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Chapter 4

COMPUTING HYPERINCURSIVE DISCRETE RELATIVISTIC QUANTUM MAJORANA AND DIRAC EQUATIONS AND QUANTUM COMPUTATION

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ABSTRACT

Nobody understands quantum mechanics, said Richard Feynman. So, this paper will begins by a step by step presentation of the second order hyperincursive discrete harmonic oscillator that bifurcates to two incursive discrete oscillators with the conservation of a constant of motion. Then, we extend this formalism to the hyperincursive discrete Klein-Gordon equation bifurcates to the Majorana real 4-spinors and to the Dirac complex 4-spinors. Naturally, the hyperincursive discrete equations defines the

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relativistic quantum mechanics. When the time and space intervals of the discrete systems tend to zero, all these systems tend to 4 first order differential equations, representing spinors. In the Dirac generic equation, one discovers the Pauli spin matrices. The Pauli matrices X, Y, Z, are used as quantum gates for which the square are equal to the unit matrix I. The Pauli X-gate acts on a single qubit and is the quantum equivalent of the NOT gate for the classical computer. The square root of NOT defines also a quantum gate. More interesting is the Hadamard matrix that is the normalized sum of the X and Z Pauli matrices. Indeed, with the addition of the Hadamard gate to the classical computations the full quantum computation power is obtained.

Keywords: quantum computing, Majorana real spinors, Dirac complex spinors, hyperincursive discrete equations, incursive discrete equations

1. INTRODUCTION

This chapter deals with the continuous and discrete equations of the Harmonic Oscillator, and the Relativistic Quantum Majorana and Dirac equations.

We begin in section 2 with the presentation step by step of the two incursive discrete harmonic oscillator following my fundamental paper (Dubois, 1995) up-dated in my recent paper (Dubois, 2019f). I define a generalized forward-backward discrete derivative, depending on a weight with 3 values, applied to the time-dependent position and velocity of the harmonic oscillator. I deduce the first and the second incursive discrete harmonic oscillators, and the hyperincursive harmonic oscillator. Then I obtain what I called "the second order hyperincursive discrete harmonic oscillator" depending only on the time-dependent position.

The section 3 introduces the two dimensionless incursive discrete harmonic oscillators. Then I present the analytical synchronous solutions of these incursive discrete harmonic oscillators that are related to their constants of motion (Dubois, 2019f).

The section 4 deals with a rotation on the position and velocity of the incursive discrete harmonic oscillators, which gives rise to recursive discrete harmonic oscillators (Dubois, 2019c). This rotation matrix, with an angle of $\pi/4$, defines the second order Hadamard matrix, which is a fundamental gate in quantum computer. The two recursive discrete harmonic oscillators are then transformed to differential equations for small value of the interval of time. In defining a complex vector, we obtain the complex harmonic oscillator, with the Pauli matrix σ_y, which corresponds to the second Pauli quantum gate in quantum computer. Finally, we develop the chiral representation of this complex harmonic oscillator. With the unitary matrix U, the 2 recursive discrete harmonic oscillators are transformed to a complex recursive discrete harmonic oscillator. The same development was applied to the quantum Majorana equation (Dubois, 2019d).

The hyperincursive discrete equations were applied to various quantum systems (Dubois, 2016, 2018).

The section 5 deals with the bifurcation of the hyperincursive second order discrete Klein-Gordon equation to the discrete Majorana quantum relativistic equations and the real 4-spinors Majorana differential equations are obtained when the spacetime intervals tend to zero (Dubois, 2019a). Then we demonstrate, with an original method based on real 2-spinors matrices that the Majorana real 4-spinors equations bifurcate simply to the Dirac real 8-spinors equations, which are transformed to the original Dirac complex 4-spinors equations (Dubois, 2019b). We present the 4 complex hyperincursive discrete Dirac equations. Let us notice that the real 2-spinors matrices are related to the three Pauli gates defined in technology of quantum computer.

The section 6 shows that the natural number of discrete wave functions of the hyperincursive second order discrete Klein-Gordon equation is equal to 16 discrete spacetime wave functions, instead of the classical 4 functions of Majorana and Dirac equations. My hyperincursive second order discrete Klein-Gordon equations are in agreement with the 16 wave functions of Dirac by Proca (Dubois, 2019b). Proca (1932) classified the 16 equations in 4 groups of 4 functions. There are 4 fundamental equations and the other 3x4 equations are similar to these 4 equations. But formally, only a theory with 16 solutions is the correct one, confirming the power of my hyperincursive second order discrete equations formalism.

Then the section 7 deals with the chiral representation of the Majorana equations in 2 components (Dubois, 2019d) with the same Hadamard matrix and Unitary matrix U used for the harmonic oscillator (Dubois, 2019c).

Next the section 8 gives the solutions of the non-relativistic chiral Majorana equation compared to the solution of the non-relativistic quantum Dirac equation (Dubois, 2019d).

Section 9 deals with the 2 coupled Majorana equations in one spatial dimension (1D), with the 3 Pauli matrices (Dubois, 2019e).

Then, the section 10 gives a remarkable relation between the Majorana and the Dirac equations in 1D (the y component), with just the inversion of the Dirac matrices α_y and β, based on the Pauli matrices σ_y and σ_0.

Next, the section 11 deals with the relation between the solutions of the non-relativistic Majorana and Dirac equations, which is given by a transformation relation given, surprisingly, by an invariant function (Dubois, 2019e) depending on the Pauli matrix σ_x.

Finally, the section 12 deals with a survey of the reversible gates used in quantum computation. The quantum Pauli gates X, Y, Z, that operate on one-qubit, are given by

$$\mathrm{X} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, = \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \text{ and } Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

More interesting is the rotation matrix

$$\mathrm{R}_1(\theta) = \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix},$$

that generates the Pauli X, Z gates and the Hadamard H_2 gate:

$$\mathrm{R}_1(0) = \mathrm{X}\,,\ \mathrm{R}_1\left(\frac{\pi}{2}\right) = \mathrm{Z}\,, \text{ and } \mathrm{R}_1\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} = H_2$$

with the addition of the reversible logic Toffoli gate to the Hadamard gate, the full quantum computation power of a quantum computer is obtained.

2. Presentation Step by Step of the Two Incursive Discrete Harmonic Oscillators

The harmonic oscillator can be represented by the two ordinary differential equations:

$$\mathrm{d}x(t)/\mathrm{d}t = v(t) \text{ and } \mathrm{d}v(t)/\mathrm{d}t = -\omega^2 x(t) \qquad (2.1\text{-a-b})$$

where $\mathrm{x}(t)$ is the position and $\mathrm{v}(t)$ the velocity as functions of the time t, and the pulsation ω is related to the spring constant k and the mass m by $\omega^2 = k/m$. The solution is given by

$$x(t) = \mathrm{x}(0)\cos(\omega t) + (v(0)/\omega)\sin(\omega t),$$

$$v(t) = -\omega x(0)\sin(\omega t) + v(0)\cos(\omega t) \qquad (2.1\text{-c-d})$$

with the initial conditions $x(0)$ and $v(0)$. The period of oscillations is given by $T = 2\pi/\omega$. The energy $e(t)$ of the harmonic oscillator is constant and is given by

$$e(t) = k\,x^2(t)/2 + m\,v^2(t)/2 = k\,x^2(0)/2 + m\,v^2(0)/2 = e(0) = e_0 \qquad (2.1\text{-e})$$

In the discrete form, there are the discrete current time t and the interval of time $\Delta\mathrm{t} = \mathrm{h}$. The discrete time is defined as $\mathrm{t_k} = \mathrm{t_0} + \mathrm{kh}, \mathrm{k} = 0,1,2, \ldots$ where t_0 is the initial value of the time and k is the counter of the number of intervals of time h. The discrete position and velocity variables are defined as $\mathrm{x(k)} = \mathrm{x(t_k)}$ and $\mathrm{v(k)} = \mathrm{v(t_k)}$.

In my paper (Dubois, 1995), up-dated in my recent paper (Dubois, 2019f), I defined a generalized forward-backward discrete derivative

$$\mathrm{D}_w = w\,\mathrm{D_f} + (1-w)\,\mathrm{D_b} \qquad (2.2)$$

where w is a weight taking the values between 0 and 1, and where the discrete forward and backward derivatives on a function f are defined by

$$D_f(f) = \Delta^+ f / \Delta t = (f(k+1) - f(k)) / h,$$
$$D_b(f) = \Delta^- f / \Delta t = (f(k) - f(k-1)) / h$$

The generalized incursive discrete harmonic oscillator is given by (Dubois, 1995) as:

$$(1 - w)\, x(k+1) + (2w - 1)\, x(k) - w\, x(k-1) = h\, v(k)$$

$$w\, v(k+1) + (1 - 2w)\, v(k) + (w - 1)\, v(k-1) = -\, h\, \omega^2 x(k) \tag{2.3-a-b}$$

When $w = 0$, $D_0 = D_b$, this gives the first incursive equations:

$$x(k+1) - x(k) = h\, v(k)$$

$$v(k) - v(k-1) = -\, h\, \omega^2 x(k) \tag{2.4-a-b}$$

When $w = 1$, $D_1 = D_f$, this gives the second incursive equations:

$$x(k) - x(k-1) = h\, v(k)$$

$$v(k+1) - v(k) = -\, h\, \omega^2 x(k) \tag{2.5-a-b}$$

When $w = 1/2$, $D_{1/2} = D_s = [D_f + D_b]/2$, this gives the hyperincursive equations:

$$x(k+1) - x(k-1) = +\, 2h\, v(k)$$
$$v(k+1) - v(k-1) = -\, 2h\, \omega^2 x(k) \tag{2.6-a-b}$$

where $D_s(f) = D_{1/2}(f) = [\, f(k+1) - f(k-1)]/2h$ defines a time-symmetric derivative, D_s.

In putting the velocity, v(k), of the first equation (2.6-a),

$$v(k) = [x(k+1) - x(k-1)]/2h,$$

to the second equation (2.6-b), one obtains

$$x(k+2) - 2x(k) + x(k-2) = -\,4h^2\omega^2 x(k) \tag{2.7-a}$$

what I called “the second order hyperincursive discrete harmonic oscillator” (Dubois, 2019f), corresponding to the second order differential equation of the harmonic oscillator, from equations (2.1-a-b), given by:

$$d^2x(t)/dt^2 = -\omega^2 x(t) \tag{2.7-b}$$

3. The Two Dimensionless Incursive Discrete Harmonic Oscillators

A series of papers were published on the incursive and hyperincursive discrete harmonic oscillator (Antippa and Dubois, 2004, 2006a, 2006b, 2007, 2008a, 2008b, 2010a, 2010b, 2010c).

For the discrete harmonic oscillator, let us use the dimensionless variables, X and V, for the variables, x and v, as follows (Antippa and Dubois, 2010c) : $X(k) = (k/2)^{1/2}\, x(k)$, $V(k) = (m/2)^{1/2}\, v(k)$, with the dimensionless time, $\tau = \omega t$, where the pulsation is given by $\omega = (k/m)^{1/2}$ and with the dimensionless interval of time given by $\Delta\tau = \omega\, \Delta t = \omega\, h = H$.

So, the equations (2.4-a-b) and (2.5-a-b) of the two incursive discrete harmonic oscillators are given respectively by the following two dimensionless incursive discrete equations

$$X_1(k+1) = X_1(k) + HV_1(k) \tag{3.1-a}$$

$$V_1(k+1) = V_1(k\,) - HX_1(k+1) \tag{3.1-b}$$

$$V_2(k+1) = V_2(k) - HX_2(k) \tag{3.2-a}$$

$$X_2(k+1) = \; X_2(k) + HV_2(k+1) \tag{3.2-b}$$

and the equations (2.6-a-b) of the hyperincursive discrete harmonic oscillator are given by the following dimensionless hyperincursive discrete equation

$$X(k+1) = X(k-1) + 2HV(k) \tag{3.3-a}$$

$$V(k+1) = V(k-1) - 2HX(k) \tag{3.3-b}$$

Let us recall that this hyperincursive discrete harmonic oscillator is a recursive computing system that is separable into the two incursive discrete harmonic oscillators (Dubois, 2019f).

It was demonstrated (Dubois, 2019f) that the following expression

$$K_1(k) = X_1(k)X_1(k+1) + V_1(k)V_1(k) = X_1^2(k) + V_1^2(k) + HX_1(k)V_1(k) \tag{3.4}$$

is a constant of motion of the first incursive equations (3.1-a-b), and that the following expression

$$K_2(k) = X_2(k)X_2(k) + V_2(k+1)V_2(k) = X_2^2(k) + V_2^2(k) - HX_2(k)V_2(k) \tag{3.5}$$

is a constant of motion of the second incursive equations (3.2-a-b).

These constants of motion differ with the inversion of the sign of the discrete time interval, H. The analytical synchronous solutions of the equations (3.1-a-b) and (3.2-a-b) are given by

$$X_1(k) = \cos(2k\pi/N) \text{ and } V_1(k) = -\sin((2k+1)\pi/N) \qquad (3.6\text{-a-b})$$

$$X_2(k) = \cos((2k+1)\pi/N) \text{ and } V_2(k) = -\sin(2k\pi/N) \qquad (3.6\text{-c-d})$$

where N is the number of iterations for a cycle of the oscillator, with the index of iterations $k = 0, 1, 2, 3, \ldots$, for which the interval of discrete time H depends of N, $H = 2\sin(\pi/N)$.

4. Rotation of the Incursive Discrete Oscillators to Recursive Discrete Oscillators

In the recent paper (Dubois, 2019c), it was demonstrated that rotations on the position and velocity variables give rise to a pure quadratic expression of the constants of motion (3.4, 3.5), similarly to the constant of energy of the classical continuous harmonic oscillator.

The constant of motion (3.4) is an expression of a quadratic curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (4.1)$$

with $A = 1, B = H, C = 1, D = 0, E = 0, F = -K_1, x = X_1(k), y = V_1(k)$

The discriminant, $\Delta = B^2 - 4AC =$ INV, is an invariant under rotations.

The discriminant of the constant of motion (3.4):

$$\Delta = B^2 - 4AC = H^2 - 4 < 0\,,$$

defines an ellipse.

This inequality gives the maximum value of the discrete interval of time, $H = \omega\, \Delta t \ < 2$, with $\mathrm{H} = 2\sin(\pi/\mathrm{N})$.

The equations for the rotation are given by

$$\mathrm{X_1(k) = \cos(\theta)\, u_1(k) - \sin(\theta)\, v_1(k)} \tag{4.2-a}$$

$$\mathrm{V_1(k) = \sin(\theta)\, u_1(k) + \cos(\theta)\, v_1(k)} \tag{4.2-b}$$

or, in matrix form, the rotation matrix $\mathrm{R_1(\theta)}$ is given by

$$\begin{pmatrix} \mathrm{V_1(k)} \\ \mathrm{X_1(k)} \end{pmatrix} = \mathrm{R_1(\theta)} \begin{pmatrix} \mathrm{u_1(k)} \\ \mathrm{v_1(k)} \end{pmatrix} = \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} \mathrm{u_1(k)} \\ \mathrm{v_1(k)} \end{pmatrix} \tag{4.3-a}$$

with $A = C$, $\theta = \pi/4$, so $\cos(\pi/4) = 2^{-1/2} = \rho$, $\sin(\pi/4) = 2^{-1/2} = \rho$.

So the equations (4.2-a-b) of the rotation are transformed to

$$\mathrm{X_1(k)} = \big(\mathrm{u_1(k) - v_1(k)}\big)/\sqrt{2} \text{ and } \mathrm{V_1(k)} = \big(\mathrm{u_1(k) + v_1(k)}\big)/\sqrt{2} \tag{4.2-c-d}$$

or, in matrix form, the rotation matrix $\mathrm{R_1}(\pi/4) = \mathrm{H_2}$, is given by

$$\begin{pmatrix} \mathrm{V_1(k)} \\ \mathrm{X_1(k)} \end{pmatrix} = \mathrm{H_2} \begin{pmatrix} \mathrm{u_1(k)} \\ \mathrm{v_1(k)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{u_1(k)} \\ \mathrm{v_1(k)} \end{pmatrix} \tag{4.3-b}$$

with the 2×2 Hadamard matrix H_2, for which, $H_2 H_2 = I_2$,

$$\mathrm{H_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix},$$

$$\mathrm{H_2 H_2} = \frac{1}{2} \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} = \begin{pmatrix} +1 & +0 \\ +0 & +1 \end{pmatrix} = I_2 = 1$$

where I_2 is the 2-Identity matrix.

So, with equations (4.2-a-b), the constant of motion (3.4) becomes a pure quadratic form

$$u_1^2(k) + v_1^2(k) + H(u_1^2(k) - v_1^2(k))/2 = K_1(k) = K_1 \qquad (4.4\text{-a})$$

where $u_1(k)$ and $v_1(k)$ are defined by adding and subtracting the equations (4.2-c-d)

$u_1(k) = (X_1(k) + V_1(k))/\sqrt{2}$ and $v_1(k) = (V_1(k) - X_1(k))/\sqrt{2}$,

or, in matrix form,

$$\begin{pmatrix} u_1(k) \\ v_1(k) \end{pmatrix} = H_2 \begin{pmatrix} V_1(k) \\ X_1(k) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \begin{pmatrix} V_1(k) \\ X_1(k) \end{pmatrix} \qquad (4.3\text{-c})$$

Now let us make the rotation to the first incursive oscillator (3.1-a-b)

$$\big(u_1(k+1) - v_1(k+1)\big) = (u_1(k) - v_1(k)) + H(u_1(k) + v_1(k))$$
$$\big(u_1(k+1) + v_1(k+1)\big) = \big(u_1(k) + v_1(k)\big) - H(u_1(k) - v_1(k)) - H^2(u_1(k) + v_1(k))$$

In adding and subtracting these two equations, the first incursive discrete oscillator becomes:

$$u_1(k+1) = u_1(k) + H\, v_1(k) - H^2(u_1(k) + v_1(k))/2 \qquad (4.5\text{-a})$$

$$v_1(k+1) = v_1(k) - H\, u_1(k) - H^2(u_1(k) + v_1(k))/2 \qquad (4.5\text{-b})$$

defining the first recursive discrete oscillator. For the second incursion, the constant of motion (3.5) is obtained by inversion the sign of H:

$$u_2^2(k) + v_2^2(k) - H(u_2^2(k) - v_2^2(k))/2 = K_2(k) = K_2 \qquad (4.4\text{-b})$$

that is also a pure quadratic function. Indeed, with a similar rotation

$$X_2(k) = \sin(\theta)\, u_2(k) + \cos(\theta)\, v_2(k) \qquad (4.6\text{-a})$$

$$V_2(k) = \cos(\theta)\, u_2(k) - \sin(\theta)\, v_2(k) \qquad (4.6\text{-b})$$

or, in matrix form, the rotation matrix $R_2(\theta)$ is given by

$$\begin{pmatrix} X_2(k) \\ V_2(k) \end{pmatrix} = R_2(\theta)\begin{pmatrix} u_2(k) \\ v_2(k) \end{pmatrix} = \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix}\begin{pmatrix} u_2(k) \\ v_2(k) \end{pmatrix} \qquad (4.7\text{-a})$$

For $\theta = \pi/4$, so $\cos(\pi/4) = 2^{-1/2} = \rho$ and $\sin(\pi/4) = 2^{-1/2} = \rho$

$X_2(k) = \big(u_2(k) + v_2(k)\big)\sqrt{2}$ and $V_2(k) = \big(u_2(k) - v_2(k)\big)/\sqrt{2}$

(4.6-c-d)

or, in matrix form the rotation $R_2(\pi/4) = H_2$ is given by

$$\begin{pmatrix} X_2(k) \\ V_2(k) \end{pmatrix} = H_2\begin{pmatrix} u_2(k) \\ v_2(k) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} u_2(k) \\ v_2(k) \end{pmatrix} \qquad (4.7\text{-b})$$

with the Hadamard matrix. So, by adding and subtracting the equations (4.6-c-d), we obtain

$u_2(k) = (X_2(k) + V_2(k))/\sqrt{2}$ and $v_2(k) = (X_2(k) - V_2(k))\sqrt{2}$,

or, in matrix form

$$\begin{pmatrix} u_2(k) \\ v_2(k) \end{pmatrix} = H_2\begin{pmatrix} X_2(k) \\ V_2(k) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}\begin{pmatrix} X_2(k) \\ V_2(k) \end{pmatrix} \qquad (4.7\text{-c})$$

Now let us make the rotation to the second incursive oscillator (3.2-a-b)

$\big(u_2(k+1) - v_2(k+1)\big) = \big(u_2(k) - v_2(k)\big) - H\big(u_2(k) + v_2(k)\big)$

$\big(u_2(k+1) + v_2(k+1)\big) = \big(u_2(k) + v_2(k)\big) + H\big(u_2(k) - v_2(k)\big) - H^2\big(u_2(k) + v_2(k)\big)$

and the sum and the difference of which give the second recursive discrete oscillator

$$u_2(k+1) = u_2(k) - H\, v_2(k) - H^2(u_2(k) + v_2(k))/2 \qquad (4.8\text{-a})$$

$$v_2(k+1) = v_2(k) + H\, u_2(k) - H^2(u_2(k) + v_2(k))/2 \qquad (4.8\text{-b})$$

These equations are the same as the equations of the first oscillator by inversion of the sign of H.

The discrete equations (4.5-a-b, 4.8-a-b) can be transformed to differential equations for small value of the interval of time

$$\partial_t u_1(t) = +\omega v_1(t) \text{ and } \partial_t v_1(t) = -\omega u_1(t) \qquad (4.9\text{-a-b})$$

$$\partial_t u_2(t) = -\omega v_2(t) \text{ and } \partial_t v_2(t) = +\omega u_2(t) \qquad (4.10\text{-a-b})$$

where $\partial_t u(t) = \partial u(t)/\, \partial t$ is the time derivative.

And the conversion to the original variables, with the equations (4.3-c) and (4.7-c), are given by

$\partial_t(X_1(t) + V_1(t)) = \omega(V_1(t) - X_1(t))$,
$\partial_t(V_1(t) - X_1(t)) = -\omega(X_1(t) + V_1(t))$

$\partial_t(X_2(t) + V_2(t)) = -\omega(X_2(t) - V_2(t))$,
$\partial_t(X_2(t) - V_2(t)) = \omega(X_2(t) + V_2(t)$

then, the sum and the difference of these equations give

$$\partial_t V_1(t) = -\omega X_1(t) \text{ and } \partial_t X_1(t)) = +\omega V_1(t) \qquad (4.11\text{-a-b})$$

$$\partial_t X_2(t) = +\omega V_2(t) \text{ and } \partial_t V_2(t) = -\omega X_2(t) \qquad (4.12\text{-a-b})$$

In defining the complex variables

$$u(t) = (u_1(t) - iu_2(t))/\sqrt{2} \text{ and } v(t) = (v_2(t) + iv_1(t))/\sqrt{2} \qquad (4.13\text{-a-b})$$

the 4 real equations are reduced to 2 complex equations

$$\partial_t(u_1(t) - iu_2(t)) = \omega(v_1(t) + iv_2(t)) \text{ and}$$

$$\partial_t(v_2(t) + iv_1(t)) = \omega(u_2(t) - iu_1(t)), \text{ or}$$

$$\partial_t u(t) = +i\omega v^*(t) \text{ and } \partial_t v(t) = -i\omega u^*(t) \quad (4.14\text{-a-b})$$

where the star sign corresponds to the complex conjugate

$$u^*(t) = (u_1(t) + iu_2(t))/\sqrt{2} \text{ and } v^*(t) = (v_2(t) - iv_1(t))/\sqrt{2}$$

We then obtain the second time derivative of the complex harmonic oscillator

$$\partial_t^2 u(t) = -\omega^2 u(t) \text{ and } \partial_t^2 v(t) = -\omega^2 v(t) \quad (4.15\text{-a-b})$$

In defining

$$w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad (4.16)$$

we obtain, with the Pauli matrix

$$\sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix},$$

the two equations

$$\partial_t w(t) = -\omega\sigma_y w^*(t) \text{ and } \partial_t w^*(t) = +\omega\sigma_y w(t) \quad (4.17\text{-a-b})$$

and the second time derivative is given by

$$\partial_t^2 w(t) = -\omega^2 w(t) \tag{4.17-c}$$

the solution of which being

$$w(t) = \cos(\omega t)\ w(0) - \sin(\omega t)\, \sigma_y\, w^*(0) \tag{4.17-d}$$

In defining a unitary matrix $U = U_R + iU_I$, we can write the transformations of the position and velocity of the discrete harmonic oscillator as follows

$$U_R \begin{pmatrix} V_1 \\ X_2 \\ V_2 \\ X_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} +1 & 0 & 0 & +1 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ +1 & 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} V_1 \\ X_2 \\ V_2 \\ X_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} +X_1 + V_1 \\ +X_2 - V_2 \\ -X_2 + V_2 \\ +X_1 + V_1 \end{pmatrix} =$$
$$\frac{1}{\sqrt{2}} \begin{pmatrix} +u_1 \\ +v_2 \\ -v_2 \\ +u_1 \end{pmatrix} = W_R \tag{4.18-a}$$

$$U_I \begin{pmatrix} V_1 \\ X_2 \\ V_2 \\ X_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -1 & -1 & 0 \\ +1 & 0 & 0 & -1 \\ +1 & 0 & 0 & -1 \\ 0 & +1 & +1 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ X_2 \\ V_2 \\ X_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -X_2 - V_2 \\ -X_1 + V_1 \\ -X_1 + V_1 \\ +X_2 + V_2 \end{pmatrix} =$$
$$\frac{1}{\sqrt{2}} \begin{pmatrix} -u_2 \\ +v_1 \\ +v_1 \\ +u_2 \end{pmatrix} = W_I \tag{4.18-b}$$

The chiral representation is related to the unitary matrix U

$$U = U_R + iU_I = \frac{1}{2} \begin{pmatrix} \sigma_0 + \sigma_y & -i(\sigma_0 - \sigma_y) \\ i(\sigma_0 - \sigma_{y)} & \sigma_0 + \sigma_y \end{pmatrix} \tag{4.19}$$

with the property $UU^* = U^*U = 1$.
So, we obtain the complex chiral representation

$$\mathrm{U}\begin{pmatrix} \mathrm{V}_1 \\ \mathrm{X}_2 \\ \mathrm{V}_2 \\ \mathrm{X}_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} +1 & -\mathrm{i} & -\mathrm{i} & +1 \\ +\mathrm{i} & +1 & -1 & -\mathrm{i} \\ +\mathrm{i} & -1 & +1 & -\mathrm{i} \\ +1 & +\mathrm{i} & +\mathrm{i} & +1 \end{pmatrix}\begin{pmatrix} \mathrm{V}_1 \\ \mathrm{X}_2 \\ \mathrm{V}_2 \\ \mathrm{X}_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} +\mathrm{u}_1 - \mathrm{iu}_2 \\ +\mathrm{v}_2 - \mathrm{iv}_1 \\ -\mathrm{v}_2 - \mathrm{iv}_1 \\ +\mathrm{u}_1 + \mathrm{iu}_2 \end{pmatrix} = \begin{pmatrix} +\mathrm{u} \\ +\mathrm{v} \\ -\mathrm{v}^* \\ +\mathrm{u}^* \end{pmatrix} = W \tag{4.18-c}$$

where we define the general function W separated in the left $\mathrm{w_L}$ and right $\mathrm{w_R}$ chiral functions

$$W = \begin{pmatrix} \mathrm{w_L} \\ \mathrm{w_R} \end{pmatrix} = \begin{pmatrix} \mathrm{w} \\ -i\sigma_y \mathrm{w}^* \end{pmatrix} \tag{4.20-a}$$

$$\mathrm{w_L} = \mathrm{w} = \begin{pmatrix} +\mathrm{u} \\ +\mathrm{v} \end{pmatrix} \text{ and } \mathrm{w_R} = -i\sigma_y \mathrm{w}^* = \begin{pmatrix} -\mathrm{v}^* \\ +\mathrm{u}^* \end{pmatrix} \tag{4.20-b-c}$$

The analytical solutions of the first incursive discrete equations are given by

$$\mathrm{X}_1(\mathrm{k}) = \cos(2\mathrm{k}\pi/\mathrm{N}) \text{ and } \mathrm{V}_1(\mathrm{k}) = -\sin((2\mathrm{k}+1)\pi/\mathrm{N}) \tag{4.21-a-b}$$

so, with the relations

$$\mathrm{u}_1(\mathrm{k}) = (\mathrm{X}_1(\mathrm{k}) + \mathrm{V}_1(\mathrm{k}))/\sqrt{2} \text{ and } \mathrm{v}_1(\mathrm{k}) = (\mathrm{V}_1(\mathrm{k}) - \mathrm{X}_1(\mathrm{k}))/\sqrt{2} \tag{4.22-a-b}$$

the functions $\mathrm{u}_1(\mathrm{k})$ and $\mathrm{v}_1(\mathrm{k})$ become

$$\begin{aligned} \mathrm{u}_1(\mathrm{k}) &= [+\cos(2\mathrm{k}\pi/\mathrm{N}) - \sin((2\mathrm{k}+1)\pi/\mathrm{N})]/\sqrt{2} \\ &= +\sqrt{2}\cos(\pi/4 + \pi/2\mathrm{N})\sin(\pi/4 - 2\mathrm{k}\pi/\mathrm{N} - \pi/2\mathrm{N}) \end{aligned} \tag{4.23-a}$$

$$\begin{aligned} \mathrm{v}_1(\mathrm{k}) &= [-\sin((2\mathrm{k}+1)\pi/\mathrm{N}) - \cos(2\mathrm{k}\pi/\mathrm{N})]/\sqrt{2} \\ &= -\sqrt{2}\sin\,(\pi/4 + \pi/2\mathrm{N})\cos(\pi/4 - 2\mathrm{k}\pi/\mathrm{N} - \pi/2\mathrm{N}) \end{aligned} \tag{4.23-b}$$

and the analytical solutions of the second incursive discrete equations are given by

$$X_2(k) = \cos((2k+1)\pi/N) \text{ and } V_2(k) = -\sin(2k\pi/N) \quad (4.21\text{-c-d})$$

so, with the relations

$$u_2(k) = (X_2(k) + V_2(k))/\sqrt{2} \text{ and } v_2(k) = (X_2(k) - V_2(k))/\sqrt{2} \quad (4.22\text{-c-d})$$

the functions $u_2(k)$ and $v_2(k)$ become

$$u_2(k) = \left[\cos\left((2k+1)\,\pi/N\right) - \sin(2k\pi/N)\right]/\sqrt{2}$$

$$= +\sqrt{2}\cos(\pi/4 - \pi/2N)\sin(\pi/4 - 2k\pi/N - \pi/2N) \quad (4.23\text{-c})$$

$$v_2(k) = [\cos((2k+1)\pi/N) + \sin(2k\pi/N)]/\sqrt{2}$$

$$= +\sqrt{2}\sin(\pi/4 - \pi/2N)\cos(\pi/4 - 2k\pi/N - \pi/2N) \quad (4.23\text{-d})$$

Finally, with

$$u(k) = u_1(k) - iu_2(k) \text{ and } v(k) = v_2(k) + iv_1(k) \quad (4.24\text{-a-b})$$

the discrete recursive harmonic oscillators are written as follows

$$u_1(k+1) - iu_2(k+1) = u_1(k) - iu_2(k) + H\,(v_1(k) + i\,v_2(k)) - H^2(u_1(k) - iu_2(k) + v_1(k) - iv_2(k))/2 \quad (4.25\text{-a})$$

$$v_2(k+1) + iv_1(k+1) = v_2(k) + iv_1(k) + H\,(u_2(k) - iu_1(k)) - H^2(u_2(k) + iu_1(k) + v_2(k) + iv_1(k))/2 \quad (4.25\text{-b})$$

So we obtain the complex discrete recursive harmonic oscillator

$$\mathrm{u(k+1) = u(k) + iHv^*(k) + iH^2(iu(k) + v(k))/2} \quad (4.26\text{-a})$$

$$\mathrm{v(k+1) = v(k) - iHu^*(k) - H^2(iu(k) + v(k))/2} \quad (4.26\text{-b})$$

In conclusion, we have demonstrated the transformation, by rotation with the Hadamard and unitary matrices, of the incursive discrete harmonic oscillators to recursive discrete harmonic oscillators. The same rotation will be applied to the Majorana equation (Dubois, 2019d).

5. The Hyperincursive Discrete Klein-Gordon Equation Bifurcates to the Majorana and Dirac Relativistic Quantum Equations

The Klein-Gordon equation (Oskar Klein, 1926, Walter Gordon, 1926) of the function $\varphi = \varphi(\mathbf{r}, \mathrm{t})$ in three spatial dimensions $\mathbf{r} = (\mathrm{x}, \mathrm{y}, \mathrm{z})$ and time t is given by

$$-\hbar^2\, \partial_{\mathrm{t}}^2 \varphi(\mathbf{r}, \mathrm{t}) = -\hbar^2\, c^2 \nabla^2 \varphi(\mathbf{r}, \mathrm{t}) + \mathrm{m}^2 \mathrm{c}^4 \varphi(\mathbf{r}, \mathrm{t}) \quad (5.1)$$

where $\partial_\mu \varphi = \partial\varphi/\partial\mu$, or, in the explicit form of the nabla operator ∇,

$$-\hbar^2\, \partial_{\mathrm{t}}^2 \varphi(\mathbf{r}, \mathrm{t}) = -\hbar^2\, c^2\, \partial_{\mathrm{x}}^2 \varphi(\mathbf{r}, \mathrm{t}) - \hbar^2\, c^2\, \partial_{\mathrm{y}}^2 \varphi(\mathbf{r}, \mathrm{t}) - \hbar^2\, c^2\, \partial_{\mathrm{z}}^2 \varphi(\mathbf{r}, \mathrm{t}) + \mathrm{m}^2 \mathrm{c}^4 \varphi \quad (5.2)$$

where $\partial_\mu^2 \varphi = \partial^2\varphi/\partial\mu^2$, $\hbar$ is the constant of Plank, c is the speed of light, and m the mass. From the Klein-Gordon equation, the relativistic quantum Dirac and Majorana equations can be deduced (Dirac, 1928, Majorana, 1937).

As we will consider the discrete Klein-Gordon equation, we make the following usual change of variables

$$q(\mathbf{r},\mathrm{t}) \;=\; \varphi(\mathbf{r},\mathrm{t}) \tag{5.3}$$

$$a = \omega = mc^2/\hbar \tag{5.4}$$

where ω is a frequency, so the Klein-Gordon equation (5.2) becomes

$$\partial^2 \mathrm{q}(\mathbf{r},\mathrm{t})/\,\partial \mathrm{t}^2 = +c^2\,\partial^2 \mathrm{q}(\mathbf{r},\mathrm{t})/\,\partial x^2 + c^2\,\partial^2 \mathrm{q}(\mathbf{r},\mathrm{t})/\,\partial y^2 + c^2\,\partial^2\,\mathrm{q}(\mathbf{r},\mathrm{t})/\,\partial z^2 - \,\mathrm{a}^2 \mathrm{q}(\mathbf{r},\mathrm{t}) \tag{5.5}$$

From the Klein-Gordon equation (5.5), the second order hyperincursive discrete Klein-Gordon equation (32, 35) is given by

$$\begin{aligned}&\mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}+2\Delta\mathrm{t}) - 2\mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}) + \mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}-2\Delta\mathrm{t}) = \\ &+\mathrm{B}^2[\mathrm{q}(\mathrm{x}+2\Delta\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}) - 2\mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}) + \mathrm{q}(\mathrm{x}-2\Delta\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t})] \\ &+\mathrm{C}^2[\mathrm{q}(\mathrm{x},\mathrm{y}+2\Delta\mathrm{y},\mathrm{z},\mathrm{t}) - 2\mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}) + \mathrm{q}(\mathrm{x},\mathrm{y}-2\Delta\mathrm{y},\mathrm{z},\mathrm{t})] \\ &+\mathrm{D}^2[\mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z}+2\Delta\mathrm{z},\mathrm{t}) - 2\mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}) + \mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z}-2\Delta\mathrm{z},\mathrm{t})] - \\ &\mathrm{A}^2\mathrm{q}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t})\end{aligned} \tag{5.6}$$

where the following parameters $\mathrm{A}, \mathrm{B}, \mathrm{C}, \text{and}, \mathrm{D}$,

$$\begin{aligned}&\mathrm{A} = \mathrm{a}\ (2\Delta\mathrm{t}), \mathrm{B} = \mathrm{c}\ (2\Delta\mathrm{t})/(2\Delta\mathrm{x}),\ \mathrm{C} = \mathrm{c}\ (2\Delta\mathrm{t})/(2\Delta\mathrm{y}), \\ &\mathrm{D} = \mathrm{c}\ (2\Delta\mathrm{t})/(2\Delta\mathrm{z})\end{aligned} \tag{5.7}$$

depend on the discrete interval of time $\Delta\mathrm{t}$, and the discrete intervals of space, $\Delta\mathrm{x}, \Delta\mathrm{y}, \Delta\mathrm{z}$, respectively.

As usually made in computer science, let us now introduce the discrete time $\mathrm{t_k}$, and the discrete spaces $\mathrm{x_l}, \mathrm{y_m}, \mathrm{z_n}$, as follows

$$\mathrm{t_k} = \mathrm{t_0} + \mathrm{k}\Delta\mathrm{t}, \mathrm{k} = 0,1,2,\ldots, \tag{5.8}$$

where k is the integer time increment, and $\mathrm{x_l} = \mathrm{x_0} + \mathrm{l}\Delta\mathrm{x}, \mathrm{l} = 0,1,2,\ldots, \mathrm{y_m} = \mathrm{y_0} + \mathrm{m}\Delta\mathrm{y}, \mathrm{m} = 0,1,2,\ldots, \mathrm{z_n} = \mathrm{z_0} + \mathrm{n}\Delta\mathrm{z}, \mathrm{n} = 0,1,2,\ldots$ (5.9)
where $\mathrm{l}, \mathrm{m}, \mathrm{n}$, are the integer space increments.

So, with these time and space increments, the second order hyperincursive discrete Klein-Gordon equation (5.6) becomes

$$q(l,m,n,k+2) - 2q(l,m,n,k) + q(l,m,n,k-2) = \\ +B^2[q(l+2,m,n,k) - 2q(l,m,n,k) + q(l-2,m,n,k)] \\ +C^2[q(l,m+2,n,k) - 2q(l,m,n,k) + q(l,m-2,n,k)] \\ +D^2[q(l,m,n+2,k) - 2q(l,m,n,k) + q(l,m,n-2,k)] - \\ A^2 q(l,m,n,k) \quad (5.10)$$

This equation without spatial components, corresponding to a particle at rest, is similar to the harmonic oscillator.

As presented in my recent paper (Dubois, 2019a), where the functions

$\tilde{q}_j = \tilde{q}_j(x,y,z,t) = \tilde{q}_j(l,m,n,k), j = 1,2,3,4,$

define discrete Majorana functions, the 4 discrete hyperincursive equations of the functions $\tilde{q}_j, j = 1,2,3,4$, are obtained as

$$\tilde{q}_1(l,m,n,k+1) = \tilde{q}_1(l,m,n,k-1) + \widetilde{B}[\tilde{q}_4(l+1,m,n,k) - \tilde{q}_4(l-1,m,n,k)] - \tilde{C}[\tilde{q}_1(l,m+1,n,k) - \tilde{q}_1(l,m-1,n,k)] + \widetilde{D}[\tilde{q}_3(l,m,n+1,k) - \tilde{q}_3(l,m,n-1,k)] - \widetilde{A}\tilde{q}_4(l,m,n,k)$$

$$\tilde{q}_2(l,m,n,k+1) = \tilde{q}_2(l,m,n,k-1) + \widetilde{B}[\tilde{q}_3(l+1,m,n,k) - \tilde{q}_3(l-1,m,n,k)] - \tilde{C}[\tilde{q}_2(l,m+1,n,k) - \tilde{q}_2(l,m-1,n,k)] - \widetilde{D}[\tilde{q}_4(l,m,n+1,k) - \tilde{q}_4(l,m,n-1,k)] + \widetilde{A}\tilde{q}_3(l,m,n,k)$$

$$\tilde{q}_3(l,m,n,k+1) = \tilde{q}_3(l,m,n,k-1) + \widetilde{B}[\tilde{q}_2(l+1,m,n,k) - \tilde{q}_2(l-1,m,n,k)] + \tilde{C}[\tilde{q}_3(l,m+1,n,k) - \tilde{q}_3(l,m-1,n,k)] + \widetilde{D}[\tilde{q}_1(l,m,n+1,k) - \tilde{q}_1(l,m,n-1,k)] - \widetilde{A}\tilde{q}_2(l,m,n,k)$$

$$\tilde{q}_4(l,m,n,k+1) = \tilde{q}_4(l,m,n,k-1) + \widetilde{B}[\tilde{q}_1(l+1,m,n,k) - \tilde{q}_1(l-1,m,n,k)] + \tilde{C}[\tilde{q}_4(l,m+1,n,k) - \tilde{q}_4(l,m-1,n,k)] - \widetilde{D}[\tilde{q}_2(l,m,n+1,k) - \tilde{q}_2(l,m,n-1,k)] + \widetilde{A}\tilde{q}_1(l,m,n,k)$$

(5-11-a-b-c-d)

with

$$\widetilde{\mathrm{A}} = \mathrm{A} = \mathrm{a}(2\Delta\mathrm{t}), \widetilde{\mathrm{B}} = \mathrm{B} = \mathrm{c}\,\Delta\mathrm{t}/\Delta\mathrm{x}, \quad (5\text{-}12\text{-a-b})$$

$$\widetilde{\mathrm{C}} = \mathrm{C} = \mathrm{c}\,\Delta\mathrm{t}/\Delta\mathrm{y}\,, \widetilde{\mathrm{D}} = \mathrm{D} = \mathrm{c}\,\Delta\mathrm{t}/\Delta\mathrm{z} \quad (5\text{-}12\text{-c-d})$$

where Δt and $\Delta\mathrm{x}, \Delta\mathrm{y}, \Delta\mathrm{z}$ are the discrete intervals of time and space respectively.

From the discrete equations, when the spacetime intervals tend to zero, we obtained the following 4 first order partial differential equations (Dubois, 2019a)

$$+\,\partial\widetilde{\Psi}_1/\partial t = +c\,\partial\widetilde{\Psi}_4/\partial x - c\,\partial\widetilde{\Psi}_1/\partial y + c\,\partial\widetilde{\Psi}_3/\partial z - (mc^2/\hbar)\widetilde{\Psi}_4$$

$$+\,\partial\widetilde{\Psi}_2/\partial t = +c\,\partial\widetilde{\Psi}_3/\partial x - c\,\partial\widetilde{\Psi}_2/\partial y - c\,\partial\widetilde{\Psi}_4/\partial z + (mc^2/\hbar)\widetilde{\Psi}_3$$

$$+\,\partial\widetilde{\Psi}_3/\partial t = +c\,\partial\widetilde{\Psi}_2/\partial x + c\,\partial\widetilde{\Psi}_3/\partial y + c\,\partial\widetilde{\Psi}_1/\partial z - (mc^2/\hbar)\widetilde{\Psi}_2$$

$$+\,\partial\widetilde{\Psi}_4/\partial t = +c\,\partial\widetilde{\Psi}_1/\partial x + c\,\partial\widetilde{\Psi}_4/\partial y - c\,\partial\widetilde{\Psi}_2/\partial z + (mc^2/\hbar)\widetilde{\Psi}_1$$

(5.13-a-b-c-d)

which are identical to the original Majorana equations (Majorana, 1937), e.g., equations (4-a-b-c-d) in Pessa (Pessa, 2006).

Recently, we demonstrated that Majorana 4-spinors equations bifurcate simply to the Dirac real 8-spinors equations (Dubois, 2019b).

First, let us consider the inverse parity space, in inversing the sign of the space variables in the Majorana equations (5.13-a-b-c-d),

$$\begin{aligned}
+\,\partial\widetilde{\Psi}_1/\partial t &= -c\,\partial\widetilde{\Psi}_4/\partial x + c\,\partial\widetilde{\Psi}_1/\partial y - c\,\partial\widetilde{\Psi}_3/\partial z - (mc^2/\hbar)\widetilde{\Psi}_4\\
+\,\partial\widetilde{\Psi}_2/\partial t &= -c\,\partial\widetilde{\Psi}_3/\partial x + c\,\partial\widetilde{\Psi}_2/\partial y + c\,\partial\widetilde{\Psi}_4/\partial z + (mc^2/\hbar)\widetilde{\Psi}_3\\
+\,\partial\widetilde{\Psi}_3/\partial t &= -c\,\partial\widetilde{\Psi}_2/\partial x - c\,\partial\widetilde{\Psi}_3/\partial y - c\,\partial\widetilde{\Psi}_1/\partial z - (mc^2/\hbar)\widetilde{\Psi}_2\\
+\,\partial\widetilde{\Psi}_4/\partial t &= -c\,\partial\widetilde{\Psi}_1/\partial x - c\,\partial\widetilde{\Psi}_4/\partial y + c\,\partial\widetilde{\Psi}_2/\partial z + (mc^2/\hbar)\widetilde{\Psi}_1
\end{aligned}$$

(5.14-a-b-c-d)

In defining the 2-spinors real functions,

$$\varphi_a = \begin{pmatrix} \widetilde{\Psi}_1 \\ \widetilde{\Psi}_2 \end{pmatrix}, \varphi_b = \begin{pmatrix} \widetilde{\Psi}_3 \\ \widetilde{\Psi}_4 \end{pmatrix}, \qquad \text{(5-15-a-b)}$$

the two equations (5.14-a-b) and (5.14-c-d) are transformed to the two 2-spinors real equations

$$+\partial\varphi_a/\partial t = -c\,\sigma_1\,\partial\varphi_b/\partial x + c\,\sigma_0\,\partial\varphi_a/\partial y - c\,\sigma_3\,\partial\varphi_b/\partial z + (mc^2/\hbar)\sigma_2\varphi_b$$

$$+\partial\varphi_b/\partial t = -c\,\sigma_1\,\partial\varphi_a/\partial x - c\,\sigma_0\,\partial\varphi_b/\partial y - c\,\sigma_3\,\partial\varphi_a/\partial z + (mc^2/\hbar)\sigma_2\varphi_a \qquad \text{(5.16-a-b)}$$

where the real 2-spinors matrices $\sigma_1, \sigma_2, \sigma_3$, are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{(5.17-a-b-c)}$$

and 2-Identity $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ (5.17-d)

With the inversion between σ_0 and σ_2, in introducing the tensor product by $-\sigma_2$, the functions $\widetilde{\Psi}_j$

$$\widetilde{\Psi}_j = \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix}, j = 1,2,3,4, \qquad \text{(5.18)}$$

bifurcate to two functions

$$-\sigma_2\,\Psi_j = -\sigma_2 \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} = \begin{pmatrix} +\Psi_{j,2} \\ -\Psi_{j,1} \end{pmatrix}, j = 1,2,3,4 \qquad \text{(5.19)}$$

So the Majorana real 4-spinors equation bifurcates into the Dirac real 8-spinors equations

$$+\,\partial\Psi_{1,1}/\partial t = -c\,\partial\Psi_{4,1}/\partial x - c\,\partial\Psi_{4,2}/\partial y - c\,\partial\Psi_{3,1}/\partial z + (mc^2/\hbar)\Psi_{1,2}$$

$$+\,\partial\Psi_{2,1}/\partial t = -c\,\partial\Psi_{3,1}/\partial x + c\,\partial\Psi_{3,2}/\partial y + c\,\partial\Psi_{4,1}/\partial z + (mc^2/\hbar)\Psi_{2,2}$$

$$+\,\partial\Psi_{3,1}/\partial t = -c\,\partial\Psi_{2,1}/\partial x - c\,\partial\Psi_{2,2}/\partial y - c\,\partial\Psi_{1,1}/\partial z - (mc^2/\hbar)\Psi_{3,2}$$

$$+\,\partial\Psi_{4,1}/\partial t = -c\,\partial\Psi_{1,1}/\partial x + c\,\partial\Psi_{1,2}/\partial y + c\,\partial\Psi_{2,1}/\partial z - (mc^2/\hbar)\Psi_{4,2}$$

(5.20-a-b-c-d)

$$+\,\partial\Psi_{1,2}/\partial t = -c\,\partial\Psi_{4,2}/\partial x + c\,\partial\Psi_{4,1}/\partial y - c\,\partial\Psi_{3,2}/\partial z - (mc^2/\hbar)\Psi_{1,1}$$

$$+\,\partial\Psi_{2,2}/\partial t = -c\,\partial\Psi_{3,2}/\partial x - c\,\partial\Psi_{3,1}/\partial y + c\,\partial\Psi_{4,2}/\partial z - (mc^2/\hbar)\Psi_{2,1}$$

$$+\,\partial\Psi_{3,2}/\partial t = -c\,\partial\Psi_{2,2}/\partial x + c\,\partial\Psi_{2,1}/\partial y - c\,\partial\Psi_{1,2}/\partial z + (mc^2/\hbar)\Psi_{3,1}$$

$$+\,\partial\Psi_{4,2}/\partial t = -c\,\partial\Psi_{1,2}/\partial x - c\,\partial\Psi_{1,1}/\partial y + c\,\partial\Psi_{2,2}/\partial z + (mc^2/\hbar)\Psi_{4,1}$$

(5.21-a-b-c-d)

These 2 x 4 = 8 real first order partial differential equations represent real 8-spinors equations that are similar to the Dirac complex 4-spinors equations (Dirac, 1964).

In defining the wave function

$$\Psi_{\mathrm{j}}(x,y,z,\mathrm{t}) = \Psi_{\mathrm{j}} = \Psi_{\mathrm{j},1} + \mathrm{i}\Psi_{\mathrm{j},2}, \mathrm{j} = 1,2,3,4, \tag{5.22}$$

with the imaginary number i, we obtain the original Dirac equations (Dirac, 1928):

$$+\partial\Psi_1/\partial t = -c\,\partial\Psi_4/\partial x + \mathrm{i}c\,\partial\Psi_4/\partial y - c\,\partial\Psi_3/\partial z - \mathrm{i}(mc^2/\hbar)\Psi_1$$

$$+\partial\Psi_2/\partial t = -c\,\partial\Psi_3/\partial x - \mathrm{i}c\,\partial\Psi_3/\partial y + c\,\partial\Psi_4/\partial z - \mathrm{i}(mc^2/\hbar)\Psi_2$$

$$+\partial\Psi_3/\partial t = -c\,\partial\Psi_2/\partial x + \mathrm{i}c\,\partial\Psi_2/\partial y - c\,\partial\Psi_1/\partial z + \mathrm{i}(mc^2/\hbar)\Psi_3$$

$$+\partial\Psi_4/\partial t = -c\,\partial\Psi_1/\partial x - \mathrm{i}c\,\partial\Psi_1/\partial y + c\,\partial\Psi_2/\partial z + \mathrm{i}(mc^2/\hbar)\Psi_4$$

(5.23-a-b-c-d)

Let us define the discrete Dirac wave function

$$\mathrm{Q_j(l,m,n,k) = Q_j = Q_{j,1} + i\,Q_{j,2}\,, j = 1,2,3,4,} \tag{5.24}$$

corresponding to the Dirac wave function (5.22).
The 4 hyperincursive discrete Dirac equations of the discrete wave function are then given by

$$\begin{aligned}&\mathrm{Q_1(l,m,n,k+1) =}\\&\mathrm{Q_1(l,m,n,k-1) - B[Q_4(l+1,m,n,k) - Q_4(l-1,m,n,k)]}\\&\mathrm{+\ iC[Q_4(l,m+1,n,k) - Q_4(l,m-1,n,k)]}\\&\mathrm{-D[Q_3(l,m,n+1,k) - Q_3(l,m,n-1,k)] -\ i\,AQ_1(l,m,n,k)}\end{aligned}$$

$$\begin{aligned}&\mathrm{Q_2(l,m,n,k+1) =}\\&\mathrm{Q_2(l,m,n,k-1) - B[Q_3(l+1,m,n,k) - Q_3(l-1,m,n,k)]}\\&\mathrm{-i\ C[Q_3(l,m+1,n,k) - Q_3(l,m-1,n,k)]}\\&\mathrm{+D[Q_4(l,m,n+1,k) - Q_4(l,m,n-1,k)] -\ i\,AQ_2(l,m,n,k)}\\&\mathrm{Q_3(l,m,n,k+1) =}\\&\mathrm{Q_3(l,m,n,k-1) - B[Q_2(l+1,m,n,k) - Q_2(l-1,m,n,k)]}\end{aligned}$$

$$+\mathrm{i}\ \mathrm{C}[\mathrm{Q}_2(\mathrm{l},\mathrm{m}+1,\mathrm{n},\mathrm{k})-\mathrm{Q}_2(\mathrm{l},\mathrm{m}-1,\mathrm{n},\mathrm{k})]$$
$$-\ \mathrm{D}[\mathrm{Q}_1(\mathrm{l},\mathrm{m},\mathrm{n}+1,\mathrm{k})-\mathrm{Q}_1(\mathrm{l},\mathrm{m},\mathrm{n}-1,\mathrm{k})]+\mathrm{i}\ \mathrm{AQ}_3(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k})$$

$$\mathrm{Q}_4(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k}+1)=$$
$$\mathrm{Q}_4(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k}-1)-\ \mathrm{B}[\mathrm{Q}_1(\mathrm{l}+1,\mathrm{m},\mathrm{n},\mathrm{k})-\mathrm{Q}_1(\mathrm{l}-1,\mathrm{m},\mathrm{n},\mathrm{k})]$$
$$-\ \mathrm{i}\ \mathrm{C}[\mathrm{Q}_1(\mathrm{l},\mathrm{m}+1,\mathrm{n},\mathrm{k})-\mathrm{Q}_1(\mathrm{l},\mathrm{m}-1,\mathrm{n},\mathrm{k})]$$
$$+\mathrm{D}[\mathrm{q}_2(\mathrm{l},\mathrm{m},\mathrm{n}+1,\mathrm{k})-\mathrm{Q}_2(\mathrm{l},\mathrm{m},\mathrm{n}-1,\mathrm{k})]+\mathrm{i}\ \mathrm{AQ}_4(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k})$$
(5.25-a-b-c-d)

with

$$\mathrm{A}=2\omega\Delta\mathrm{t},\ \mathrm{B}=\mathrm{c}\ \Delta\mathrm{t}/\Delta\mathrm{x},\ \mathrm{C}=\mathrm{c}\ \Delta\mathrm{t}/\Delta\mathrm{y}\ ,\mathrm{D}=\mathrm{c}\ \Delta\mathrm{t}/\Delta\mathrm{z} \tag{5.26}$$

where Δt and $\Delta\mathrm{x},\Delta\mathrm{y},\Delta\mathrm{z}$ are the discrete intervals of time and space respectively.

6. THE HYPERINCURSIVE DISCRETE KLEIN-GORDON EQUATION BIFURCATES TO THE 16 PROCA EQUATIONS

Let us show that there are 16 complex functions associated to this second order hyperincursive discrete Klein-Gordon equation. This equation without spatial components, corresponding to a particle at rest, is similar to the harmonic oscillator. For a particle at rest, the Klein-Gordon equation (5.10), with the function $\mathrm{q}(\mathrm{t})$ depending only on the time variable, is given by

$$\partial^2\mathrm{q}(\mathrm{t})/\ \partial\mathrm{t}^2=-\ a^2\mathrm{q}(\mathrm{t}) \tag{6.1}$$

with the frequency, $a=\omega=mc^2/\hbar$, given by the equation (5.4).
This equation (6.1) is formally similar to the equation of the harmonic oscillator for which $\mathrm{q}(\mathrm{t})$ would represent the position $\mathrm{x}(\mathrm{t})$, and $\partial q(t)/\partial t$ would represent the velocity $v(t)=\partial x(t)/\partial t$.

So, with only the temporal component, the second order hyperincursive discrete Klein-Gordon equation (5.10) becomes

$$q(k+2) - 2q(k) + q(k-2) = -A^2 q(k) \qquad (6.2)$$

that is similar to the second order hyperincursive discrete equation of the harmonic oscillator, as shown in section 2. This hyperincursive equation (6.2) is separable into a first discrete incursive oscillator depending on two functions defined by $q_1(k), q_2(k)$, and a second incursive oscillator depending on two other functions defined by $q_3(k), q_4(k)$, given by first order discrete equations.

So the first incursive equations are given by:

$$q_1(2k) = q_1(2k-2) + Aq_2(2k-1)$$

$$q_2(2k+1) = q_2(2k-1) - Aq_1(2k) \qquad (6.3\text{-a-b})$$

where $q_1(2k)$ is defined of the even steps of the time, and $q_2(2k+1)$ is defined on the odd steps of the time.
And the second incursive equations are given by:

$$q_3(2k) = q_3(2k-2) - Aq_4(2k-1)$$

$$q_4(2k+1) = q_4(2k-1) + Aq_3(2k) \qquad (6.4\text{-a-b})$$

where $q_3(2k)$ is defined of the even steps of the time, and $q_4(2k+1)$ is defined on the odd steps of the time.
The second incursive system is the time reverse of the first incursive system in making the discrete time inversion ***T***

$$\boldsymbol{T}: \Delta t \rightarrow -\Delta t \qquad (6.5)$$

which gives an oscillator and its anti-oscillator.

In defining the following 2 complex functions, where i is the imaginary number,

$$q_{13}(2\mathrm{k}) = \mathrm{q}_1(2\mathrm{k}) + \mathrm{i}\ \mathrm{q}_3(2\mathrm{k})$$

$$q_{24}(2\mathrm{k}+1) = \mathrm{q}_2(2\mathrm{k}+1) - \mathrm{i}\ \mathrm{q}_4(2\mathrm{k}+1) \qquad (6.6\text{-a-b})$$

the 4 real incursive equations (6.3-a-b) and (6.4-a-b) are transformed to 2 complex incursive equations

$$q_{13}(2\mathrm{k}) = q_{13}(2\mathrm{k}-2) + \mathrm{A}q_{24}(2\mathrm{k}-1)$$

$$q_{24}(2\mathrm{k}+1) = q_{24}(2\mathrm{k}-1) - \mathrm{A}q_{13}(2\mathrm{k}) \qquad (6.7\text{-a-b})$$

So the hyperincursive equation for a particle at rest shows a temporal bifurcation into an oscillatory equation and an anti-oscillatory equation.

For a moving particle, the 3 discrete space-symmetric terms in equation (5.10)

$$\mathrm{q}(\mathrm{l}+2,\mathrm{m},\mathrm{n},\mathrm{k}) - 2\mathrm{q}(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k}) + \mathrm{q}(\mathrm{l}-2,\mathrm{m},\mathrm{n},\mathrm{k})$$
$$\mathrm{q}(\mathrm{l},\mathrm{m}+2,\mathrm{n},\mathrm{k}) - 2\mathrm{q}(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k}) + \mathrm{q}(\mathrm{l},\mathrm{m}-2,\mathrm{n},\mathrm{k})$$
$$\mathrm{q}(\mathrm{l},\mathrm{m},\mathrm{n}+2,\mathrm{k}) - 2\mathrm{q}(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k}) + \mathrm{q}(\mathrm{l},\mathrm{m},\mathrm{n}-2,\mathrm{k})$$

are similar to the discrete time-symmetric term

$$\mathrm{q}(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k}+2) - 2\mathrm{q}(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k}) + \mathrm{q}(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{k}-2)$$

The two complex functions bifurcate for even and odd steps of space x, giving 4 complex functions depending on 4 discrete incursive equations. These 4 complex functions bifurcate for even and odd steps of space y, giving 8 complex functions depending on 8 discrete incursive equations. Finally, these 8 complex functions bifurcate for even and odd steps of space z, giving 16 complex functions depending on 16 incursive discrete equations.

But if we consider the space variable as a set of the 3 space variables

$$\boldsymbol{r} = (x, y, z) \tag{6.8}$$

the two complex functions bifurcate for even and odd steps of the space variable $\boldsymbol{r} = (x, y, z)$, giving 4 complex functions depending on 4 discrete incursive equations, which correspond to a discrete parity inversion $\boldsymbol{P}$

$$\boldsymbol{P}: \Delta\boldsymbol{r} \rightarrow -\Delta\boldsymbol{r} \tag{6.9}$$

In conclusion, with the discrete time inversion and the parity, we define a group of 4 incursive discrete equations with 4 functions. This is in agreement with the thesis of Proca. Indeed, as demonstrated by Proca (Proca, 1930, 1932) in 1930 and 1932, the Klein-Gordon equation admits in the general case a total of 16 functions. Classically, for the well-known Dirac equation, there are 4 complex wave functions.

Proca demonstrated that there are 4 fundamental equations of 4 wave functions for the Dirac equation

$$\varphi_{r,s} \; for \; r = 1,2,3,4, and \; s = 1 \tag{6.10}$$

and the other 3 x 4 other equations are similar to these 4 equations.
Proca classified the 16 equations in 4 groups of 4 functions:

1. 4 equations of the 4 functions $\varphi_{r,s} \; for \; r = 1,2,3,4, and \; s = 1$
2. 4 equations of the 4 functions $\varphi_{r,s} \; for \; r = 1,2,3,4, and \; s = 2$
3. 4 equations of the 4 functions $\varphi_{r,s} \; for \; r = 1,2,3,4, and \; s = 3$
4. 4 equations of the 4 functions $\varphi_{r,s} \; for \; r = 1,2,3,4, and \; s = 4$

In each group, the 4 equations depend on 4 functions which are not separable except in particular cases.

In this chapter we restricted our analysis to the first group of 4 functions in studying the case of the Majorana and Dirac equations.

7. CHIRAL REPRESENTATION OF THE MAJORANA EQUATIONS IN 2 COMPONENTS

In the preceding section 4, we have presented the rotation of the incursive discrete harmonic oscillators by the Hadamard matrix and unitary matrix U. The incursive discrete equations are transformed to recursive discrete equations, what is a remarkable result (Dubois, 2019c).

The rotation of the relativistic quantum Majorana equations with the same Hadamard matrix and unitary matrix U, gives rise to the transformation of the Majorana equations in 2 components (Dubois, 2019d).

Indeed, we will give the Chiral representation of Majorana equations from the unitary matrix, $\mathrm{U} = \mathrm{U_R} + \mathrm{iU_I}$,

$$\mathrm{U} = \mathrm{U_R} + \mathrm{iU_I} = \frac{1}{2}\begin{pmatrix} \sigma_0 + \sigma_y & -\mathrm{i}(\sigma_0 - \sigma_y) \\ \mathrm{i}(\sigma_0 - \sigma_y) & \sigma_0 + \sigma_y \end{pmatrix} \tag{7.1}$$

which can be defined with the Pauli matrix σ_y and with the unit matrix, $\sigma_0 = I_2 = 1$, with the property $\mathrm{UU^*} = \mathrm{U^*U} = 1$. An excellent introduction to the properties of the unitary matrix is given by Palash (Palash, 2011).

So the real and imaginary parts of this unitary matrix are applied to the Majorana real 4-spinors as follows

$$U_R\tilde{\Psi} = \frac{1}{2}\begin{pmatrix} +1 & 0 & 0 & +1 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ +1 & 0 & 0 & +1 \end{pmatrix}\begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \\ \tilde{\Psi}_3 \\ \tilde{\Psi}_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} +\tilde{\Psi}_1 + \tilde{\Psi}_4 \\ +\tilde{\Psi}_2 - \tilde{\Psi}_3 \\ -\tilde{\Psi}_2 + \tilde{\Psi}_3 \\ +\tilde{\Psi}_1 + \tilde{\Psi}_4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} +\breve{\Psi}_{11} \\ +\breve{\Psi}_{21} \\ -\breve{\Psi}_{21} \\ +\breve{\Psi}_{11} \end{pmatrix} \tag{7.2-a}$$

$$\mathrm{U_I}\tilde{\Psi} = \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 0 \\ +1 & 0 & 0 & -1 \\ +1 & 0 & 0 & -1 \\ 0 & +1 & +1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \\ \tilde{\Psi}_3 \\ \tilde{\Psi}_4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\tilde{\Psi}_2 - \tilde{\Psi}_3 \\ +\tilde{\Psi}_1 - \tilde{\Psi}_4 \\ +\tilde{\Psi}_1 - \tilde{\Psi}_4 \\ +\tilde{\Psi}_2 + \tilde{\Psi}_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\breve{\Psi}_{12} \\ +\breve{\Psi}_{22} \\ +\breve{\Psi}_{22} \\ +\breve{\Psi}_{12} \end{pmatrix} \tag{7.2-b}$$

So the application of the unitary matrix to the Majorana real 4-spinors is given by

$$\mathrm{U}\tilde{\Psi} = \frac{1}{2}\begin{pmatrix} +1 & -\mathrm{i} & -\mathrm{i} & +1 \\ +\mathrm{i} & +1 & -1 & -\mathrm{i} \\ +\mathrm{i} & -1 & +1 & -\mathrm{i} \\ +1 & +\mathrm{i} & +\mathrm{i} & +1 \end{pmatrix}\begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \\ \tilde{\Psi}_3 \\ \tilde{\Psi}_1 \end{pmatrix} = \begin{pmatrix} +\breve{\Psi}_{11} - \mathrm{i}\breve{\Psi}_{12} \\ +\breve{\Psi}_{21} + \mathrm{i}\breve{\Psi}_{22} \\ -\breve{\Psi}_{21} + \mathrm{i}\breve{\Psi}_{22} \\ +\breve{\Psi}_{11} + \mathrm{i}\breve{\Psi}_{12} \end{pmatrix} = \begin{pmatrix} \breve{\Psi}_1 \\ \breve{\Psi}_2 \\ \breve{\Psi}_3 \\ \breve{\Psi}_4 \end{pmatrix} = \breve{\Psi} \tag{7.2-c}$$

The general function $\breve{\Psi}$ can be separated in the top left chiral function, $\breve{\Psi}_{\mathrm{L}}$, and in the bottom right chiral function, $\breve{\Psi}_{\mathrm{R}}$ chiral function,

$$\breve{\Psi} = \begin{pmatrix} \breve{\Psi}_{\mathrm{L}} \\ \breve{\Psi}_{\mathrm{R}} \end{pmatrix}, \breve{\Psi}_{\mathrm{L}} = \begin{pmatrix} \breve{\Psi}_1 \\ \breve{\Psi}_2 \end{pmatrix} \text{ and } \breve{\Psi}_{\mathrm{R}} = \begin{pmatrix} \breve{\Psi}_3 \\ \breve{\Psi}_4 \end{pmatrix} \tag{7.4-a-b-c}$$

and the bottom right function can be deduced directly from the top left function as follows

$$\breve{\Psi}_{\mathrm{R}} = -i\sigma_y \breve{\Psi}_{\mathrm{L}}^* = \begin{pmatrix} -\breve{\Psi}_2^* \\ +\breve{\Psi}_1^* \end{pmatrix} \tag{7.5}$$

with the rotation $2x2$ Hadamard matrix, H_2,

$$H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \tag{7.6}$$

let us transform the Majorana 2-spinors, as follow

$$\begin{pmatrix} \breve{\Psi}_{11} \\ \breve{\Psi}_{22} \end{pmatrix} = H_2 \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{\Psi}_1 + \tilde{\Psi}_4 \\ \tilde{\Psi}_1 - \tilde{\Psi}_4 \end{pmatrix}$$

$$\begin{pmatrix} \breve{\Psi}_{12} \\ \breve{\Psi}_{21} \end{pmatrix} = H_2 \begin{pmatrix} \tilde{\Psi}_2 \\ \tilde{\Psi}_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \tilde{\Psi}_2 + \tilde{\Psi}_3 \\ \tilde{\Psi}_2 - \tilde{\Psi}_3 \end{pmatrix} \tag{7.7-a-b}$$

and let us apply these rotations to the Majorana equations as follow

$$+\,\partial\breve{\Psi}_{11}/\partial t = +c\,\partial\breve{\Psi}_{11}/\partial x - c\,\partial\breve{\Psi}_{22}/\partial y - c\,\partial\breve{\Psi}_{21}/\partial z + (mc^2/\hbar)\breve{\Psi}_{22}$$
$$+\,\partial\breve{\Psi}_{12}/\partial t = +c\,\partial\breve{\Psi}_{12}/\partial x - c\,\partial\breve{\Psi}_{21}/\partial y + c\,\partial\breve{\Psi}_{22}/\partial z - (mc^2/\hbar)\breve{\Psi}_{21}$$
$$+\,\partial\breve{\Psi}_{21}/\partial t = -c\,\partial\breve{\Psi}_{21}/\partial x - c\,\partial\breve{\Psi}_{12}/\partial y - c\,\partial\breve{\Psi}_{11}/\partial z + (mc^2/\hbar)\breve{\Psi}_{12}$$
$$+\,\partial\breve{\Psi}_{22}/\partial t = -c\,\partial\breve{\Psi}_{22}/\partial x - c\,\partial\breve{\Psi}_{11}/\partial y + c\,\partial\breve{\Psi}_{12}/\partial z - (mc^2/\hbar)\breve{\Psi}_{11} \qquad (7.8\text{-a-b-c-d})$$

Again with the Hadamard matrix, let us transform the 2-spinors (7.7-a-b) as follows

$$\begin{pmatrix}\breve{\Psi}_4\\ \breve{\Psi}_1\end{pmatrix} = H_2\begin{pmatrix}\breve{\Psi}_{11}\\ \mathrm{i}\breve{\Psi}_{12}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}+\breve{\Psi}_{11} + \mathrm{i}\breve{\Psi}_{12}\\ +\breve{\Psi}_{11} - \mathrm{i}\breve{\Psi}_{12}\end{pmatrix},$$

$$\begin{pmatrix}\breve{\Psi}_2\\ \breve{\Psi}_3\end{pmatrix} = H_2\begin{pmatrix}\mathrm{i}\breve{\Psi}_{22}\\ \breve{\Psi}_{21}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}+\breve{\Psi}_{21} + \mathrm{i}\breve{\Psi}_{22}\\ -\breve{\Psi}_{21} + \mathrm{i}\breve{\Psi}_{22}\end{pmatrix} \qquad (7.9\text{-a-b})$$

which are the same transformations as in the unitary matrix (7.2-c).
Let us give the partial differential equations of the 2 left chiral functions $\breve{\Psi}_1$ and $\breve{\Psi}_2$, as

$$+\,\partial\breve{\Psi}_1/\partial t = +c\,\partial\breve{\Psi}_1/\partial x + \mathrm{i}\,c\,\partial\breve{\Psi}_2/\partial y - c\,\partial\breve{\Psi}_2/\partial z + \mathrm{i}\,(mc^2/\hbar)\breve{\Psi}_2^*$$
$$+\,\partial\breve{\Psi}_2/\partial t = -c\,\partial\breve{\Psi}_2/\partial x - \mathrm{i}\,c\,\partial\breve{\Psi}_1/\partial y - c\,\partial\breve{\Psi}_1/\partial z - \mathrm{i}\,(mc^2/\hbar)\breve{\Psi}_1^* \qquad (7.10\text{-a-b})$$

Let us write the chiral left Majorana equation with the left chiral function, $\breve{\Psi}_L$ (7.4-b):

$$\partial_t\breve{\Psi}_L = +c\sigma_z\partial_x\breve{\Psi}_L - c\sigma_y\partial_y\breve{\Psi}_L - c\sigma_x\partial_z\breve{\Psi}_L - (mc^2/\hbar)\sigma_y\breve{\Psi}_L^* \qquad (7.10\text{-c})$$

The chiral right Majorana equation with the right chiral function, $\breve{\Psi}_{\mathrm{R}}$ (7.4-c), is easy to write. For a particle at rest, the left non-relativistic Majorana equation is given by

$$\partial_t \breve{\Psi}_{\mathrm{L}} = -(mc^2/\hbar)\sigma_y \breve{\Psi}_{\mathrm{L}}^* \tag{7.11}$$

In conclusion, in this section, we have considered the chiral representation of the Majorana equations.

8. Solutions of the Non-Relativistic Quantum Majorana and Dirac Equations

This section is written following our recent paper (Dubois, 2019d).

In the non-relativistic limit $p \ll mc$, the particles are at rest, with a momentum $p \cong 0$.

In this limit, the Majorana equations (7.10-a-b) are given by

$$\begin{aligned} &+\partial\breve{\Psi}_1/\partial t = +\,\mathrm{i}\,(mc^2/\hbar)\breve{\Psi}_2^* \\ &+\partial\breve{\Psi}_2/\partial t = -\,\mathrm{i}\,(mc^2/\hbar)\breve{\Psi}_1^* \end{aligned} \tag{8.1-a-b}$$

$$\text{With } \breve{\Psi}(\mathrm{t}) = \begin{pmatrix} \breve{\Psi}_1(\mathrm{t}) \\ \breve{\Psi}_2(\mathrm{t}) \end{pmatrix} \tag{8.2-a}$$

these equations (8.1-a-b) become

$$\partial_t \breve{\Psi} = -(mc^2/\hbar)\sigma_y \breve{\Psi}^* \tag{8.3-a}$$

where $\partial_t = \partial/\partial t$, and $\sigma_y = \mathrm{i}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is a Pauli matrix.

The complex conjugate of equation (8.3-a) is given by

$$\partial_t \breve{\Psi}^* = +(mc^2/\hbar)\sigma_y \breve{\Psi} \tag{8.3-b}$$

With the two equations (8.3-a-b), one obtains a second order equation

$$\partial_t^2\breve{\Psi}(\mathrm{t}) = -(mc^2/\hbar)^2\breve{\Psi}(\mathrm{t}) \tag{8.4}$$

that is the temporal Klein-Gordon equation.

The solution of equation (8.4) is given by

$$\breve{\Psi}(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\breve{\Psi}(0) - \sin(mc^2\mathrm{t}/\hbar)\,\sigma_y\breve{\Psi}^*(0) \tag{8.5}$$

or in explicit form

$$\breve{\Psi}_1(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\breve{\Psi}_1(0) + \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\breve{\Psi}_2^*(0) \tag{8.6-a}$$

$$\breve{\Psi}_2(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\breve{\Psi}_2(0) - \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\breve{\Psi}_1^*(0) \tag{8.6-b}$$

Now let us consider the following Dirac 2-spinors

$$\hat{\Psi}(\mathrm{t}) = \begin{pmatrix}\Psi_1(\mathrm{t})\\ \Psi_4(\mathrm{t})\end{pmatrix}, \tag{8.7}$$

for which the temporal non-relativistic Dirac equation is given by

$$\partial_t\hat{\Psi}(\mathrm{t}) = -i(mc^2/\hbar)\sigma_z\hat{\Psi}(\mathrm{t}) \tag{8.8}$$

where $\partial_t = \partial/\partial t$, and $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, is a the Pauli matrix.

The analytical solution of the non-relativistic Dirac equation (8.8) is given by

$$\hat{\Psi}(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\hat{\Psi}(0) - \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\,\sigma_z\hat{\Psi}(0) \tag{8.9}$$

or in explicit form

$$\Psi_1(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\Psi_1(0) - \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\Psi_1(0) \tag{8.10-a}$$

$$\Psi_4(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\Psi_4(0) + \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\Psi_4(0) \qquad (8.10\text{-b})$$

In conclusion, in this section, we have considered the solutions of the non-relativistic chiral Majorana equation and the Dirac equation.

9. The Generic Majorana 4-Spinors Equation

With the Majorana wave functions, $\widetilde{\Psi}_{\mathrm{j}} = \widetilde{\Psi}_{\mathrm{j}}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{t}), \mathrm{j} = 1,2,3,4,$ we have given the 4 Majorana partial differential equations (5.13-a-b-c-d).

Let us define the two Majorana bi-spinors wave functions

$$\widetilde{\Psi}_{\mathrm{a}} = \begin{pmatrix} \widetilde{\Psi}_1 \\ \widetilde{\Psi}_2 \end{pmatrix}, \widetilde{\Psi}_{\mathrm{b}} = \begin{pmatrix} \widetilde{\Psi}_3 \\ \widetilde{\Psi}_4 \end{pmatrix}, \qquad (9.1\text{-a-b})$$

The Majorana equations (5.13a-b-c-d), for the bi-spinors, become:

$$\partial_t\widetilde{\Psi}_{\mathrm{a}} = +c\sigma_x\partial_x\widetilde{\Psi}_{\mathrm{b}} - c\sigma_0\partial_y\widetilde{\Psi}_{\mathrm{a}} + c\sigma_z\partial_z\widetilde{\Psi}_{\mathrm{b}} - i(mc^2/\hbar)\sigma_y\widetilde{\Psi}_{\mathrm{b}}$$

$$\partial_t\widetilde{\Psi}_{\mathrm{b}} = +c\sigma_x\partial_x\widetilde{\Psi}_{\mathrm{a}} + c\sigma_0\partial_y\widetilde{\Psi}_{\mathrm{b}} + c\sigma_z\partial_z\widetilde{\Psi}_{\mathrm{a}} - i(mc^2/\hbar)\sigma_y\widetilde{\Psi}_{\mathrm{a}}$$

(9.2-a-b)

where $\partial_\mu = \partial/\partial\mu$ and, σ_x , σ_y , σ_z, are the Pauli 2x2 matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}_2$$

(9.3-a-b-c-d)

and where σ_0, is the 2x2 unit matrix I_2 .

Let us define the Majorana 4-spinors wave function from the two bi-spinors (9.1-a-b):

$$\widetilde{\Psi} = \begin{pmatrix} \widetilde{\Psi}_a \\ \widetilde{\Psi}_b \end{pmatrix} \tag{9.1-c}$$

The Majorana equations (9.2-a-b) for the 4-spinors become the following generic Majorana equation:

$$\partial_t \widetilde{\Psi} = c\alpha_x \partial_x \widetilde{\Psi} - c\beta \partial_y \widetilde{\Psi} + c\alpha_z \partial_z \widetilde{\Psi} - i(mc^2/\hbar)\alpha_y \widetilde{\Psi} \tag{9.4-c}$$

where the 4x4 matrices, $\alpha_x, \alpha_y, \alpha_z$, are defined with the Pauli matrices by

$$\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} \tag{9.5-a-b-c}$$

and β is defined with the unit matrix by

$$\beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \tag{9.5-d}$$

In the next section we will give the generic Dirac 4-spinors equation and its relation to the Majorana equation.

10. The Generic Dirac 4-Spinors Equation

In defining the Dirac wave function by $\Psi_j = \Psi_j\ (x, y, z, t), j = 1,2,3,4$, we have given the 4 Dirac partial differential equations (5.23-a-b-c-d).

Let us define the Dirac bi-spinors wave functions

$$\Psi_a = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \Psi_b = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix}, \tag{10.1-a-b}$$

The Dirac equations (5.23a-b-c-d), for the bi-spinors, become:

$$\partial_t \Psi_\mathrm{a} = -c\sigma_x \partial_x \Psi_\mathrm{b} - c\sigma_y \partial_y \Psi_\mathrm{b} - c\sigma_z \partial_z \Psi_\mathrm{b} - i(mc^2/\hbar)\sigma_0 \Psi_\mathrm{a}$$

$$\partial_t \Psi_\mathrm{b} = -c\sigma_x \partial_x \Psi_\mathrm{a} - c\sigma_y \partial_y \Psi_\mathrm{a} - c\sigma_z \partial_z \Psi_\mathrm{a} + i(mc^2/\hbar)\sigma_0 \Psi_\mathrm{b}$$

(10.2-a-b)

where $\partial_\mu = \partial/\partial\mu$ and $\sigma_x, \sigma_y, \sigma_z$, are the Pauli 2x2 matrices (9.3-a-b-c), and where σ_0, is the 2x2 unit matrix I_2 (9.3-d).

Let us define the Dirac 4-spinors from the two bi-spinors (10.1-a-b):

$$\Psi = \begin{pmatrix} \Psi_\mathrm{a} \\ \Psi_\mathrm{b} \end{pmatrix} \tag{10.3}$$

The Dirac equations (10.2-a-b) for the 4-spinors become the following generic Dirac equation:

$$\partial_t \Psi = -c\alpha_x \partial_x \Psi - c\alpha_y \partial_y \Psi - c\alpha_z \partial_z \Psi - i(mc^2/\hbar)\beta\Psi \tag{10.4}$$

where the 4x4 matrices, $\alpha_x, \alpha_y, \alpha_z$, were defined in equations (9.5-a-b-c), and β was defined in equation (9.5-d).

In comparing the Dirac equation (10.4-c) with the Majorana equation (9.4-c), we see that there is an inversion of the two matrices, β, and, α_y, with an inversion of signs of the space variables, x, and, z.

This is in agreement with my demonstration, given in the preceding section 5, of the bifurcation of the Majorana real equation to the Dirac complex equations (Dubois, 2019b).

Let us remark that the Pauli matrices represent logical quantum gates in quantum compution.

Let us first recall the properties of the Pauli 2x2 matrices, $\sigma_x, \sigma_y, \sigma_z$:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10.5}$$

The square of the Pauli gates are equal to the 2x2 unit gate.

The square of the unit gate is equal to itself:

$$\sigma_0^2 = I_2 = \sigma_0 \tag{10.6}$$

The Pauli gates do not commute, and show the following properties:

$$\begin{aligned}
&\sigma_y\sigma_z - \sigma_z\,\sigma_x = i\sigma_x\\
&\sigma_z\sigma_x - \sigma_x\,\sigma_z = i\sigma_y\\
&\sigma_x\sigma_y - \sigma_y\,\sigma_x = i\sigma_z
\end{aligned} \tag{10.7-a-b-c}$$

With the Kronecker product, ⊗, it is possible to create the 4x4 matrices $\alpha_x, \alpha_y, \alpha_z$ and, β, with the product of two Pauli 2x2 matrices, as follows:

$$\alpha_x = \sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \alpha_y = \sigma_x \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_z = \sigma_x \otimes \sigma_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{10.8-a-b-c}$$

,

$$\beta = \sigma_z \otimes \sigma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{10.8-d}$$

The square of the matrices, $\alpha_x, \alpha_y, \alpha_z$, β, are equal to the unit matrix, I_4:

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.9}$$

The matrices, $\alpha_x, \alpha_y, \alpha_z$, β, show the following important properties:

$$\alpha_y\alpha_z + \alpha_z\,\alpha_x = 0$$

$$\alpha_z\alpha_x + \alpha_x\,\alpha_z = 0$$
$$\alpha_x\alpha_y + \alpha_y\,\alpha_x = 0 \qquad (10.10\text{-a-b-c})$$

$$\alpha_y\beta + \beta\alpha_y = 0$$
$$\alpha_z\beta + \beta\alpha_z = 0$$
$$\alpha_x\beta + \beta\alpha_x = 0 \qquad (10.11\text{-a-b-c})$$

The next section deals with a fundamental invariant related to the Pauli matrix, σ_x.

11. A New Invariant of the Non-Relativistic Quantum Majorana and Dirac Wave Functions

This section gives the comparison of the solutions of the non-relativistic quantum Majorana and Dirac equations after (Dubois, 2019e).

In the limit, $p \ll mc$, the particles are at rest, with a momentum $p \cong 0$. In the preceding section, we have given the following 2-components chiral Majorana equations (7.10-a-b);

$$+\,\partial\breve{\Psi}_1/\partial t = +c\,\partial\breve{\Psi}_1/\partial x + \mathrm{i}\,c\,\partial\breve{\Psi}_2/\partial y - c\,\partial\breve{\Psi}_2/\partial z + \mathrm{i}\,(mc^2/\hbar)\breve{\Psi}_2^*$$

$$+\,\partial\breve{\Psi}_2/\partial t = -c\,\partial\breve{\Psi}_2/\partial x - \mathrm{i}\,c\,\partial\breve{\Psi}_1/\partial y - c\,\partial\breve{\Psi}_1/\partial z - \mathrm{i}\,(mc^2/\hbar)\breve{\Psi}_1^* \qquad (11.1\text{-a-b})$$

In the non-relativistic limit, these 2-components Majorana equations are given by

$$+\,\partial\breve{\Psi}_1/\partial t = +\,\mathrm{i}\,(mc^2/\hbar)\breve{\Psi}_2^* \qquad (11.1\text{-c})$$

$$+\,\partial\breve{\Psi}_2/\partial t = -\,\mathrm{i}\,(mc^2/\hbar)\breve{\Psi}_1^* \qquad (11.1\text{-d})$$

with

$$\breve{\Psi}(\mathrm{t}) = \begin{pmatrix} \breve{\Psi}_1(\mathrm{t}) \\ \breve{\Psi}_2(\mathrm{t}) \end{pmatrix}, \tag{11.2}$$

these Majorana equations become

$$\partial_t \breve{\Psi} = -(mc^2/\hbar)\sigma_y \breve{\Psi}^* \tag{11.3-a}$$

where $\partial_t = \partial/\partial t$, with the Pauli matrix, $\sigma_y = \mathrm{i}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

The complex conjugate of equation (11.3-a) is given by

$$\partial_t \breve{\Psi}^* = +(mc^2/\hbar)\sigma_y \breve{\Psi} \tag{11.3-b}$$

These 2 equations (11.3-a-b) transform to the following second order equation

$$\partial_t^2 \breve{\Psi}(\mathrm{t}) = -(mc^2/\hbar)^2 \breve{\Psi}(\mathrm{t}) \tag{11.4}$$

which is identical to the second order derivative of the Klein-Gordon equation for a particle at rest, with a 2-spinors complex Majorana function $\breve{\Psi}(\mathrm{t})$.

The analytical solution of the equation (11.4) is given by

$$\breve{\Psi}(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\breve{\Psi}(0) - \sin(mc^2\mathrm{t}/\hbar)\,\sigma_y \breve{\Psi}^*(0) \tag{11.5}$$

or, in explicit form

$$\breve{\Psi}_1(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\breve{\Psi}_1(0) + \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\breve{\Psi}_2^*(0) \tag{11.6-a}$$

$$\breve{\Psi}_2(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\breve{\Psi}_2(0) - \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\breve{\Psi}_1^*(0) \tag{11.6-b}$$

Now let us consider the following Dirac 2-spinors

$$\hat{\Psi}(\mathrm{t}) = \begin{pmatrix} \Psi_1(\mathrm{t}) \\ \Psi_4(\mathrm{t}) \end{pmatrix}, \tag{11.7}$$

The non-relativistic Dirac equation is given by

$$\partial_t \hat{\Psi}(\mathrm{t}) = -i(mc^2/\hbar)\sigma_z \hat{\Psi}(\mathrm{t}) \tag{11.8}$$

where $\partial_t = \partial/\partial t$, and with the Pauli matrix, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The analytical solution of the Dirac equation (11.8) is given by

$$\hat{\Psi}(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\hat{\Psi}(0) - \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\,\sigma_z \hat{\Psi}(0) \tag{11.9}$$

or, in explicit form

$$\Psi_1(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\Psi_1(0) - \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\Psi_1(0) \tag{11.10-a}$$

$$\Psi_4(\mathrm{t}) = \cos(mc^2\mathrm{t}/\hbar)\Psi_4(0) + \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\Psi_4(0) \tag{11.10-b}$$

Now, we will show the relation between the solutions of the Dirac equations from the solutions of the Majorana equations with the method of Lamata et al. (Lamata et al., 2012). So, let us consider the sum of the forward and backward solutions (11.9) of the Dirac equation

$$\left[\hat{\Psi}(+\mathrm{t}) + \hat{\Psi}(-\mathrm{t})\right]/2 = \cos(mc^2\mathrm{t}/\hbar)\,\hat{\Psi}(0)$$

and the difference of the forward and backward solutions

$$\left[\hat{\Psi}^*(+\mathrm{t}) - \hat{\Psi}^*(-\mathrm{t})\right]/2 = \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\,\sigma_z\,\hat{\Psi}^*(0)$$

In multiplying by the Pauli matrix, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the relation becomes

$$\mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\sigma_x\,\sigma_z\,\widehat{\Psi}^*(0) = \sin(mc^2\mathrm{t}/\hbar)\sigma_y\ \widehat{\Psi}^*(0)$$

so we obtain the following relation between the solution of the Dirac equation and the solution of the Majorana equation

$$\breve{\Psi}(\mathrm{t}) = \left[\widehat{\Psi}(+\mathrm{t}) + \widehat{\Psi}(-\mathrm{t})\right]/2 - \sigma_x\left[\widehat{\Psi}^*(+\mathrm{t}) - \widehat{\Psi}^*(-\mathrm{t})\right]/2 \qquad (11.11)$$

that is equal to the solution (11.5) of the Majorana equation.

We obtain the same result as Lamata et al, but they have not given the inverse equation for obtaining the Majorana solution from the Dirac solution.

Let us now make the inverse in giving the Dirac solution as a function of the Majorana solution, after (Dubois, 2019e). So, let us start from the solution (11.5) of the Majorana equation. Let us consider the sum of the forward and backward solutions (11.5)

$$\left[\,\breve{\Psi}(\mathrm{t}) + \breve{\Psi}(-\mathrm{t})\right]/2 = \cos(mc^2\mathrm{t}/\hbar)\breve{\Psi}(0)$$

and the difference of the forward and backward solutions

$$\left[\breve{\Psi}^*(\mathrm{t}) - \breve{\Psi}^*(-\mathrm{t})\right]/2 = \sin(mc^2\mathrm{t}/\hbar)\,\sigma_y\breve{\Psi}(0)$$

Let us multiply this relation by the Pauli matrix, σ_x,

$$\sin(mc^2\mathrm{t}/\hbar)\,\sigma_x\sigma_y\breve{\Psi}(0) = \mathrm{i}\sin(mc^2\mathrm{t}/\hbar)\,\sigma_z\breve{\Psi}(0)$$

So we obtain the relation between the solution of the Majorana equation and the solution of the Dirac equation (11.9) as follows

$$\widehat{\Psi}(\mathrm{t}) = \left[\,\breve{\Psi}(\mathrm{t}) + \breve{\Psi}(-\mathrm{t})\right]/2 - \sigma_x\left[\breve{\Psi}^*(\mathrm{t}) - \breve{\Psi}^*(-\mathrm{t})\right]/2 \qquad (11.12)$$

Surprisingly, the transformation relation is invariant, the relation (11.12), which gives the Dirac wave function from the Majorana wave function, is identical to the relation (11.11), which gives the Majorana wave function from the Dirac wave function. At our knowledge, this is a new invariant of the non-relativistic quantum Majorana and Dirac equations. This invariant is based on the Pauli matrix σ_x, that is the quantum gate X, which is the "spin flip" or the NOT gate, a reversible gate in quantum computation.

12. Quantum Computation with Reversible Gates

In this chapter, we have used some reversible quantum gates as defined for developing quantum computers. The quantum Pauli gates X, Y, Z, that operate on one-qubit, are based on Pauli matrices:

$$\mathrm{X} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12.1}$$

which is a "spin flip" or NOT gate,

$$Y = \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{12.2}$$

$$Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12.3}$$

that is a phase shift gate with $\varphi = \pi$.

Only the X and Z are necessary, because the Y can be deduced from them:

$$Y = \mathrm{iXZ} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{12.4}$$

The square of each Pauli gate is the identity matrix I

$$I^2 = X^2 = Y^2 = Z^2 = -iXYZ = I \tag{12.5}$$

The quantum Hadamard gate H_2 is defined by

$$H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \tag{12.6}$$

which is a rotation gate, that gives a basis change.

The Hadamard gate can be deduced from the X and the Z gates:

$$H_2 = \frac{1}{\sqrt{2}}(\mathrm{X} + \mathrm{Z}) = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} \tag{12.7}$$

In the section 4, the Hadamard matrix was deduced from the rotation matrix

$$\mathrm{R}_1(\theta) = \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix} \tag{12.8}$$

for the angle $\theta = \pi/4$, as

$$\mathrm{R}_1(\pi/4) = \begin{pmatrix} \sin(\pi/4) & \cos(\pi/4) \\ \cos(\pi/4) & -\sin(\pi/4) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix} = H_2 \tag{12.9}$$

In this section 4, we have demonstrated a remarkable result: by the rotation of the position and velocity of the two incursive discrete equations of the harmonic oscillator, with the Hadamard matrix gate, we have transformed the incursive discrete equations to recursive discrete equations of the harmonic oscillator.

Let us remark that the X ans Z gates can be deduced from this rotation matrix for the angles $\theta = 0$ and $\theta = \pi/2$ respectively

$$\mathrm{R}_1(0) = \begin{pmatrix} \sin(0) & \cos(0) \\ \cos(0) & -\sin(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{X} \tag{12.10}$$

$$R_1(\pi/2) = \begin{pmatrix} \sin(\pi/2) & \cos(\pi/2) \\ \cos(\pi/2) & -\sin(\pi/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z \tag{12.11}$$

In the technology of quantum computers, many quantum gates are also defined, for example, the phase gate, the square root of the NOT gate, the CNOT gate and the CCNOT gate.

The phase gate is given by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{12.12}$$

This phase gate can also be deduced from the Z gate,

$$S = \sqrt[2]{Z} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{12.13}$$

indeed, it is the square root of Z,

$$SS = Z = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12.14}$$

The square root of the NOT gate is written as

$$\sqrt[2]{X} = \sqrt[2]{NOT} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \tag{12.15}$$

The XOR (exclusive OR) gate, the Controlled NOT gate CNOT, is a two-qubit operation defined by

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{12.16}$$

And finally, the reversible Toffoli gate, the Controlled-Controlled NOT gate CCNOT, is a three-qubit operation defined by

$$CCNOT = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{12.17}$$

The Hadamard and Toffoli gates are quantum universal gates (Aharonov, 2003).

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