Selecting directed cycles: a polyhedral study

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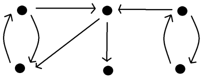
Cycle Selection Problem

Given a directed graph $G = (V, A)$.

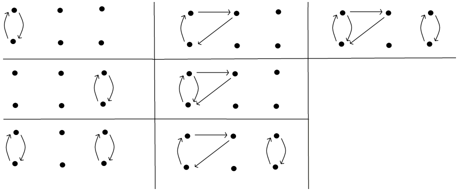


We are interested in subsets $B \subseteq A$

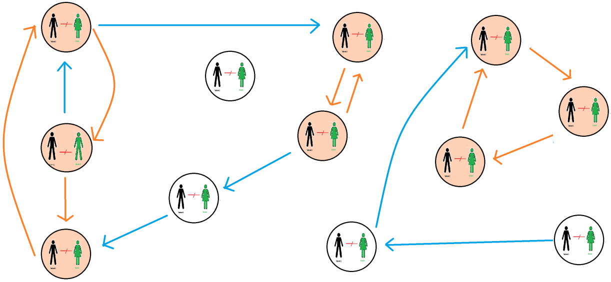
- forming a union of directed cycles in $G_B = (V, B)$

or equivalently,

- such that each arc in B is in a directed cycle in G_B



Kidney Exchange Problem



The *Maximum Weighted Cycle Selection* (**MWCS**) problem: given a directed graph $G = (V, A)$ and a weight $w_{i,j} \in \mathbb{R}$ for each arc $(i, j) \in A$, find a selection B which maximizes $w(B) = \sum_{(i,j)\in B} w_{ij}$.

Result

The MWCS problem is strongly NP-hard.

Goals

- Describe the linear model associated with the problem and in particular the set of solutions S.
- Polyhedral study of the convex hull of the set of solutions S.
- Test numerically the resolution of the MWCS problem.

Formulations

- Arc formulation

Extended (non compact) formulation:

- Cycle formulation

Extended compact formulations:

- Extended arc formulation
- Position indexed formulation

Relative strength of the formulations

In terms of tightness of the linear relaxation:

- the Arc formulation equivalent to the Extended arc formulation
- the Arc formulation dominates the Cycle formulation
- the Arc formulation dominates the Position indexed formulation

$\rightarrow$ Focus on the arc formulation

Arc formulation

Variables

$\beta_{i,j} = 1$ if arc (i,j) is selected, 0 otherwise, for all $(i,j) \in A$.

Objective function

$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j} \tag{1}$$

Constraints

$$\beta_{i,j} \in \mathbb{Z} \qquad \forall (i,j) \in A \tag{2}$$

$$\beta_{i,j} \leq 1 \qquad \forall (i,j) \in A \tag{3}$$

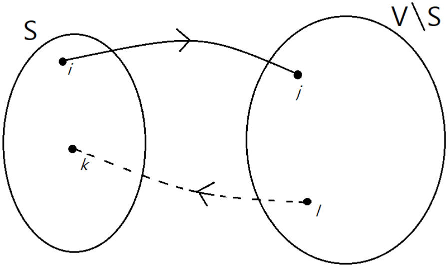
$$\beta_{i,j} \geq 0 \qquad \forall (i,j) \in A \tag{4}$$

$$\beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S \tag{5}$$

Constraints

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j) \in A$$

$$\beta_{i,j} \leq \sum_{(l,k)\in A: l \in V\setminus S, k\in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S$$

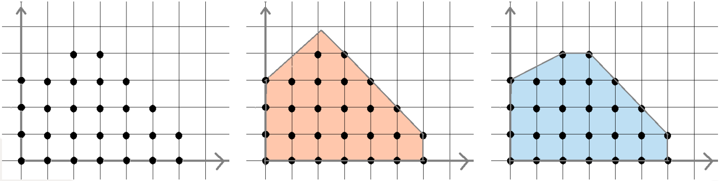


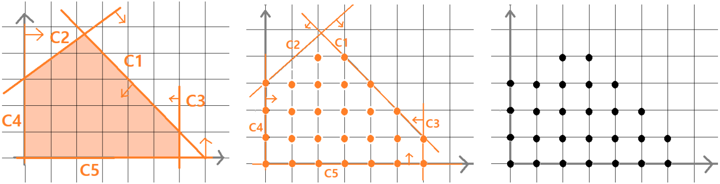
Notation

$$S = \left\{ \beta \in \{0,1\}^{|A|} : \beta_{i,j} \leq \sum_{(l,k)\in A: l \in V\setminus S, k\in S} \beta_{l,k} \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

$$SL = \left\{ \beta \in [0,1]^{|A|} : \beta_{i,j} \leq \sum_{(l,k)\in A: l \in V\setminus S, k\in S} \beta_{l,k} \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

$$S* = conv(S)$$



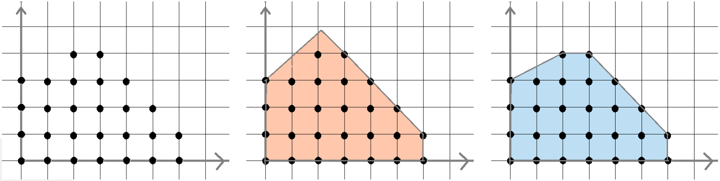

C2
C1
C3
C4
C5
C2
C1
C3
C4
C5

Notation

$$S = \left\{ \beta \in \{0,1\}^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \forall S \subseteq V, \ \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

$$SL = \left\{ \beta \in [0,1]^{|A|} : \beta_{i,j} \leq \sum_{(l,k) \in A : l \in V \setminus S, k \in S} \beta_{l,k} \forall S \subseteq V, \ \forall (i,j) \in A : i \in S, j \in V \setminus S \right\}$$

$$S* = conv(S)$$

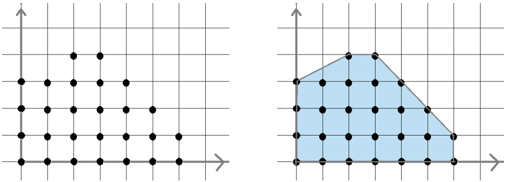


Recall

Theorem

For any set S in $\mathbb{R}^n$ and $c \in \mathbb{R}^n$ then

$$\max\{c^T x : x \in S\} = \max\{c^T x : x \in conv(S)\}$$



Polyhedral study

For complete directed graph.

Result

S^* is full dimensional

Result

The trivial inequalities and return inequalities are facet defining for S^*

$$\beta_{i,j} \leq 1 \qquad \forall (i,j) \in A \tag{6}$$

$$\beta_{i,j} \geq 0 \qquad \forall (i,j) \in A \tag{7}$$

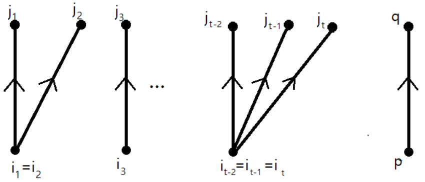
$$\beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j) \in A : i \in S, j \in V \setminus S \tag{8}$$

Additional facet defining inequalities

- Let $E = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\}$ be a subset of arcs, and let $I = \{i_1, i_2, \ldots, i_t\}$, $J = \{j_1, j_2, \ldots, j_t\}$. Assume that $I \cap J = \emptyset$ and $|I| \leq |J| = t$. Let p and q be two distinct vertices not in $I \cup J$. We define the *out-star inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p,q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}, \tag{9}$$

$$\sum_{l=1}^{t} \beta_{i_l, j_l} + \beta_{p,q} \leq \sum_{k \in V \setminus I} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q,k}. \tag{10}$$



Result

The out-star inequalities are facet defining for S^*

- Symmetrically, if we assume that that $|J| \leq |I| = t$, we define the *in-star inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V \setminus J} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}, \tag{11}$$

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V \setminus J} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{q,k}. \tag{12}$$

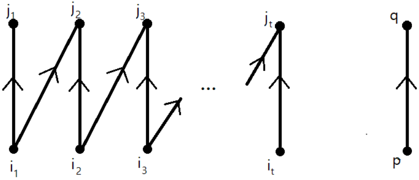
Result

The in-star inequalities are facet defining for S^*

Additional facet defining inequalities

- Let $I = \{i_1, i_2, \dots, i_t\}$ and $J = \{j_1, j_2, \dots, j_t\}$ be two subsets of vertices with $I \cap J = \emptyset$ and $|I| = |J| = t$. Let p and q be two distinct vertices not in $I \cup J$, we define the *path inequalities*:

$$\sum_{l=1}^{t} \beta_{i_l,j_l} + \sum_{l=1}^{t-1} \beta_{i_l,j_{l+1}} + \beta_{p,q} \leq \sum_{k \in V} \sum_{i \in I} \beta_{k,i} + \sum_{j \in J} \sum_{k \in V} \beta_{j,k} + \sum_{k \in V \setminus (I \cup J)} \beta_{k,p}. \tag{13}$$



Result

The path inequalities are facet defining for S^*

Numerical tests

Formulation

$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j}$$

$$s.t. \quad \beta_{i,j} \leq \sum_{(l,k)\in A: l\in V\setminus S, k\in S} \beta_{l,k} \qquad \forall S \subseteq V,\ \forall (i,j)\in A : i\in S, j\in V\setminus S$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j)\in A$$

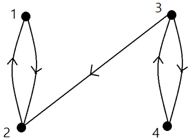
$$\max \sum_{(i,j)\in A} w_{i,j}\beta_{i,j} \tag{14}$$

$$s.t. \quad \beta_{i,j} \leq \sum_{k\in V, k\neq i} \beta_{k,i} \qquad \forall (i,j)\in A \tag{15}$$

$$\beta_{i,j} \leq \sum_{k\in V, k\neq j} \beta_{j,k} \qquad \forall (i,j)\in A \tag{16}$$

$$\beta_{i,j} \in \{0,1\} \qquad \forall (i,j)\in A \tag{17}$$

for a fixed arc (i,j), $S = \{i\}$ and $S = V \setminus \{j\}$



$$\beta_{3,2} \leq \beta_{1,3} + \beta_{1,4} + \beta_{2,3} + \beta_{2,4}$$

$$(i,j) = (3,2) \; S = \{3,4\}$$

Result

The return inequalities are separable in polynomial time.

$\rightarrow$ Use of **callbacks** on CPLEX

Numerical tests

Density	[5,10,20,50,70]
Number of vertices	[50,100,150,200,250,300]

- d% of the arcs have a positive weight uniformly distributed in $[0, 1]$ $(100 - d)$% of the arcs have a negative weight uniformly distributed in $[-1, 0]$
- Mean over 20 instances

Nb vert	Density	Cuts	Calls	Nodes
50	10	0	1	0
100	10	0	1	0
150	10	0	1	0
200	10	0	1	0
250	10	0	1	0
300	10	0	1	0

$\rightarrow$ Natural formulation is very strong on random graphs and KE graphs.

Add new constraints:

- On the cardinality of a selection: $|B| \leq \mathbf{b}$
- On the length of the cycles, the cycles in B can not exceed a certain length **K**

Variant

Variant of the initial cycle selection problem.
Additional constraint on the number of selected arcs:

$$\sum_{(i,j)\in A} \beta_{i,j} \leq \mathbf{b}$$

where $\mathbf{b} \in \mathbb{N}$.

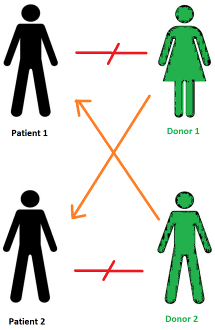
Most of the theoretical results remain valid.

Density	[5,10,20,50,70]
Number of vertices	[50,100,150,200,250,300]
Maximum number of selected arcs	$[n, \frac{n}{2}, \frac{n}{4}, \frac{n ln(n)}{10}]$

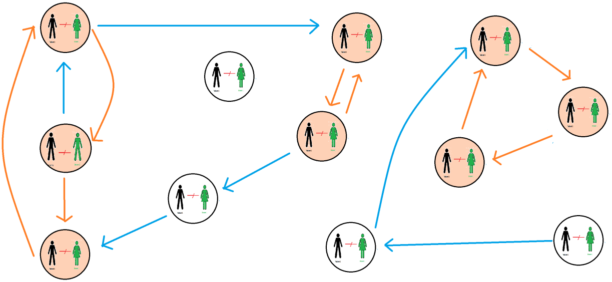
Nb vert	B	Density	Cuts	Calls	Nodes	Nb unsolved
50	12	10	1,47	2,13	10,27	0
100	25	10	15,27	4,20	24,53	0
150	37	10	133,40	20,67	138,93	0
200	50	10	65,80	7,33	44,67	0
250	62	10	390,60	32,13	186,53	0
300	75	10	227,60	18,00	89,93	1
300	171	50	243,87	20,00	33,53	4

What's next ?

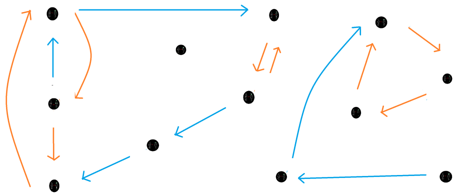
Kidney Exchange Problem


Patient 1
Donor 1
Patient 2
Donor 2

Possible exchanges



Compatibility graph



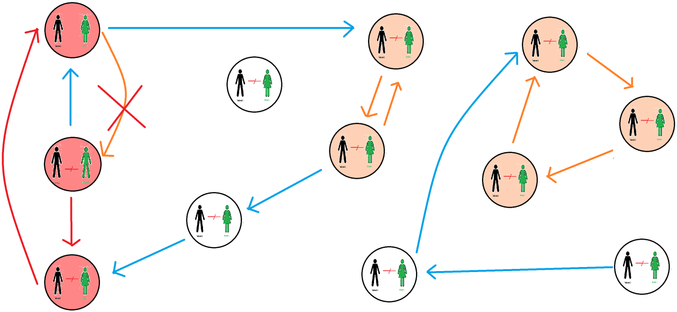
G=(V,A,w) where:

- $V = \{1, ..., n\}$ set of vertices, consisting of all patient-donor pairs.
- A, the set of arcs, designating compatibilities between the vertices. Two vertices i and j are connected by arc (i, j) if the donor in pair i is compatible with the patient in pair j.
- Weight function w such that for each arc $(i, j) \in A$, $w_{i,j}$ represents the weight/the utility of a transplant between donor i and patient j.

Definition

An exchange is a set of disjoint cycles in the directed graph. It is feasible if every cycle length does not exceed a given limit K.

Uncertainty



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