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Selecting directed cycles: a polyhedral study

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Problem definition

Given a directed graph $G = (V, A)$, a **cycle selection** of G is:

- a subset of arcs $B \subseteq A$ forming a union of directed cycles in $G_B = (V, B)$

or equivalently,

- a subset of arcs $B \subseteq A$ such that each arc in B is in a directed cycle in G_B

Maximum Weighted Cycle Selection (MWCS): given a directed graph $G = (V, A)$ and a weight $w_{i,j} \in \mathbb{R}$ for each arc $(i, j) \in A$, find a cycle selection B which maximizes $w(B) = \sum_{(i,j)\in B} w_{ij}$.

These definitions are motivated by an application to the kidney exchange problem.

Result

MWCS is strongly NP-hard.

Formulations

Four IP formulations for the cycle selection problem:

- Natural formulation
- Cycle formulation
- Extended edge formulation
- Position indexed formulation (based on inequalities described in [1])

Results

- Natural formulation equivalent to extended edge formulation
- Natural formulation dominates cycle formulation and position indexed formulation

Polyhedral study of the cycle selection polytope

Focus on the **natural formulation**. Let $\beta_{i,j} = 1$ if arc (i, j) is selected and 0 otherwise, for all $(i, j) \in A$ and consider the following set of constraints:

$$0 \leq \beta_{i,j} \leq 1 \qquad \forall (i,j) \in A \qquad (1)$$

$$\beta_{i,j} \text{ integer} \qquad \forall (i,j) \in A \qquad (2)$$

$$\beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall (i,j) \in A, \forall S \subseteq V : i \in S, j \in V \setminus S. \qquad (3)$$

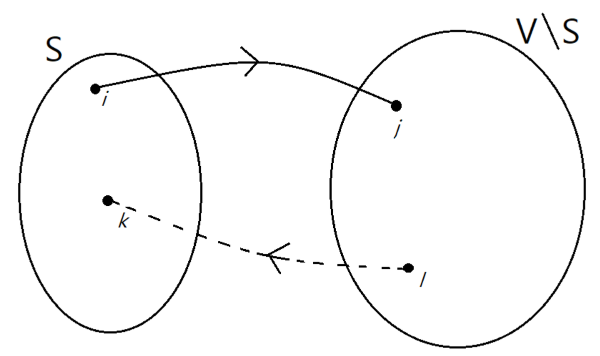

Figure 1: Illustration of constraint (3)

- P: set of 0-1 vectors associated with cycle selections
- PL: linear relaxation of P
- P^*: convex hull of P

Next,

- Let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\} \subseteq A$, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $|I| \leq |J| = t$. Let p and $q \in V, p \neq q$, $\{p, q\} \not\subseteq I \cup J$. Define the *out-star inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k\in V\setminus I} \sum_{i\in I} \beta_{k,i} + \sum_{j\in J} \sum_{k\in V} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{k,p}, \qquad (4)$$

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k\in V\setminus I} \sum_{i\in I} \beta_{k,i} + \sum_{j\in J} \sum_{k\in V} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{q,k}. \qquad (5)$$

Symmetrically, if $|J| \leq |I| = t$, define the *in-star inequalities*.

- Let $I = \{i_1, i_2, \ldots, i_t\}$ and $J = \{j_1, j_2, \ldots, j_t\} \subseteq V$ with $I \cap J = \emptyset$ and $|I| = |J| = t$. Let p and $q \in V, p \neq q$, $\{p, q\} \not\subseteq I \cup J$, define the *path inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \sum_{l=1}^{t-1} \beta_{i_l,j_{l+1}} + \beta_{p,q} \leq \sum_{k\in V} \sum_{i\in I} \beta_{k,i} + \sum_{j\in J} \sum_{k\in V} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{k,p}. \qquad (6)$$

Results

For *complete directed graphs*:

- P^* is full dimensional.
- Constraints (3) are separable in polynomial time.
- Constraints (1) and (3) are facet defining for P^*.
- The out-star ((4) (5)) and in-star inequalities are facet defining for P^*.
- The path inequalities (6) are facet defining for P^*.

Variant

Variant of the initial cycle selection problem:

- Add a constraint on the number of selected arcs:

$$\sum_{(i,j)\in A} \beta_{i,j} \leq \mathbf{b}$$

where $\mathbf{b} \in \mathbb{N}$

Except for the path inequalities defining facets of P^*, all results proved for the original cycle selection problem remain valid when $\mathbf{b} \geq 7$.

Numerical tests

- Despite the theoretical complexity of the problem the ILP formulation of the cycle selection problem is very quickly solved for many random instances.
- ILP formulation of the constrained variant is harder to solve.

References

[1] Bart Smeulders, Valentin Bartier, Yves Crama, and Frits C. R. Spieksma. Recourse in kidney exchange programs. *Working paper HEC - University of Liege*, 2019.