Nonlinearity

PAPER

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To cite this article: L Loosveldt and S Nicolay 2021 *Nonlinearity* **34** 6561

IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **34** (2021) 6561–6586 https://doi.org/10.1088/1361-6544/ac1724

Generalized spaces of pointwise regularity: toward a general framework for the WLM

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Received 14 May 2020, revised 20 July 2021
Accepted for publication 22 July 2021
Published 9 August 2021

Abstract

In this work we generalize the spaces T_u^p introduced by Calderón and Zygmund using pointwise conditions emanating from generalized Besov spaces. We give conditions binding the functions belonging to these spaces and their wavelet coefficients. Next, we propose a multifractal formalism based on such spaces which generalizes the so-called wavelet leaders method and show that it is satisfied on a prevalent set.

Keywords: pointwise regularity spaces, multifractal formalisms, wavelets
Mathematics Subject Classification numbers: 42C40, 26A16, 28A78.

1. Introduction

The Hölder-regularity can be seen as a notion that fills gaps between being 'n times continuously differentiable' and '$n+1$ times continuously differentiable'. More precisely, a function f from $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to the space $T^p_u(x_0)$ (with $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and $u > 0$) if there exist a polynomial P_{x_0} of degree strictly less than u and a positive constant C such that

$$r^{-u}\|f - P_{x_0}\|_{L^p(B(x_0,r))} \leqslant C, \tag{1}$$

for $r > 0$, where $B(x_0, r)$ denotes the open ball centered at x_0 with radius r (see [2]); $T^\infty_u(x_0)$ is called a Hölder space (and is usually denoted by $\Lambda^u(x_0)$ [16]). These spaces are embedded and the Hölder exponent of f at x_0 is defined as

$$h_\infty(x_0) := \sup\{u > 0 : f \in T^\infty_u(x_0)\}. \tag{2}$$

The discrete wavelet transform provides a useful tool for studying the Hölder spaces, since the condition on f at x_0 can be transposed to a condition on some wavelet coefficients near x_0

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Recommended by Dr Mark Pollicott.

1361-6544/21/096561+26$33.00

(for more details, see [11, 15] for example), the so-called wavelet leaders (see definition 3.1 with $p = \infty$). Indeed, if a function belongs to a space $T_u^\infty(x_0)$, the wavelet leaders of x_0 satisfy an inequality somehow similar to (1). Conversely, if this condition on the wavelet leaders is met, the corresponding function belongs to a space close to $T_u^\infty(x_0)$. More precisely, in this case one has

$$\theta_u^{-1}(r)\|f - P_{x_0}\|_{L^\infty(B(x_0,r))} \leqslant C, \tag{3}$$

with $\theta_u(r) = r^u|\ln(r)|$. In other words, f belongs to $T_u^\infty(x_0)$ up to a logarithmic correction. If such results hold, we will say that we have a quasi-characterization of the space ($T_u^\infty(x_0)$ in this case). Such a quasi-characterization provides an exact characterization of the Hölder-regularity, i.e. of the Hölder exponent $h_\infty(x_0)$.

This notion of regularity can be generalized in several ways. First, one can replace the expression r^{-u} appearing in (1) with a general function $\theta_u(r)$ satisfying some requirements, as in inequality (3). Such ideas go back at least to the eighties where several similar generalizations of the Besov spaces have been proposed (see e.g. [4, 23]); these are still investigated nowadays (see [21]). By doing so, one defines spaces that are able to make subtle distinctions between functions associated to the same Hölder exponent, giving tools for detecting the presence of a Brownian motion in the signal. Such spaces have been studied in [18], where a quasi-characterization is obtained. Another idea consists in replacing the Hölder space appearing in (2) with a general T_u^p space, in order to study non-locally bounded functions (see [13] for such an application). This approach has been undertaken in [12], where generalized wavelet leaders, called p-leaders, are introduced. However, this definition is not a direct generalization of the usual leaders and fails to quasi-characterize the $T_u^p(x_0)$ spaces, although they still can be used to study the corresponding generalized Hölder exponent.

The first part of this paper consists in combining these two points of view, by considering the spaces of functions satisfying the condition

$$\theta_u^{-1}(r)\|f - P_{x_0}\|_{L^p(B(x_0,r))} \leqslant C. \tag{4}$$

Indeed, we consider an even larger class of spaces called here spaces of generalized pointwise smoothness (see definition 2.2). Their functional properties have been studied in [20] (albeit with slightly different definitions, see remark 2.12) and they correspond in some way to a pointwise version of the generalized Besov spaces introduced in [21]. We obtain a quasi-characterization of such spaces by introducing a variant definition of the p-leaders that naturally extends the classical case where $p = \infty$.

The second part of this paper aims at providing a multifractal formalism suited for the spaces introduced here. A multifractal formalism is an empirical method that allows to estimate the quantity

$$\dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_p(x_0) = h\},$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension with the convention $\dim_{\mathcal{H}}(\emptyset) = -\infty$ (see [7] for example) and $h_p(x_0)$ is the generalized Hölder exponent obtained by replacing $T_u^\infty(x_0)$ with $T_u^p(x_0)$ in (2). Usually, one requires such a method to be valid for a large class of functions. Such a multifractal formalism was first presented in [26] in the context of the analysis of fully developed turbulence velocity data and it can be shown that, from a prevalent point of view (see section 4.3), almost every function belonging to a given Besov space satisfies this formalism (see [14]). We show here that, from the prevalence point of view, almost every function belonging to a space of generalized smoothness satisfies a multifractal formalism derived from the formalism relying on the wavelet leaders. By doing so, we show that the generalized Besov spaces (see [21] and definition 4.2) provide a natural framework for supporting this theory, reinforcing the idea that the spaces of generalized smoothness are a natural pointwise version of these spaces.

This paper can be seen as a generalization of the ideas and techniques employed in [8, 11, 14, 18].

The notations used here are rather standard. Throughout this paper, we will use Euler's notation for the derivatives, i.e. $D_j f$ designates the derivative of f following the jth component.

2. Generalized spaces of pointwise smoothness

2.1. Admissible sequences

Let us recall the notion of admissible sequence (see e.g. [17] and references therein).

Definition 2.1. A sequence $\boldsymbol{\sigma} = (\sigma_j)_j$ of real positive numbers is called admissible if there exists a positive constant C such that

$$C^{-1}\sigma_j \leqslant \sigma_{j+1} \leqslant C\sigma_j,$$

for any $j \in \mathbb{N}_0$.

If $\boldsymbol{\sigma}$ is such a sequence, we set

$$\underline{\sigma}_j = \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j = \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k}$$

and define the lower and upper Boyd indices as follows,

$$\underline{s}(\boldsymbol{\sigma}) = \lim_j \frac{\log_2 \underline{\sigma}_j}{j} \quad \text{and} \quad \overline{s}(\boldsymbol{\sigma}) = \lim_j \frac{\log_2 \overline{\sigma}_j}{j}.$$

Since $(\log \underline{\sigma}_j)_j$ is a subadditive sequence, such limits always exist. The following relations about such sequences are well known (see e.g. [17]). If $\boldsymbol{\sigma}$ is an admissible sequence, let $\varepsilon > 0$; there exists a positive constant C such that

$$C^{-1}2^{j(\underline{s}(\boldsymbol{\sigma})-\varepsilon)} \leqslant \underline{\sigma}_j \leqslant \frac{\sigma_{j+k}}{\sigma_k} \leqslant \overline{\sigma}_j \leqslant C2^{j(\overline{s}(\boldsymbol{\sigma})+\varepsilon)},$$

for any $j, k \in \mathbb{N}_0$. In this paper, $\boldsymbol{\sigma}$ will always stand for an admissible sequence and, given $u > 0$, we set $\boldsymbol{u} = (2^{ju})_j$. Of course, we have $\underline{s}(\boldsymbol{u}) = \overline{s}(\boldsymbol{u}) = u$.

2.2. Definition of the generalized spaces of pointwise smoothness

Definition 2.2. Let $p, q \in [1, \infty]$, $f \in L^p_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$ whenever

$$(\sigma_j 2^{jd/p} \sup_{|h|\leqslant 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma)\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))})_j \in \ell^q,$$

where, given $r > 0$,

$$B_h(x_0, r) = \{x : [x, x + (\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor + 1)h] \subset B(x_0, r)\}.$$

It is easy to check that $T^{\boldsymbol{\sigma}}_{\infty,\infty}(x_0)$ is the generalized Hölder space $\Lambda^{\boldsymbol{\sigma}}(x_0)$ introduced in [18]. These spaces can also be seen as a generalization of the spaces $T^p_u(x_0)$ introduced by Calderón and Zygmund in [2], as corollary 2.11 will show.

Let us give an alternative definition of $T^{\sigma}_{p,q}(x_0)$.

Proposition 2.3. *Let* $p, q \in [1, \infty]$, $f \in L^p_{\mathrm{loc}}$, $x_0 \in \mathbb{R}^d$ *and* $\boldsymbol{\sigma}$ *be an admissible sequence such that* $\underline{s}(\boldsymbol{\sigma}) > 0$. *We have* $f \in T^{\sigma}_{p,q}(x_0)$ *if and only if there exists a sequence of polynomials* $(P_{j,x_0})_j$ *of degree less or equal to* $\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor$ *such that*

$$(\sigma_j 2^{jd/p} \|f - P_{j,x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^q. \tag{5}$$

Proof. The necessity of the condition being a consequence of the Whitney theorem, let us check the sufficiency. Let $j \in \mathbb{N}_0$; for any polynomial P of degree less or equal to $n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor$, we have, given $x, h \in \mathbb{R}^d$,

$$|\Delta_h^{n+1} f(x)| \leqslant |\Delta_h^{n+1} (f(x) - P(x))| \leqslant C_n \sum_{k=0}^{n+1} |f(x + kh) - P(x + kh)|,$$

for a constant C_n. Therefore, for $|h| \leqslant 2^{-j}$ and $x \in B_h(x_0, 2^{-j})$, we get

$$\|\Delta_h^{n+1} f\|_{L^p(B_h(x_0,2^{-j}))} \leqslant C_n(n + 2)\|f - P\|_{L^p(B(x_0,2^{-j}))},$$

hence the conclusion. □

2.3. Independence of the polynomial from the scale

Under some additional assumptions on the admissible sequence $\boldsymbol{\sigma}$, the sequence of polynomials $(P_{j,x_0})_j$ appearing in inequality (5) can be replaced by a unique polynomial P_{x_0} independent from the scale j: $P_{x_0} = P_{j,x_0}$.

We first need some preliminary results. Let us first state a somehow standard result about inequalities on polynomials; we sketch a proof for the sake of completeness.

Lemma 2.4. *Given* $x_0 \in \mathbb{R}^d$, *a radius* $r > 0$, $p \in (0, \infty]$ *and a maximum degree* n, *there exist two constants* C, C′ > 0 *only depending on* n *and* p *such that, for any polynomial* P *of degree lower or equal to* n,

$$\|D^{\alpha} P\|_{L^p(B(x_0,r))} \leqslant C r^{-|\alpha|} \|P\|_{L^p(B(x_0,r))},$$

for any multi-index α *and*

$$\sup_{x \in B(x_0,r)} |P(x)| \leqslant C' r^{-d/p} \|P\|_{L^p(B(x_0,r))}.$$

Proof. For the first inequality, let us recall that the Markov inequality affirms that, given a convex bounded set E of $\mathbb{R}^d$, there exists a constant $C_{E,p} > 0$ such that for any $n \in \mathbb{N}_0$ and $k \in \{1, \ldots, d\}$, we have

$$\|D_k P\|_{L^p(E)} \leqslant C_{E,p}(n + 1)^2 \|P\|_{L^p(E)},$$

for any polynomial P of degree less or equal to n. As a consequence, given $r > 0$, there exists a constant $C > 0$ depending on n and p such that, for any multi-index α, we have

$$\|D^{\alpha} P\|_{L^p(B(x_0,r))} \leqslant C r^{-|\alpha|} \|P\|_{L^p(B(x_0,r))}.$$

That being done, using Sobolev's inequality, we can now write

$$\sup_{x \in B(x_0,r)} |P(x)| \leqslant C' r^{-d/p} \|P\|_{L^p(B(x_0,r))},$$

for a constant $C' > 0$ which only depends on n and p. □

Lemma 2.5. *Let $m \in \mathbb{N}_0$, $\boldsymbol{\sigma}$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}^{-1}) > m$ and $\boldsymbol{\varepsilon} \in \ell^q$ with $q \in [1, \infty]$; there exists a sequence $\boldsymbol{\xi} \in \ell^q$ such that*

$$\sum_{j=J}^{\infty} \varepsilon_j 2^{jm} \sigma_j \leqslant \xi_J 2^{Jm} \sigma_J,$$

for all $J \in \mathbb{N}_0$.

Proof. Let δ, $\delta' > 0$ be such that $-2\delta' > m + \overline{s}(\sigma) + \delta$; given $J \in \mathbb{N}_0$, we have, using Hölder's inequality,

$$\sum_{j=J}^{\infty} \varepsilon_j 2^{jm} \sigma_j \leqslant C \sum_{j=J}^{\infty} \varepsilon_j 2^{(j-J)(m+\overline{s}(\sigma)+\delta)} 2^{Jm} \sigma_J$$
$$\leqslant C \left(\sum_{j=J}^{\infty} (\varepsilon_j 2^{-\delta'(j-J)})^q \right)^{1/q} \left(\sum_{j=J}^{\infty} 2^{-q'\delta'(j-J)} \right)^{1/q'} 2^{Jm} \sigma_J,$$

where q' is the conjugate exponent of q (with the usual modification if one of the indices is ∞). It remains to check that the sequence $\boldsymbol{\xi}$ defined by

$$\xi_j = C \left(\sum_{k=j}^{\infty} (\varepsilon_j 2^{-\delta'(j-J)})^q \right)^{1/q}$$

belongs to ℓ^q, which is easy. □

In the same way, we can get the following result.

Lemma 2.6. *Let $m \in \mathbb{N}_0$, $\boldsymbol{\sigma}$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}^{-1}) < m$ and $\boldsymbol{\varepsilon} \in \ell^q$ with $q \in [1, \infty]$; there exists a sequence $\boldsymbol{\xi} \in \ell^q$ such that*

$$\sum_{j=0}^{J} \varepsilon_j 2^{jm} \sigma_j \leqslant \xi_J 2^{Jm} \sigma_J,$$

for all $J \in \mathbb{N}_0$.

Remark 2.7. Lemma 2.5 generalizes the relation $\sum_{j=J}^{\infty} \sigma_j \leqslant C\sigma_J$, satisfied whenever $\underline{s}(\boldsymbol{\sigma}^{-1}) > 0$, while lemma 2.6 should be compared with $\sum_{j=0}^{J} 2^{jm} \sigma_j \leqslant C 2^{Jm} \sigma_J$, holding for $\overline{s}(\boldsymbol{\sigma}^{-1}) < m$ ($m \in \mathbb{N}_0$) (see e.g. [17] for more details).

The main theorem of this section relies on the following lemma.

Lemma 2.8. *Let $p, q \in [1, \infty]$, $f \in L^p_{\mathrm{loc}}$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that $0 \leqslant n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor < \underline{s}(\boldsymbol{\sigma})$. If f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$, the sequence of polynomials $(P_{j,x_0})_j$ satisfying (5) is such that, given a multi-index α for which $|\alpha| \leqslant n$, there exists a sequence $\boldsymbol{\xi} \in \ell^q$ satisfying*

$$2^{-|\alpha| j} \sigma_j |D^\alpha (P_{j,x_0} - P_{k,x_0})(x_0)| \leqslant \xi_j,$$

whenever $j < k$.

In particular, under the same hypothesis, the sequence $(D^\alpha P_{j,x_0}(x_0))_j$ is Cauchy and its limit does not depend on the chosen sequence of polynomials satisfying (5).

Proof. Let $\boldsymbol{\varepsilon} \in \ell^q$ be such that

$$\sigma_j 2^{jd/p} \|f - P_{j,x_0}\|_{L^p(B(x_0,2^{-j}))} \leqslant \varepsilon_j,$$

for any $j \in \mathbb{N}_0$. Given a multi-index α satisfying the hypothesis and $j \in \mathbb{N}_0$, we know that there exists a constant $C > 0$ such that

$$\begin{aligned} &\|D^\alpha(P_{j,x_0} - P_{j+1,x_0})\|_{L^p(B(x_0,2^{-(j+1)}))} \\ &\quad\leqslant C2^{|\alpha|(j+1)}\|P_{j,x_0} - P_{j+1,x_0}\|_{L^p(B(x_0,2^{-(j+1)}))} \\ &\quad\leqslant C2^{|\alpha|(j+1)}\|P_{j,x_0} - f\|_{L^p(B(x_0,2^{-(j+1)}))} + \|f - P_{j+1,x_0}\|_{L^p(B(x_0,2^{-(j+1)}))} \\ &\quad\leqslant C2^{|\alpha|(j+1)}(\varepsilon_j 2^{-jd/p}\sigma_j^{-1} + \varepsilon_{j+1}2^{-(j+1)d/p}\sigma_{j+1}^{-1}), \end{aligned}$$

which implies, from what we have obtained so far,

$$|D^\alpha(P_{j,x_0} - P_{j+1,x_0})(x_0)| \leqslant C'(\varepsilon_j + \varepsilon_{j+1})2^{|\alpha|j}\sigma_j^{-1}.$$

For $j < k$, lemma 2.5 then implies

$$|D^\alpha(P_{j,x_0} - P_{k,x_0})(x_0)| \leqslant \xi_j 2^{|\alpha|j}\sigma_j^{-1},$$

for the appropriate sequence $\boldsymbol{\xi} \in \ell^q$.

It remains to show that the limit $\mathscr{D}^\alpha f(x_0)$ of the sequence $(D^\alpha P_{j,x_0}(x_0))_j$ is independent of the peculiar choice of the sequence $(D^\alpha P_{j,x_0}(x_0))_j$; let $(Q_{j,x_0})_j$ be another sequence of polynomials satisfying (5). With the same reasoning as before, we get

$$|D^\alpha(P_{j,x_0} - Q_{j,x_0})(x_0)| \leqslant C2^{|\alpha|j}\sigma_j^{-1},$$

for j large enough, which is sufficient to assert that

$$|D^\alpha Q_{j,x_0}(x_0) - \mathscr{D}^\alpha f(x_0)|$$

tends to zero as j tends to infinity. □

We are now able to show the existence of the unique polynomial P_{x_0} introduced in the beginning of this section.

Theorem 2.9. *Let $p, q \in [1,\infty]$, $f \in L^p_{\mathrm{loc}}$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that $0 \leqslant n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor < \underline{s}(\boldsymbol{\sigma})$. The following assertions are equivalent:*

- *f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$,*
- *There exists a unique polynomial P_{x_0} of degree less or equal to n such that*

$$(\sigma_j 2^{jd/p}\|f - P_{x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^q. \tag{6}$$

Proof. We need to prove that the first assertion implies the second one. As f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$, there exists a sequence of polynomials $(P_{j,x_0})_j$ of degree less or equal to n such that

$$(\sigma_j 2^{jd/p}\|f - P_{j,x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^q.$$

Given a multi-index α satisfying $|\alpha| \leqslant n$, let us set

$$\mathscr{D}^{\alpha} f(x_0) := \lim_j D^{\alpha} P_{j,x_0}(x_0)$$

and define the polynomial

$$P_{x_0} : x \mapsto \sum_{|\alpha| \leqslant n} \mathscr{D}^{\alpha} f(x_0) \frac{(x - x_0)^{\alpha}}{|\alpha|!}. \tag{7}$$

One directly gets

$$\|P_{j,x_0} - P_{x_0}\|_{L^p(B(x_0,2^{-j}))} \leqslant \sum_{|\alpha| \leqslant n} |D^{\alpha} P_{j,x_0}(x_0) - \mathscr{D}^{\alpha} f(x_0)| \, 2^{-j(|\alpha|+d/p)}.$$

That being said, we know from the previous lemma that, given α, there exists a sequence $\boldsymbol{\xi}^{(\alpha)} \in \ell^q$ such that

$$|D^{\alpha} P_{j,x_0}(x_0) - \mathscr{D}^{\alpha} f(x_0)| \leqslant \xi_j^{(\alpha)} 2^{|\alpha| j} \sigma_j^{-1}.$$

We thus have

$$(\sigma_j 2^{jd/p} \|P_{j,x_0} - P_{x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^q,$$

which proves the first part of the theorem.

Concerning the uniqueness of the polynomial, the idea of the proof is the same as the one given in [2] for the spaces $T_u^p(x_0)$. Let P and Q be two polynomials satisfying a relation of type (6); one directly gets $P(x_0) = Q(x_0)$. That being said, let us define

$$L := \sum_{|\alpha| = m} c_{\alpha} (\cdot - x_0)^{\alpha},$$

where m is the lowest degree of $P - Q$, with

$$c_{\alpha} := \frac{D^{\alpha}(P - Q)(x_0)}{|\alpha|!}.$$

If $m < \sup\{l \in \mathbb{Z} : l < \underline{s}(\boldsymbol{\sigma})\}$, one can write

$$\|L\|_{L^1(B(x_0,1))} \leqslant C(2^{-mj} \sigma_j^{-1} + 2^{-j}),$$

for a constant C, which means $L = 0$. For $m = \sup\{l \in \mathbb{Z} : l < \underline{s}(\boldsymbol{\sigma})\}$, we simply get $\|L\|_{L^1(B(x_0,1))} \leqslant C 2^{-mj} \sigma_j^{-1}$, which implies $L = P - Q = 0$. □

Remark 2.10. In the previous result, if $\boldsymbol{\sigma}$ is the usual sequence $\boldsymbol{u}$ with $u \in \mathbb{N}_0$, it is easy to check that the polynomial P_{x_0} is unique if one requires its degree to be strictly smaller than n.

The link with the spaces of Calderón and Zygmund introduced in [2] is now obvious.

Corollary 2.11. *Given $p \in [1, \infty]$, $u > 0$ and $x \in \mathbb{R}^d$, we have $T_{p,\infty}^{\boldsymbol{u}}(x_0) = T_u^p(x_0)$, where as usual $T_u^p(x_0)$ denotes the class of functions $f \in L^p$ such that there exists a polynomial P_{x_0} of degree strictly less than u with the property that*

$$r^{-d/p} \|f - P_{x_0}\|_{L^p(B(x_0,r))} \leqslant C r^u,$$

for a constant $C > 0$.

Remark 2.12. In [20], the spaces T_u^p have been generalized by replacing the index u with a Boyd function ϕ. If $\phi(r) = r^u$, these spaces are the usual spaces from [2]. Given a Boyd function ϕ, $j \mapsto \phi(2^{-j})$ defines an admissible sequence. Conversely, from any admissible sequence $\boldsymbol{\sigma}$ one can define a Boyd function ϕ such that, for all $j \in \mathbb{N}_0$, $\phi(2^{-j}) = \sigma_j$ (see [17]), so that this approach is equivalent to the one proposed here. The concept of admissible sequence is more appropriate for the discrete character of the wavelet leaders method.

3. Spaces of generalized smoothness and wavelets

Various function spaces can be characterized by wavelets. We combine here the approaches adopted for the classical T_u^p spaces, the generalized Besov spaces and the generalized pointwise Hölder spaces to obtain a 'nearly' characterization of the spaces $T_{p,q}^{\boldsymbol{\sigma}}$.

3.1. Definitions

Let us briefly recall some definitions and notations about wavelets (for more precisions, see e.g. [6, 22, 24]). Under some general assumptions, there exist a function φ and $2^d - 1$ functions $(\psi^{(i)})_{1\leqslant i<2^d}$, called wavelets, such that

$$\{\varphi(x-k) : k \in \mathbb{Z}^d\} \cup \{\psi^{(i)}(2^j x - k) : 1 \leqslant i < 2^d, k \in \mathbb{Z}^d, j \in \mathbb{N}_0\}$$

form an orthogonal basis of L^2. Any function $f \in L^2$ can be decomposed as follows,

$$f(x) = \sum_{k\in\mathbb{Z}^d} C_k \varphi(x-k) + \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^d} \sum_{1\leqslant i<2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int_{\mathbb{R}^d} f(x) \psi^{(i)}(2^j x - k)\, \mathrm{d}x$$

and

$$C_k = \int_{\mathbb{R}^d} f(x) \varphi(x-k)\, \mathrm{d}x. \tag{8}$$

Let us remark that we do not choose the L^2 normalization for the wavelets, but rather an L^∞ normalization, which is better fitted to the study of the Hölderian regularity. For the sake of simplicity, we will only consider here compactly supported wavelet basis of regularity $r > \overline{s}(\boldsymbol{\sigma}^{-1})$; such wavelets are considered in [5].

Let $\lambda_{j,k}^{(i)}$ denote the dyadic cube

$$\lambda_{j,k}^{(i)} := \frac{i}{2^{j+1}} + \frac{k}{2^j} + [0, \frac{1}{2^{j+1}})^d.$$

In the sequel, we will often omit any reference to the indices i, j and k for such cubes by writing $\lambda = \lambda_{j,k}^{(i)}$. We will also index the wavelet coefficients of a function f with the dyadic cubes λ so that c_λ will refer to the quantity $c_{j,k}^{(i)}$. The notation Λ_j will stand for the set of dyadic cubes λ of $\mathbb{R}^d$ with side length 2^{-j} and the unique dyadic cube from Λ_j containing the point $x_0 \in \mathbb{R}^d$ will be denoted $\lambda_j(x_0)$. The set of the dyadic cubes is $\Lambda := \cup_{j\in\mathbb{N}_0} \Lambda_j$. Two dyadic cubes λ and λ' are adjacent if there exists $j \in \mathbb{N}_0$ such that $\lambda, \lambda' \in \Lambda_j$ and $\mathrm{dist}(\lambda, \lambda') = 0$. The set of the 3^d dyadic cubes adjacent to λ will be denoted by 3λ.

As for the wavelet-based study of the pointwise Hölder spaces, we will work with wavelet leaders [11]. However, as we work here with L^p norms, we need to introduce a generalized version.

Definition 3.1. Given a dyadic cube $\lambda \in \Lambda_j$ at scale j, the p-wavelet leader of λ ($p \in [1, \infty]$) is defined by

$$d_\lambda^p = \sup_{j' \geqslant j} \left(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} (2^{(j-j')d/p} |c_{\lambda'}|)^p \right)^{1/p}.$$

Given $x_0 \in \mathbb{R}^d$, we set

$$d_j^p(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda^p.$$

Remark 3.2. The definition of the wavelet leaders given here is different from the one presented in [19]. The quantities introduced here are easier to work with and naturally generalize the usual wavelet leaders $d_j(x_0)$ introduced in [11], since we have $d_j(x_0) = d_j^\infty(x_0)$.

We will need the following definition of the so-called Xu spaces [25] (which are some kind of local Besov space) to ensure a minimum regularity for the function.

Definition 3.3. Given $x_0 \in \mathbb{R}^d$, a function f defined on $\mathbb{R}^d$ belongs to the Xu space $\dot{X}^u_{p,q}(x_0)$ ($u \in \mathbb{R}$, $p, q \in [1, \infty]$) if there exists a constant $C_* > 0$ such that

$$\left(\sum_{|k - 2^j x_0| < C_* 2^j} (2^{(u-d/p)j} |c_{\lambda^{(i)}_{j,k}}|)^p \right)^{1/p} \in \ell^q.$$

3.2. A wavelet characterization

Let us give a quasi-characterization of the $T^\sigma_{p,q}(x_0)$ spaces using the wavelets defined in the previous section. In this context, j_0 is a natural number such that the support of each wavelet is contained in $B(0, 2^{j_0})$.

Let us first give a necessary condition for a function to belong to such a space.

Theorem 3.4. *If f belongs to the space $T^\sigma_{p,q}(x_0)$, then*

$$(\sigma_j d_j^p(x_0))_j \in \ell^q.$$

Proof. Let $\varepsilon \in \ell^q$ and $(P_j)_j$ be a sequence of polynomials of degree less or equal to $\bar{s}(\boldsymbol{\sigma})$ such that

$$\sigma_j 2^{jd/p} \|f - P_j\|_{L^p(B(x_0, 2^{-j}))} \leqslant \varepsilon_j,$$

for all $j \in \mathbb{N}_0$. Let us set choose $j_1 \in \mathbb{N}_0$ such that $2\sqrt{d} + 2^{j_0} \leqslant 2^{j_1}$ and fix $n \geqslant j_1$. For $\lambda^{(i)}_{j,k} \subset 3\lambda_n(x_0)$, we have

$$|\frac{k}{2^j} - x_0| \leqslant 2\sqrt{d} 2^{-n}.$$

By setting

$$\Lambda_{j,n} := \{\lambda^{(i)}_{j,k} \in \Lambda_j : |k - 2^j x_0| \leqslant 2\sqrt{d} 2^{j-n}\},$$

for $\lambda \in 3\lambda_n(x_0)$, we can write

$$\sum_{\lambda' \in \Lambda_j, \lambda' \subset \lambda} 2^{(n-j)d} |c_{\lambda'}|^p \leqslant \sum_{\lambda' \in \Lambda_{j,n}} 2^{(n-j)d} |c_{\lambda'}|^p,$$

whenever $p \neq \infty$. In this case, let us set

$$s_{n,j} := \sum_{\lambda' \in \Lambda_{j,n}} |c_{\lambda'}|^p$$

and define

$$g_{n,j} := \sum_{\lambda' \in \Lambda_{j,n}} |c_{\lambda'}|^{p-1} \operatorname{sign}(c_{\lambda'}) \psi_{\lambda'}.$$

One easily check the support of $g_{n,j}$ is contained in $B(x_0, 2^{j_1-n})$ and

$$s_{n,j} = 2^{jd} \langle f, g_{n,j} \rangle = 2^{jd} \int_{B(x_0, 2^{j_1-n})} (f(x) - P_{n-j_1}(x)) \overline{g_{n,j}(x)} \, \mathrm{d}x,$$

so that, if we denote by p' the conjugate exponent of p,

$$s_{n,j} \leqslant 2^{jd} \| f - P_{n-j_1} \|_{L^p(B(x_0, 2^{j_1-n}))} \| g_{n,j} \|_{L^{p'}}.$$

To estimate $\|g_{n,j}\|_{L^{p'}}$, let us remark that there exists a constant $C_* \in \mathbb{N}$ that does not depend on λ nor the scale j such that the cardinal of

$$\{\lambda' \in \Lambda_j : \operatorname{supp}(\psi_\lambda) \cap \operatorname{supp}(\psi_{\lambda'}) \neq \emptyset\}$$

is bounded by C_*. Therefore, given $j \in \mathbb{N}_0$, we can choose a partition $E_1, \ldots, E_{C_*}$ of Λ_j such that $\lambda', \lambda'' \in E_m$ $(1 \leqslant m \leqslant C_*)$ and

$$\operatorname{supp}(\psi_{\lambda'}) \cap \operatorname{supp}(\psi_{\lambda''}) \neq \emptyset$$

implies $\lambda' = \lambda''$. For $p \neq 1$, we easily get

$$|g_{n,j}|^{p'} \leqslant C_*^{p'} \sum_{\lambda' \in \Lambda_{j,n}} |c_{\lambda'}|^p |\psi_{\lambda'}|^{p'}$$

and thus

$$\|g_{n,j}\|_{L^{p'}} \leqslant C_* 2^{-jd/p'} s_{n,j}^{1/p'} \max_{1 \leqslant i < 2^d} \|\psi^{(i)}\|_{L^{p'}}. \tag{9}$$

If $p = 1$, one easily checks that one has

$$\|g_{n,j}\|_{L^\infty} \leqslant C_* 2^{-jd/q} \max_{1 \leqslant i < 2^d} \|\psi^{(i)}\|_{L^\infty},$$

so that (9) is still satisfied in this case.

That being done, since we have

$$s_{n,j}^{1/p} \leqslant C \varepsilon_{n-j_1} 2^{(j-n)d/p} \sigma_n^{-1},$$

for a constant $C > 0$, we get

$$\sum_{\lambda' \in \Lambda_j, \lambda' \subset \lambda} 2^{(n-j)d} |c_{\lambda'}|^p \leqslant 2^{(n-j)d} s_{n,j} \leqslant C \varepsilon_{n-j_1}^p \sigma_n^{-p},$$

which is sufficient to conclude in the case $p \neq \infty$.

Finally, let us consider the case $p = \infty$. Indeed, the conclusion is straightforward since, given $\lambda \subset 3\lambda_n(x_0)$, one easily check that, using an analogous reasoning, we can write

$$|c_\lambda| \leqslant C \varepsilon_{n-j_1} \sigma_n,$$

for a constant $C > 0$. □

For the sufficient condition, we need the following definition.

Definition 3.5. Let $p, q \in [1, \infty]$, $x_0 \in \mathbb{R}^d$ and f be a function from L^p_{loc}; if $\boldsymbol{\sigma}$ is an admissible sequence such that $2^{-jd/p} \sigma_j^{-1}$ tends to 0 as j tends to ∞, we says that f belongs to $T^{\boldsymbol{\sigma}}_{p,q,\log}(x_0)$ if there exists $J \in \mathbb{N}_0$ for which

$$\left(\frac{2^{jd/p} \sigma_j}{\log_2(2^{-jd/p} \sigma_j^{-1})} \sup_{|h| \leqslant 2^{-j}} \| \Delta_h^{\lfloor \overline{s}(\sigma^{-1}) \rfloor + 1} f \|_{L^p(B_h(x_0, 2^{-j}))} \right)_{j \geqslant J} \in \ell^q.$$

Theorem 3.6. *Let $p, q \in [1, \infty]$, $x_0 \in \mathbb{R}^d$ and f be a function from L^p_{loc}; let also $\boldsymbol{\sigma}$ be an admissible sequence such that $2^{-jd/p} \sigma_j^{-1}$ tends to 0 as j tends to ∞ and $\underline{\sigma}_1 > 2^{-d/p}$. If f belongs to $\dot{X}^\eta_{p,q}(x_0)$ for some $\eta > 0$, then*

$$(\sigma_j d_j^p(x_0))_j \in \ell^q$$

implies $f \in T^{\boldsymbol{\sigma}}_{p,q,\log}(x_0)$.

Proof. Let us first suppose that $\overline{s}(\boldsymbol{\sigma}) \geqslant 0$ and set $n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor$. We need to define some quantities. First, choose $m \in \mathbb{N}_0$ such that $k/2^j \in B(x, r)$ implies $\lambda^{(i)}_{j,k} \subset B(x, 2^m r)$, for any $x \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$, $j \in \mathbb{N}_0$ and $r \geqslant 2^{-j}$. Let also $m' \in \mathbb{N}_0$ be such that, for any $x \in \mathbb{R}^d$ and any $j \in \mathbb{N}_0$, $B(x, 2^{-j})$ is included in some dyadic cube of side length $2^{m'-j}$ and define $J_0 := j_0 + m + m'$. Let $C_* > 0$ be such that

$$\left(\sum_{|k - 2^j x_0| \leqslant C_* 2^j} \left(2^{(\eta - d/p)j} |c_{\lambda^{(i)}_{j,k}}| \right)^p \right)^{1/p} \in \ell^q$$

and choose a number $J_1 \in \mathbb{N}_0$ for which we have $(1 + 2^{j_0}) \leqslant C_* 2^{J_1}$. We also need a sequence $\boldsymbol{\varepsilon} \in \ell^q$ satisfying $\sigma_j d_j^p(x_0) \leqslant \varepsilon_j$, for all $j \in \mathbb{N}_0$. Finally, given $J \geqslant \max\{J_0, J_1\}$, define

$$P_J := \sum_{|\alpha| \leqslant n} \left(\frac{(\cdot - x_0)^\alpha}{|\alpha|!} \sum_{j=-1}^{J} D^\alpha f_j(x_0) \right),$$

where

$$f_{-1} := \sum_{k \in \mathbb{Z}^d} C_k \varphi_k \quad \text{and} \quad f_j := \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda,$$

for $j \geqslant 0$. We have

$$2^{Jd/p}\|f - P_J\|_{L^p(B(x_0,2^{-J}))}$$

$$\leqslant \sum_{j=-1}^{J} 2^{Jd/p}\|f_j - \sum_{|\alpha|\leqslant n} \frac{(\cdot - x_0)^\alpha}{|\alpha|!} D^\alpha f_j(x_0)\|_{L^p(B(x_0,2^{-J}))} \tag{10}$$

$$+ \sum_{j=J+1}^{\infty} 2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))}. \tag{11}$$

Let us fix $y \in B(x_0, 2^{-J})$ and $|\alpha| = n + 1$. We will first consider the case $p \neq \infty$. We have $D^\alpha \psi_{\lambda^{(i)}_{j,k}}(y) \neq 0$ only if $k/2^j$ belongs to $B(y, 2^{j_0-j})$; for $J_0 \leqslant j \leqslant J$, we have

$$\lambda^{(i)}_{j,k} \subset B(y, 2^{m-j-j_0}) \subset \lambda_{j-J_0}(x_0),$$

so that we can write, using the same reasoning as in the previous proof,

$$\begin{aligned} |D^\alpha f_j(y)| &\leqslant C2^{jp(n+1)} \sum_{\lambda\in\Lambda_j} |c_\lambda|^p |D^\alpha \psi_\lambda(y)|^p \\ &\leqslant C2^{jp(n+1)} \sum_{\lambda\in\Lambda_j, \lambda\subset\lambda_{j-J_0}(x_0)} |c_\lambda|^p |D^\alpha \psi_\lambda(y)|^p \\ &\leqslant C2^{jp(n+1)} \varepsilon^p_{j-J_0} \sigma_j^{-p}, \end{aligned}$$

since $\boldsymbol{\sigma}$ is an admissible sequence. Moreover, as the wavelet coefficients are finite and there exists a constant C_d which only depends on d such that

$$\#\{k \in \mathbb{Z}^d : k \in B(y, 2^{j_0})\} \leqslant C_d, \qquad \#\{k \in \mathbb{Z}^d : k/2^j \in B(y, 2^{j_0-j})\} \leqslant C_d,$$

we also have

$$|D^\alpha f_j(y)|^p \leqslant C2^{jp(n+1)} \sigma_j^{-p},$$

for all $j \in \{-1, \ldots, J_0 - 1\}$. As a consequence, we can write, for any $j \in \{-1, \ldots, J\}$,

$$\|f_j - \sum_{|\alpha|\leqslant n} \frac{(\cdot - x_0)^\alpha}{|\alpha|!} D^\alpha f_j(x_0)\|_{L^p(B(x_0,2^{-J}))} \leqslant \theta_j 2^{-J(n+1+d/p)} 2^{j(n+1)} \sigma_j^{-1},$$

for some sequence $\boldsymbol{\theta} \in \ell^q$. A similar reasoning gives the same inequality for $p = \infty$. Now, since $\overline{s}(\boldsymbol{\sigma}) < n + 1$, (10) is upper bounded by

$$C' 2^{-J(n+1)} \sum_{j=-1}^{J} \theta_j 2^{j(n+1)} \sigma_j^{-1} \leqslant C' \xi_J \sigma_J^{-1},$$

for some constant $C' > 0$, where the sequence $\boldsymbol{\xi}$ is given by lemma 2.5.

For the second term, let us fix $j \geqslant J + 1$ and $p \neq \infty$ to define

$$\Lambda_{j,J} := \{\lambda^{(i)}_{j,k} \in \Lambda_j : B(k/2^j, 2^{j_0}/2^j) \cap B(x_0, 2^{-J}) \neq \emptyset\}.$$

By proceeding as before for $x \in B(x_0, 2^{-J})$, we get

$$\|f_j(x)\|^p_{L^p(B(x_0,2^{-J}))} \leqslant C \sum_{\lambda \in \Lambda_{j,J}} 2^{-dj} |c_\lambda|^p, \tag{12}$$

for some constant C, which gives

$$2^{Jd/p} \|f_j(x)\|_{L^p(B(x_0,2^{-J}))} \leqslant C\varepsilon_{J-J_0} \sigma_J^{-1}.$$

Moreover, since the coefficient $c_{\lambda^{(i)}_{j,k}}$ does not vanish in the sum (12) only if the condition $|k - 2^j x_0| \leqslant C_* 2^j$ is satisfied, we also have

$$\|f_j(x)\|^p_{L^p(B(x_0,2^{-J}))} \leqslant \delta_j^p 2^{-\eta p j},$$

for a sequence $\boldsymbol{\delta} \in \ell^q$, as f belongs to the space $\dot{X}^\eta_{p,q}(x_0)$. Let us obtain upper bounds for the case $p = \infty$; for $x \in B(x_0, 2^{-J})$, $k/2^j \in B(x, 2^{j_0 - j})$ implies $\lambda^{(i)}_{j,k} \subset \lambda_{J-j_0}(x_0)$, so that we have $|c_{\lambda^{(k)}_{j,l}}| \leqslant C\varepsilon_{J-J_0}\sigma_N$. The same reasoning as before leads to

$$\|f_j(x)\|_{L^\infty(B(x_0,2^{-J}))} \leqslant C\delta_j 2^{-\eta j}.$$

Let us now set $j_*(J) := \lceil |\log_2(2^{-Jd/p}\sigma_J^{-1})| / \eta \rceil$ and choose η small enough in order to insure that we have $\log_2(2^{d/p}\underline{\sigma}_1)/\eta > 1$. With such a definition, we have $j_*(J) = j_*(J')$ if and only if $J = J'$ and we can write

$$\begin{aligned}
&\sum_{j=J+1}^{\infty} 2^{Jd/p} \|f_j\|_{L^p(B(x_0,2^{-J}))} \\
&\quad = \sum_{j=J+1}^{j_*(J)} 2^{Jd/p} \|f_j\|_{L^p(B(x_0,2^{-J}))} + \sum_{j=j_*(J)+1}^{\infty} 2^{Jd/p} \|f_j\|_{L^p(B(x_0,2^{-J}))} \\
&\quad \leqslant C \sum_{j=J+1}^{j_*(J)} \varepsilon_{J-J_0}\sigma_J^{-1} + C 2^{Jd/p} \sum_{j=j_*(J)+1}^{\infty} \delta_j 2^{-\eta j} \\
&\quad \leqslant C(\varepsilon_{J-J_0} + \xi_{j_1(J)})\, |\log_2(2^{-Jd/p}\sigma_J^{-1})| \sigma_J^{-1},
\end{aligned}$$

for J large enough, where the sequence $(\xi_{j_*(J)})_J$ belongs to ℓ^q.

It only remains to consider the situation where $\underline{s}(\boldsymbol{\sigma}) < 0$. In this case, let us set $P_J = 0$ whenever $J \geqslant \max\{J_0, J_1\}$. Once again, there exists a sequence $\boldsymbol{\xi} \in \ell^q$ such that

$$|f_j(y)| \leqslant \xi_j \sigma_J^{-1},$$

for $y \in B(x_0, 2^{-J})$, any $J \geqslant \max\{J_0, J_1\}$ and any $j \in \{-1, \dots, J\}$. As done previously, we get

$$\begin{aligned}
&2^{Jd/p} \|f - P_J\|_{L^p(B(x_0,2^{-J}))} \\
&\quad \leqslant C \sum_{j=-1}^{J} 2^{Jd/p} \|f_j\|_{L^p(B(x_0,2^{-J}))} + \sum_{j=J+1}^{\infty} 2^{Jd/p} \|f_j\|_{L^p(B(x_0,2^{-J}))} \\
&\quad \leqslant \delta_J |\log_2(2^{-Jd/p}\sigma_J^{-1})| \sigma_J^{-1},
\end{aligned}$$

with $\delta \in \ell^q$. □

Remark 3.7. It is well known that theorem 3.4 has no converse: the 'logarithmic correction' appearing in theorem 3.6 is necessary in the classical case (see e.g. [11]).

Remark 3.8. The regularity condition related to the Xu spaces is less restrictive than the one involving Besov spaces in [14].

4. A multifractal formalism

We show here that the generalized Besov spaces provide a natural framework for the multifractal formalism based on the $T^{\sigma}_{p,q}$ spaces.

4.1. Definitions

As the wavelet leaders method (WLM) involves the oscillation spaces $\mathcal{O}^{s,s'}_{p}$ (see [11, 14]), we will temporarily use them in our general framework.

Definition 4.1. Let $p, q, r \in [1, \infty]$; a function f belongs to $\mathcal{O}^{\sigma}_{p,r,q}$ if the sequence $(C_k)_k$ defined by (8) belongs to ℓ^q and if

$$\left(\sum_{j\in\mathbb{N}_0}\left(\sum_{\lambda\in\Lambda_j}(\sigma_j 2^{-dj/r} d^p_\lambda)^r\right)^{q/r}\right)^{1/q} \leqslant C,$$

for some positive constant C.

We will show that these spaces are closely related to the generalized Besov spaces, introduced in the following definition.

Definition 4.2. Let $(\varphi_j)_j$ be a sequence of functions belonging to the Schwartz class $\mathcal{S}$ such that

- $\mathrm{supp}(\varphi_0) \subset \{x \in \mathbb{R}^d : |x| \leqslant 2\}$,
- $\mathrm{supp}(\varphi_j) \subset \{x \in \mathbb{R}^d : 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\}$ $(j \geqslant 1)$,
- For any multi-index α, there exists a constant $C_\alpha > 0$ such that we have $\sup_{x\in\mathbb{R}^d} |D^\alpha \varphi_j(x)| \leqslant C_\alpha 2^{-j|\alpha|}$,
- $\sum_{j\in\mathbb{N}_0} \varphi_j = 1$.

Given $p, q \in [1, \infty]$, the space $B^{\sigma}_{p,q}$ is defined as

$$B^{\sigma}_{p,q} := \left\{ f \in \mathcal{S}' : \left(\sum_{j\in\mathbb{N}_0} \|\sigma_j \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|^q_{L^p}\right)^{1/q} < \infty \right\},$$

where $\mathcal{F}$ denotes the Fourier transform.

One can show that these spaces are obtained by replacing the dyadic sequence appearing in the usual definition with an admissible sequence (see e.g. [21]). We will use the wavelet characterization of such spaces (see [1]).

We first need the following definition to introduce a multifractal formalism.

Definition 4.3. Let $p, q \in [1, \infty]$; if, given $h > -d/p$, $\gamma^{(h)}$ is an admissible sequence, the family of admissible sequences $h \mapsto \gamma^{(h)}$ is (p, q)-decreasing if it satisfies $\underline{s}(\gamma^{(h)}) > -d/p$, $\underline{\gamma}_1^{(h)} > 2^{-d/p}$ for any $h > -d/p$ and if $-d/p < h < h'$ implies

$$T_{p,q}^{\gamma^{(h)}}(x_0) \subset T_{p,q}^{\gamma^{(h')}}(x_0).$$

In the sequel, we will only consider families of admissible sequences $\gamma^{(\cdot)}$ that are implicitly (p, q)-decreasing. This notion was introduced in [18], where criteria to obtain such families are presented.

A multifractal formalism is an empirical method that allows to estimate the quantity

$$\dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h(x_0) = h\},$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension with the convention $\dim_{\mathcal{H}}(\emptyset) = -\infty$ (see [7] for example) and $h(x_0)$ is some kind of generalized Hölder exponent. Usually, one requires such a method to be valid for a large class of functions. We aim at providing here a multifractal formalism for the exponents defined from the $T_{p,q}^{\sigma}$ spaces (see (13) in the next definition), thus generalizing the WLM [11, 14].

Definition 4.4. Given $p, q \in [1, \infty]$ and a family of admissible sequences $\gamma^{(\cdot)}$, the generalized (p, q)-Hölder exponent associated to $f \in L^p_{\mathrm{loc}}$ and $\gamma^{(\cdot)}$ at $x_0 \in \mathbb{R}^d$ is defined by

$$h_{p,q}(x_0) := \sup\{h > -d/p : f \in T_{p,q}^{\gamma^{(h)}}(x_0)\}. \tag{13}$$

The most natural family of admissible sequences is $h \mapsto (2^{jh})_j$. In this case, $h_{\infty,\infty}(x_0)$ is the usual Hölder exponent [11], while $h_{p,\infty}(x_0)$ is the p-exponent considered in [14].

Given $p, q \in [1, \infty]$, a family of admissible sequences $\gamma^{(\cdot)}$ and a function $f \in L^p_{\mathrm{loc}}$, we set

$$D_{p,q}(h) := \dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) = h\}.$$

4.2. Preliminary results

In this section, we will implicitly work with indices $p, q, r \in [1, \infty]$, a function f that belongs to L^p_{loc}, a point $x_0 \in \mathbb{R}^d$, a family of admissible sequences $\gamma^{(\cdot)}$ and an admissible sequence $\boldsymbol{\sigma}$.

Lemma 4.5. *If*

$$\gamma_j^{(h)} 2^{\eta j} d_j^p(x_0) \in \ell^q,$$

for some $\eta > 0$ such that $\lfloor \overline{s}(\gamma^{(h)}) + \eta \rfloor = \lfloor \overline{s}(\gamma^{(h)}) \rfloor$, then $h_{p,q}(x_0) \geqslant h$.

Proof. We know that there exists a sequence of polynomials $(P_j)_j$ of degree at most $\overline{s}(\gamma^{(h)})$ and a sequence $\varepsilon \in \ell^q$ such that

$$\gamma_j^{(h)} 2^{jd/p} \|f - P_j\|_{L^p(B(x_0, 2^{-j}))} \leqslant C\varepsilon_j 2^{-\eta j} |\log_2(2^{-\eta j - jd/p}/\gamma_j^{(h)})|,$$

for j large enough, which implies $f \in T_{p,q}^{\gamma^{(h)}}(x_0)$. □

Proposition 4.6. *If the function f belongs to both $B^{\eta}_{p,q}$ for some $\eta > 0$ and $\mathcal{O}^{\sigma}_{p,r,q}$, then*

$$\dim_{\mathcal{H}}(x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\} \leqslant d + r\overline{s}\left(\frac{\gamma^{(h)}}{\boldsymbol{\sigma}}\right).$$

Proof. Let $\boldsymbol{\varepsilon} \in \ell^q$ be such that $\varepsilon_j \neq 0$ and

$$\left(\sum_{\lambda\in\Lambda_j}(\sigma_j 2^{-jd/r} d_\lambda^p)^r\right)^{1/r} \leqslant \varepsilon_j,$$

for all $j \in \mathbb{N}_0$. Let us first consider the case $r = \infty$; if $\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}) < 0$, there exists $\delta > 0$ such that $\gamma_j^{(h)} 2^{\delta j} d_j^p(x_0) \leqslant C\varepsilon_j$ for any j and $h_{p,q}(x_0) \geqslant h$ for all $x_0 \in \mathbb{R}^d$. As a consequence, we have

$$\dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\} = -\infty = d + r\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}).$$

On the other hand, if $\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}) \geqslant 0$,

$$\dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\} \leqslant d \leqslant d + r\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}).$$

Now, suppose $r < \infty$, fix $h > -d/p$ and define, given $j \in \mathbb{N}_0$ and $\delta > 0$ sufficiently small,

$$E_{j,\delta}^h := \{\lambda \in \Lambda_j : d_\lambda^p \geqslant \varepsilon_j 2^{-\delta j}/\gamma_j^{(h)}\}$$

and set $n = \# E_{j,\delta}^h$. As $f \in \mathcal{O}_{p,r,q}^{\boldsymbol{\sigma}}$, we have

$$\sigma_j^r 2^{-jd} n (2^{-\delta j}/\gamma_j^{(h)})^r \leqslant \varepsilon_j^{-r}\sigma_j^r 2^{-jd} \sum_{\lambda\in E_{j,\delta}^h} (d_\lambda^p)^r \leqslant 1,$$

so that

$$n \leqslant 2^{jd}(2^{-\delta j}/\gamma_j^{(h)})^{-r}/\sigma_j^r.$$

Now, define $\Lambda_{j,\delta}^h$ as the set of the dyadic cubes $\lambda \in \Lambda_j$ such that there exists a neighbor $\lambda' \in 3\lambda$ that belongs to $E_{j,\delta}^h$. Finally, define

$$F_\delta^h := \limsup_j \{x_0 \in \mathbb{R}^d : \lambda_j(x_0) \in \Lambda_{j,\delta}^h\}.$$

If x_0 does not belong to F_δ^h, then there exists $J \in \mathbb{N}_0$ such that $j \geqslant J$ implies $\lambda_j(x_0) \notin \Lambda_{j,\delta}^h$ and, from what we have obtained for n, there exists a constant $C > 0$ for which $j \geqslant J$ implies

$$2^{\delta j}\gamma_j^{(h)} d_j^p(x_0) \leqslant C\varepsilon_j$$

and therefore

$$\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\} \subset F_\delta^h. \tag{14}$$

Let $\alpha > 0$, set $j_1 := \inf\{j : \sqrt{d}2^{-j} < \alpha\}$ and

$$E_\delta := \{\lambda \in \Lambda_{j,\delta}^h : j \geqslant j_1\}.$$

It is easy to check that E_δ is an α-covering of F_δ^h; given $s, \eta > 0$, we have

$$\begin{aligned}\sum_{\lambda\in E_\delta}\operatorname{diam}(\lambda)^s &\leqslant \sum_{j\geqslant j_0} \# F_j^h (\sqrt{d}2^{-j})^s \leqslant C \sum_{j\geqslant j_0} 2^{(d-s)j}(2^{-\delta j}/\gamma_j^{(h)})^{-r}/\sigma_j^r \\ &\leqslant C' \sum_{j\in\mathbb{N}_0} 2^{rj(\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma})+\delta+\eta} 2^{(d-s)j}.\end{aligned}$$

As a consequence, we have

$$\dim_{\mathcal{H}}(F^h_\delta) \leqslant d + r\left(\overline{s}\left(\frac{\gamma^{(h)}}{\boldsymbol{\sigma}}\right) + \delta + \eta\right),$$

for any $\eta > 0$ and we can conclude thanks to (14). □

Of course, for the classic examples of families of admissible sequences, the application $h \mapsto \overline{s}(\gamma^{(h)})$ is continuous (see [18]); in such a case, the previous result can be improved.

Remark 4.7. If there exists a sequence ε converging to 0^+ such that

$$\overline{s}\left(\frac{\gamma^{(h+\varepsilon_j)}}{\boldsymbol{\sigma}}\right) \to \overline{s}\left(\frac{\gamma^{(h)}}{\boldsymbol{\sigma}}\right),$$

we have

$$\dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) \leqslant h\} \leqslant d + r\overline{s}\left(\frac{\gamma^{(h)}}{\boldsymbol{\sigma}}\right).$$

Proposition 4.8. *If $\boldsymbol{\sigma}$ is an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > 0$ and $\underline{s}(\boldsymbol{\sigma}) - d/r > -d/p$, we have $\mathcal{O}^{\boldsymbol{\sigma}}_{p,r,q} = B^{\boldsymbol{\sigma}}_{r,q}$.*

Proof. We obviously have $\mathcal{O}^{\boldsymbol{\sigma}}_{p,r,q} \hookrightarrow B^{\boldsymbol{\sigma}}_{r,q}$. If f belongs to $B^{\boldsymbol{\sigma}}_{r,q}$, we have

$$\left(\sum_{\lambda\in\Lambda_j}(\sigma_j 2^{-jd/r} d^p_\lambda)^r\right)^{q/r} \leqslant \left(\sum_{\lambda\in\Lambda_j}(\sigma_j 2^{-jd/r})^q \sum_{j'\geqslant j}\left(\sum_{\lambda'\in\Lambda_{j'},\lambda'\subset\lambda}(2^{(j-j')d/p}|c_{\lambda'}|)^p\right)^{r/p}\right)^{q/r}, \quad (15)$$

for any $j \in \mathbb{N}_0$.

Let us first suppose that $r \leqslant p$; in this case, (15) is bounded by

$$\left(\sum_{j'\geqslant j}(\sigma_j\sigma_{j'}^{-1}2^{(j-j')d/p}2^{(j'-j)d/r})^r \sum_{\lambda'\in\Lambda_{j'}}(\sigma_{j'}2^{-j'd/r}|c_{\lambda'}|)^r\right)^{q/r}.$$

Let $\varepsilon > 0$ be such that $\underline{s}(\boldsymbol{\sigma}) - \varepsilon - d/r > -d/p$; there exists a constant $C_\varepsilon > 0$ such that

$$\sigma_j\sigma_{j'}^{-1} < C_\varepsilon 2^{(\underline{s}(\boldsymbol{\sigma})-\varepsilon)(j-j')}.$$

If $q \leqslant r$, (15) is bounded by

$$C\left(\sum_{j'\geqslant j}(2^{(\underline{s}(\boldsymbol{\sigma})+\varepsilon-d/r-d/r)(j-j')})^q\left(\sum_{\lambda'\in\Lambda_{j'}}(\sigma_{j'}2^{-j'd/r}|c_{\lambda'}|)^r\right)^{q/r}\right).$$

As f belongs to $B^{\sigma}_{r,q}$, we can write

$$\left(\sum_{j\in\mathbb{N}_0}\left(\sum_{\lambda\in\Lambda_j}(\sigma_j 2^{-jd/r}d_\lambda^p)^r\right)^{q/r}\right)^{1/q} \leqslant C\left(\sum_{j'\in\mathbb{N}_0}\left(\sum_{\lambda'\in\Lambda_{j'}}(\sigma_{j'}2^{-j'd/r}|c_{\lambda'}|)^r\right)^{q/r}\right)^{1/q},$$

which implies $f\in\mathcal{O}^{\sigma}_{p,r,q}$. If $r<q$, by denoting s the conjugate exponent of q/r, we can use Hölder's inequality to bound (15) by

$$C\left(\sum_{j'\geqslant j}\left(2^{-\underline{s}(\sigma)+\varepsilon-d/p-d/r}\right)^{(j'-j)}\right)^{rs/2}\Bigg)^{q/(rs)}$$

$$\times\left(\sum_{j'\geqslant j}(2^{-\underline{s}(\sigma)+\varepsilon-d/p-d/r)(j'-j)})^{q/(2r)}\left(\sum_{\lambda'\in\Lambda_{j'}}(\sigma_{j'}2^{-j'd/r}|c_{\lambda'}|)^r\right)^{q/r}\right)$$

$$\leqslant C\left(\sum_{j'\geqslant j}(2^{-\underline{s}(\sigma)+\varepsilon-d/p-d/r)(j'-j)})^{q/(2r)}\left(\sum_{\lambda'\in\Lambda_{j'}}(\sigma_{j'}2^{-j'd/r}|c_{\lambda'}|)^r\right)^{q/r}\right),$$

so that f belongs to $\mathcal{O}^{\sigma}_{p,r,q}$, as in the previous case.

We still have to consider the case $p<r$; by Jensen's inequality, we can bound (15) by

$$\left(\sum_{\lambda\in\Lambda_j}(\sigma_j 2^{-jd/r})^r\sum_{j\geqslant j'}\sum_{\lambda'\in\Lambda_{j'},\lambda'\subset\lambda}2^{(j-j')d}|c_{\lambda'}|^r\right)^{q/r} \leqslant \left(\sum_{j'\geqslant j}(\sigma_j/\sigma_{j'})^r\sum_{\lambda'\in\Lambda_{j'}}(\sigma_{j'}2^{-j'd/r}|c_{\lambda'}|)^r\right)^{q/r},$$

so that we can conclude as in the other cases. □

4.3. The notion of prevalence

In this section, we very briefly introduce the notion of prevalence (see [3, 9, 10] for more details).

In $\mathbb{R}^d$, it is well known that if one can associate a probability measure μ to a Borel set B such that $\mu(B+x)$ vanishes for very $x\in\mathbb{R}^d$, then the Lebesgue measure $\mathcal{L}(B)$ of B also vanishes. For the notion of prevalence, this property is turned into a definition in the context of infinite-dimensional spaces.

Definition 4.9. Let E be a complete metric vector space; a Borel set B of E is Haar-null if there exists a compactly-supported probability measure μ such that $\mu(B+x)=0$, for every $x\in E$. A subset of E is Haar-null if it is contained in a Haar-null Borel set; the complement of a Haar-null set is a prevalent set.

If E is finite-dimensional, B is Haar-null if and only if $\mathcal{L}(B)=0$; if E is infinite-dimensional, the compact sets of E are Haar-null. Moreover, it can be shown that a translated of a Haar-null set is Haar-null and that a prevalent set is dense in E. Finally, the intersection of a countable collection of prevalent sets is prevalent.

Let us make some remarks about how to show that a set is Haar-null. A common choice for the measure in definition 4.9 is the Lebesgue measure on the unit ball of a finite-dimensional subset E' of E. For such a choice, one has to show that $\mathcal{L}(B \cap (E' + x))$ vanishes for every x. If E is a function space, one can choose a random process X whose sample paths almost surely belong to E. In this case, one can show that a property only holds on a Haar-null set by showing that the sample path X is such that, for any $f \in E$, $X_t + f$ almost surely does not satisfy the property.

If a property holds on a prevalent set, we will say that it holds almost everywhere from the prevalence point of view.

4.4. A multifractal formalism associated to the generalized Besov spaces

We propose here the following formula to estimate the spectrum $D_{p,q}$ related to a function $f \in B^{\boldsymbol{\sigma}}_{r,s}$:

$$D_{p,q}(h) = d + r\overline{s}\left(\frac{\gamma^{(h)}}{\boldsymbol{\sigma}}\right)$$

and show that, under natural smooth conditions, this equality is satisfied almost everywhere from a prevalent point of view.

Definition 4.10. An admissible sequence $\boldsymbol{\sigma}$ and a family of admissible sequences $\gamma^{(\cdot)}$ are compatible for $p, q, r, s \in [1, \infty]$ with $s \leqslant q$ if

- $\underline{s}(\boldsymbol{\sigma}) > 0$,
- $\underline{s}(\boldsymbol{\sigma}) - d/r > -d/p$,
- The function ζ defined on $(-d/p, \infty)$ by

  $$\zeta(h) := \underline{s}\left(\frac{\gamma^{(h)}}{\boldsymbol{\sigma}}\right) = \overline{s}\left(\frac{\gamma^{(h)}}{\boldsymbol{\sigma}}\right)$$

  is non decreasing, continuous and such that

  $$\{h > -d/p : \zeta(h) < -d/r\} \neq \emptyset.$$

We call ζ the ratio function. We will also frequently use the quantity

$$h_{\min}(r) := \sup\{h > -d/p : \zeta(h) < -d/r\}.$$

The following remark stresses the importance of $h_{\min}$.

Remark 4.11. Suppose that $\boldsymbol{\sigma}$ and $\gamma^{(\cdot)}$ are compatible as in the previous definition. If f belongs to $B^{\boldsymbol{\sigma}}_{p,q}$, there exists $\eta > 0$ such that $B^{\boldsymbol{\sigma}}_{p,q} \hookrightarrow B^{\eta}_{p,q}$. For $\lambda \in \Lambda_j$ and $j' \geqslant j$, we have

$$\begin{aligned}\left(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} (2^{(j-j')d/p}|c_{\lambda'}|)^p\right)^{1/p} &\leqslant 2^{jd/p}\left(\sum_{\lambda' \in \Lambda_{j'}} (\sigma_{j'}2^{-j'd/p}|c_{\lambda'}|)^p\right)^{1/p} \sigma_{j'}^{-1} \\ &\leqslant 2^{jd/p}\varepsilon_{j'}\sigma_{j'}^{-1},\end{aligned}$$

for a sequence $\boldsymbol{\varepsilon} \in \ell^q$. As a consequence, there exists $\eta > 0$ and a sequence $\boldsymbol{\xi} \in \ell^q$ given by lemma 2.5 such that, for $\lambda \in \Lambda_j$,

$$d_\lambda^p \leqslant C\sum_{j' \geqslant j} 2^{jd/p}\varepsilon_{j'}\sigma_{j'}^{-1} \leqslant C\xi_j 2^{-\eta j}/\gamma_j^{(h)},$$

for all $h > -d/p$ such that $\overline{s}(\gamma^{(h)}/\sigma) < -d/p$. Therefore, one has $h_{p,q}(x_0) \geqslant h_{\min}(p)$, for any $x_0 \in \mathbb{R}^d$.

In the same spirit, for $r \leqslant p$, one has $B^{\sigma}_{r,q} \hookrightarrow B^{\theta}_{p,q}$, where θ is the admissible sequence defined by $\theta_j := 2^{(d/p-d/r)j}\sigma_j$ ($j \in \mathbb{N}_0$). As $\underline{s}(\sigma) - d/r > -d/p$ implies $\underline{s}(\theta) > 0$, there exists $\eta > 0$ such that $B^{\sigma}_{r,q} \hookrightarrow B^{\eta}_{p,q}$ and $h_{p,q}(x_0) \geqslant h_{\min}(r)$, for any $x_0 \in \mathbb{R}^d$.

That being done, if $p < r$ then, for any $f \in B^{\sigma}_{r,q}$,

$$h_{p,q}(x_0) \geqslant h_{r,q}(x_0) \geqslant h_{\min}(r).$$

Thus, if $f \in B^{\sigma}_{r,s}$, we have $f \in B^{\sigma}_{r,q}$ and $h_{p,q}(x_0) \geqslant h_{\min}(r)$.

From what we have done so far, we get the following corollary.

Corollary 4.12. *Let $p, q, r, s \in [1, \infty]$, σ be an admissible sequence and $\gamma^{(\cdot)}$ be a family of admissible sequences such that σ and $\gamma^{(\cdot)}$ are compatible. If f belongs to $B^{\sigma}_{r,s}$, then*

- $\{x \in \mathbb{R}^d : h_{p,q}(x_0) \leqslant h\} = \emptyset$ *for any* $h < h_{\min}(r)$,
- $\dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : h_{p,q}(x_0) \leqslant h\}) \leqslant d + r\zeta(h)$ *for any* $h \geqslant h_{\min}(r)$.

To show that, under some general hypothesis, the last upper bound is optimal for a prevalent set of functions in $B^{\sigma}_{r,s}$, we need the following definition.

Definition 4.13. Let $x_0 \in \mathbb{R}^d$ and $r > 0$; the strict cone of influence above x_0 of width r is

$$\mathcal{C}_{x_0}(r) := \left\{(j,k) \in \mathbb{N}_0 \times \mathbb{Z}^d : \|\frac{k}{2^j} - x_0\|_\infty < \frac{r}{2^j}\right\},$$

where $\|x - y\|_\infty$ is the Chebyshev distance between x and y ($x, y \in \mathbb{R}^d$):

$$\|x - y\|_\infty := \max_{1 \leqslant n \leqslant d} |x_n - y_n|.$$

This definitions is related to the wavelets as follows: in this context, we set

$$\mathcal{K}_{x_0}(r) := \{\lambda^{(i)}_{j,k} \in \Lambda : (j,k) \in \mathcal{C}_{x_0}(r)\}.$$

The following result explains why $\mathcal{K}_{x_0}$ can be seen as a cone of influence for the wavelets.

Proposition 4.14. *If f belongs to $T^{\sigma}_{p,q}(x_0)$, then*

$$\left(\sigma_j \left(\sum_{\lambda \in \Lambda_j \cap \mathcal{K}_{x_0}(r)} |c_\lambda|^p\right)^{1/p}\right)_j \in \ell^q.$$

Proof. Choose $j_1 \in \mathbb{N}_0$ such that $\sqrt{d}r + 2^{j_0} \leqslant 2^{j_1}$; for $j \geqslant j_1$, if $\lambda \in \Lambda_j$ also belongs to $\mathcal{K}_{x_0}(r)$, then the support of ψ_λ is included in $B(x_0, 2^{j_1 - j})$. From the proof of theorem 3.4, we know that there exists a sequence $\varepsilon \in \ell^q$ such that

$$\sigma_j \left(\sum_{\lambda \in \Lambda_j \cap \mathcal{K}_{x_0}(r)} |c_\lambda|^p\right)^{1/p} \leqslant \varepsilon_j,$$

for any $j \geqslant j_1$. The conclusion then comes from the Archimedean property of the real line. □

Given a dyadic cube $\lambda = \lambda^{(i)}_{j,k}$, let us denote by $k(\lambda)$ and $j(\lambda)$ the numbers such that $k(\lambda)/2^{j(\lambda)}$ is the dyadic irreducible form of $k/2^j$. For $\alpha \in [1, \infty]$, let us set

$$h_*(\alpha) := \zeta^{-1}\left(\frac{d}{\alpha r} - \frac{d}{r}\right).$$

We have $h_*(\alpha) \geqslant h_{\min}(r) = h_*(\infty)$. If $\zeta(h) > d/\alpha r - d/r$, choose $\varepsilon_0 > 0$ such that $\zeta(h) - \varepsilon_0 > d/\alpha r - d/r$ and let $m_0 \in \mathbb{N}_0$ be such that

$$d - \left(\frac{d}{\alpha r} - \frac{d}{r} - \zeta(h) + \varepsilon_0\right) 2^{dm_0}\alpha < 0. \tag{16}$$

Let us split each cube $\lambda \in \Lambda_j$ into 2^{dm_0} cubes at the scale $j + m_0$ and for each $n \in \{1, \ldots, 2^{dm_0}\}$, choose the unique subcube $\lambda^{(n)}$ of λ such that $n \neq n'$ implies $\lambda^{(n)} \neq \lambda^{(n')}$. From this, we can consider a function $g^{(n)}$ such that its wavelet coefficients c_λ satisfy the following conditions:

$$c_{\lambda^{(n)}} := j^{-a_0} 2^{jd/r} 2^{-j(\lambda)d/r} \sigma_j^{-1} \quad \text{if } \lambda \in \Lambda_j \cap [0, 1]^d,$$

with $a_0 := 1 + 1/r + 1/s$ and $c_\lambda := 0$ if λ is not of the form $\lambda^{(n)}$ for some n.

Proposition 4.15. *For all $n \in \{1, \ldots, 2^{dm_0}\}$, $g^{(n)}$ belongs to $B^{\sigma}_{r,s}$.*

Proof. For $j \geqslant 1$, we have

$$\left(\sum_{\lambda\in\Lambda_{j+m_0}} (\sigma_{j+m_0} 2^{-(j+m_0)d/r} |c_\lambda|)^r\right)^{1/r}$$
$$= \left(\sum_{l=0}^{j} \sum_{\substack{\lambda\in\Lambda_j \cap [0,1]^d \\ j(\lambda)=l}} (\sigma_{j+m_0} 2^{-(j+m_0)d/r} j^{-a_0} 2^{jd/r} 2^{-ld/r} \sigma_j^{-1})^r\right)^{1/r}$$

and

$$\left(\sum_{\lambda\in\Lambda_{j+m_0}} (\sigma_{j+m_0} 2^{-(j+m_0)d/r} |c_\lambda|)^r\right)^{1/r} \leqslant \left(\sum_{l=0}^{j} (\overline{\sigma}_{m_0} 2^{-(j+m_0)d/r} j^{-a_0})^r\right)^{1/r}$$
$$\leqslant C j^{-a_0+1/r}.$$

As $a_0 > 1/r + 1/s$, we get

$$\left(\sum_{j\geqslant 1}\left(\sum_{\lambda\in\Lambda_{j+m_0}} (\sigma_{j+m_0} 2^{-(j+m_0)d/r} |c_\lambda|)^r\right)^{s/r}\right)^{1/s} \leqslant C\left(\sum_{j\geqslant 1} j^{-s(a_0-1/r)}\right)^{1/s} < \infty,$$

which is sufficient to conclude the assertion. □

Definition 4.16. Let $\alpha \geqslant 1$; a point $x_0 \in [0, 1]^d$ is α-approximable by dyadics if there exists two sequences $\boldsymbol{k}$ and $\boldsymbol{j}$ of natural numbers with $k_n < 2^{j_n}$ for any $n \in \mathbb{N}_0$ such that

$$\|x_0 - \frac{k_n}{j_n}\|_\infty \leqslant \frac{1}{2^{\alpha j_n}},$$

for any $n \in \mathbb{N}_0$.

Let us denote the set of points of $[0,1]^d$ which are α-approximable by dyadics by E^α and define

$$E_j^\alpha := \left\{ x_0 \in [0,1]^d : \exists\, k \in \{0,\ldots,2^j-1\}^d \text{ such that } \|x_0 - \frac{k}{2^j}\|_\infty \leqslant \frac{1}{2^{\alpha j}} \right\},$$

so that $E^\alpha = \limsup_j E_j^\alpha$. We also define

$$E_{j,k}^\alpha := \left\{ x_0 \in [0,1]^d : \|x_0 - \frac{k}{2^j}\|_\infty \leqslant \frac{1}{2^{\alpha j}} \right\},$$

for $k \in \{0,\ldots,2^j-1\}^d$, in order to have

$$E_j^\alpha = \bigcup_{l \in \{0,\ldots,2^j-1\}^d} E_{j,k}^\alpha.$$

Finally, set $E^\infty = \cap_{\alpha \geqslant 1} E^\alpha$; this set in not empty since it contains the dyadic numbers.

Proposition 4.17. *Given $C > 0$, $j \in \mathbb{N}_0$ and $k \in \{0,\ldots,2^j-1\}^d$, the set*

$$F_{j,k}^{\alpha,C}(h) := \{ f \in B_{r,s}^{\boldsymbol{\sigma}} : (\exists\, x \in E_{j,k}^\alpha : \forall\, n \in \mathbb{N}_0\, \forall\, \lambda \in \Lambda_n \cap \mathcal{K}_x(2^{m_0+1}), |c_\lambda| \leqslant C/\gamma_n^{(h)}) \}$$

is closed in $B_{r,s}^{\boldsymbol{\sigma}}$.

Proof. Let $(f_l)_l$ be a sequence of functions of $F_{j,k}^{\alpha,C}$ such that $f_l \to f$ in $B_{r,s}^{\boldsymbol{\sigma}}$ and denote by $c_\lambda^{(l)}$ (resp. c_λ) the wavelet coefficients of f_l (resp. f). Since

$$B_{r,s}^{\overline{s}(\boldsymbol{\sigma})+\gamma} \hookrightarrow B_{r,s}^{\boldsymbol{\sigma}} \hookrightarrow B_{r,s}^{\underline{s}(\boldsymbol{\sigma})-\gamma},$$

for any $\gamma > 0$ and as the application which associates to a function its wavelet coefficients is continuous in the Besov spaces, we have $c_\lambda^{(l)} \to c_\lambda$ for all $\lambda \in \Lambda$.

For $l \in \mathbb{N}_0$, let $x_l \in E_{j,k}^\alpha$ be such that, for all $n \in \mathbb{N}_0$ and $\lambda \in \Lambda_n \cap \mathcal{K}_{x_l}(2^{m_0+1})$, we have $|c_\lambda^{(l)}| \leqslant C/\gamma_n^{(h)}$. As $E_{j,k}^\alpha$ is compact, we can suppose that the sequence $(x_j)_l$ converges to a point x_0 of $E_{j,k}^\alpha$. Now, let us fix $N \in \mathbb{N}_0$ and $\delta > 0$; if l is sufficiently large, we have $\mathcal{K}_{x_0}(2^{m_0+1}) \subset K_{x_l}(2^{m_0+1})$ and, for $n \leqslant N$, we have, for $\lambda \in \Lambda_n \cap \mathcal{K}_{x_l}(2^{m_0+1})$, $|c_\lambda^{(l)} - c_\lambda| \leqslant \delta/\gamma_n^{(h)}$ as $c_\lambda^{(l)}$ converges to c_λ. Also, we have $|c_\lambda^{(l)}| \leqslant C/\gamma_n^{(h)}$ for $\lambda \in \Lambda_n \cap \mathcal{K}_{x_l}(2^{m_0+1})$. As a consequence, $\lambda \in \Lambda_n \cap \mathcal{K}_{x_0}(2^{m_0+1})$ implies

$$|c_\lambda| \leqslant (C+\delta)/\gamma_n^{(h)},$$

for all $n \leqslant N$. Taking the limit for $N \to \infty$ and $\delta \to 0^+$ leads to $f \in F_{j,k}^{\alpha,C}(h)$. □

Let us set

$$F_j^{\alpha,C}(h) := \bigcup_{k \in \{0,\ldots,2^j-1\}^d} F_{j,k}^{\alpha,C}(h)$$

and $F^{\alpha,C}(h) := \limsup_j F_j^{\alpha,C}(h)$. All these sets are obviously Borel sets.

Proposition 4.18. *The set $F^{\alpha,C}(h)$ is a Haar-null Borel set.*

Proof. Set $m_1 := 2^{m_0 d}$ and let us fix $j \in \mathbb{N}_0$ and $k \in \{0, \ldots, 2^j - 1\}$; for $f \in B^{\boldsymbol{\sigma}}_{r,s}$, suppose that there exist two points of $\mathbb{R}^{m_1}$, $a^{(1)} = (a^{(1)}_1, \ldots, a^{(1)}_{m_1})$ and $a^{(2)} = (a^{(2)}_1, \ldots, a^{(2)}_{m_1})$, such that

$$f_l := f + \sum_{m=1}^{m_1} a^{(l)}_m g^{(m)}$$

belongs to $F^{\alpha,C}_{j,k}(h)$ $(l \in \{1, 2\})$. For $l \in \{1, 2\}$, let us also denote by $c^{(l)}_\lambda$ the wavelet coefficient of f_l associated to the dyadic cube $\lambda \in \Lambda$ and let x_l be a point of $E^\alpha_{j,k}$ such that $\lambda \in \Lambda_{\lfloor \alpha j \rfloor} \cap \mathcal{K}_{x_l}(2^{m_0+1})$ implies $|c^{(l)}_\lambda| \leqslant C/\gamma^{(h)}_{\lfloor \alpha j \rfloor}$.

For $\lambda' \in \Lambda_{\lfloor \alpha j \rfloor + m_0}$ satisfying $\lambda' \subset \lambda^{(i)}_{\lfloor \alpha j \rfloor, k}$, we have

$$|c^{(l)}_{\lambda'}| \leqslant C/\gamma^{(h)}_{\lfloor \alpha j \rfloor + m_0}.$$

As a consequence, we get, by denoting $c'^{(m)}_\lambda$ the wavelet coefficient of $g^{(m)}$ associated to λ,

$$|a^{(1)}_m - a^{(2)}_m| = |a^{(1)}_m - a^{(2)}_m|\, |c'^{(m)}_{\lambda^{(m)}}|/|c'^{(m)}_{\lambda^{(m)}}| \leqslant 2C/(\gamma^{(h)}_{\lfloor \alpha j \rfloor + m_0} |c'^{(m)}_{\lambda^{(m)}}|),$$

for any $m \in \{1, \ldots, m_1\}$. On the other hand, for $j \geqslant j(\lambda)$, we have

$$|c'^{(m)}_{\lambda^{(n)}}| = \lfloor \alpha j \rfloor^{-a_0} 2^{\lfloor \alpha j \rfloor d/q} 2^{-j(\lambda) d/q} \sigma^{-1}_{\lfloor \alpha j \rfloor} \geqslant C' \lfloor \alpha j \rfloor^{-a_0} 2^{\lfloor \alpha j \rfloor d/q} 2^{-\lfloor \alpha j \rfloor d/\alpha q} \sigma^{-1}_{\lfloor \alpha j \rfloor},$$

so that there exists a constant $C'' > 0$ for which

$$\|a^{(1)} - a^{(2)}\|_\infty \leqslant C'' \lfloor \alpha j \rfloor^{-a_0} 2^{\lfloor \alpha j \rfloor (d/\alpha q - d/q)} \sigma_{\lfloor \alpha j \rfloor}/\gamma^{(h)}_{\lfloor \alpha j \rfloor}. \tag{17}$$

That being done, for $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have

$$\begin{aligned}\{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}(h)\} &\subset \bigcup_{j \geqslant J} \{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}_j(h)\} \\ &\subset \bigcup_{j \geqslant J} \bigcup_{k \in \{0,\ldots,2^j-1\}^d} \{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}_{j,k}(h)\},\end{aligned}$$

for any $J \in \mathbb{N}_0$. Thus, from (17), we get

$$\begin{aligned}\mathcal{L}(\{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}(h)\}) &\leqslant \sum_{j \geqslant J} 2^{jd} (C'' \lfloor \alpha j \rfloor^{a_0} 2^{\lfloor \alpha j \rfloor (d/\alpha q - d/q)} \sigma_{\lfloor \alpha j \rfloor}/\gamma^{(h)}_{\lfloor \alpha j \rfloor})^M \\ &\leqslant C''' \sum_{j \geqslant J} \lfloor \alpha j \rfloor^{a_0 m_1} 2^{j(d - m_1 \alpha(\zeta(h) - d/\alpha q - d/q - \varepsilon_0))}.\end{aligned}$$

Letting J going to ∞, (16) implies

$$\mathcal{L}(\{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}(h)\}) = 0,$$

hence the conclusion. □

Theorem 4.19. *Let $p, q, r, s \in [1, \infty]$ with $s \leqslant q$, $\boldsymbol{\sigma}$ be an admissible sequence and $\boldsymbol{\gamma}^{(\cdot)}$ be a family of admissible sequences compatible with $\boldsymbol{\sigma}$. From the prevalence point of view, for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, $D_{p,q}$ is defined on $I = [\zeta^{-1}(-d/r), \zeta^{-1}(0)]$ and*

$$D_{p,q}(h) = d + r\zeta(h),$$

for any $h \in I$.

Moreover, for almost every $x_0 \in \mathbb{R}^d$*, we have* $h_{p,q}(x_0) = \zeta^{-1}(0)$.

Proof. We know that

$$\{f \in B^{\boldsymbol{\sigma}}_{r,s} : (\exists\, x_0 \in E^\alpha : f \in T^{\boldsymbol{\sigma}^{(h)}}_{p,q}(x_0)\} \subset \bigcup_{l \in \mathbb{N}_0} F^{\alpha,l}(h),$$

so that, for any $\alpha \geqslant 1$ and any $h > h_*(\alpha)$, for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have $h_{p,q}(x_0) \leqslant h$ for every $x_0 \in E^\alpha$. By countable intersection, we thus get that for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have $h_{p,q}(x_0) \leqslant h(\alpha)$ for every $x_0 \in E^\alpha$. Let $f \in B^{\boldsymbol{\sigma}}_{r,s}$ be such that the preceding assertion holds.

First, let us fix $\alpha \in (1, \infty)$; if $\boldsymbol{\alpha}$ is an increasing sequence of rational numbers converging to α, the sequence $(E^{\alpha_n})_n$ is decreasing and $E^\alpha \subset \cup_n E^{\alpha_n}$. If x_0 belongs to E^{α_n}, we have $h_{p,q}(x_0) \leqslant h_*(\alpha_n)$ and thus $h_{p,q}(x_0) \leqslant h_*(\alpha)$, for every $x_0 \in E^\alpha$. Let μ_α be a measure such that

- $\mathrm{supp}(\mu_\alpha) \subset E^\alpha$,
- $\mu_\alpha(E^\alpha) > 0$,
- $\mu_\alpha(F) = 0$ whenever $\dim_{\mathcal{H}}(F) < d/\alpha$,

let us define

$$F^\alpha := \{x_0 \in [0,1]^d : h_{p,q}(x_0) < h_*(\alpha)\}$$

and, for $n \in \mathbb{N}_0$,

$$F^\alpha_n := \{x_0 \in [0,1]^d : h_{p,q}(x_0) < h_*(\alpha) - 1/n\}.$$

For n large enough, we have $h(\alpha) - 1/n \geqslant -d/p$ and thus $\dim_{\mathcal{H}}(F^\alpha_n) < d/\alpha$. Since F^α is included in a countable union of μ_α-measurable null sets, we have $\mu_\alpha(F^\alpha) = 0$. As a consequence, we have

$$\mu_\alpha(E^\alpha \backslash F^\alpha) \geqslant d + r\zeta(h_*(\alpha)).$$

Since

$$E^\alpha \backslash F^\alpha \subset \{x_0 \in [0,1]^d : h_{p,q}(x_0) = h_*(\alpha)\},$$

we get

$$D(h_*(\alpha)) = d + r\zeta(h_*(\alpha)).$$

If $\alpha = \infty$, we know that $x_0 \in E^\infty$ implies $h_{p,q}(x_0) \leqslant h_*(\alpha_n)$ for any $n \in \mathbb{N}_0$ and thus $h_{p,q}(x_0) \leqslant h_{\min}(r)$. As a consequence, the set

$$\{x_0 \in [0,1]^d : h_{p,q}(x_0) = h_{\min}(r)\}$$

is not empty.

It remains to consider the case $\alpha = 1$. In this case, $E^1 = [0,1]^d$ and μ_1 can be chosen to be the Lebesgue measure restricted on $[0,1]^d$. For $x_0 \in E^1$, $h_{p,q}(x_0) \leqslant h_*(1)$ and by the same argument as in the first case, we get

$$\mu_1(\{x_0 \in [0,1]^d : h_{p,q}(x_0) < h_*(1)\}) = 0,$$

so that E^1 is equal to $E^1 \backslash F^1$ almost everywhere.

As the proof can be easily adapted to any translated of $[0,1]^d$, the conclusion follows by countable intersection. □

Theorem 4.20. *Let $p, q, r, s \in [1, \infty]$ with $s \leqslant q$, $\boldsymbol{\sigma}$ be an admissible sequence and $\boldsymbol{\gamma}^{(\cdot)}$ be a family of admissible sequences compatible with $\boldsymbol{\sigma}$. Let x_0 be a point of $\mathbb{R}^d$; from the prevalence point of view, for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have $h_{p,q}(x_0) = \zeta^{-1}(-d/r)$.*

Proof. Given $n \in \mathbb{N}_0$, let us define the admissible sequence $\boldsymbol{\theta}^{(n)}$ by

$$\theta_j^{(n)} := \frac{1}{\gamma_j^{(\zeta^{-1}(-d/r)+1/n)}} \frac{1}{(j+1)^{1+1/s}},$$

$j \in \mathbb{N}_0$. We can now define the function $g^{(n)}$ which is a function whose wavelet coefficients are

$$c_\lambda^{(n)} := \begin{cases} \theta_j^{(n)} & \text{if} \quad \lambda \in \Lambda_j \quad \text{and} \quad \lambda = \lambda_j(x_0) \\ 0 & \text{if} \quad \lambda \in \Lambda_j \quad \text{and} \quad \lambda \neq \lambda_j(x_0) \end{cases}.$$

Since, for $n \in \mathbb{N}_0$, there exists $C_n > 0$ such that

$$\left(\sum_{\lambda \in \Lambda_j} (\sigma_j 2^{jd/r} |c_\lambda^{(n)}|)^r \right)^{1/r} \leqslant C_n/(j+1)^{1+1/s},$$

$g^{(n)}$ belongs to $B^{\boldsymbol{\sigma}}_{r,s}$.

Let us fix $n_0 \in \mathbb{N}_0$ and define

$$F_{n_0} := \{ f \in B^{\boldsymbol{\sigma}}_{r,s} : \forall j \in \mathbb{N}_0 \ \forall \ \lambda \in \Lambda_j \cap \mathcal{K}_{x_0}(2), |c_\lambda| \leqslant n_0 \theta_j^{(n)}/j \}.$$

As shown before, F_{n_0} is a Borel set. For $f \in B^{\boldsymbol{\sigma}}_{r,s}$ and $a, a' \in \mathbb{R}$ satisfying $f + a g^{(n)} \in F_{n_0}$ and $f + a' g^{(n)} \in F_{n_0}$, we get

$$|a - a'| \leqslant 2n_0/j,$$

so that the Lebesgue measure of $\{a \in \mathbb{R} : f + a g^{(n)} \in F_{n_0}\}$ vanishes, implying that F_{n_0} is Haar-null. As we have

$$\{ f \in B^{\boldsymbol{\sigma}}_{r,s} : f \in T^{\boldsymbol{\theta}^{(n)}}_{p,q}(x_0) \} \subset \bigcup_{l \in \mathbb{N}_0} F_l,$$

for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have $h_{p,q}(x_0) \leqslant \zeta^{-1}(-d/r) + 1/n$, which leads to the conclusion. □

Acknowledgments

Laurent Loosveldt is supported by a grant of the FNRS.

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