Selecting directed cycles: a polyhedral study

Marie BARATTO, Yves CRAMA

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Cycle Selection Problem

Given a directed graph $G = (V, A)$. We are interested in subsets $B \subseteq A$

- forming a union of directed cycles in $G_B = (V, B)$

or equivalently,

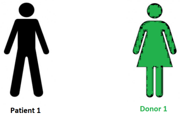
- such that each arc in B is in a directed cycle in G_B

The *Maximum Weighted Cycle Selection* (**MWCS**) problem: given a directed graph $G = (V, A)$ and a weight $w_{i,j} \in \mathbb{R}$ for each arc $(i, j) \in A$, find a cycle selection B which maximizes $w(B) = \sum_{(i,j)\in B} w_{ij}$.

Kidney exchange problem

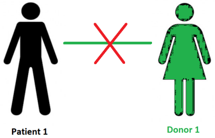
Patient with a kidney disease that requires a kidney transplant.

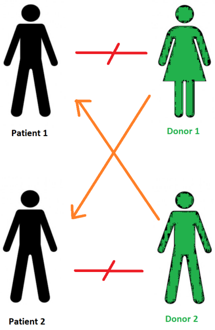
- Dialysis
- Deceased donor waiting list
- Willing donor



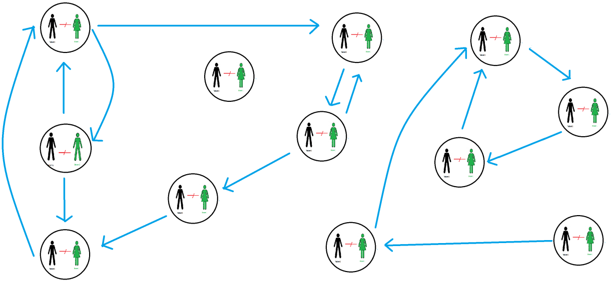
Might not be compatible with its donor:

- Blood incompatibility
- Tissue type incompatibility




Patient 1
Donor 1
Patient 2
Donor 2

Pool of pairs



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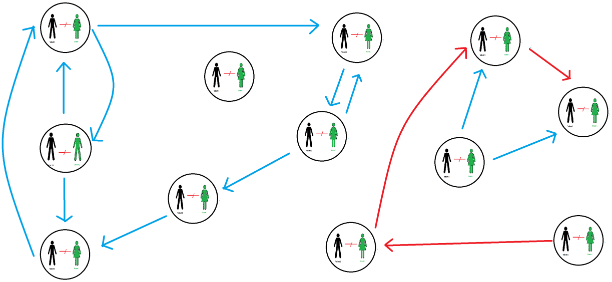
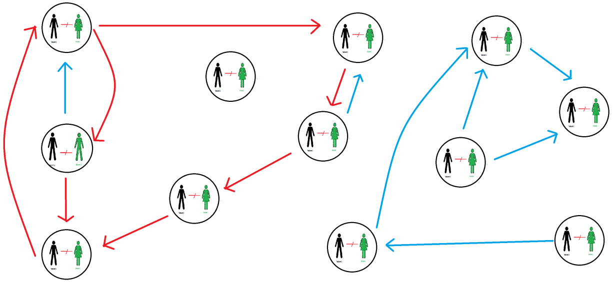
Marie Baratto

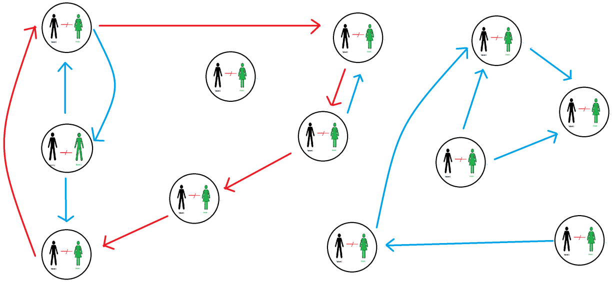
Constraints



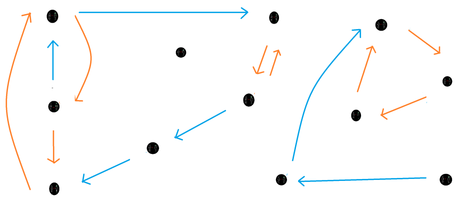
Constraints



Constraints



Compatibility graph



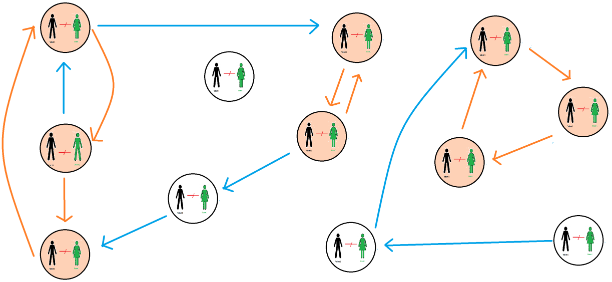
G=(V,A,w) where:

- $V = \{1, ..., n\}$ set of vertices, consisting of all patient-donor pairs.
- A, the set of arcs, designating compatibilities between the vertices. Two vertices i and j are connected by arc (i, j) if the donor in pair i is compatible with the patient in pair j.
- Weight function w such that for each arc $(i, j) \in A$, $w_{i,j}$ represents the weight/the utility of a transplant between donor i and patient j.

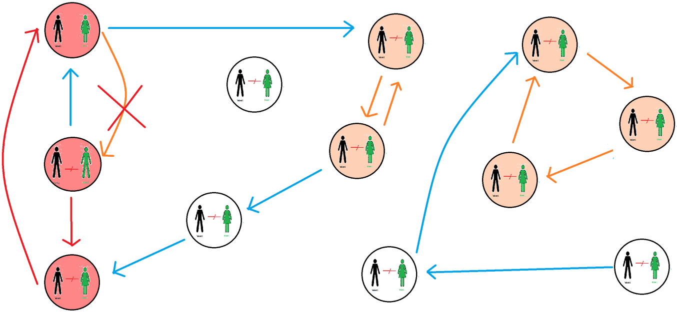
Definition

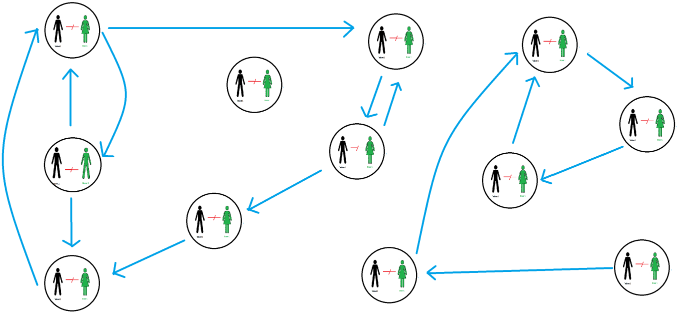
An exchange is a set of disjoint cycles in the directed graph. It is feasible if every cycle length does not exceed a given limit K.

Possible exchanges



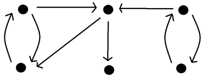
Uncertainty





Cycle Selection Problem

Given a directed graph $G = (V, A)$.

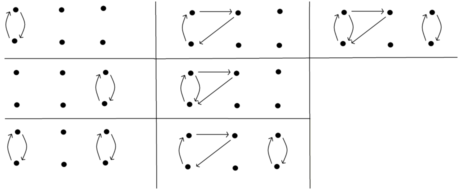


We are interested in subsets $B \subseteq A$

- forming a union of directed cycles in $G_B = (V, B)$

or equivalently,

- such that each arc in B is in a directed cycle in G_B



Goals:

- Study the complexity of the maximum weighted cycle selection (MWCS) problem
- Describe the linear model associated with the problem and in particular the set of solutions S.
- Polyhedral study of the convex hull of the set of solutions S.
- Test numerically the resolution of the MWCS problem.

Add new constraints:

- On the cardinality of a selection: $|B| \leq \mathbf{b}$
- On the length of the cycles, the cycles in B can not exceed a certain length $\mathbf{K}$

The *Maximum Weighted Cycle Selection* (**MWCS**) problem: given a directed graph $G = (V, A)$ and a weight $w_{i,j} \in \mathbb{R}$ for each arc $(i, j) \in A$, find a selection B which maximizes $w(B) = \sum_{(i,j)\in B} w_{ij}$.

Result

The MWCS problem is strongly NP-hard.

$\hookrightarrow$ Separation problem on the convex hull of the solutions is NP-hard.

$\hookrightarrow$ Description of the facets of the polytope associated with that optimization problem is hard.

Goals

- Study the complexity of the maximum weighted cycle selection (MWCS) problem $\surd$
- Describe the linear model associated with the problem and in particular the set of solutions S.
- Polyhedral study of the convex hull of the set of solutions S.
- Test numerically the resolution of the MWCS problem.

Formulations

- Arc formulation

Extended (non compact) formulation:

- Cycle formulation

Extended compact formulations:

- Extended arc formulation
- Position indexed formulation

Arc formulation

Variables

$\beta_{i,j} = 1$ if arc (i,j) is selected, 0 otherwise, for all $(i,j) \in A$.

Objective function

$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j} \qquad (1)$$

Constraints

$$\beta_{i,j} \in \mathbb{Z} \qquad \forall (i,j) \in A \qquad (2)$$

$$\beta_{i,j} \leq 1 \qquad \forall (i,j) \in A \qquad (3)$$

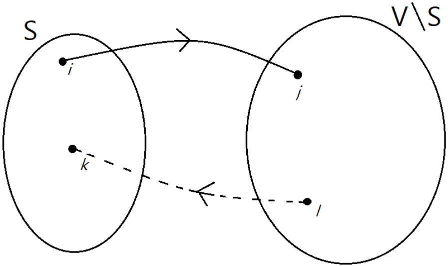
$$\beta_{i,j} \geq 0 \qquad \forall (i,j) \in A \qquad (4)$$

$$\beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S \qquad (5)$$

Constraints

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j) \in A$$

$$\beta_{i,j} \leq \sum_{(l,k)\in A: l \in V\setminus S, k \in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S$$



Cycle formulation

Variables:

- $\beta_{i,j} = 1$ if arc (i, j) is selected, 0 otherwise$\forall (i, j) \in A$.
- $x_c = 1$ if cycle c is selected, 0 otherwise, $\forall c \in \Gamma$, where Γ is the set of all possible cycles in G.

Constraints:

$$x_c \leq \beta_{i,j} \qquad \forall c \in \Gamma, \forall (i,j) \in c \tag{6}$$

$$\beta_{i,j} \leq \sum_{c \in \Gamma : (i,j) \in c} x_c \qquad \forall (i,j) \in A \tag{7}$$

$$x_c \in \{0, 1\} \qquad \forall c \in \Gamma \tag{8}$$

$$\beta_{i,j} \in \{0, 1\} \qquad \forall (i,j) \in A \tag{9}$$

$$\tag{10}$$

Extended arc formulation

- View a selection B as a collection of circulations $C^1, \dots, C^L$ such that $B = \cup_{l=1}^{L} C^l$.
- Associate each arc $(i,j) \in B$ with one of the circulations C^l, hence we will need $|A|$ circulations C^l.
- Use one copy of the graph for each circulation.

Intuitively, we want to express that $(i,j) \in B$ is contained in some circulation $C^{(i,j)}$ and we view $C^{(i,j)}$ as the representative circulation associated with arc (i,j).

Variables:

$$x_{i,j}^{(u,v)} = 1 \text{ if } (i,j) \in C^{(u,v)}, \qquad \forall (i,j) \in A, \forall (u,v) \in A$$

Constraints:

$$x_{i,j}^{(u,v)} \leq x_{i,j}^{(i,j)} \qquad \forall (i,j) \in A, \forall (u,v) \in A \qquad (11)$$

$$\sum_{h:(h,i)\in A} x_{h,i}^{(u,v)} = \sum_{h:(i,h)\in A} x_{i,h}^{(u,v)} \qquad \forall i \in V, \forall (u,v) \in A \qquad (12)$$

$$\sum_{h:(i,h)\in A} x_{i,h}^{(u,v)} \leq 1 \qquad \forall i \in V, \forall (u,v) \in A \qquad (13)$$

$$x_{i,j}^{(u,v)} \in \{0,1\} \qquad \forall (i,j) \in A, \forall (u,v) \in A \qquad (14)$$

Position indexed formulation

Variables:

- $\phi^l_{i,j,k}$ = 1 if arc (i,j) is in position k in a cycle in graph copy G^l, 0 otherwise.
  $$\forall (i,j) \in A, l \in V, k \in \kappa(i,j,l) \text{ where } \kappa(i,j,l) = \begin{cases} \{1\} \text{ if } i = l \\ \{2, ..., n\} \text{ if } j = l \\ \{2, ..., n-1\} \text{ if } i,j > l \end{cases}$$
- $\beta_{i,j}$ = 1 if arc (i,j) is selected, 0 otherwise. $\quad \forall (i,j) \in A$

Constraints:

$$\beta_{i,j} \leq \sum_{l \in V} \sum_{k \in \lambda(i,j,l)} \phi^l_{i,j,k} \qquad \forall (i,j) \in A \qquad (15)$$

$$\phi^l_{i,j,k} \leq \beta_{i,j} \qquad \forall l \in V, (i,j) \in A^l, k \in \kappa(i,j,l) \qquad (16)$$

$$\phi^l_{i,j,k} \leq \sum_{h:(h,i) \in A^l \wedge k-1 \in \kappa(h,i,l)} \phi^l_{h,i,k-1} \qquad \forall l \in V, (i,j) \in A^l, k \in \kappa(i,j,l), k > 1 \qquad (17)$$

$$\phi^l_{i,j,k} \leq \sum_{h:(j,h) \in A^l \wedge k+1 \in \kappa(j,h,l)} \phi^l_{j,h,k+1} \qquad \forall l \in V, (i,j) \in A^l, j \neq l, k \in \kappa(i,j,l) \qquad (18)$$

+ Binary constraints (19)

Relative strength of the formulations

- Arc formulation = Extended arc formulation
- Arc formulation < Cycle formulation
- Arc formulation < Position indexed formulation

$\rightarrow$ Focus on the arc formulation

Goals

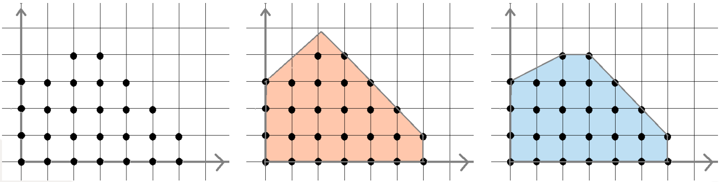
- Study the complexity of the maximum weighted cycle selection (MWCS) problem $\surd$
- Describe the linear model associated with the problem and in particular the set of solutions S. $\surd$
- Polyhedral study of the convex hull of the set of solutions S.
- Test numerically the resolution of the MWCS problem.

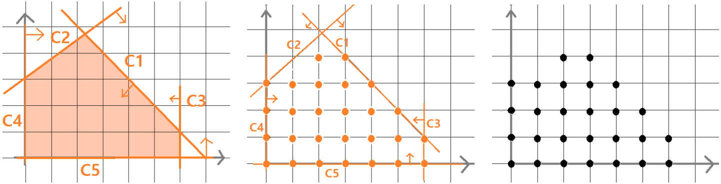
Notation

$$S = \left\{ \beta \in \{0,1\}^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \forall S \subseteq V, \ \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

$$SL = \left\{ \beta \in [0,1]^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \forall S \subseteq V, \ \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

$$S* = conv(S)$$




C2
C1
C3
C4
C5
C2
C1
C3
C4
C5

Notation

$$S = \left\{ \beta \in \{0,1\}^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

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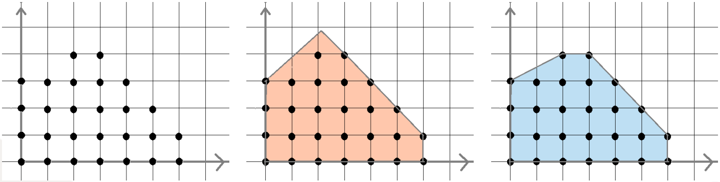
$$S* = conv(S)$$

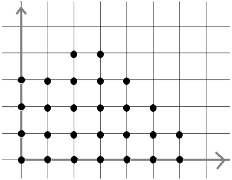


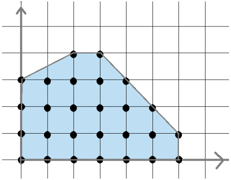
Recall

Theorem

For any set S in $\mathbb{R}^n$ and $c \in \mathbb{R}^n$ then

$$\max\{c^T x : x \in S\} = \max\{c^T x : x \in conv(S)\}$$





Polyhedral study

For complete directed graph.

Result

S^* is full dimensional

Result

The trivial inequalities and return inequalities are facet defining for S^*

$$\beta_{i,j} \leq 1 \qquad \forall (i,j) \in A \tag{20}$$

$$\beta_{i,j} \geq 0 \qquad \forall (i,j) \in A \tag{21}$$

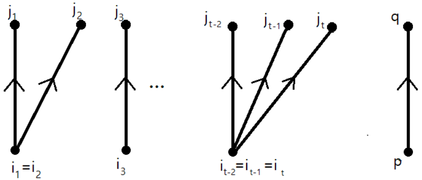
$$\beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S \tag{22}$$

Additional facet defining inequalities

- Let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\}$ be a subset of arcs, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $|I| \leq |J| = t$. Let p and q be two distinct vertices not in $I \cup J$. We define the *out-star inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p,q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}, \tag{23}$$

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p,q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q,k}. \tag{24}$$



Result

The out-star inequalities are facet defining for S^*

- Symmetrically, if we assume that that $|J| \leq |I| = t$, we define the *in-star inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V \setminus J} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}, \tag{25}$$

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V \setminus J} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q,k}. \tag{26}$$

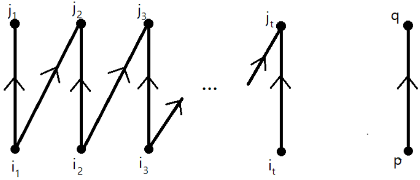
Result

The in-star inequalities are facet defining for S^*

Additional facet defining inequalities

- Let $I = \{i_1, i_2, \ldots, i_t\}$ and $J = \{j_1, j_2, \ldots, j_t\}$ be two subsets of vertices with $I \cap J = \emptyset$ and $|I| = |J| = t$. Let p and q be two distinct vertices not in $I \cup J$, we define the *path inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \sum_{l=1}^{t-1} \beta_{i_l,j_{l+1}} + \beta_{p,q} \leq \sum_{k\in V} \sum_{i\in I} \beta_{k,i} + \sum_{j\in J} \sum_{k\in V} \beta_{j,k} + \sum_{k\in V\setminus(I\cup J)} \beta_{k,p}. \tag{27}$$



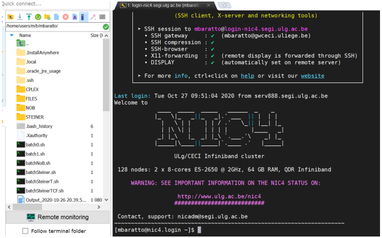
Result

The path inequalities are facet defining for S^*

Goals

- Study the complexity of the maximum weighted cycle selection (MWCS) problem √
- Describe the linear model associated with the problem and in particular the set of solutions S. √
- Polyhedral study of the convex hull of the set of solutions S. √
- Test numerically the resolution of the MWCS problem.

Numerical tests



Formulation

$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j}$$

$$s.t. \quad \beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j)\in A : i \in S, j \in V\setminus S$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j) \in A$$

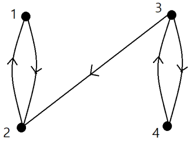
$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j} \tag{28}$$

$$s.t. \quad \beta_{i,j} \leq \sum_{k\in V, k\neq i} \beta_{k,i} \qquad \forall (i,j)\in A \tag{29}$$

$$\beta_{i,j} \leq \sum_{k\in V, k\neq j} \beta_{j,k} \qquad \forall (i,j)\in A \tag{30}$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j) \in A \tag{31}$$

for a fixed arc (i,j), $S = \{i\}$ and $S = V \setminus \{j\}$



$$\beta_{3,2} \leq \beta_{1,3} + \beta_{1,4} + \beta_{2,3} + \beta_{2,4}$$

$$(i, j) = (3, 2) \; S = \{3, 4\}$$

Result

The return inequalities are separable in polynomial time.

$\rightarrow$ Use of **callbacks** on CPLEX

Numerical tests

Density	[5,10,20,50,70]
Number of vertices	[50,100,150,200,250,300]

- d% of the arcs have a positive weight uniformly distributed in $[0, 1]$ $(100 - d)$% of the arcs have a negative weight uniformly distributed in $[-1, 0]$
- Mean over 20 instances

Nb vert	Density	Cuts	Calls	Nodes
50	10	0	1	0
100	10	0	1	0
150	10	0	1	0
200	10	0	1	0
250	10	0	1	0
300	10	0	1	0

$\rightarrow$ Natural formulation is very strong on random graphs and KE graphs.

Remarks

→ Natural formulation is very strong on random graphs and KE graphs.

- Can be explained by theoretical properties of random graphs.
- No additional cuts

Goals:

- Study the complexity of the maximum weighted cycle selection (MWCS) problem $\surd$
- Describe the linear model associated with the problem and in particular the set of solutions S. $\surd$
- Polyhedral study of the convex hull of the set of solutions S. $\surd$
- Test numerically the resolution of the MWCS problem. $\surd$

Add new constraints:

- On the cardinality of a selection: $|B| \leq \mathbf{b}$
- On the length of the cycles, the cycles in B can not exceed a certain length $\mathbf{K}$

Variant

Variant of the initial cycle selection problem.
Additional constraint on the number of selected arcs:

$$\sum_{(i,j)\in A} \beta_{i,j} \leq \mathbf{b}$$

where $\mathbf{b} \in \mathbb{N}$.

- Most of the results remain valid.

Theorem	**b**
Dimension	4
$\beta_{i,j} \leq 0$	5
$\beta_{i,j} \geq 1$	5
Return inequalities	5
Out-star and in-star inequalities	7
Path inequalities	$5t-1$

Density	[5,10,20,50,70]
Number of vertices	[50,100,150,200,250,300]
Maximum number of selected arcs	$[n, \frac{n}{2}, \frac{n}{4}, \frac{n\ln(n)}{10}]$

Nb vert	B	Density	Cuts	Calls	Nodes	Nb unsolved
50	12	10	1,47	2,13	10,27	0
100	25	10	15,27	4,20	24,53	0
150	37	10	133,40	20,67	138,93	0
200	50	10	65,80	7,33	44,67	0
250	62	10	390,60	32,13	186,53	0
300	75	10	227,60	18,00	89,93	1
300	171	50	243,87	20,00	33,53	4

What's next ?

Kidney Exchange Problem

More especially to the stochastic integer programming formulation of the Kidney exchange problem proposed by Smeulders et al. (2019).

A *two-stage selection problem* which, given a *testing budget* b, identifies (in stage 1) a subset of arcs $B \subseteq A$ with $|B| \leq b$ such that the expected number of transplants in the graph (V, B) (in stage 2) is maximized.

- Benders decomposition
- In the master problem the information about cycles is lost (i.e. the set B must be a cycle selection)
- Strengthen the formulation using the constraints of the ILP formulation of the selection problem.

Smeulders, B., Bartier, V., Crama, Y., and Spieksma, F. C. R. (2019). Recourse in kidney exchange programs.
Working paper HEC - University of Liege

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