Selecting directed cycles: a polyhedral study
Marie Baratto

Definition
Motivation
MCWS problem
Formulations
Numerical tests
Variant
What's next?
References

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Marie Baratto, Yves Crama

May 21th 2021
Cycle Selection Problem

Given a directed graph $G = (V, A)$. We are interested in subsets $B \subseteq A$
- forming a union of directed cycles in $G_B = (V, B)$
or equivalently,
- such that each arc in $B$ is in a directed cycle in $G_B$

The Maximum Weighted Cycle Selection (MWCS) problem: given a directed graph $G = (V, A)$ and a weight $w_{i,j} \in \mathbb{R}$ for each arc $(i, j) \in A$, find a cycle selection $B$ which maximizes $w(B) = \sum_{(i,j) \in B} w_{ij}$. 
Kidney exchange problem

Patient with a kidney disease that requires a kidney transplant.

- Dialysis
- Deceased donor waiting list
- Willing donor

Might not be compatible with its donor:

- Blood incompatibility
- Tissue type incompatibility
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Definition
Motivation
MCWS problem
Formulations
Numerical tests
Variant
What’s next?
References
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Definition
Motivation
MCWS problem
Formulations
Numerical tests
Variant
What's next?
References

Pool of pairs
Compatibility graph

G=\((V,A,w)\) where:

- \(V = \{1, \ldots, n\}\) set of vertices, consisting of all patient-donor pairs.
- \(A\), the set of arcs, designating compatibilities between the vertices. Two vertices \(i\) and \(j\) are connected by arc \((i,j)\) if the donor in pair \(i\) is compatible with the patient in pair \(j\).
- Weight function \(w\) such that for each arc \((i,j) \in A\), \(w_{i,j}\) represents the weight/the utility of a transplant between donor \(i\) and patient \(j\).

**Definition**

An exchange is a set of disjoint cycles in the directed graph. It is feasible if every cycle length does not exceed a given limit \(K\).
Possible exchanges
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Definition
Motivation
MCWS problem
Formulations
Numerical tests
Variant
What's next?
References
Cycle Selection Problem

Given a directed graph \( G = (V, A) \).

We are interested in subsets \( B \subseteq A \)
  - forming a union of directed cycles in \( G_B = (V, B) \)
  or equivalently,
  - such that each arc in \( B \) is in a directed cycle in \( G_B \)
Goals:

- Study the complexity of the maximum weighted cycle selection (MWCS) problem
- Describe the linear model associated with the problem and in particular the set of solutions $S$.
- Polyhedral study of the convex hull of the set of solutions $S$.
- Test numerically the resolution of the MWCS problem.

Add new constraints:

- On the cardinality of a selection: $|B| \leq b$
- On the length of the cycles, the cycles in $B$ cannot exceed a certain length $K$
The **Maximum Weighted Cycle Selection (MWCS)** problem: given a directed graph \( G = (V, A) \) and a weight \( w_{i,j} \in \mathbb{R} \) for each arc \((i, j) \in A\), find a selection \( B \) which maximizes \( w(B) = \sum_{(i,j) \in B} w_{ij} \).

**Result**

The MWCS problem is strongly NP-hard.

\[ \text{\rightarrow} \text{ Separation problem on the convex hull of the solutions is NP-hard.} \]

\[ \text{\rightarrow} \text{ Description of the facets of the polytope associated with that optimization problem is hard.} \]
Goals

• Study the complexity of the maximum weighted cycle selection (MWCS) problem ✓
• Describe the linear model associated with the problem and in particular the set of solutions $S$.
• Polyhedral study of the convex hull of the set of solutions $S$.
• Test numerically the resolution of the MWCS problem.
Formulations

• Arc formulation

Extended (non compact) formulation:

• Cycle formulation

Extended compact formulations:

• Extended arc formulation

• Position indexed formulation
Arc formulation

**Variables**

\[ \beta_{i,j} = 1 \text{ if arc } (i, j) \text{ is selected, } 0 \text{ otherwise, for all } (i, j) \in A. \]

**Objective function**

\[
\max \sum_{(i,j) \in A} w_{i,j} \beta_{i,j}
\]

**Constraints**

\[
\beta_{i,j} \in \mathbb{Z} \quad \forall (i, j) \in A \tag{2}
\]

\[
\beta_{i,j} \leq 1 \quad \forall (i, j) \in A \tag{3}
\]

\[
\beta_{i,j} \geq 0 \quad \forall (i, j) \in A \tag{4}
\]

\[
\beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \quad \forall S \subseteq V, \forall (i, j) \in A : i \in S, j \in V \setminus S \tag{5}
\]
**Constraints**

\[ \beta_{i,j} \in \{0, 1\} \quad \forall (i, j) \in A \]

\[ \beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \quad \forall S \subseteq V, \forall (i,j) \in A : i \in S, j \in V \setminus S \]

![Diagram showing constraints](image)
Cycle formulation

Variables:
• $\beta_{i,j} = 1$ if arc $(i, j)$ is selected, 0 otherwise $\forall (i, j) \in A$.
• $x_c = 1$ if cycle $c$ is selected, 0 otherwise, $\forall c \in \Gamma$, where $\Gamma$ is the set of all possible cycles in $G$.

Constraints:
\begin{align*}
    x_c & \leq \beta_{i,j} & \forall c \in \Gamma, \forall (i, j) \in c & \quad (6) \\
    \beta_{i,j} & \leq \sum_{c \in \Gamma : (i, j) \in c} x_c & \forall (i, j) \in A & \quad (7) \\
    x_c & \in \{0, 1\} & \forall c \in \Gamma & \quad (8) \\
    \beta_{i,j} & \in \{0, 1\} & \forall (i, j) \in A & \quad (9) \\
\end{align*}
Extended arc formulation

- View a selection $B$ as a collection of circulations $C^1, \ldots, C^L$ such that $B = \bigcup_{i=1}^L C^i$.
- Associate each arc $(i,j) \in B$ with one of the circulations $C^i$, hence we will need $|A|$ circulations $C^i$.
- Use one copy of the graph for each circulation.

Intuitively, we want to express that $(i,j) \in B$ is contained in some circulation $C^{(i,j)}$ and we view $C^{(i,j)}$ as the representative circulation associated with arc $(i,j)$.

Variables:

$$x_{i,j}^{(u,v)} = 1 \text{ if } (i,j) \in C^{(u,v)}, \quad \forall (i,j) \in A, \forall (u,v) \in A$$

Constraints:

$$x_{i,j}^{(u,v)} \leq x_{i,j}^{(i,j)} \quad \forall (i,j) \in A, \forall (u,v) \in A \quad (11)$$

$$\sum_{h:(h,i) \in A} x_{h,i}^{(u,v)} = \sum_{h:(i,h) \in A} x_{i,h}^{(u,v)} \quad \forall i \in V, \forall (u,v) \in A \quad (12)$$

$$\sum_{h:(i,h) \in A} x_{i,h}^{(u,v)} \leq 1 \quad \forall i \in V, \forall (u,v) \in A \quad (13)$$

$$x_{i,j}^{(u,v)} \in \{0, 1\} \quad \forall (i,j) \in A, \forall (u,v) \in A \quad (14)$$
Position indexed formulation

Variables:

- \( \phi^l_{i,j,k} = 1 \) if arc \((i, j)\) is in position \(k\) in a cycle in graph copy \(G^l\), 0 otherwise.
  \[ \forall (i, j) \in A, l \in V, k \in \kappa(i, j, l) \text{ where } \kappa(i, j, l) = \begin{cases} \{1\} & \text{if } i = l \\ \{2, \ldots, n\} & \text{if } j = l \\ \{2, \ldots, n - 1\} & \text{if } i, j > l \end{cases} \]
  
- \( \beta_{i,j} = 1 \) if arc \((i, j)\) is selected, 0 otherwise. \( \forall (i, j) \in A \)

Constraints:

\[ \beta_{i,j} \leq \sum_{l \in V} \sum_{k \in \lambda(i,j,l)} \phi^l_{i,j,k} \quad \forall (i, j) \in A \quad (15) \]
\[ \phi^l_{i,j,k} \leq \beta_{i,j} \quad \forall l \in V, (i, j) \in A^l, k \in \kappa(i, j, l) \quad (16) \]
\[ \phi^l_{i,j,k} \leq \sum_{h: (h,i) \in A^l \land k-1 \in \kappa(h,i,l)} \phi^l_{h,i,k-1} \quad \forall l \in V, (i, j) \in A^l, k \in \kappa(i, j, l), k > 1 \quad (17) \]
\[ \phi^l_{i,j,k} \leq \sum_{h: (j,h) \in A^l \land k+1 \in \kappa(j,h,l)} \phi^l_{j,h,k+1} \quad \forall l \in V, (i, j) \in A^l, j \neq l, k \in \kappa(i, j, l) \quad (18) \]

+ Binary constraints (19)
Relative strength of the formulations

- Arc formulation = Extended arc formulation
- Arc formulation < Cycle formulation
- Arc formulation < Position indexed formulation

→ Focus on the arc formulation
Goals

- Study the complexity of the maximum weighted cycle selection (MWCS) problem ✓
- Describe the linear model associated with the problem and in particular the set of solutions $S$. ✓
- Polyhedral study of the convex hull of the set of solutions $S$.
- Test numerically the resolution of the MWCS problem.
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Notation

\[ S = \left\{ \beta \in \{0, 1\}^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \forall S \subseteq V, \forall (i,j) \in A : i \in S, j \in V \setminus S \right\} \]

\[ SL = \left\{ \beta \in [0, 1]^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \forall S \subseteq V, \forall (i,j) \in A : i \in S, j \in V \setminus S \right\} \]

\[ S^* = \text{conv}(S) \]
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Definition
Motivation
MCWS problem
Formulations
Numerical tests
Variant
What’s next?
References
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\[ S^* = \text{conv}(S) \]
Recall

Theorem

For any set $S$ in $\mathbb{R}^n$ and $c \in \mathbb{R}^n$ then

$$\max\{c^T x : x \in S\} = \max\{c^T x : x \in \text{conv}(S)\}$$
For complete directed graph.

<table>
<thead>
<tr>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^*$ is full dimensional</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>The trivial inequalities and return inequalities are facet defining for $S^*$</td>
</tr>
</tbody>
</table>

\[
\beta_{i,j} \leq 1 \quad \forall (i,j) \in A \tag{20}
\]

\[
\beta_{i,j} \geq 0 \quad \forall (i,j) \in A \tag{21}
\]

\[
\beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \quad \forall S \subseteq V, \forall (i,j) \in A : i \in S, j \in V \setminus S \tag{22}
\]
Additional facet defining inequalities

- Let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\}$ be a subset of arcs, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $|I| \leq |J| = t$. Let $p$ and $q$ be two distinct vertices not in $I \cup J$. We define the out-star inequalities:

\[
\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p, q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k, i} + \sum_{j \in J} \sum_{k \in V} \beta_{j, k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k, p}, \tag{23}
\]

\[
\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p, q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k, i} + \sum_{j \in J} \sum_{k \in V} \beta_{j, k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q, k}. \tag{24}
\]

Result

The out-star inequalities are facet defining for $S^*$
• Symmetrically, if we assume that $|J| \leq |I| = t$, we define the in-star inequalities:

$$
\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V \setminus J} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p},
$$

(25)

$$
\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V \setminus J} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q,k}.
$$

(26)

**Result**

The in-star inequalities are facet defining for $S^*$
Additional facet defining inequalities

- Let $I = \{i_1, i_2, \ldots, i_t\}$ and $J = \{j_1, j_2, \ldots, j_t\}$ be two subsets of vertices with $I \cap J = \emptyset$ and $|I| = |J| = t$. Let $p$ and $q$ be two distinct vertices not in $I \cup J$, we define the path inequalities:

$$
\sum_{l=1}^{t} \beta_{i_l,j_l} + \sum_{l=1}^{t-1} \beta_{i_l,j_{l+1}} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \backslash (I \cup J)} \beta_{k,p}.
$$

(27)

Result

The path inequalities are facet defining for $S^*$
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Numerical tests
Selecting directed cycles: a polyhedral study

Marie Baratto

Definition

Motivation

MCWS problem

Formulations

Numerical tests

Variant

What's next ?

References

Formulation

\[
\text{max } \sum_{(i,j) \in A} w_{i,j} \beta_{i,j} \\
\text{s.t. } \beta_{i,j} \leq \sum_{(l,k) \in A: l \in V \setminus S, k \in S} \beta_{l,k} \quad \forall S \subseteq V, \ (i,j) \in A : i \in S, j \in V \setminus S
\]

\[
\beta_{i,j} \in \{0, 1\} \quad \forall (i,j) \in A
\]

\[
\text{max } \sum_{(i,j) \in A} w_{i,j} \beta_{i,j}
\]

\[
\text{s.t. } \beta_{i,j} \leq \sum_{k \in V, k \neq i} \beta_{k,i} \quad \forall (i,j) \in A
\]

\[
\beta_{i,j} \leq \sum_{k \in V, k \neq j} \beta_{j,k} \quad \forall (i,j) \in A
\]

\[
\beta_{i,j} \in \{0, 1\} \quad \forall (i,j) \in A
\]

for a fixed arc \((i,j)\), \(S = \{i\}\) and \(S = V \setminus \{j\}\)
Result
The return inequalities are separable in polynomial time.

→ Use of **callbacks** on CPLEX
Numerical tests

<table>
<thead>
<tr>
<th>Density</th>
<th>5, 10, 20, 50, 70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of vertices</td>
<td>50, 100, 150, 200, 250, 300</td>
</tr>
</tbody>
</table>

- $d\%$ of the arcs have a positive weight uniformly distributed in $[0, 1]$
- $(100 - d)\%$ of the arcs have a negative weight uniformly distributed in $[-1, 0]$
- Mean over 20 instances

<table>
<thead>
<tr>
<th>Nb vert</th>
<th>Density</th>
<th>Cuts</th>
<th>Calls</th>
<th>Nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>10</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>150</td>
<td>10</td>
<td>0</td>
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<tr>
<td>200</td>
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<td>0</td>
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<tr>
<td>250</td>
<td>10</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>300</td>
<td>10</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

→ Natural formulation is very strong on random graphs and KE graphs.
Natural formulation is very strong on random graphs and KE graphs.

- Can be explained by theoretical properties of random graphs.
- No additional cuts
Goals:

• Study the complexity of the maximum weighted cycle selection (MWCS) problem ✓
• Describe the linear model associated with the problem and in particular the set of solutions $S$. ✓
• Polyhedral study of the convex hull of the set of solutions $S$. ✓
• Test numerically the resolution of the MWCS problem. ✓

Add new constraints:

• On the cardinality of a selection: $|B| \leq b$
• On the length of the cycles, the cycles in $B$ can not exceed a certain length $K$
Variant of the initial cycle selection problem. Additional constraint on the number of selected arcs:

\[ \sum_{(i,j) \in A} \beta_{i,j} \leq b \]

where \( b \in \mathbb{N} \).

- Most of the results remain valid.

<table>
<thead>
<tr>
<th>Theorem</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension</td>
<td>4</td>
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<tr>
<td>( \beta_{i,j} \leq 0 )</td>
<td>5</td>
</tr>
<tr>
<td>( \beta_{i,j} \geq 1 )</td>
<td>5</td>
</tr>
<tr>
<td>Return inequalities</td>
<td>5</td>
</tr>
<tr>
<td>Out-star and in-star inequalities</td>
<td>7</td>
</tr>
<tr>
<td>Path inequalities</td>
<td>( 5t - 1 )</td>
</tr>
</tbody>
</table>
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### Definition

**Motivation**

**MCWS problem**

**Formulations**

**Numerical tests**

**Variant**

**What’s next ?**

**References**

<table>
<thead>
<tr>
<th>Nb vert</th>
<th>B</th>
<th>Density</th>
<th>Cuts</th>
<th>Calls</th>
<th>Nodes</th>
<th>Nb unsolved</th>
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</thead>
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<tr>
<td>100</td>
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<td>20,00</td>
<td>33,53</td>
<td>4</td>
</tr>
</tbody>
</table>

**Density**

**Number of vertices**

**Maximum number of selected arcs**

**[5,10,20,50,70]**

**[50,100,150,200,250,300]**

**[n, \( \frac{n}{2} \), \( \frac{n \ln(n)}{4} \), \( \frac{10}{10} \)]**
Kidney Exchange Problem

More especially to the stochastic integer programming formulation of the Kidney exchange problem proposed by Smeulders et al. (2019).

A two-stage selection problem which, given a testing budget \( b \), identifies (in stage 1) a subset of arcs \( B \subseteq A \) with \( |B| \leq b \) such that the expected number of transplants in the graph \( (V, B) \) (in stage 2) is maximized.

- Benders decomposition
- In the master problem the information about cycles is lost (i.e. the set \( B \) must be a cycle selection)
- Strengthen the formulation using the constraints of the ILP formulation of the selection problem.

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Variant
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