{C \in \mathcal{C}^{(u,v)}(i,j,k,l)} \lambda^{(u,v)}_C \\
&= \sum_{C \in \mathcal{C}^{(u,v)}(h(C),i,k-1,l)} \lambda^{(u,v)}_C \\
&\leq \sum_{h \in V^l} \sum_{C \in \mathcal{C}^{(u,v)}(h,i,k-1,l)} \lambda^{(u,v)}_C \\
&\leq \sum_{h \in V^l} \phi^l_{h,i,k-1},
\end{aligned}$$

as required.

The case of inequalities (23) is similar. Finally, the bounds (24) are implied by (30) and by (21). Thus, we conclude that (β, ϕ) satisfies all inequalities (20)-(25), and that PL is indeed contained in the projection of P_{PI}.

To prove strict inclusion for complete digraphs with $n \geq 4$ vertices, consider the following assignment for the (β, ϕ) variables (we only list the variables with nonzero value):

- $\beta_{1,3} = \beta_{3,4} = \beta_{4,3} = 1, \beta_{3,1} = 0.5$,
- $\phi^1_{1,3,1} = 1, \phi^1_{3,1,2} = \phi^1_{3,1,4} = \phi^1_{3,4,2} = \phi^1_{4,3,3} = 0.5$,
- $\phi^3_{3,4,1} = \phi^3_{4,3,2} = 0.5$.

These values satisfy all constraints of the relaxed position-indexed formulation. However, with $S = \{1\}$, $i = 1$ and $j = 3$, the return inequality

$$\beta_{i,j} \leq \sum_{(l,k) \in A: l \in V \setminus S, k \in S} \beta_{l,k}$$

is violated by the given assignment since $\beta_{1,3} > \beta_{3,1}$. $\square$

3.3 Cycle formulation

Let us define Γ_G (or simply, Γ) as the set of all directed cycles in the digraph G. Like [Abraham et al., 2007] and [Roth et al., 2007] for **CCMC**, we next propose a formulation for cycle selections based on the cycle variables z_C, $C \in \Gamma$, where $z_C = 1$ if cycle C is selected and 0 otherwise. Then, together with the arc variables $\beta_{i,j}$, the cycle selection problem can be formulated as follows:

$$z_C \le \beta_{i,j} \qquad \forall C \in \Gamma, \forall (i,j) \in C \tag{31}$$

$$\beta_{i,j} \le \sum_{C \in \Gamma : (i,j) \in C} z_C \qquad \forall (i,j) \in A \tag{32}$$

$$0 \le z_C \le 1 \qquad \forall C \in \Gamma \tag{33}$$

$$0 \le \beta_{i,j} \le 1 \qquad \forall (i,j) \in A \tag{34}$$

$$z_C \text{ integer} \qquad \forall C \in \Gamma \tag{35}$$

$$\beta_{i,j} \text{ integer} \qquad \forall (i,j) \in A \tag{36}$$

Constraints (31) enforce that if cycle C is selected then all arcs $(i,j) \in C$ must be selected. Constraints (32) enforce that if arc $(i,j) \in A$, is selected then at least one cycle containing (i,j) must be selected as well.

The constraints (31)-(36) provide a valid formulation of the cycle selection problem. We refer to it as the *cycle formulation*. Note that it is an exponential formulation due to the number of potential cycles ($|\Gamma| = O(2^{|m|})$) in graph G.

Denote by P_C the linear relaxation (31)-(34) of the cycle formulation. This relaxation is again weaker than the relaxation of the arc formulation:

Theorem 8. *The polytope PL is included in the projection of the polytope P_C on the space $\mathbb{R}^A$ of the variables $\beta_{i,j}$, $(i,j) \in A$, and the inclusion is strict for complete digraphs on $n \ge 4$ vertices.*

Proof. In view of Theorem 4, it suffices to prove that given a feasible solution $x \in P_{EA}$, there exists a solution (β, z) in P_C with $\beta_{i,j} = x_{i,j}^{(i,j)}$ for all $(i,j) \in A$. With the same notations as in the proof of Theorem 5, consider the positive linear combination

$$x^{(i,j)} = \sum_{C \in \mathcal{C}^{(i,j)}} \lambda_C^{(i,j)} \xi_C, \tag{37}$$

and the associated expression of $\beta_{i,j}$:

$$\beta_{i,j} = \sum_{C \in \mathcal{C}^{(i,j)}} \lambda_C^{(i,j)}. \tag{38}$$

For all $C \in \Gamma$, define

$$z_C = \max_{(u,v) \in A} \lambda_C^{(u,v)}. \tag{39}$$

Consider constraint (31) for a given cycle $C^* \in \Gamma$ and an arc $(i,j) \in C^*$. The maximum in the right-hand side of (39) is achieved for some arc (u,v), say, $z_{C^*} = \lambda_{C^*}^{(u,v)}$. So,

$$z_{C^*} = \lambda_{C^*}^{(u,v)} \le \sum_{C \in \mathcal{C}^{(u,v)} : (i,j) \in C} \lambda_C^{(u,v)} \le \beta_{i,j}.$$

The last inequality holds by construction of the circulation $x^{(u,v)}$: the sum of the weights of the cycles involved in $\mathcal{C}^{(u,v)}$ cannot exceed the upper bound $\beta_{s,t}$ on any arc (s,t). In particular, it cannot exceed $\beta_{i,j}$ on arc (i,j).

Next, consider constraint (32) for an arc $(i,j) \in A$. In view of equations (38)-(39),

$$\beta_{i,j} = \sum_{C \in \mathcal{C}^{(i,j)}} \lambda_C^{(i,j)} \leq \sum_{C \in \mathcal{C}^{(i,j)}} z_C \leq \sum_{C \in \Gamma : (i,j) \in C} z_C .$$

This shows that PL is contained in the projection of P_C, as required. To prove that the containment is strict when $n \geq 4$, consider the following assignment (only the nonzero values are displayed):

- $\beta_{1,2} = 1.0$, $\beta_{2,3} = \beta_{3,4} = \beta_{4,1} = \beta_{3,1} = 0.5$,
- $z_{(1,2),(2,3),(3,4),(4,1)} = z_{(1,2),(2,3),(3,1)} = 0.5$.

The point (β, z) in in P_C, but $\beta \notin PL$ since β does not satisfy the return inequality

$$\beta_{1,2} \leq \sum_{k \in V \setminus \{2\}} \beta_{2,k}$$

associated with $i = 1$, $j = 2$ and $S = V \setminus \{2\}$. □

3.4 Relative strength of formulations

In conclusion, six different formulations of the selection problem have been proposed in this section.

The relative strength of the linear relaxation of these formulations can be described as follows:

- (Theorem 4, Theorem 5.) The arc formulation is equivalent to the extended arc formulation and to the modified extended arc formulation, in the sense that PL is equal to the projection of P_{EA} and of P_{MEA} on the space of the β variables.
- (Theorem 6, Theorem 7, Theorem 8.) The arc formulation is strictly tighter than the simple extended arc formulation, the position-indexed formulation, and the cycle formulation, in the sense that PL is strictly contained in the projection of P_{SEA}, of P_{PI} and of P_C on the space of the β variables.

In view of these results, we focus for the rest of the article on the arc formulation of the selection problem.

4 Polyhedral structure

Note that any instance of **MWCS** on an incomplete digraph $G = (V, A)$ can be transformed into an instance on a complete digraph by setting a large negative weight $w_{i,j}$ on all pairs $(i, j) \notin A$. Therefore, from now on, we restrict our attention to the case of a complete directed graph $G = (V, A)$, where $|V| = n$ and A contains $m = n(n-1)$ arcs. Our objective is to investigate the polyhedral structure of the cycle selection polytope P^*, which only depends on n in this case.

4.1 Dimension

When $|V| = 2$, say $V = \{1, 2\}$, the directed graph only has two arcs $(1, 2), (2, 1)$. The only two feasible cycle selections are the empty cycle selection and the 2-cycle $\{(1, 2), (2, 1)\}$. In this case the dimension of P^* is 1. For the rest of the document, we assume that $|V| \geq 3$.

Theorem 9. *When $|V| \geq 3$, $P^* = conv(P)$ is full dimensional, that is, $dim(P^*) = n(n-1)$.*

Proof. Suppose that P^* is contained in a hyperplane defined by the equation

$$\sum_{(u,v)\in A} b_{u,v}\beta_{u,v} = b_0. \tag{40}$$

We are going to show that the equation (40) is of the form: $0 = 0$, which implies that P^* is full dimensional.

1. Since $0 \in P^*$, we get $b_0 = 0$.
2. Let i, j, k be three distinct vertices in V. Consider the point β^1 with $\beta^1_{i,j} = \beta^1_{j,k} = \beta^1_{k,i} = 1$ and $\beta^1_{u,v} = 0$ for all others arcs $(u, v) \in A$. Since $\beta^1 \in P^*$, it follows that $b_{i,j} + b_{j,k} + b_{k,i} = 0$.
3. For the same three vertices i, j, k as above, let β^2 be such that $\beta^2_{i,j} = \beta^2_{j,k} = \beta^2_{k,i} = \beta^2_{j,i} = 1$ and $\beta^2_{u,v} = 0$ for all others arcs $(u, v) \in A$. Again, $\beta^2 \in P^*$, and the previous conclusions imply that $b_{j,i} = 0$.

It follows that $b_{u,v} = 0$ for all arcs $(u, v) \in A$, as claimed. □

4.2 Facets

In this section, we are going to show that the constraints of the arc formulation (3) are facet-defining for the cycle selection polytope P^*. As mentioned before, we assume that $|V| \geq 3$.

4.2.1 Lower bound inequalities

Theorem 10. *For all $(i,j) \in A$, the inequality $\beta_{i,j} \geq 0$ defines a facet of P^*.*

Proof. Fix $(i,j) \in A$, and let F be the face of P^* defined as

$$F = \{\beta \in P^* : \beta_{i,j} = 0\}.$$

Suppose that F is included in a hyperplane defined by the equation

$$\sum_{(u,v)\in A} b_{u,v}\beta_{u,v} = b_0 \tag{41}$$

and consider the following points $\beta^1, \dots, \beta^6$ which are all in F.

1. $\beta^1 = 0 \in F$, hence $b_0 = 0$.
2. For each $(l,k) \notin \{(i,j),(j,i)\}$, let $\beta^2 \in F$ be defined by $\beta^2_{l,k} = \beta^2_{k,l} = 1$ and $\beta^2_{u,v} = 0$ for all others arcs $(u,v) \in A$. From equation (41), We obtain: $b_{l,k} = -b_{k,l}$.
3. For $l \notin \{i,j\}$, let $\beta^3 \in F$ be such that $\beta^3_{j,i} = \beta^3_{l,i} = \beta^3_{i,l} = \beta^3_{l,j} = \beta^3_{j,l} = 1$ and $\beta^3_{u,v} = 0$ for all others arcs $(u,v) \in A$. This yields $b_{j,i} = 0$.
4. For $l \notin \{i,j\}$, let $\beta^4 \in F$ be such that $\beta^4_{j,i} = \beta^4_{l,i} = \beta^4_{i,l} = \beta^4_{l,j} = 1$ and $\beta^4_{u,v} = 0$ for all others arcs $(u,v) \in A$. From (41), we get $b_{l,j} = 0$, and together with point 2 here above, $b_{j,l} = 0$.
5. For $l \notin \{i,j\}$, let $\beta^5 \in F$ be such that $\beta^5_{j,i} = \beta^5_{i,l} = \beta^5_{l,j} = 1$ and $\beta^5_{u,v} = 0$ for all others arcs $(u,v) \in A$. We deduce $b_{i,l} = 0$ and from point 2, $b_{l,i} = 0$.
6. If $|V| \geq 4$, fix $l,k \notin \{i,j\}$, and define $\beta^6 \in F$ by $\beta^6_{j,l} = \beta^6_{l,k} = \beta^6_{k,j} = 1$ and $\beta^6_{u,v} = 0$ for all others arcs $(u,v) \in A$. We then obtain $b_{l,k} = 0$.

In conclusion, we find that the equation (41) is identical to $b_{i,j}\beta_{i,j} = 0$, and hence F is a facet of the convex hull polytope P^*. □

4.2.2 Upper bound inequalities

Theorem 11. *For all $(i,j) \in A$, the inequality $\beta_{i,j} \leq 1$ defines a facet of P^*.*

Proof. Fix $(i,j) \in A$ and define the face $F = \{\beta \in P^* : \beta_{i,j} = 1\}$. Assume that F is contained in a hyperplane of the form (41) and consider the binary points $\beta^1, \dots, \beta^6$ below, which are all in $P \cap F$. (From now on, for the sake of brevity, we only explicitly list the nonzero components of each such point.)

1. Let β^1 be such that $\beta^1_{i,j} = \beta^1_{j,i} = 1$.
2. Fix $(l,k) \notin \{(i,j),(j,i)\}$ and let β^2 be such that $\beta^2_{i,j} = \beta^2_{j,i} = \beta^2_{l,k} = \beta^2_{k,l} = 1$.

3. Fix $l \notin \{i,j\}$ and let β^3 be such that $\beta^3_{i,j} = \beta^3_{j,i} = \beta^3_{l,i} = \beta^3_{j,l} = \beta^3_{l,j} = 1$.
4. Fix $l \notin \{i,j\}$ and let β^4 such that $\beta^4_{i,j} = \beta^4_{j,i} = \beta^4_{l,i} = \beta^4_{i,l} = \beta^4_{l,j} = 1$.
5. Fix $l \notin \{i,j\}$ and let β^5 be such that $\beta^5_{i,j} = \beta^5_{j,l} = \beta^5_{l,i} = 1$.
6. If $|V| \geq 4$, fix $l,k \notin \{i,j\}$ and let β^6 be such that $\beta^6_{i,j} = \beta^6_{j,l} = \beta^6_{l,k} = \beta^6_{k,j} = 1$.

By successively substituting these points in (41), one concludes that the equation of the hyperplane is of the form $\beta_{i,j} = 1$, up to a multiplicative constant. $\square$

4.2.3 Return inequalities

Theorem 12. *For all $(i,j) \in A$ and for all $S \subseteq V$ such that $i \in S, j \in V \setminus S$, the return inequality*

$$\beta_{i,j} \leq \sum_{(l,k) \in A: l \in V \setminus S, k \in S} \beta_{l,k}$$

defines a facet of P^*.

Proof. Fix $(i,j) \in A$ and S such that $i \in S, j \in V \setminus S$. Let F be the face of P^* defined as

$$F = \left\{ \beta \in P^* : \beta_{i,j} = \sum_{(l,k) \in A: l \in V \setminus S, k \in S} \beta_{l,k} \right\},$$

and consider the following points $\beta^1, \ldots, \beta^{14}$:

1. $\beta^1 = 0$.
2. Let β^2 be such that $\beta^2_{i,j} = \beta^2_{j,i} = 1$.
3. If $|S| \geq 2$, fix $k \in S, k \neq i$, let β^3 be such that $\beta^3_{i,j} = \beta^3_{j,k} = \beta^3_{k,i} = 1$, and let $\beta^{3'}$ be such that $\beta^{3'}_{i,j} = \beta^{3'}_{j,k} = \beta^{3'}_{k,i} = \beta^{3'}_{i,k} = 1$.
4. If $|S| \geq 2$, fix $k \in S, k \neq i$, and let β^4 be such that $\beta^4_{i,k} = \beta^4_{k,i} = 1$.
5. If $|S| \geq 3$, fix $h,k \in S, k \neq i, h \neq i$, and let β^5 be such that $\beta^5_{i,k} = \beta^5_{k,h} = \beta^5_{h,i} = 1$.
6. If $|V \setminus S| \geq 2$, fix $l \in V \setminus S, l \neq j$, let $\beta^6_{i,j} = \beta^6_{j,l} = \beta^6_{l,i} = 1$, and let $\beta^{6'}_{i,j} = \beta^{6'}_{j,l} = \beta^{6'}_{l,j} = \beta^{6'}_{l,i} = 1$.
7. If $|V \setminus S| \geq 2$, fix $l \in V \setminus S, l \neq j$, and let $\beta^7_{j,l} = \beta^7_{l,j} = 1$.
8. If $|V \setminus S| \geq 3$, fix $l,k \in V \setminus S, l \neq j, k \neq j$, and let $\beta^8_{j,l} = \beta^8_{l,k} = \beta^8_{k,j} = 1$.
9. If $|S| \geq 2$, fix $k \in S, k \neq i$, and let $\beta^9_{i,j} = \beta^9_{j,i} = \beta^9_{i,k} = \beta^9_{k,j} = 1$.

10. If $|V \setminus S| \geq 2$, fix $l \in V \setminus S, l \neq j$, and let $\beta^{10}_{i,j} = \beta^{10}_{j,i} = \beta^{10}_{i,l} = \beta^{10}_{l,j} = 1$.
11. If $|S| \geq 2$ and $|V \setminus S| \geq 2$, fix $k \in S, k \neq i$, fix $l \in V \setminus S, l \neq j$, and let $\beta^{11}_{i,j} = \beta^{11}_{j,i} = \beta^{11}_{i,k} = \beta^{11}_{k,l} = \beta^{11}_{l,j} = 1$.
12. If $|S| \geq 2$, fix $k \in S, k \neq i$, and let $\beta^{12}_{i,j} = \beta^{12}_{j,k} = \beta^{12}_{k,i} = 1$.
13. If $|V \setminus S| \geq 2$, fix $l \in V \setminus S, l \neq j$, and let $\beta^{13}_{i,j} = \beta^{13}_{j,l} = \beta^{13}_{l,i} = 1$.
14. If $|S| \geq 2$ and $|V \setminus S| \geq 2$, fix $k \in S, k \neq i$, fix $l \in V \setminus S, l \neq j$, and let $\beta^{14}_{i,j} = \beta^{14}_{j,l} = \beta^{14}_{l,k} = \beta^{14}_{k,i} = 1$.

Note that all the points $\beta^1, \ldots, \beta^{14}$ are in F. Suppose now that F is included in a hyperplane defined by the equation

$$\sum_{(u,v) \in A} b_{u,v} \beta_{u,v} = b_0. \tag{42}$$

By successively substituting the points $\beta^1, \ldots, \beta^{14}$ in this equation, one can easily conclude that, up to a multiplicative constant, (42) is equivalent to the equation defining F. This proves that F is a facet of P^*. □

We have numerically verified that when $|V| = 3$, the bound inequalities and the return inequalities completely describe the cycle selection polytope P^*. In the following sections, we introduce several additional classes of facet-defining inequalities for the case where $|V| \geq 4$.

4.2.4 Out-star inequalities

Let $t \in \mathbb{N}$, let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\}$ be a subset of arcs, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $1 \leq |I| \leq |J| = t$ (meaning that $j_1, j_2, \ldots, j_t$ are pairwise distinct, but $i_1, i_2, \ldots, i_t$ are not necessarily distinct), so that (V, E) is a collection of disjoint out-stars: in (V, E), each vertex of I has indegree 0 and outdegree at least 1, whereas each vertex of J has indegree 1 and outdegree 0. Let p and q be two distinct vertices not in $I \cup J$. Then, we can define two *out-star inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p,q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}, \tag{43}$$

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p,q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q,k}. \tag{44}$$

As an illustration, Figure 3 displays an example of the structure of the arcs involved in the left-hand side of inequalities (43) and (44).

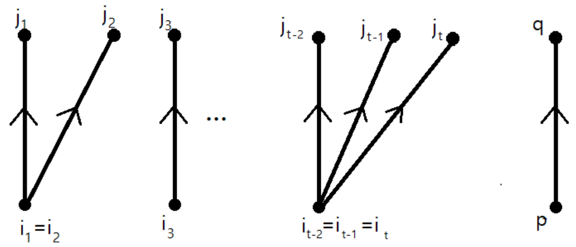

Figure 3: Structure of the arcs involved in the left-hand side of inequalities (43)-(44).

Remark 2. When the out-star inequalities are formally written for $t = 0$, they boil down to special cases of the return inequalities. On the other hand, when $t = 1$, the following point

$$\beta_{12} = \beta_{13} = \beta_{23} = \beta_{41} = \beta_{42} = 0.5, \beta_{34} = 1$$

satisfies all return inequalities of the arc formulation, but not the out-star inequality (43) with $i_1 = 1$,$j_1 = 2$, $p = 3$, $q = 4$. □

Let us now focus on the out-star inequality (43). We are first going to prove that it is valid for the cycle selection polytope P^*, and next that it is facet defining for P^*. When stating these results, we implicitly assume that $|V| \geq |I| + |J| + 2 \geq 4$ since (43) is not defined without this assumption.

Theorem 13. *Let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\} \subseteq A$, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $1 \leq |I| \leq |J| = t$. Let $p, q \in V \setminus (I \cup J)$, $p \neq q$. Then, the out-star inequality* (43) *is valid for P^*.*

Proof. Consider any cycle selection B containing exactly s arcs of E, say, the arcs $(i_l, j_l) \in H$ with $H = E \cap B$, $|H| = s$. So, the left-hand side of (43) is at most $s + 1$.

If $s \geq 1$, then the arcs in H cover a subset of vertices $I_H \subseteq I$ and a subset of vertices $J_H \subseteq J$, with $|I_H| \geq 1$ and $|J_H| = s$. For each vertex $j \in J_H$, the cycle selection B must contain an arc leaving j, that is, an arc of the form (j, h). All these arc are distinct, hence there are exactly s of them. Moreover, since $I \cap J = \emptyset$, every arc of H leaves I. Hence, there must also be (at least) one arc of B entering I, that is, an arc of the form (k, i) for $k \in V \setminus I$ and $i \in I$. So, in total, the right-hand side of (43) is at least $s + 1$, and the inequality holds.

If $s = 0$, then the left-hand side of (43) is exactly $\beta_{p,q}$. Assume that $\beta_{p,q} = 1$ (otherwise, the inequality is trivially satisfied). There must be an arc of B entering p, say (h, p). The vertex h is either in I, or in J, or in $V \setminus (I \cup J)$. In the first case, since (h, p) leaves I, there must be an arc entering I, that is, an arc of the form (k, i), $k \notin I$, $i \in I$. So, in all three cases, the right-hand side of (43) is at least 1, and this completes the proof. □

Theorem 14. *Let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\} \subseteq A$, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $1 \leq |I| \leq |J| = t$. Let $p, q \in V \setminus (I \cup J)$. Then, the out-star inequality* (43) *defines a facet of P^*.*

Proof. Consider the equation

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} = \sum_{k\in V\setminus I} \sum_{i\in I} \beta_{k,i} + \sum_{j\in J} \sum_{k\in V} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{k,p}. \tag{45}$$

Let F be the face of P^* defined as $F = \{\beta \in P^* : \beta \text{ satisfies } (45)\}$, and suppose that F is included in a hyperplane defined by the equation

$$\sum_{(u,v)\in A} b_{u,v}\beta_{u,v} = b_0. \tag{46}$$

We must show that, up to a multiplicative constant, the equation (46) is identical to (45).

Hereunder, as usual, we consider a list of points $\beta \in P$ defined by their nonzero components. We denote as $e^{(u,v)}$ the unit vector with $e^{(u,v)}_{u,v} = 1$.

1. Since $\beta^1 = 0 \in F$, we get $b_0 = 0$.
2. Let β^2 be such that $\beta^2_{p,q} = \beta^2_{q,p} = 1$. The point β^2 is in F, and therefore it satisfies equation (46). This implies that $b_{p,q} = -b_{q,p}$.

Fix $l \in V, l \notin I \cup J \cup \{p, q\}$.

3. Let $\beta^3_{q,l} = \beta^3_{l,q} = 1$. Since $\beta^3 \in F$, $b_{q,l} = -b_{l,q}$.
4. Let $\beta^4_{p,q} = \beta^4_{q,l} = \beta^4_{l,p} = 1$. This point is in F because it defines a 3-cycle and because it makes both sides of equation (45) equal to 1. Next, let $\beta^{4'} = \beta^4 + e^{(l,q)}$. Again, $\beta^{4'}$ is in F. Since β^4 and $\beta^{4'}$ only differ in their component (l, q), equation (46) implies that $b_{l,q} = 0$. From point 3 above, we also obtain $b_{q,l} = 0$, and from (46), $b_{l,p} = -b_{p,q}$.

So, from points 3 and 4, we know that $b_{q,l} = b_{l,q} = 0$ and $b_{l,p} = b_{q,p} = -b_{p,q}$ for all $l \notin I \cup J \cup \{p, q\}$. Now, fix a pair $(i_s, j_s), 1 \leq s \leq t$, and fix $l \notin I \cup J \cup \{p, q\}$.

5. Let $\beta^5_{i_s,j_s} = \beta^5_{j_s,p} = \beta^5_{p,q} = \beta^5_{q,l} = \beta^5_{l,i_s} = 1$. The point β^5 is in F because it defines a 5-cycle and it makes both sides of (45) equal to 2. Hence it satisfies equation (46).
   a) The point $\beta^5 + e^{(i_s,l)}$ is in F and by comparison with β^5, it immediately follows that $b_{i_s,l} = 0$ for all $i_s \in I$, for all $l \notin I \cup J \cup \{p, q\}$.
   b) The point $\beta^5 + e^{(l,j_s)}$ is in F (it defines a cycle selection which is the union of a 5-cycle and a 4-cycle), and hence $b_{l,j_s} = 0$ for all $l \notin I \cup J \cup \{p, q\}$, for all $j_s \in J$.
   c) The point $\beta^5 + e^{(i_s,p)}$ is in F, and hence $b_{i_s,p} = 0$ for all $i_s \in I$.
   d) The point $\beta^5 + e^{(i_s,q)}$ is in F, and hence $b_{i_s,q} = 0$ for all $i_s \in I$.
   e) The point $\beta^5 + e^{(p,j_s)}$ is in F, and hence $b_{p,j_s} = 0$ for all $j_s \in J$.
   f) The point $\beta^5 + e^{(q,j_s)}$ is in F, and hence $b_{q,j_s} = 0$ for all $j_s \in J$.

g) The point $\beta^5 + e^{(p,l)}$ is in F, and hence $b_{p,l} = 0$ for all $l \notin I \cup J \cup \{p, q\}$.

Point 5 has established that all coefficients $b_{i_s,l}$ and b_{l,j_s} are zero, except possibly when $l \in I \cup J$. The coefficients of the variables $\beta_{i_s,i_r}, \beta_{i_s,j_r}, \beta_{j_r,j_s}, \beta_{j_r,i_s}$ for $r \neq s$ will be taken care of at the end of the proof.

6. Let $\beta^6_{i_s,p} = \beta^6_{p,q} = \beta^6_{q,l} = \beta^6_{l,i_s} = 1$. The point β^6 is in F. Since we already know that $b_{i_s,p} = b_{q,l} = 0$, we can conclude $b_{l,i_s} = -b_{p,q}$ for all $l \notin I \cup J \cup \{p, q\}$, for all $i_s \in I$.

7. Let $\beta^7_{i_s,p} = \beta^7_{p,q} = \beta^7_{q,i_s} = 1$. Again, the point β^7 is in F and since $b_{i_s,p} = 0$, we obtain $b_{q,i_s} = -b_{p,q}$ for all $i_s \in I$.

8. Let $\beta^8_{j_s,p} = \beta^8_{p,q} = \beta^8_{q,j_s} = 1$. Since β^8 is in F, we obtain $b_{j_s,p} = -b_{p,q}$ for all $j_s \in J$.

9. Let $\beta^9_{i_s,j_s} = \beta^9_{j_s,p} = \beta^9_{p,q} = \beta^9_{q,i_s} = 1$. The point β^9 is in F because it defines a 4-cycle and it makes both sides of equation (45) equal to 2.

   Since we know that $b_{j_s,p} = b_{q,i_s} = -b_{p,q}$, it follows that $b_{i_s,j_s} = b_{p,q}$ for all pairs (i_s, j_s).

10. Let $\beta^{10}_{i_s,p} = \beta^{10}_{i_s,j_s} = \beta^{10}_{j_s,l} = \beta^{10}_{l,i_s} = \beta^{10}_{p,q} = \beta^{10}_{q,j_s} = 1$. The point β^{10} is in F. Since $b_{l,i_s} = -b_{p,q}$, $b_{i_s,j_s} = b_{p,q}$, and $b_{i_s,p} = b_{q,j_s} = 0$, we conclude $b_{j_s,l} = -b_{p,q}$ for all $l \notin I \cup J \cup \{p, q\}$, for all $j_s \in J$.

11. Let $\beta^{11}_{i_s,j_s} = \beta^{11}_{j_s,q} = \beta^{11}_{q,i_s} = \beta^{11}_{i_s,p} = \beta^{11}_{p,q} = \beta^{11}_{p,j_s} = 1$. Again, the point β^{11} is in F, and since $b_{i_s,j_s} = b_{p,q}$, $b_{q,i_s} = -b_{p,q}$, $b_{i_s,p} = b_{p,j_s} = 0$, we obtain $b_{j_s,q} = -b_{p,q}$ for all $j_s \in J$.

12. Let $\beta^{12}_{i_s,j_s} = \beta^{12}_{j_s,p} = \beta^{12}_{p,q} = \beta^{12}_{q,j_s} = \beta^{12}_{p,i_s} = 1$. The point β^{12} is in F and $b_{i_s,j_s} = b_{p,q}$, $b_{j_s,p} = -b_{p,q}$, $b_{q,j_s} = 0$, so that $b_{p,i_s} = -b_{p,q}$ for all $i_s \in I$.

13. Let $\beta^{13}_{i_s,j_s} = \beta^{13}_{j_s,i_s} = \beta^{13}_{q,j_s} = \beta^{13}_{p,q} = \beta^{13}_{i_s,p} = 1$. The point β^{13} is in F (note that $\beta^{13}_{j_s,i_s}$ contributes for two units to the right-hand side of equation (45)). Since $b_{i_s,p} = b_{q,j_s} = 0$ and $b_{i_s,j_s} = b_{p,q}$, we derive $b_{j_s,i_s} = -2b_{p,q}$ for all pairs (i_s, j_s).

Fix now $k, l \in V \setminus (I \cup J \cup \{p, q\})$.

14. Let $\beta^{14}_{l,k} = \beta^{14}_{k,l} = 1$. The point β^{14} is in F and it follows that $b_{l,k} = -b_{k,l}$ for all $k, l \in V \setminus (I \cup J \cup \{p, q\})$.

15. Let $\beta^{15}_{i_s,j_s} = \beta^{15}_{j_s,p} = \beta^{15}_{p,k} = \beta^{15}_{k,l} = \beta^{15}_{l,i_s} = \beta^{15}_{p,q} = \beta^{15}_{q,j_s} = 1$. The point β^{15} is in F, and since $b_{i_s,j_s} = b_{p,q}$, $b_{j_s,p} = b_{l,i_s} = -b_{p,q}$, $b_{p,k} = b_{q,j_s} = 0$, we obtain $b_{k,l} = 0$ for all $k, l \in V \setminus (I \cup J \cup \{p, q\})$.

For the rest of the proof, let $r, s \in \{1, \ldots, t\}$ with $r \neq s$.

16. Let i_r, i_s be two distinct vertices in I, and let $\beta^{16}_{i_r,i_s} = \beta^{16}_{i_s,p} = \beta^{16}_{p,q} = \beta^{16}_{q,i_r} = 1$. The point β^{16} is in F and $b_{i_s,p} = 0$, $b_{q,i_r} = -b_{p,q}$. Therefore, $b_{i_r,i_s} = 0$ for all distinct $i_r, i_s \in I$.

17. Let (i_s, j_s) be an arc in E, and let $i_r \in I$, $i_r \neq i_s$. Let $\beta^{17}_{i_r,i_s} = \beta^{17}_{i_s,j_s} = \beta^{17}_{i_r,j_s} = \beta^{17}_{j_s,p} = \beta^{17}_{p,q} = \beta^{17}_{q,i_r} = 1$. The point β^{17} defines the union of a 5-cycle and of a 4-cycle, and it is in F. From $b_{i_r,i_s} = 0$, $b_{i_s,j_s} = b_{p,q}$, $b_{j_s,p} = b_{q,i_r} = -b_{p,q}$, we derive $b_{i_r,j_s} = 0$ for all $i_r \in I, j_s \in J$, $i_r \neq i_s$.

18. Let (i_s, j_s) be an arc in E, and let $i_r \in I$, $i_r \neq i_s$. Let $\beta^{18}_{i_s,j_s} = \beta^{18}_{j_s,i_r} = \beta^{18}_{i_r,p} = \beta^{18}_{p,q} = \beta^{18}_{q,j_s} = \beta^{18}_{i_r,i_s} = 1$. The point β^{18} is in F because $\beta^{18}_{j_s,i_r}$ contributes for two units to the right-hand side of equation (45). From $b_{i_s,j_s} = b_{p,q}$ and $b_{i_r,p} = b_{q,j_s} = b_{i_r,i_s} = 0$, we deduce $b_{j_s,i_r} = -2b_{p,q}$ for all $r \neq s$.

19. Finally, let j_r, j_s be two distinct vertices in J, with $(i_r, j_r) \in E$, $(i_s, j_s) \in E$. If $i_r \neq i_s$, let $\beta^{19}_{i_s,j_s} = \beta^{19}_{j_s,j_r} = \beta^{19}_{j_r,p} = \beta^{19}_{p,q} = \beta^{19}_{q,i_r} = \beta^{19}_{i_r,i_s} = \beta^{19}_{i_r,j_r} = 1$. The point β^{19} is in F: it defines the a union of a 6-cycle and of a 4-cycle, and it makes both sides of equation (45) equal to 3. Since $b_{i_s,j_s} = b_{i_r,j_r} = b_{p,q}$, $b_{j_r,p} = b_{q,i_s} = -b_{p,q}$, and $b_{i_r,i_s} = 0$, we obtain $b_{j_s,j_r} = -b_{p,q}$.
If $i_r = i_s$, the same reasoning applies by simply disregarding the arc (i_r, i_s) in the definition of β^{19}. So, in all cases, $b_{j_s,j_r} = -b_{p,q}$ for all distinct $j_s, j_r \in J$.

The previous observations imply that the equation (46) is identical to (45), up to a multiplicative constant $b_{p,q}$, and hence F is a facet of the convex hull polytope P^*. $\square$

Theorem 13 and Theorem 14 can be extended to deal with the case of the out-star inequalities (44).

Theorem 15. *Let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\} \subseteq A$, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $1 \leq |I| \leq |J| = t$. Let $p, q \in V \setminus (I \cup J)$. Then, the out-star inequality* (44) *is valid and defines a facet of P^*.*

Proof. The proof is similar to the previous ones. In particular, points 1-2-7-8-9-11-12-13-14-15-16-17-18-19 in the proof of Theorem 14 can be handled in exactly the same way. The remaining cases involve predecessors of p other than vertices in $I \cup J \cup \{q\}$ and/or successors of q other than vertices in $I \cup J \cup \{p\}$. To deal with these cases, denote as F the face of P^* defined by equation (44). Fix $l \in V, l \notin I \cup J \cup \{p, q\}$.

3'. With $\beta^3_{p,l} = \beta^3_{l,p} = 1$, we can conclude that $b_{p,l} = -b_{l,p}$.

4'. Let $\beta^4_{p,q} = \beta^4_{q,l} = \beta^4_{l,p} = 1$, and let $\beta^{4'}_{p,q} = \beta^{4'}_{q,l} = \beta^{4'}_{l,p} = \beta^{4'}_{p,l} = 1$. Both β^4 and $\beta^{4'}$ are in the face F. It easily follows that $b_{p,l} = b_{l,p} = 0$ and $b_{q,l} = -b_{p,q}$ for all $l \notin I \cup J \cup \{p, q\}$.

Now, fix a pair $(i_s, j_s), 1 \leq s \leq t$, and fix $l \notin I \cup J \cup \{p, q\}$.

5'. Let $\beta^5_{i_s,j_s} = \beta^5_{j_s,l} = \beta^5_{l,p} = \beta^5_{p,q} = \beta^5_{q,i_s} = 1$. The point β^5 is in F.

a) Since $\beta^5 + e^{(i_s,l)} \in F$, it follows that $b_{i_s,l} = 0$ for all $i_s \in I$, for all $l \notin I \cup J \cup \{p, q\}$.

b) $\beta^5 + e^{(i_s,p)} \in F$, hence $b_{i_s,p} = 0$ for all $i_s \in I$.

c) $\beta^5 + e^{(i_s,q)} \in F$, hence $b_{i_s,q} = 0$ for all $i_s \in I$

d) $\beta^5 + e^{(l,j_s)} \in F$, hence $b_{l,j_s} = 0$ for all $j_s \in J$, for all $l \notin I \cup J \cup \{p,q\}$.

e) $\beta^5 + e^{(p,j_s)} \in F$, hence $b_{p,j_s} = 0$ for all $j_s \in J$.

f) $\beta^5 + e^{(q,j_s)} \in F$, hence $b_{q,j_s} = 0$ for all $j_s \in J$.

g) $\beta^5 + e^{(l,q)} \in F$, hence $b_{l,q} = 0$ for all $l \notin I \cup J \cup \{p,q\}$.

6'. With $\beta^6_{l,p} = \beta^6_{p,q} = \beta^6_{q,j_s} = \beta^6_{j_s,l} = 1$, we can conclude that $b_{j_s,l} = -b_{p,q}$ for all $l \notin I \cup J \cup \{p,q\}$, for all $j_s \in J$.

10'. Finally, let $\beta^{10}_{i_s,p} = \beta^{10}_{i_s,j_s} = \beta^{10}_{j_s,l} = \beta^{10}_{l,i_s} = \beta^{10}_{p,q} = \beta^{10}_{q,j_s} = 1$ (as in the proof of Theorem 14). Since $\beta^{10} \in F$, we can conclude that $b_{l,i_s} = -b_{p,q}$ for all $i_s \in I$, for all $l \notin I \cup J \cup \{p,q\}$. ☐

4.2.5 In-star inequalities

Symmetrically with the case of out-star inequalities, we can introduce the class of *in-star inequalities*. With the same notations as in Section 4.2.4, assume that $1 \leq |J| \leq |I| = t$ (meaning that $i_1, i_2, \ldots, i_t$ are all distinct, but $j_1, j_2, \ldots, j_t$ are not necessarily distinct), so that (V,E) is a collection of disjoint in-stars: in (V,E), each vertex of I has indegree 0 and outdegree 1, each vertex of J has outdegree 0 and indegree at least 1. Then, the in-star inequalities are defined as

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k\in V}\sum_{i\in I} \beta_{k,i} + \sum_{j\in J}\sum_{k\in V\setminus J} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{k,p}, \tag{47}$$

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k\in V}\sum_{i\in I} \beta_{k,i} + \sum_{j\in J}\sum_{k\in V\setminus J} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{q,k}. \tag{48}$$

Theorem 16. *Let $E = \{(i_1,j_1),(i_2,j_2),\ldots,(i_t,j_t)\} \subseteq A$, and let $I = \{i_1,i_2,\ldots,i_t\}$, $J = \{j_1,j_2,\ldots,j_t\}$. Assume that $I \cap J = \emptyset$ and $1 \leq |J| \leq |I| = t$. Let $p,q \in V \setminus (I \cup J)$. Then, the in-star inequalities* (47)-(48) *are valid and define facets of P^*.*

This theorem can be established by mimicking the proofs in Section 4.2.4. But we prefer to propose here a more insightful argument.

Proof. Define the bijection rev which associates with each $\beta \in \mathbb{R}^{|A|}$ another point $rev(\beta) = \beta^r \in \mathbb{R}^{|A|}$ such that $\beta^r_{i,j} = \beta_{j,i}$ for all $(i,j) \in A$. Intuitively, when $\beta \in \{0,1\}^{|A|}$, then rev simply reverses the direction of each arc in the support of β (remember that we consider here a complete digraph $G = (V,A)$). In particular, if β defines a cycle selection, then so does $rev(\beta)$. As a consequence, rev maps P and P^* onto themselves: $rev(P) = P$ and $rev(P^*) = P^*$.

Consider now the in-star inequality (47) associated with E and (p,q), and let $F^{(E,p,q)}$ be the face that it defines. Moreover, consider the out-star inequality (44) associated with $E^r = \{(j_1,i_1),(j_2,i_2),\ldots,(j_t,i_t)\}$ and with the arc (q,p), that is:

$$\sum_{l=1}^{t} \beta_{j_l,i_l} + \beta_{q,p} \leq \sum_{k\in V\setminus J}\sum_{j\in J} \beta_{k,j} + \sum_{i\in I}\sum_{k\in V} \beta_{i,k} + \sum_{k\in V\setminus(J\cup I)} \beta_{p,k}. \tag{49}$$

Let $F^{(E^r,q,p)}$ be the face of P^* defined by (49).

If β is in $F^{(E,p,q)}$, that is, if β satisfies (47) as an equality, then it is immediately obvious that $rev(\beta)$ satisfies (49) as an equality, and hence $rev(\beta)$ is in $F^{(E^r,q,p)}$. The converse relation holds as well, meaning that $rev(F^{(E,p,q)}) = F^{(E^r,q,p)}$.

Since we know from Theorem 15 that $F^{(E^r,q,p)}$ is a facet of P^*, it follows that $F^{(E,p,q)}$ also is a facet of P^*. □

4.2.6 Path inequalities

Let $I = \{i_1,i_2,\ldots,i_t\}$ and $J = \{j_1,j_2,\ldots,j_t\}$ be two subsets of pairwise distinct vertices, $I\cap J = \emptyset$ and $|I| = |J| = t$. Let p and q be two distinct vertices not in $I\cup J$. Then, we define the *path inequality*

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \sum_{l=1}^{t-1} \beta_{i_l,j_{l+1}} + \beta_{p,q} \leq \sum_{k\in V}\sum_{i\in I} \beta_{k,i} + \sum_{j\in J}\sum_{k\in V} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{k,p}. \tag{50}$$

Observe that except for (p,q), the arcs involved in the left-hand side of (50) define a non directed path $\pi = (j_1,i_1,j_2,i_2,\ldots,j_t,i_t)$. In this path, each arc leaves a vertex of I and enters a vertex of J. As an illustration, Figure 4 displays an example of the structure of the arcs involved in the left-hand side of inequality (50).

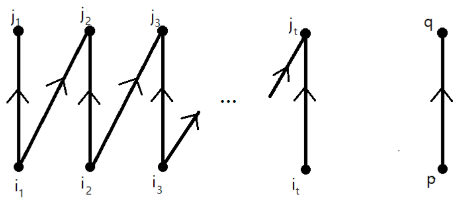

Figure 4: Structure of the arcs involved in the left-hand side of inequality (50).

We will prove that inequality (50) is valid and that it is facet defining for P^*.

Theorem 17. *Let $I = \{i_1,i_2,\ldots,i_t\}$ and $J = \{j_1,j_2,\ldots,j_t\}$ be two subsets of vertices with $I\cap J = \emptyset$ and $1 \leq |I| = |J| = t$. Let $p,q \in V\setminus(I\cup J)$, $p\neq q$. Then, the path inequality* (50) *is valid for P^*.*

Proof. Assume that we have a cycle selection B containing exactly s arcs of the path π, say, the arcs $(k,l) \in H$, $|H| = s$. So, the left-hand side of (50) is at most $s+1$.

If $s \geq 1$, then the arcs in H form a collection of disjoint subpaths. These subpaths contain a subset of vertices $I_H \subseteq I$ and a subset of vertices $J_H \subseteq J$, and there holds $|I_H| + |J_H| \geq s+1$. For each vertex $i \in I_H$, the cycle selection B must contain an arc entering i. And for each vertex $j \in J_H$, B must contain an arc leaving j. So, the right-hand side of (50) is at least $|I_H| + |J_H| \geq s+1$, which implies that (50) is satisfied.

If $s = 0$, then the left-hand side of (50) is exactly $\beta_{p,q}$. So, assume that $\beta_{p,q} = 1$. There must be an arc of B entering p, say (h,p). The vertex h is either $i_l \in I$ (in which case B must also contain an arc (k, i_l) entering i_l), or $j_l \in J$, or h is not in $I \cup J$. In all three cases, the right-hand side of (50) is at least 1, which completes the proof. □

Theorem 18. *Let $I = \{i_1, i_2, \ldots, i_t\}$ and $J = \{j_1, j_2, \ldots, j_t\}$ be two subsets of vertices with $1 \leq I \cap J = \emptyset$ and $|I| = |J| = t$. Let $p, q \in V \setminus (I \cup J)$, $p \neq q$. Then, the path inequality (50) defines a facet of P^*.*

Proof. Let F be the face of P^* defined by

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \sum_{l=1}^{t-1} \beta_{i_l,j_l+1} + \beta_{p,q} = \sum_{k\in V} \sum_{i\in I} \beta_{k,i} + \sum_{j\in J} \sum_{k\in V} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{k,p}, \tag{51}$$

and suppose that F is included in a hyperplane

$$\sum_{(u,v)\in A} b_{u,v}\beta_{u,v} = b_0. \tag{52}$$

The first 15 conclusions in the proof of Theorem 14 can be drawn here in exactly the same way. To see this, it is enough to notice that the cycle selections B^n defined by the points β^n $(n = 1, \ldots, 15)$ in the proof of Theorem 14 do not contain any arc (i_s, j_r) with $s \neq r$. Hence the left-hand sides of (45) and (51) are equal for all these points. Moreover, the cycle selections B^n do not contain any arc of the form (i_s, i_r) with $s \neq r$, so that the right-hand sides of (45) and (51) are also equal in all cases.

For the remainder of the proof, we have to determine the coefficients of the variables representing the arcs with both vertices in $I \cup J$ (except for the pairs (i_s, j_s) and (j_s, i_s), since we already know the coefficients b_{i_s,j_s} and b_{j_s,i_s} as functions of $b_{p,q}$). Let us consider two distinct pairs (i_s, j_s) and (i_r, j_r) with $s < r$ (note that their order matters, because of the definition of the path inequalities). We have to find the value of the coefficients b_{i_s,j_r}, b_{j_r,i_s}, b_{i_s,i_r}, b_{j_s,j_r}, b_{j_r,j_s}, b_{j_s,i_r}, b_{i_r,j_s}, b_{i_r,i_s}.

For the coefficients b_{i_s,j_r} and b_{j_r,i_s}, we further have to deal with two subcases:

(i) s and r are consecutive, i.e., $r = s+1$; in that case, we must show that $b_{i_s,j_{s+1}} = b_{p,q}$ and $b_{j_{s+1},i_s} = -2b_{p,q}$;

(ii) s and r are not consecutive, i.e., $r > s+1$; in that case, we must show that $b_{i_s,j_r} = 0$ and $b_{j_r,i_s} = -2b_{p,q}$.

For all the other coefficients, a single analysis will cover both situations.

Case $r = s + 1$.

16. Let β^{16} be such that $\beta^{16}_{i_s,j_{s+1}} = \beta^{16}_{j_{s+1},p} = \beta^{16}_{p,q} = \beta^{16}_{q,i_s} = 1$ and $\beta^{16}_{u,v} = 0$ for all other arcs $(u, v) \in A$. This point is in F because it makes both sides of (51) equal to 2. From (52), we get: $b_{i_s,j_{s+1}} + b_{j_{s+1},p} + b_{p,q} + b_{q,i_s} = b_0$. Since we already know that $b_0 = 0$ and $b_{j_{s+1},p} = b_{q,i_s} = -b_{p,q}$, we can conclude that $b_{i_s,j_{s+1}} = b_{p,q}$.
17. Next, let $\beta^{17}_{i_s,j_{s+1}} = \beta^{17}_{j_{s+1},i_s} = \beta^{17}_{q,j_{s+1}} = \beta^{17}_{p,q} = \beta^{17}_{i_s,p} = 1$. The point β^{17} is in F (it makes again both sides of (51) equal to 2). Hence: $b_{i_s,j_{s+1}} + b_{j_{s+1},i_s} + b_{q,j_{s+1}} + b_{p,q} + b_{i_s,p} = 0$. Since $b_{i_s,p} = b_{q,j_{s+1}} = 0$ and $b_{i_s,j_{s+1}} = b_{p,q}$, we can conclude $b_{j_{s+1},i_s} = -2b_{p,q}$.

Case $r > s + 1$.

18. Let $\beta^{18}_{p,q} = \beta^{18}_{j_r,i_s} = \beta^{18}_{i_h,j_h} = \beta^{18}_{i_s,j_{s+1}} = \beta^{18}_{i_h,j_{h+1}} = \beta^{18}_{q,i_h} = \beta^{18}_{j_h,p} = 1$ for all h with $s < h < r$.

    The point β^{18} defines a cycle selection which is the union of the following 4-cycles: $\{(i_h, j_h), (j_h, p), (p, q), (q, i_h)\}$ for $s < h < r$, $\{(i_h, j_{h+1}), (j_{h+1}, p), (p, q), (q, i_h)\}$ for $s < h < r-1$, and of the 6-cycle $\{(i_s, j_{s+1}), (j_{s+1}, p), (p, q), (q, i_{r-1}), (i_{r-1}, j_r), (j_r, i_s)\}$. Moreover, β^{18} is in F as well, as it makes both sides of (51) equal to $2(r - s)$.

    From (52), we get:

$$b_{p,q} + b_{j_r,i_s} + \sum_{h=s+1}^{r-1} b_{i_h,j_h} + \sum_{h=s}^{r-1} b_{i_h,j_{h+1}} + \sum_{h=s+1}^{r-1} b_{q,i_h} + \sum_{h=s+1}^{r-1} b_{j_h,p} = 0.$$

    Since we know that $b_{i_h,j_h} = b_{i_h,j_{h+1}} = -b_{q,i_h} = -b_{j_h,p} = b_{p,q}$ for $s \le h < r$, we can conclude that $b_{j_r,i_s} = -2b_{p,q}$.

    Moreover, the point $\beta^{18} + e^{(i_s,j_r)}$ also is in F, and hence $b_{i_s,j_r} = 0$.

Remaining coefficients b_{i_s,i_r}, b_{j_s,j_r}, b_{j_r,j_s}, b_{j_s,i_r}, b_{i_r,j_s}, b_{i_r,i_s}.

19. Let $\beta^{19}_{p,q} = \beta^{19}_{i_s,i_r} = 1$, $\beta^{19}_{i_h,j_h} = \beta^{19}_{j_h,p} = 1$ for $s \le h \le r$, $\beta^{19}_{i_h,j_{h+1}} = \beta^{19}_{q,i_h} = 1$ for $s \le h < r$.

    The point β^{19} defines a cycle selection which is the union of the following 4-cycles and 5-cycle: $\{(i_h, j_h), (j_h, p), (p, q), (q, i_h)\}$ for $s \le h < r$, $\{(i_h, j_{h+1}), (j_{h+1}, p), (p, q), (q, i_h)\}$ for $s \le h < r$, and $\{(i_s, i_r), (i_r, j_r), (j_r, p), (p, q), (q, i_s)\}$. Moreover, $\beta^{19} \in F$.

    From (52), we get:

$$b_{p,q} + b_{i_s,i_r} + \sum_{h=s}^{r} b_{i_h,j_h} + \sum_{h=s}^{r} b_{j_h,p} + \sum_{h=s}^{r-1} b_{i_h,j_{h+1}} + \sum_{h=s}^{r-1} b_{q,i_h} = 0.$$

    Since $b_{i_h,j_h} = b_{i_h,j_{h+1}} = -b_{q,i_h} = -b_{j_h,p} = b_{p,q}$ for $s \le h \le r$, we obtain $b_{i_s,i_r} = -b_{p,q}$.

20. Let $\beta^{20}_{p,q} = \beta^{20}_{j_s,j_r} = 1$, $\beta^{20}_{i_h,j_h} = \beta^{20}_{i_h,j_{h+1}} = \beta^{20}_{q,i_h} = 1$ for $s \leq h < r$, $\beta^{20}_{j_h,p} = 1$ for $s < h \leq r$.

    The point β^{20} defines a cycle selection which is the union of the following cycles: $\{(i_h, j_h), (j_h, p), (p, q), (q, i_h)\}$ for $s < h < r$, $\{(i_h, j_{h+1}), (j_{h+1}, p), (p, q), (q, i_h)\}$ for $s \leq h < r$, and $\{(i_s, j_s), (j_s, j_r), (j_r, p), (p, q), (q, i_s)\}$. Since β^{20} is in F, we obtain

    $$b_{p,q} + b_{j_s,j_r} + \sum_{h=s}^{r-1} b_{i_h,j_h} + \sum_{h=s}^{r-1} b_{i_h,j_{h+1}} + \sum_{h=s}^{r-1} b_{q,i_h} + \sum_{h=s+1}^{r} b_{j_h,p} = 0.$$

    We know that $b_{i_h,j_h} = b_{i_h,j_{h+1}} = -b_{q,i_h} = -b_{j_{h+1},p} = b_{p,q}$ for $s \leq h < r$, and hence $b_{j_s,j_r} = -b_{p,q}$.

21. Let $\beta^{21}_{p,q} = \beta^{21}_{j_r,j_s} = \beta^{21}_{q,j_r} = \beta^{21}_{i_h,j_h} = \beta^{21}_{i_h,j_{h+1}} = \beta^{21}_{p,i_h} = \beta^{21}_{j_h,p} = 1$ for $s \leq h < r$.

    Then, β^{21} defines a cycle selection, as the union of the following cycles: $\{(i_h, j_h), (j_h, p), (p, i_h)\}$ for $s \leq h < r$, $\{(i_h, j_{h+1}), (j_{h+1}, p), (p, i_h)\}$ for $s \leq h < r$, $\{(p, q), (q, j_r), (j_r, j_s), (j_s, p)\}$.

    Since $\beta^{21} \in F$, we get:

    $$b_{p,q} + b_{j_r,j_s} + b_{q,j_r} + \sum_{h=s}^{r-1} b_{i_h,j_h} + \sum_{h=s}^{r-1} b_{i_h,j_{h+1}} + \sum_{h=s}^{r-1} b_{p,i_h} + \sum_{h=s}^{r-1} b_{j_h,p} = 0.$$

    From $b_{i_h,j_h} = b_{i_h,j_{h+1}} = -b_{p,i_h} = -b_{j_h,p} = b_{p,q}$ for $s \leq h < r$ and $b_{q,j_r} = 0$, we conclude that $b_{j_r,j_s} = -b_{p,q}$.

22. Let $\beta^{22}_{p,q} = \beta^{22}_{j_s,i_r} = 1$, $\beta^{22}_{i_h,j_h} = 1$ for $s \leq h \leq r$, $\beta^{22}_{i_h,j_{h+1}} = \beta^{22}_{q,i_h} = 1$ for $s \leq h < r$, $\beta^{22}_{j_h,p} = 1$ for $s < h \leq r$. The point β^{22} defines a cycle selection which is the union of the cycles: $\{(i_h, j_h), (j_h, p), (p, q), (q, i_h)\}$ for $s < h < r$, $\{(i_h, j_{h+1}), (j_{h+1}, p), (p, q), (q, i_h)\}$ for $s \leq h < r$, and $\{(i_s, j_s), (j_s, i_r), (i_r, j_r), (j_r, p), (p, q), (q, i_s)\}$. Because $\beta^{22} \in F$, there holds:

    $$b_{p,q} + b_{j_s,i_r} + \sum_{h=s}^{r} b_{i_h,j_h} + \sum_{h=s}^{r-1} b_{i_h,j_{h+1}} + \sum_{h=s}^{r-1} b_{q,i_h} + \sum_{h=s+1}^{r} b_{j_h,p} = 0.$$

    Since $b_{i_h,j_h} = -b_{q,i_h} = -b_{j_h,p} = b_{p,q}$ for $s \leq h \leq r$, and $b_{i_h,j_{h+1}} = b_{p,q}$ for $s \leq h < r$, we obtain $b_{j_s,i_r} = -2b_{p,q}$.

23. Considering the point $\beta^{23} = \beta^{22} + e^{(i_r,j_s)} \in F$, we further derive $b_{i_r,j_s} = 0$.

24. Let $\beta^{24}_{p,q} = \beta^{24}_{i_r,i_s} = \beta^{24}_{j_r,q} = 1$, $\beta^{24}_{i_h,j_h} = 1$ for $s \leq h \leq r$, $\beta^{24}_{i_h,j_{h+1}} = \beta^{24}_{j_h,i_{h+1}} = \beta^{24}_{q,j_h} = 1$ for $s \leq h < r$, $\beta^{24}_{i_h,p} = 1$ for $s < h \leq r$.

    The point β^{24} defines a cycle selection as union of the cycles:
    $\{(i_h, j_h), (j_h, i_{h+1}), (i_{h+1}, p), (p, q), (q, j_{h-1}), (j_{h-1}, i_h)\}$ for $s < h < r$,
    $\{(i_h, j_{h+1}), (j_{h+1}, i_{h+2}), (i_{h+2}, p), (p, q), (q, j_{h-1}), (j_{h-1}, i_h)\}$ for $s < h < r - 1$,

$\{(i_s, j_s), (j_s, i_{s+1}), (i_{s+1}, p), (p, q), (q, j_{r-1}), (j_{r-1}, i_r), (i_r, i_s)\}$,
$\{(i_r, j_r), (j_r, q), (q, j_{r-1}), (j_{r-1}, i_r)\}$, and $\{(i_{r-1}, j_r), (j_r, q), (q, j_{r-2}), (j_{r-2}, i_{r-1})\}$.
Since $\beta^{24} \in F$, there holds

$$b_{p,q} + b_{i_r,i_s} + b_{j_r,q} + \sum_{h=s}^{r} b_{i_h,j_h} + \sum_{h=s}^{r-1} b_{i_h,j_{h+1}} + \sum_{h=s}^{r-1} b_{j_h,i_{h+1}} + \sum_{h=s}^{r-1} b_{q,j_h} + \sum_{h=s+1}^{r} b_{i_h,p} = 0.$$

Using $b_{i_h,j_h} = -b_{j_r,q} = b_{p,q}$ for $s \leq h \leq r$, $b_{i_h,j_{h+1}} = b_{p,q}$ for $s \leq h < r$, $b_{i_h,p} = b_{q,j_h} = 0$ for $s \leq h \leq r$, $b_{j_h,i_{h+1}} = -2b_{p,q}$ for $s \leq h < r$, we can finally conclude that $b_{i_r,i_s} = -b_{p,q}$.

This completes the proof that F is a facet of P^*. □

Just as in the case of the out-star and in-star inequalities, the following variant of the path inequality is also valid and facet-defining for the cycle selection polytope:

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \sum_{l=1}^{t-1} \beta_{i_l,j_l+1} + \beta_{p,q} \leq \sum_{k\in V} \sum_{i\in I} \beta_{k,i} + \sum_{j\in J} \sum_{k\in V} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{q,k}. \quad (53)$$

We omit the proof of this result.

4.3 Additional valid inequalities

In this section, we describe one last class of valid inequalities.

Theorem 19. *Let i, j, p, q be four distinct vertices in V. Then, the following inequality is valid for the cycle selection polytope P^*:*

$$\beta_{i,j} + \beta_{i,q} + \beta_{p,j} \leq \sum_{k\in V} \beta_{k,i} + \sum_{k\in V} \beta_{j,k} + \sum_{k\notin\{i,j\}} \beta_{q,k} + \sum_{k\notin\{i,j\}} \beta_{k,q} + \sum_{k\notin\{i,j,q\}} \beta_{k,p} + \beta_{q,j} + \beta_{i,p}. \quad (54)$$

Proof. In order to establish this result, we rely on the Chvátal-Gomory procedure. Consider the following valid inequalities for P^* (all of them, except the last one are special instances of the return inequalities (2)):

- $\beta_{i,q} \leq \sum_{k\in V} \beta_{q,k}$,
- $\beta_{i,q} \leq \sum_{k\in V} \beta_{k,i}$,
- $\beta_{p,j} \leq \sum_{k\in V} \beta_{j,k}$,
- $\beta_{p,j} \leq \sum_{k\in V} \beta_{k,p}$,
- $\beta_{i,j} \leq \sum_{l\notin\{i,q\}} \sum_{k\in\{i,q\}} \beta_{l,k}$ (i.e., the return inequality with $S = \{i, q\}$),
- $\beta_{i,j} \leq 1$.

By adding all these inequalities, dividing the result by 2, and rounding each coefficient according to the Chvátal-Gomory procedure, we obtain the inequality (54). $\square$

Remark 3. We have verified numerically that, when $n = |V| = 4$, the inequalities (54) define facets of the cycle selection polytope P^* and that, together with the star inequalities (or equivalently, with the path inequalities), they completely describe P^*. $\square$

5 Constrained cycle selections

In [Smeulders et al., 2021], the authors consider cycle selections which contain at most $\mathbf{B}$ arcs and which are unions of directed cycles of length at most $\mathbf{K}$, where $\mathbf{B}$ and $\mathbf{K}$ are two given constants. (See Section 1.2: the cycle length restriction is customary in kidney exchange models; the bound on the cardinality of the selections represents a budget constraint on the cost of crossmatch tests.)

The cardinality constraint on the number of selected arcs is easily incorporated in the arc formulation: it simply requires that

$$\sum_{(i,j)\in A} \beta_{i,j} \leq \mathbf{B}. \tag{55}$$

The cycle length constraint, however, is less natural in this formulation. ([Smeulders et al., 2021] rely on the PI formulation to express it.) Nevertheless, we can define $P(\mathbf{B},\mathbf{K})$ to be the set of $\beta \in \{0,1\}^{|A|}$ associated with $(\mathbf{B},\mathbf{K})$-constrained cycle selections in complete digraphs, and $P(\mathbf{B},\mathbf{K})^*$ to be its convex hull.

By simple inspection of the polyhedral results established in Section 4, we can observe that these results remain valid for $P(\mathbf{B},\mathbf{K})^*$ when $\mathbf{B}$ and $\mathbf{K}$ are large enough. For example, the proof of Theorem 9 does not involve any selection containing more than 4 arcs, nor any cycle of length larger than 3. It follows that $P(\mathbf{B},\mathbf{K})^*$ is full-dimensional when $\mathbf{B} \geq 4$ and $\mathbf{K} \geq 3$.

These observations are summarized in the table below. One should read that each theorem remains valid for $P(\mathbf{B},\mathbf{K})^*$ as long as $\mathbf{B} \geq \mathbf{B}_0$ and $\mathbf{K} \geq \mathbf{K}_0$.

Valid result for $P(\mathbf{B},\mathbf{K})^*$	$\mathbf{B}_0$	$\mathbf{K}_0$
Theorem 9 (Dimension)	4	3
Theorems 10–11 (Bound inequalities)	5	3
Theorem 12 (Return inequalities)	5	4
Theorems 14–16 (Star inequalities)	7	6
Theorem 18 (Path inequalities)	$5t-1$	7

6 Conclusions and perspectives

In this paper, we have introduced the definition of cycle selections and of the associated maximum weigthed cycle selection (**MWCS**) problem. To the best of our knowledge, these concepts had not been explicitly identified earlier, in spite of their rather natural definition and of their relation with fundamental graph theoretic concepts like directed cycles and circulations. We have investigated several properties of cycle selections and of the **MWCS** problem, including their computational complexity, the relation between various integer programming formulations, and the polyhedral structure of the cycle selection polytope.

As explained in Section 1.2, [Smeulders et al., 2021] have modeled cycle selections in order to handle a probabilistic variant of the kidney exchange problem formulated as a two-stage stochastic integer programming problem. In their experiments, the latter problem turned out to be very difficult to solve. In future work, we hope to be able to rely on our improved understanding of cycle selections in order to facilitate the solution of the **MWCS** problem and of the stochastic kidney exchange problem.

Remarks

- This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

- Declarations of interest: none.

References

[Abraham et al., 2007] Abraham, D., Blum, A., and Sandholm, T. (2007). Clearing algorithms for barter exchange markets: Enabling nationwide kidney exchanges. In *Proceedings of the 8th ACM Conference on Electronic Commerce*, pages 295–304, New York. ACM.

[Balas and Oosten, 2000] Balas, E. and Oosten, M. (2000). On the cycle polytope of a directed graph. *Networks*, 36(1):34,46.

[Balas and Rüdiger, 2009] Balas, E. and Rüdiger, S. (2009). On the cycle polytope of a directed graph and its relaxations. *Networks*, 54(1):47,55.

[Bang-Jensen and Gutin, 2009] Bang-Jensen, J. and Gutin, G. Z. (2009). *Digraphs: Theory, Algorithms and Applications.* Springer Monographs in Mathematics. Springer, London, second edition.

[Bauer, 1997] Bauer, P. (1997). The circuit polytope: Facets. *Mathematics of Operations Research*, 22(1):110–145.

[Bauer et al., 2002] Bauer, P., Linderoth, J. T., and Savelsbergh, M. W. P. (2002). A branch and cut approach to the cardinality constrained circuit problem. *Mathematical Programming*, 91:307–348.

[Biró et al., 2021] Biró, P., van de Klundert, J., Manlove, D., Pettersson, W., Andersson, T., Burnapp, L., Chromy, P., Delgado, P., Dworczak, P., Haase, B., Hemke, A., Johnson, R., Klimentova, X., Kuypers, D., Nanni Costa, A., Smeulders, B., Spieksma, F., Valentín, M. O., and Viana, A. (2021). Modelling and optimisation in European kidney exchange programmes. *European Journal of Operational Research*, 291(2):447–456.

[Conforti et al., 2014] Conforti, M., Cornuéjols, G., and Zambelli, G. (2014). *Integer Programming*. Springer.

[Constantino et al., 2013] Constantino, M., Klimentova, X., Viana, A., and Rais, A. (2013). New insights on integer-programming models for the kidney exchange problem. *European Journal of Operational Research*, 231(1):57–68.

[Coullard and Pulleyblank, 1989] Coullard, C. R. and Pulleyblank, W. R. (1989). On cycle cones and polyhedra. *Linear Algebra and Its Applications*, 114(C):613,640.

[Dickerson et al., 2016] Dickerson, J., Manlove, D., Plaut, B., Sandholm, T., and Trimble, J. (2016). Position-indexed formulations for kidney exchange. *arXiv.org*.

[Grötschel et al., 1981] Grötschel, M., Lovász, L., and Schrijver, A. (1981). The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 6:169–197.

[Hartmann and Özlük, 2001] Hartmann, M. and Özlük, O. (2001). Facets of the p-cycle polytope. *Discrete Applied Mathematics*, 112(1):147–178.

[Hoffman, 1960] Hoffman, A. J. (1960). Some recent applications of the theory of linear inequalities to extremal combinatorial analysis. In Bellman, R. and Hall, M., editors, *Combinatorial Analysis: Proceedings of Symposia in Applied Mathematics*, volume 10, pages 113–127, Providence. American Mathematical Society.

[Karp, 1972] Karp, R. M. (1972). Reducibility among combinatorial problems. In *Complexity of Computer Computations*, pages 85–103. Springer.

[Lam and Mak-Hau, 2020] Lam, E. and Mak-Hau, V. (2020). Branch-and-cut-and-price for the cardinality-constrained multi-cycle problem in kidney exchange. *Computers and Operations Research*, 115:104852–.

[Mak-Hau, 2017] Mak-Hau, V. (2017). On the kidney exchange problem: cardinality constrained cycle and chain problems on directed graphs: a survey of integer programming approaches. *Journal of Combinatorial Optimization*, 33:35–59.

[Mak-Hau, 2018] Mak-Hau, V. (2018). A polyhedral study of the cardinality constrained multi-cycle and multi-chain problem on directed graphs. *Computers and Operations Research*, 99:13–26.

[Roth et al., 2004] Roth, A., Ünver, M. U., and Sönmez, T. (2004). Kidney exchange. *The Quarterly Journal of Economics*, 119(2).

[Roth et al., 2007] Roth, A. E., Sönmez, T., and Ünver, M. U. (2007). Efficient kidney exchange: Coincidence of wants in markets with compatibility-based preferences. *American Economic Review*, 97(3):828–851.

[Seymour, 1979] Seymour, P. D. (1979). Sums of circuits. In Bondy, J. A. and Murty, U. S. R., editors, *Graph Theory and Related Topics*, pages 341–355, New York. Academic Press.

[Smeulders et al., 2021] Smeulders, B., Bartier, V., Crama, Y., and Spieksma, F. C. R. (2021). Recourse in kidney exchange programs. *INFORMS Journal on Computing*, in press.

[Tarjan, 1972] Tarjan, R. E. (1972). Depth-first search and linear graph algorithms. *SIAM Journal on Computing*, 1:146–160.