An Implicit Finite Volume Method for the Solution of 3D Low Mach Number Viscous Flows Using A Local Preconditioning Technique

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Introduction

- Navier-Stokes equations and choice of variables
- Cell-centered finite volume discretization scheme
- Fully implicit pseudo-transient Newton-GMRes method
- Local preconditioning
- Results (2D and 3D test cases)
- Conclusions
Navier-Stokes equations

Choice of dimensionless primitive variables \( \mathbf{w} = (p, \mathbf{u}, T)^T \)

\[
p = \frac{p_d - p_0}{\rho_0 U_0^2} \quad \mathbf{u} = \frac{\mathbf{u}_d}{U_0} \quad T = \frac{T_d}{T_0} \quad \rho(p, T) = \frac{\rho_d(p_d, T_d)}{\rho_0} \quad h(p, T) = \frac{h_d(p_d, T_d)}{c_p T_0}
\]

Definitions of dimensionless numbers:

\[
Str = \frac{L_0}{U_0 t_0} \quad Re = \frac{\rho_0 U_0 L_0}{\mu_0} \quad M_0 = \frac{U_0}{c_0} \quad \chi = \frac{U_0^2}{c_p T_0} \quad Pe = \frac{\rho U_0 L_0 c_p}{\kappa_0}
\]

Navier-Stokes equations:

\[
Str \frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho H - \chi p \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho \mathbf{u}^T \\ \rho \mathbf{u}^T \mathbf{u} + p \mathbf{I} \\ \rho \mathbf{u}^T H \end{pmatrix} = \nabla \cdot \begin{pmatrix} 0^T \\ -\frac{1}{Pe} \mathbf{q}^T + \frac{\chi}{Re} \mathbf{u}^T \mathbf{T} \end{pmatrix}
\]

\[
H = h + \frac{1}{2} \chi \| \mathbf{u} \|^2 \quad \mathbf{T} = \mu \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T \right) - \frac{2}{3} \mu \left( \nabla \cdot \mathbf{u} \right) \mathbf{I} \quad \mathbf{q} = -\kappa \nabla T
\]
Cell-centered Finite Volume Scheme

Integrated form of the **conservative Navier-Stokes** equations:

\[
\frac{ds}{dt} + \frac{1}{V} \int_S F_a(w) \cdot d\Sigma = \frac{1}{V} \int_S F_d(w, \nabla w) \cdot d\Sigma
\]

- Primitive variables at the Gauss Points of the cell faces **reconstructed** from left (L) and right (R) neighbours by a truncated Taylor series expansion

\[
\tilde{w}_{R,L} = w_{R,L} + (r_G - r_{R,L})^T \nabla w_{R,L} + \frac{1}{2} (r_G - r_{R,L})^T H_{R,L} (r_G - r_{R,L}) + O(h^3)
\]

Quadrangular face : 4 Gauss Points

Triangular face : 3 Gauss Points

- Gradient \(\nabla w\) evaluated by a modification of Coirier’s diamond path scheme

Truncature error \(\Rightarrow O(h^2)\) on advective terms and \(O(h^1)\) on viscous terms
Cell-centered Finite Volume Scheme

Advecive fluxes computed by **AUSM+up** method designed by Liou \( \Rightarrow \tilde{F}(\tilde{w}_R, \tilde{w}_L) \)

\[
c_{1/2} = \frac{1}{2} (\tilde{c}_L + \tilde{c}_R) \quad M_{L,R} = \frac{\tilde{u}_{L,R}^T \mathbf{n}}{c_{1/2}} \quad M^2 = \frac{1}{2} \frac{||\tilde{u}_L||^2 + ||\tilde{u}_R||^2}{c_{1/2}^2}
\]

\[
M_{1/2} = M_4^+(M_L) + M_4^-(M_R) - \frac{K_p}{f_c} \max \left(1 - M^2, 0\right) \frac{(\tilde{p}_R - \tilde{p}_L)}{\left(\frac{2\rho_0}{\rho_0 U_0^2} + \tilde{p}_R + \tilde{p}_L\right)}
\]

\[
p_{1/2} = \mathcal{P}_{(5)}^+(M_L) \tilde{p}_L + \mathcal{P}_{(5)}^-(M_R) \tilde{p}_R - K_u \mathcal{P}_{(5)}^+(M_L) \mathcal{P}_{(5)}^-(M_R) (\tilde{\rho}_L + \tilde{\rho}_R) \left(f_c c_{1/2}^2\right) (M_R - M_L)
\]

\[
F_a^T \mathbf{n} = c_{1/2} M_{1/2} \begin{pmatrix}
\tilde{\rho} \\
\tilde{\rho} \tilde{u} \\
\tilde{\rho} \tilde{H}
\end{pmatrix}
\begin{cases}
L & \text{if } M_{1/2} > 0 \\
R & \text{if } M_{1/2} < 0
\end{cases}
+ \begin{pmatrix}
0 \\
p_{1/2} \mathbf{n}
\end{pmatrix}
\]

Function \( f_c \) used for scaling speed of sound is defined by the local preconditioning method.
Fully implicit pseudo-transient Newton-GMRES scheme

- **Pseudo-transient** iterations for convergence to steady flow

\[ S \left( \frac{w^{l+1} - w^l}{\Delta \tau^l} \right) + F \left( w^{l+1} \right) = 0 \quad S^l = \left( \frac{\partial s}{\partial w} \right)^l \]

- System of non-linear equations solved by **Newton’s method**

\[ \left( \frac{1}{\Delta \tau^l} S^l + J^l \right) \delta w^{(l)} = -F \left( w^{(l)} \right) \quad J^l = \left( \frac{\partial F}{\partial w} \right)^l \]

- Linear system are solved by a **matrix-free GMRes** algorithm.

\[ J \left( w^{(l)} \right) \nu \approx \frac{F \left( w^{(l)} + \epsilon \nu \right) - F \left( w^{(l)} \right)}{\epsilon} \]

- **Preconditionning** required to ensure good convergence of the GMRES solver: right preconditioning based on a block incomplete factorization (BILU(\(k\))) of an approximate Jacobian
  - constant reconstruction for advective terms
  - classical diamond scheme for viscous ones
Fully implicit pseudo-transient Newton-GMRES scheme

Problems arising for low Mach number flows

- $\Delta \tau^l = \text{local timestep computed by the Switched Evolution Relaxation method}$

\[ CFL^{(l+1)} = CFL^{(0)} \left( \frac{\| F(w^{(0)}) \|}{\| F(w^{(l)}) \|} \right)^p \]

- Poor convergence of GMRES algorithm caused by a very bad conditioning. The condition number being

\[ CN \approx \frac{1}{M} \]

⇒ Use of a numerical speed of sound by using a local preconditioning matrix $P$.

\[ P \left( \frac{w^{l+1} - w^l}{\Delta \tau^l} \right) + F(w^{l+1}) = 0 \]

\[ P = \begin{pmatrix} \rho_p^2 & 0^T & \rho_T \\ \rho_p u & \rho I & \rho_T u \\ \rho_p H - \left( \chi - \rho h_p \right) & \chi \rho u^T & \rho_T H + \rho h_T \end{pmatrix} \]
Local Preconditioning Technique

Eigenvalues analysis of the pseudo-transient equation written in quasi linear form

\[ P \frac{\partial w}{\partial \tau} + A_x \frac{\partial w}{\partial x} + A_y \frac{\partial w}{\partial y} + A_z \frac{\partial w}{\partial z} - D \left( \frac{\partial^2 w}{\partial x \partial x} + \frac{\partial^2 w}{\partial y \partial y} + \frac{\partial^2 w}{\partial z \partial z} \right) = 0 \]

\[ A_{x,y,z} = u_{x,y,z} S + \begin{pmatrix} 0 & \rho e_{x,y,z}^T & 0 \\ e_{x,y,z} & 0 & 0 \\ \chi u_{x,y,z} & \rho H e_{x,y,z}^T & 0 \end{pmatrix} \]

\[ D = \begin{pmatrix} 0 & 0^T & 0 \\ 0 & \frac{1}{Re} I & 0 \\ 0 & 0^T & \frac{1}{Re} \end{pmatrix} \]

Introducing a Fourier mode

\[ w = w_0 \exp \left( k^T x - i \omega \tau \right) \]

\[ \| k \| = \frac{\phi}{\Delta x} \quad \phi \in [0, \pi] \]

And defining a cell Reynolds number \( \overline{Re} \), a cell Peclet number \( \overline{Pe} \) and \( \overline{\chi} \) by

\[ u_k = \frac{u^T}{\| k \|} \quad \| k \| \overline{Re} = \frac{\rho u_k \Delta x}{\mu} Re \quad \overline{Pe} = \frac{\rho u_k h_T \Delta x}{\kappa} Pe \quad \overline{\chi} = \frac{u_k^2 \rho_T}{\rho h_T} \chi \]
Local Preconditioning Technique

The first two eigenvalues are found to be

\[ \lambda = \frac{\omega}{\|k\|} = u_k \left( 1 - i \frac{\phi}{Re} \right) \]

while the following equation is obtained for the three other ones \((\lambda^* = \frac{\lambda}{u_k})\)

\[
\left( i + \frac{\phi}{Re} - i \lambda^* \right) \left[ \lambda^* - \lambda^* \left( 1 + \frac{c^*}{c^2} - i \frac{\phi}{P e} \right) - \left( \frac{c^*}{c^2} + \frac{c^*}{u_k^2} + i \frac{\phi}{P e} \right) \right]
\]

\[
+ \frac{\phi}{u_k^2 Re} \left( i \bar{\chi} + \left( 1 - \frac{Re}{P e} \right) - i \lambda^* \bar{\chi} \right) = 0
\]

with

\[ c^* = \frac{\rho h_T}{\rho h_T \rho_p + \rho_T (\chi - \rho h_p)} \]
Weiss and Smith Local Preconditioning Technique

Following the idea of Weiss and Smith

\[
\rho' p = \frac{1}{c^2} - \frac{(\chi - \rho h_p) \rho_T}{\rho h_T}
\]

\[
c' = \begin{cases} 
\min (c, \max (\|u\|, u_c)) & \text{if } \overline{Re} > 1 \\
\frac{\|u\|}{\overline{Re}} & \text{otherwise}
\end{cases}
\]

For the inviscid case, the equation can easily be solved. The set of eigenvalues \(\lambda_i\) is

\[
\lambda_{1,2,3} = u_k \quad \lambda_{4,5} = \frac{1}{2} \left( 1 + M^*^2 \right) (u_k \pm c f_c)
\]

\[
f_c = \frac{\sqrt{4 M^*^2 + M^2 (1 - M^*^2)^2}}{(1 + M^*^2)} \quad \text{with } M^* = \frac{c'}{c}
\]

This function \(f_c\) is used in the AUSM+up scheme.

In most cases \(\rightarrow\) \(C'N = \frac{1+\sqrt{5}}{2}\)
Laminar flow past a backward facing step \( Re = 150 \)

Reattachment point position

<table>
<thead>
<tr>
<th>Method</th>
<th>Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kueny - Binder (experiment)</td>
<td>6</td>
</tr>
<tr>
<td>Glowinski (INRIA)</td>
<td>5.75</td>
</tr>
<tr>
<td>Ecer</td>
<td>5.9</td>
</tr>
<tr>
<td>Bourdel (ONERA-CERT)</td>
<td>6.17</td>
</tr>
<tr>
<td>Present</td>
<td>6.277</td>
</tr>
</tbody>
</table>

(a) Isobar contours \( M_0 = 10^{-5} \).

(b) Streamlines \( M_0 = 10^{-5} \).
Laminar flow in a lid driven cavity \( M_0 = 10^{-4} \) on a \( 5 \times 12 \times 10^3 \) quadrangles mesh

(c) High stretched grid.  
(d) \( Re = 1 \).  
(e) \( Re = 100 \).  
(f) \( Re = 1000 \).

(g) velocity \( u_x \) profile.  
(h) velocity \( u_y \) profile.  
(i) Convergence curves.
**Laminar flow past a sphere** \( M_0 = 10^{-3} \) on a \( 3 \times 10^5 \) tetrahedra mesh

(j) Streamlines \( Re = 118 \).

(k) Isobar contours \( \Delta p = 2 \times 10^{-2} \).

(l) Comparison with Taneda experiment.

(m) Convergence curves.
Conclusions

- A cell-centered **density-based** finite volume method has been used for the solution of low Mach number flows on **3D unstructured grids**.

- A **fully implicit** pseudo-transient preconditioned Newton-GMRes scheme has been tested.

- Weiss and Smith local preconditioning technique with Liou AUSM+up method have been found to be efficient for viscous (**low cell Reynolds number**) and **low Mach number** flows.

Future developments

- Dual time stepping approach for unsteady flows

- Analysis of boundary conditions treatment

- Implementation of turbulence models