Modelling suspended load with moment equations and linear concentration profiles

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ABSTRACT: *In numerical simulations, it is always necessary to find an optimum between the simplicity of the model and a good representation of real phenomena. In the field of hydraulic flows simulations, the models using depth-averaged and moment equations are an interesting compromise between full 3D and simple depth-averaged models. This paper presents the use of a moment equation for suspended load transport. A simple but representative model for the sediment concentration profiles is developed. This original bi-linear concentration profile is compared to the traditional Rouse-profiles and shows a good correspondence despite its great simplicity. Advective and diffusive sediment fluxes are developed analytically and lead to a concise formulation, which is an asset for practical use. A differential equation for the sediment concentration moment is also fully developed, and a special attention is cast to the source term. The finite volume scheme has been chosen to implement the model, because it is particularly well suited for highly advective transport equations, it is conservative and it makes the choice of the upwinding easier. 1D simulations show the capacity of the model to reproduce laboratory experiments described in the literature.*

KEY WORDS: *sediment transport, suspended load, moment equations, depth-averaged, finite volume.*

1. Introduction

Many problems in fluvial hydraulics include a sedimentary aspect: silting of dam reservoirs, desilting pools; evolution of riverbeds, scouring near bridge pier, silting of hydraulic constructions,… As a consequence, there is a demand for efficient solid transport models.

The suspended load, opposite to the bed load, is characterised by a vertical profile for the sediment concentration [van Rijn 1984]. This profile changes with space and time. It influences solid discharges, erosion and deposition, and is therefore an important parameter in the suspended load simulations. For a steady uniform flow, it has been shown that the resulting profile depends mainly on the ratio between the gravity and the turbulent forces. This ratio is represented by the Rouse number, $P = w_s / \beta \kappa u_*$, where w_s is the settling velocity, β a coefficient linked to the sediment diffusion, κ the Von Karman constant, and u_* the friction velocity. For more details, see [Chanson 1999; Graf and Cellino 2002; Hervouet and Villaret 2004; Julien 1995; van Rijn 1984; Yalin 1977]

In unsteady and non-uniform flows, the profile can take many different forms and must be computed on the basis of flow and sediment equations. For an exact representation of the sediment profile, this involves a vertical discretisation of the flow and the sediment concentration. But as computation power is still a limitation to be taken into account, it is essential that the chosen models require few calculation time in order to be used in large-scale practical applications. This paper will present an original profile description, which can represent correctly real sediment profiles while keeping a great simplicity.

2. Governing Equations

This paper focuses more particularly on the solid transport modelling itself. It doesn't dwell on the water flow equations, but the velocity fields will be supposed to be known. A depth-averaged model for flow is used here, and therefore the most common profile for velocities is uniform. Some more complete approaches provide a linear velocity profile by using an additional moment of momentum equation for flow [Ghamry and Steffler 2002; Steffler and Jin 1993]. The following theory is based on such a type of profile.

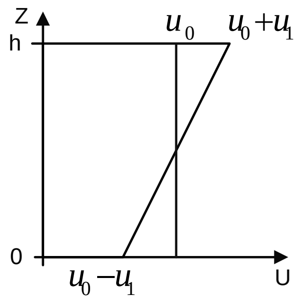

Figure 1. *Velocity profile*

The governing equation for solid transport is the result of the mass conservation principle applied to a control volume inside the flow [Julien 1995]:

$$\frac{\partial C}{\partial t}+\frac{\partial q_{sx}}{\partial x}+\frac{\partial q_{sy}}{\partial y}+\frac{\partial q_{sz}}{\partial z}=S_s\ , \qquad [1]$$

where C is the local sediment concentration, S_s is the sediment source term, and q_{si} $\left(i=x,y,z\right)$ are the solid discharges.

Developing mass fluxes, we obtain advective-diffusive equation for sediments [Pirotton et al. 2002]:

$$\frac{\partial C}{\partial t}+\frac{\partial}{\partial x}\left(uC-\varepsilon_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(vC-\varepsilon_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(wC-\varepsilon_z\frac{\partial C}{\partial z}\right)=S_s \qquad [2]$$

where u, v and w are the velocity components, and ε_i are the turbulent diffusion coefficients.

Before going further with equation [2], it is necessary to introduce the concentration profile which will be used later on. In contrast to the velocity profile, it is inadequate to use a simple linear profile, because concentration cannot be negative. Therefore, another profile will be used, which is partly linear and partly equal to zero.

It can take three different forms, as shown in Figure 2, Figure 3 and Figure 4.

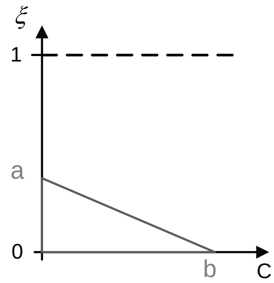

Figure 2. *(A)-Type concentration profile*

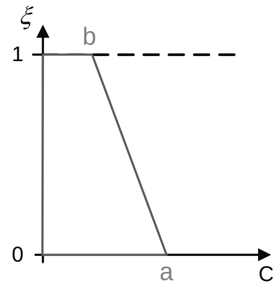

Figure 3. *(B)-type concentration profile*

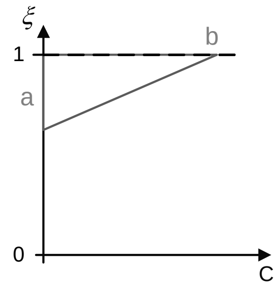

Figure 4. *(C)-type concentration profile*

This kind of profile is univocally defined if we know both

– mean value $\left(h\overline{C}\right)$, corresponding to the surface under the profile

– moment arm m_c, corresponding to the relative height of the centre of gravity:

$$m_c=\frac{1}{h\overline{C}}\int_0^h \xi C dz\ ,\ \xi=z/h$$

Indeed, the value of m_c can be worked out for each profile.

– $0<m_c\leq 1/3$ for (A)-type profiles.

– $1/3\leq m_c\leq 2/3$ for (B)-type profiles.

– $2/3\leq m_c<1$ for (C)-type profiles.

As a consequence, there is a univocal relation between the concentration profiles and unknowns $h\overline{C}$ and m_c, which allow to know the concentration profile as soon as we have the mean concentration and the moment arm.

As the quality of a particular type of concentration profile can be measured by its capacity to represent real profiles, it is therefore useful to see to what extend the "Linear-zero" profile is similar to the conventional profiles commonly met. The well known Rouse profiles, which develop in the case of a steady uniform flow, constitute a good comparison basis. The form of these profiles depends on the Rouse number. To each of these profiles correspond both a moment arm and a mean concentration, and it is therefore possible to relate Rouse profiles and "Linear-zero" profiles.

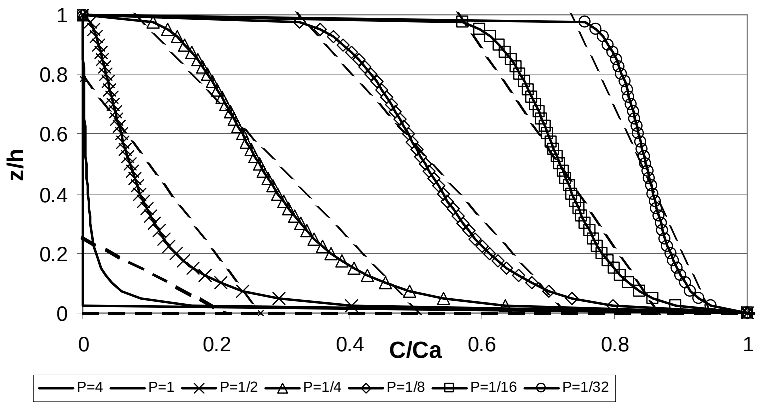

Figure 5. C*omparison between Rouse and Linear-zero profiles (Ca is the bed reference concentration)*

We can see a satisfactory concordance despite the extreme simplicity of the model. The profiles with lower Rouse numbers are better fitted, while high-Rouse profiles, which have more definite curvatures, are represented with more difficulties. In particular, we can see that the bed concentration can differ significantly. For a Rouse number equal to 1, the bed concentration of the linear profile is 5 times smaller than the Rouse profile bed concentration. Nevertheless, this high difference only exists near the bed, and is quickly reducing with distance from bed (see Figure 5). The difference between the two profiles has a minor influence on the horizontal solid fluxes but can have an effect on the source term if it is based on bed concentration.

3. Fluxes calculation

Integration of equation [2] over the depth provides the following result:

$$\frac{\partial hC}{\partial t}+\frac{\partial Q_{sx}}{\partial x}+\frac{\partial Q_{sy}}{\partial y}=\overline{S}_s\ , \qquad [3]$$

where solid discharge terms Q_{sx} and Q_{sy} (sum of advective and diffusive solid discharges) could be determined by analytic integration.

The advective part comes from the product between the velocity and the concentration profiles [Chanson 1999]:

$$q_{s,adv}=\int_a^h C(z)u(z)dz \qquad [4]$$

Thanks to the simplicity of linear-zero profiles, this term reduces to the following simple form:

$$q_{s,adv}=h\overline{C}\left[u_0+\left(2m_c-1\right)u_1\right] \qquad [5]$$

In a first approximation, the diffusive part is simply given by

$$q_{s,diff}=-\overline{\varepsilon}_i\,\frac{\partial\left(h\overline{C}\right)}{\partial x_i}\ , \qquad [6]$$

where $\overline{\varepsilon}$ is a mean coefficient of turbulent diffusivity. A better estimation would be to calculate the integral

$$q_{s,adv}=\int_a^h \frac{\partial}{\partial x_i}\left(\varepsilon_i\,\frac{\partial C}{\partial x_i}\right)dz\ , \qquad [7]$$

but as the diffusive flux will play a minor part in treated applications, a rough but simple estimation of this term (equation [6]) is sufficient. Nevertheless, development of this term remains an interesting prospect, especially for applications with important diffusion processes.

4. Moment Equation

As we add one unknown (m_c) in order to determine univocally the concentration profiles, it is necessary to add a moment equation for the sediments. This one is derived from the mass conservation equation [1], which is multiplied by the weight function $F=\xi=z/h$ [Guo and Jin 1999]. After multiplication and integration, this leads to

$$\frac{\partial}{\partial t}\int_0^h \xi C dz + \frac{\partial}{\partial x}\int_0^h \xi q_{sx} dz + \frac{\partial}{\partial y}\int_0^h \xi q_{sy} dz = M_s \qquad [8]$$

In this equation, the moment source term M_s includes

– boundary terms (coming from the integration of derivatives by Liebniz rule), which take into account the solid fluxes at the surface and the bed.

– an integral term

$$\int_0^h \xi S_s(z)\,dz \qquad [9]$$

– a term equal to

$$\int_0^h q_{sz} dz \qquad [10]$$

coming from the integration of the term $\xi \frac{\partial}{\partial z} q_{sz}$ from equation [1].

In many applications, the only sediment flux trough boundaries comes from the bed. This implies that all other boundary terms reduce to zero. The boundary term corresponding to the external flux (sediment exchange with the bed) is equal to

$$\xi q_{s,\mathrm{bed}}\,, \qquad [11]$$

but as $\xi = 0$ at the bed level, this term is also equal to zero.

Moreover, because there is no sediment source inside the flow, the term

$$\int_0^h \xi S_s(z)\,dz \qquad [12]$$

also reduces to zero.

In a steady uniform flow, the concentration profile will evolve even without any modification of suspended sediment quantity. It will tend to an equilibrium profile (Rouse profile), and this results in a spontaneous evolution of the concentration moment towards an equilibrium moment $M_{éq}$. This suggests that the term [10] can be replaced by a moment source term of the form

$$\pm \zeta h \bar{C} \left| m_{éq} - m_c \right|^{\eta}, \qquad [13]$$

where $m_{éq}$ is the moment arm corresponding to the equilibrium concentration profile; ζ and η are dimensionless parameters. These parameters need to be calibrated. This goal has been achieved by numerous 1D vertical simulations. A non equilibrium profile is set and evolution of the concentration moment is plotted. This allows to find best values of ς and η to fit resulting data. Resulting values are

$$\eta = 1.3$$

$$\zeta = 1.5 \frac{\beta u_*}{h}$$

(with all variables in SI units).

We can see in equation [8] appearing a moment of concentration

$$M_c = \int_0^h \xi C dz \qquad [14]$$

which can be used instead of the moment arm m_c , as both variables are linked by the relation

$$m_c = \frac{M_C}{h\bar{C}} \qquad [15]$$

The advective and diffusive terms

$$M_{q,adv} = \frac{\partial}{\partial x}\int_0^h \xi q_{sx,adv} dz \text{ and } M_{q,diff} = \frac{\partial}{\partial x}\int_0^h \xi q_{sx,diff} dz \qquad [16]$$

can be calculated analytically, and a concise formulation is obtained:

$$M_{q,adv} = M_c \left[u_0 + f(m_c) u_1 \right], \qquad [17]$$

where

$$f(m_c) = \begin{cases} 3m_c - 1 & \text{if} \quad m_c \le \frac{1}{3} \\ 1 - \frac{1}{3m_c} & \text{if} \quad \frac{1}{3} \le m_c \le \frac{2}{3} \\ 3m_c - 3 + \frac{1}{m_c} & \text{if} \quad m_c \ge \frac{2}{3} \end{cases} \qquad [18]$$

and

$$M_{qi,diff} = -\varepsilon_i \frac{\partial M_c}{\partial x_i} \qquad [19]$$

5. Application in a Finite Volume program

This suspended load model has been implemented in a finite volume program. An original upwinding [Mouzelard 2002] has been chosen for the flow and the sediment transport, as shown in the following simplified case (1D rectangular channel with constant width). The results can be easily extended to more complex flows.

The system of equations for the flow, the suspended load and the bed level evolution can be written

$$\frac{\partial \vec{s}}{\partial t}+\frac{\partial \vec{f}^{+}}{\partial x}+\frac{\partial \vec{f}^{-}}{\partial x}=\vec{S}\ , \qquad [20]$$

where

– $\vec{s}=\begin{Bmatrix} h \\ q \\ hC \\ Z_b \end{Bmatrix}$ is the unknowns vector, $\vec{S}=\begin{bmatrix} q_L \\ gh\left(\sin\theta - J\right) - gh\cos\theta\dfrac{\partial Z_b}{\partial x} \\ -\dfrac{\partial}{\partial x} q_{s,diff} + \overline{S}_s \\ \dfrac{-\overline{S}_s}{\left(1-p\right)} \end{bmatrix}$

includes the source terms (which don't need to be split), and $\begin{cases} \vec{f}^{+} \\ \vec{f}^{-} \end{cases}$ are the flux terms with an $\begin{cases} \text{upstream} \\ \text{downstream} \end{cases}$ upwinding.

– Z_b is the bed level, q_L is the lateral water inflow, θ is the bed slope, J is the friction slope, $q_{s,diff}$ is the diffusive term for suspended load, S_s is the sediment source term (positive for erosion) and p is the porosity of the bed.

In this model, the following splitting is used for the fluxes:

$$\vec{f}^{+}=\begin{bmatrix} q \\ \rho_\omega \dfrac{q^2}{h} \\ KqC \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{f}^{-}=\begin{bmatrix} 0 \\ g\cos\theta\dfrac{h^2}{2} \\ 0 \\ 0 \end{bmatrix}, \qquad [21]$$

where ρ_ω is the non-uniform velocity coefficient, and K is a coefficient of proportionality between qC and the advective solid discharge (see equation [4] and [5]).

In this flux vector splitting, we can see that $\vec{f}^{+}$ includes the advective terms (water discharge, momentum and suspended load discharge), while $\vec{f}^{-}$ corresponds to the pressure term.

The stability of this scheme has been proven by a Von Neumann analysis. A similar procedure for bed load can be found in [Dewals 2001].

6. Model validation

To carry out the validation of the model, two experiments have been chosen. These well-documented experiments show the capacity of the model to simulate different kinds of processes (sediment unloading and bed evolution).

Unloading experiment

The first validation simulation is based on the experiment by Wang and Ribberink. Its purpose is the measure of concentration profiles in a steady uniform flow on a perforated plate.

The channel is 30m long and 0.5m wide. It consists in three parts:

– A rigid bed to allow the development of a uniform concentration profile.

– A perforated plate, the goal of which is to catch the sediments near the bed

– An exit section with a rigid bed

A more accurate description of the experiment can be found in [Wang and Ribberink 1986].

A detailed description of the concentration profiles is provided by the author, which allow the comparison with the results of the developed model. We can see in Figure 6 that the results show a very good concordance. The relative error, defined by

$$err = \frac{C_{simul} - C_{measured}}{C_{initial}} \qquad [22]$$

is less than 6.5% for experiment 1 and less than 21% for experiment 2. But in this last one, we can see that the measure at $x = 1m$ seem erroneous.

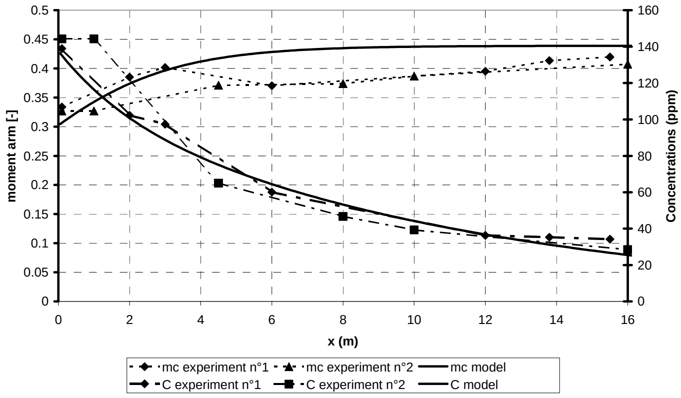

Figure 6. *Comparison of measures and simulations in the unloading experiment*

We can see in Figure 7 that concentration profiles in this experiment have an almost linear form and are therefore easily represented by the linear-zero model.

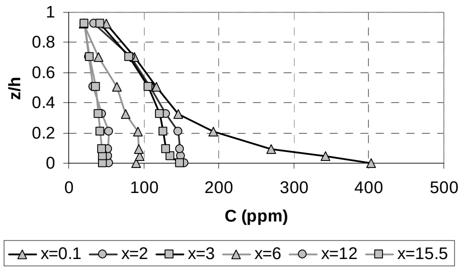

Figure 7. *Evolution of concentration profiles*

Trench experiment

This experiment has been carried out by Van Rijn in the Delft Hydraulic laboratory. The channel used is 30m long and 0.5m wide. The bed includes a trench, the evolution of which is described in the literature [van Rijn 1987]. The results show a movement and a deformation of the trench.

In the first experiment (unloading), the perforated plate reduced the sediment source term to a simple deposition term.

In this experiment, we have both erosion and deposition, and flow conditions change with time. Therefore, a more complete source term must be chosen.

This experiment involves both bed load and suspended load, in defined proportions. On the face of it, it seems that suspended load models can not be used alone. But Guo and Jin simulated this experiment by using a suspended load model with the souce term [Guo and Jin 1999]:

$$\bar{S}_s = \alpha w_s \left(C_{éq} - \bar{C} \right) \qquad [23]$$

where $C_{éq} = k\left(\frac{|\overline{u}|^3}{hw_s}\right)^m$; α , k and m are parameters.

More details on this source term can be found in the article from Guo and Jin.

Their simulations showed good results. It seems therefore that this solid transport law resembles to a total load transport, while remaining based on an advection-diffusion equation (as in suspended load models). This source term will therefore be used here.

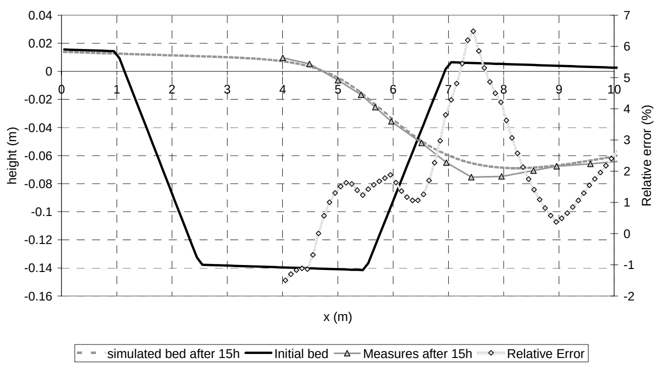

Figure 8. *Van Rijn experiment (Trench)*

The comparison between the measures and the simulated results shows that the developed model is able to properly reproduce Van Rijn's experiment when used with an adequate source term. The relative error, defined by

$$Err = \frac{Z_{\text{simulated bed}} - Z_{\text{measured bed}}}{\text{depth of trench}} \qquad [24]$$

is less than 7%.

7. Conclusion

A suspended load model using a moment equation for concentration profiles has been presented. A new model for concentration profiles both simple and representative of reality has been defined. Its simplicity led to a concise formulation for the terms in the mass conservation and concentration moment equations, and its quality has been proved on several occasions. First, by the comparison between the new concentration profiles and the classical Rouse profiles, then by the validation experiments chosen in order to highlight the quality of the model in different kinds of solid transport processes.

In order to have an efficient tool for large-scale solid transport studies, only the choice of an adequate source term is left, because we henceforth have a good model for the transport itself.

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