An Implicit Finite Volume Method for the Solution of 3D Low Mach Number Viscous Flows Using A Local Preconditioning Technique

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June 2005

Introduction

- Navier-Stokes equations and choice of variables
- Cell-centered finite volume discretization scheme
- Fully implicit pseudo-transient Newton-GMRes method
- Local preconditioning
- Results (2D and 3D test cases)
- Conclusions

Navier-Stokes equations

Choice of dimensionless primitive variables $\boldsymbol{w} = (p\,,\,\boldsymbol{u}\,,\,T)^T$

$$p = \frac{p_d - p_0}{\rho_0\,U_0^2} \qquad \boldsymbol{u} = \frac{\boldsymbol{u}_d}{U_0} \qquad T = \frac{T_d}{T_0} \qquad \rho(p,T) = \frac{\rho_d(p_d,T_d)}{\rho_0} \qquad h(p,T) = \frac{h_d(p_d,T_d)}{cp_0\,T_0}$$

Definitions of dimensionless numbers :

$$Str = \frac{L_0}{U_0\,t_0} \qquad Re = \frac{\rho_0\,U_0\,L_0}{\mu_0} \qquad M_0 = \frac{U_0}{c_0} \qquad \chi = \frac{U_0^2}{cp_0\,T_0} \qquad Pe = \frac{\rho\,U_0\,L_0\,cp_0}{\kappa_0}$$

Navier-Stokes equations :

$$Str\,\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho\,\boldsymbol{u} \\ \rho\,H - \chi\,p \end{pmatrix} + \nabla\cdot\begin{pmatrix} \rho\,\boldsymbol{u}^T \\ \rho\,\boldsymbol{u}\,\boldsymbol{u}^T + p\boldsymbol{I} \\ \rho\,\boldsymbol{u}^T\,H \end{pmatrix} = \nabla\cdot\begin{pmatrix} \boldsymbol{0}^T \\ \frac{1}{Re}\,\boldsymbol{T} \\ -\frac{1}{Pe}\,\boldsymbol{q}^T + \frac{\chi}{Re}\,\boldsymbol{u}^T\,\boldsymbol{T} \end{pmatrix}$$

$$H = h + \frac{1}{2}\chi\,\|\boldsymbol{u}\|^2 \qquad \boldsymbol{T} = \mu\,\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) - \frac{2}{3}\mu\,(\nabla\cdot\boldsymbol{u})\,\boldsymbol{I} \qquad \boldsymbol{q} = -\kappa\,\nabla T$$

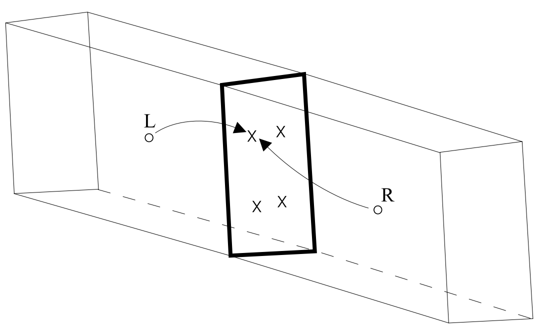
Cell-centered Finite Volume Scheme

Integrated form of the **conservative Navier-Stokes** equations :

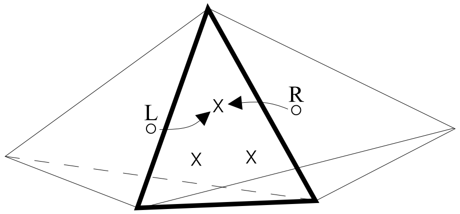
$$\frac{d\boldsymbol{s}}{dt} + \frac{1}{V}\int_S \boldsymbol{F}_a(\boldsymbol{w}) \cdot \boldsymbol{d\Sigma} = \frac{1}{V}\int_S \boldsymbol{F}_d(\boldsymbol{w}, \boldsymbol{\nabla w}) \cdot \boldsymbol{d\Sigma}$$

- Primitive variables at the Gauss Points of the cell faces **reconstructed** from left (L) and right (R) neighbours by a truncated Taylor series expansion

$$\tilde{w}_{R,L} = w_{R,L} + (\boldsymbol{r}_G - \boldsymbol{r}_{R,L})^T \boldsymbol{\nabla} w_{R,L} + \frac{1}{2}(\boldsymbol{r}_G - \boldsymbol{r}_{R,L})^T H_{R,L}(\boldsymbol{r}_G - \boldsymbol{r}_{R,L}) + \mathcal{O}\left(h^3\right)$$



Quadrangular face : 4 Gauss Points



Triangular face : 3 Gauss Points

- Gradient $\boldsymbol{\nabla w}$ evaluated by a modification of Coirier's diamond path scheme

Truncature error $\Rightarrow \mathcal{O}(h^2)$ on advective terms and $\mathcal{O}(h^1)$ on viscous terms

Cell-centered Finite Volume Scheme

Advective fluxes computed by **AUSM+up** method designed by Liou $\Rightarrow \tilde{\boldsymbol{F}}(\tilde{w}_R, \tilde{w}_L)$

$$c_{1/2} = \frac{1}{2}\left(\tilde{c}_L + \tilde{c}_R\right) \qquad M_{L,R} = \frac{\tilde{\boldsymbol{u}}_{L,R}{}^T \boldsymbol{n}}{c_{1/2}} \qquad \overline{M}^2 = \frac{1}{2}\,\frac{\|\tilde{\boldsymbol{u}}_L\|^2 + \|\tilde{\boldsymbol{u}}_R\|^2}{c_{1/2}^2}$$

$$M_{1/2} = \mathcal{M}^+_{(4)}\left(M_L\right) + \mathcal{M}^-_{(4)}\left(M_R\right) - \frac{K_p}{f_c}\max\left(1 - \overline{M}^2, 0\right)\frac{\left(\tilde{p}_R - \tilde{p}_L\right)}{\left(\frac{2\,p_0}{\rho_0\,U_0^2} + \tilde{p}_R + \tilde{p}_L\right)}$$

$$\begin{aligned} p_{1/2} = \; & \mathcal{P}^+_{(5)}\left(M_L\right)\tilde{p}_L + \mathcal{P}^-_{(5)}\left(M_R\right)\tilde{p}_R \\ & - K_u\,\mathcal{P}^+_{(5)}\left(M_L\right)\mathcal{P}^-_{(5)}\left(M_R\right)\left(\tilde{\rho}_L + \tilde{\rho}_R\right)\left(f_c\,c_{1/2}^2\right)\left(M_R - M_L\right) \end{aligned}$$

$$\boldsymbol{F}_a^T \boldsymbol{n} = c_{1/2}\,M_{1/2}\left.\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\,\tilde{\boldsymbol{u}}\\ \tilde{\rho}\,\tilde{H}\end{pmatrix}\right|_{\substack{L \;\; if \;\; M_{1/2} > 0\\ R \;\; if \;\; M_{1/2} < 0}} + \begin{pmatrix}0\\ p_{1/2}\,\boldsymbol{n}\\ 0\end{pmatrix}$$

Function f_c used for scaling speed of sound is defined by the local preconditioning method.

Fully implicit pseudo-transient Newton-GMRES scheme

❒ **Pseudo-transient** iterations for convergence to steady flow

$$\boldsymbol{S}\left(\frac{\boldsymbol{w}^{l+1}-\boldsymbol{w}^{l}}{\Delta\tau^{l}}\right)+\boldsymbol{F}\left(\boldsymbol{w}^{l+1}\right)=0 \qquad \boldsymbol{S}^{l}=\left(\frac{\partial\,\boldsymbol{s}}{\partial\,\boldsymbol{w}}\right)^{l}$$

❒ System of non-linear equations solved by **Newton's method**

$$\left(\frac{1}{\Delta\tau^{l}}\boldsymbol{S}^{l}+\boldsymbol{J}^{l}\right)\,\boldsymbol{\delta w}^{(l)}=-\boldsymbol{F}\left(\boldsymbol{w}^{(l)}\right) \qquad \boldsymbol{J}^{l}=\left(\frac{\partial\,\boldsymbol{F}}{\partial\,\boldsymbol{w}}\right)^{l}$$

❒ Linear system are solved by a **matrix-free GMRes** algorithm.

$$\boldsymbol{J}\left(\boldsymbol{w}^{(l)}\right)\boldsymbol{v}\approx\frac{\boldsymbol{F}\left(\boldsymbol{w}^{(l)}+\epsilon\boldsymbol{v}\right)-\boldsymbol{F}\left(\boldsymbol{w}^{(l)}\right)}{\epsilon}$$

❒ **Preconditionning** required to ensure good convergence of the GMRES solver : right preconditioning based on a block incomplete factorization (BILU(k)) of an approximate Jacobian

- ▶ constant reconstruction for advective terms
- ▶ classical diamond scheme for viscous ones

Fully implicit pseudo-transient Newton-GMRES scheme

Problems arising for **low Mach number flows**

- ❒ $\Delta\tau^l$ = local timestep computed by the **Switched Evolution Relaxation** method

$$CFL^{(l+1)} = CFL^{(0)} \left(\frac{\left\| \boldsymbol{F}(\boldsymbol{w}^{(0)} \right\|}{\left\| \boldsymbol{F}(\boldsymbol{w}^{(l)} \right\|} \right)^p$$

- ❒ Poor convergence of GMRES algorithm caused by a very **bad conditioning**. The condition number being

$$CN \approx \frac{1}{M}$$

$\Rightarrow$ Use of a numerical speed of sound by using a local preconditioning matrix $\boldsymbol{P}$.

$$\boldsymbol{P} \left(\frac{\boldsymbol{w}^{l+1} - \boldsymbol{w}^l}{\Delta\tau^l} \right) + \boldsymbol{F}\left(\boldsymbol{w}^{l+1}\right) = 0$$

$$\boldsymbol{P} = \begin{pmatrix} \rho'_p & \boldsymbol{0}^T & \rho_T \\ \rho'_p\, \boldsymbol{u} & \rho\, \boldsymbol{I} & \rho_T\, \boldsymbol{u} \\ \rho'_p\, H - (\chi - \rho\, h_p) & \chi\, \rho\, \boldsymbol{u}^T & \rho_T\, H + \rho\, h_T \end{pmatrix}$$

Local Preconditioning Technique

Eigenvalues analysis of the pseudo-transient equation written in quasi linear form

$$\boldsymbol{P}\frac{\partial \boldsymbol{w}}{\partial \tau} + \boldsymbol{A}_x\frac{\partial \boldsymbol{w}}{\partial x} + \boldsymbol{A}_y\frac{\partial \boldsymbol{w}}{\partial y} + \boldsymbol{A}_z\frac{\partial \boldsymbol{w}}{\partial z} - \boldsymbol{D}\left(\frac{\partial^2 \boldsymbol{w}}{\partial x \partial x} + \frac{\partial^2 \boldsymbol{w}}{\partial y \partial y} + \frac{\partial^2 \boldsymbol{w}}{\partial z \partial z}\right) = \boldsymbol{0}$$

$$\boldsymbol{A}_{x,y,z} = u_{x,y,z}\,\boldsymbol{S} + \begin{pmatrix} 0 & \rho\,\boldsymbol{e}_{x,y,z}^T & 0 \\ \boldsymbol{e}_{x,y,z} & \boldsymbol{0} & \boldsymbol{0} \\ \chi\,u_{x,y,z} & \rho\,H\,\boldsymbol{e}_{x,y,z}^T & 0 \end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix} 0 & \boldsymbol{0}^T & \boldsymbol{0} \\ 0 & \frac{1}{Re}\boldsymbol{I} & \boldsymbol{0} \\ 0 & \boldsymbol{0}^T & \frac{1}{Re} \end{pmatrix}$$

Introducing a Fourier mode

$$\boldsymbol{w} = \boldsymbol{w}_0 \,\exp\left(\boldsymbol{k}^T\boldsymbol{x} - i\,\omega\,\tau\right)$$

$$\|\boldsymbol{k}\| = \frac{\phi}{\Delta x} \qquad \phi \in [0, \pi]$$

And defining a cell Reynolds number $\overline{Re}$, a cell Peclet number $\overline{Pe}$ and $\overline{\chi}$ by

$$u_k = \frac{\boldsymbol{u}^T}{\|\boldsymbol{k}\|} \qquad \|\boldsymbol{k}\|\overline{Re} = \frac{\rho\,u_k\,\Delta x}{\mu}\,Re \qquad \overline{Pe} = \frac{\rho\,u_k\,h_T\,\Delta x}{\kappa}\,Pe \qquad \overline{\chi} = \frac{u_k^2\,\rho_T}{\rho\,h_T}\,\chi$$

Local Preconditioning Technique

The first two eigenvalues are found to be

$$\lambda = \frac{\omega}{\|\boldsymbol{k}\|} = u_k \left(1 - i\,\frac{\phi}{\overline{Re}}\right)$$

while the following equation is obtained for the three other ones ($\lambda^* = \frac{\lambda}{u_k}$)

$$\left(i + \frac{\phi}{\overline{Re}} - i\,\lambda^*\right) \left[\lambda^{*\,2} - \lambda^* \left(1 + \frac{c'^{\,2}}{c^2} - i\frac{\phi\;c'^{\,2}\,\rho'_p}{\overline{Pe}}\right) - \left(\frac{c'^{\,2}}{c^2} + \frac{c'^{\,2}}{u_k^2} + i\frac{\phi\;c'^{\,2}\,\rho_p}{\overline{Pe}}\right)\right]$$
$$+\frac{\phi\;c'^{\,2}}{u_k^2\,\overline{Re}} \left(i\,\overline{\chi} + \left(1 - \frac{\overline{Re}}{\overline{Pe}}\right) - i\,\lambda^*\,\overline{\chi}\right) = 0$$

with

$$c'^{\,2} = \frac{\rho\,h_T}{\rho\,h_T\,\rho'_p + \rho_T\;(\chi - \rho\,h_p)}$$

Weiss and Smith Local Preconditioning Technique

Following the idea of Weiss and Smith

$$\rho_p' = \frac{1}{c'^2} - \frac{(\chi - \rho\, h_p)\ \rho_T}{\rho\, h_T}$$

$$c' = \begin{cases} \min\left(c, \max\left(\|\boldsymbol{u}\|, u_\epsilon\right)\right) & if\ \overline{Re} > 1 \\ \frac{\|\boldsymbol{u}\|}{\overline{Re}} & otherwise \end{cases}$$

For the inviscid case, the equation can easily be solved. The set of eigenvalues λ_i is

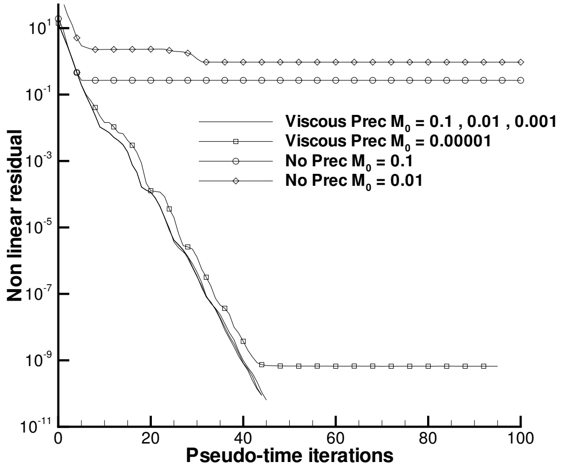
$$\lambda_{1,2,3} = u_k \qquad \lambda_{4,5} = \frac{1}{2}\left(1 + {M^*}^2\right)\ (u_k \pm c\, f_c)$$

$$f_c = \frac{\sqrt{4\ {M^*}^2 + M^2\ \left(1 - {M^*}^2\right)^2}}{\left(1 + {M^*}^2\right)} \qquad with\ M^* = \frac{c'}{c}$$

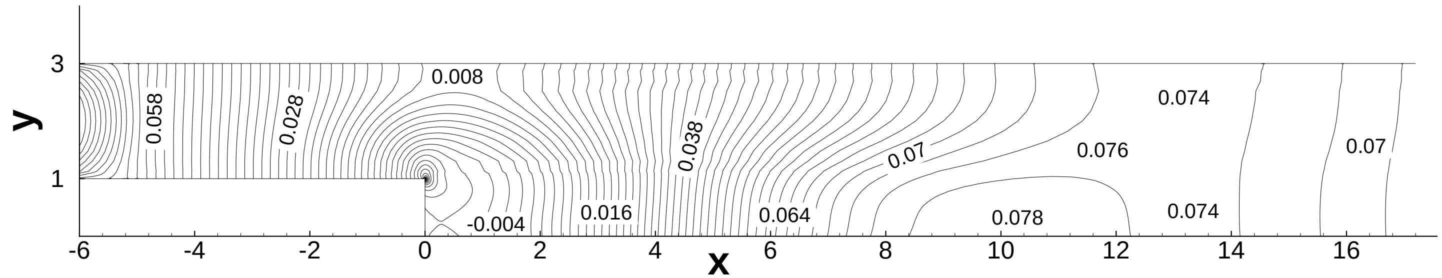
This function f_c is used in the AUSM+up scheme.

In most cases $\longrightarrow$ $CN = \frac{1+\sqrt{5}}{2}$

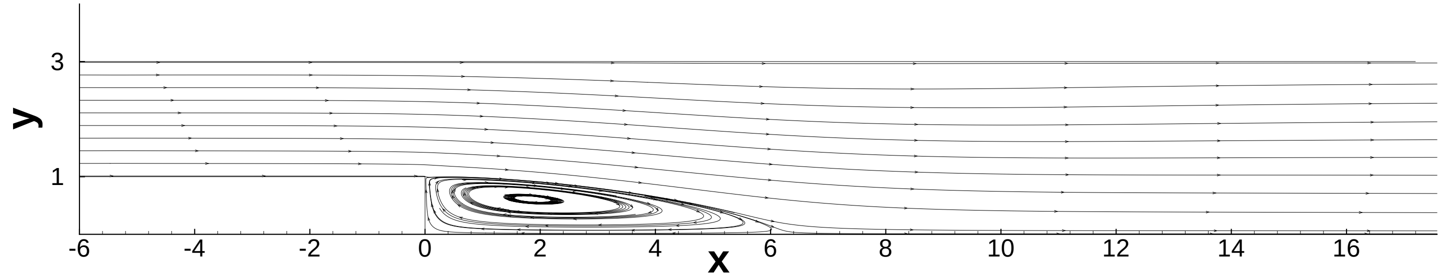
Laminar flow past a backward facing step $Re = 150$



Reattachment point position	
Kueny - Binder (experiment)	6
Glowinski (INRIA)	5.75
Ecer	5.9
Bourdel (ONERA-CERT)	6.17
Present	6.277

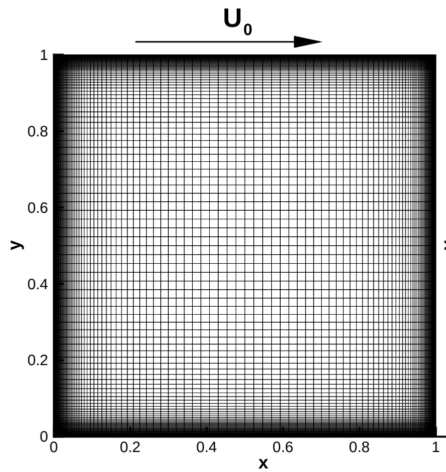


(a) Isobar contours ($M_0 = 10^{-5}$).

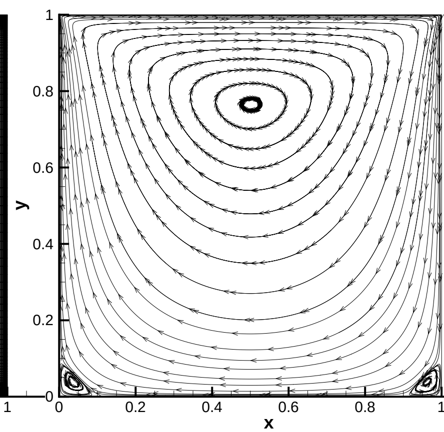


(b) Streamlines ($M_0 = 10^{-5}$).

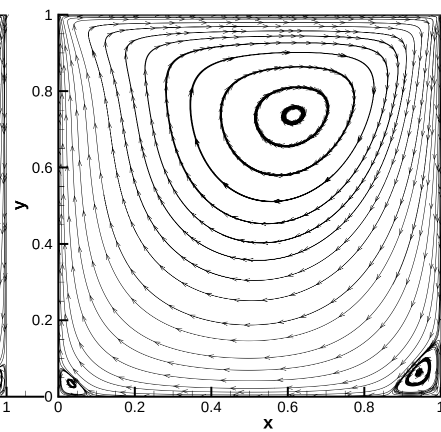
Laminar flow in a lid driven cavity $M_0 = 10^{-4}$ on a $5 \times 12\,10^3$ quadrangles mesh



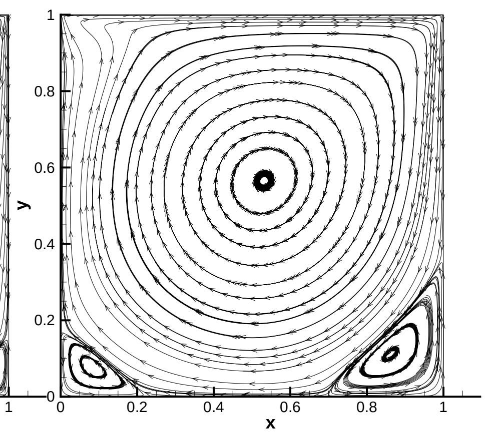
(c) High stretched grid.



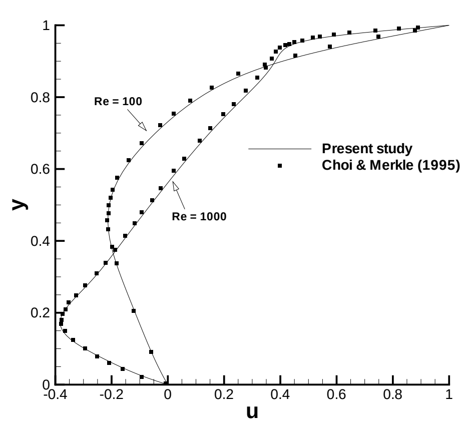
(d) $Re = 1$.



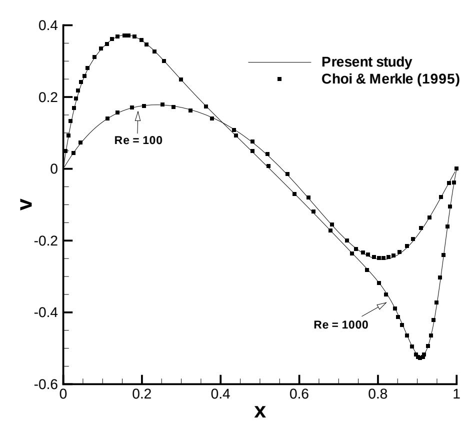
(e) $Re = 100$.



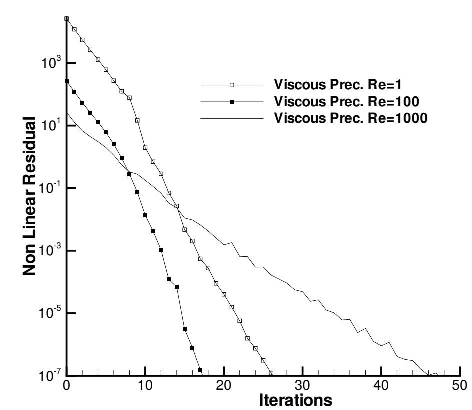
(f) $Re = 1000$.



(g) velocity u_x profile.

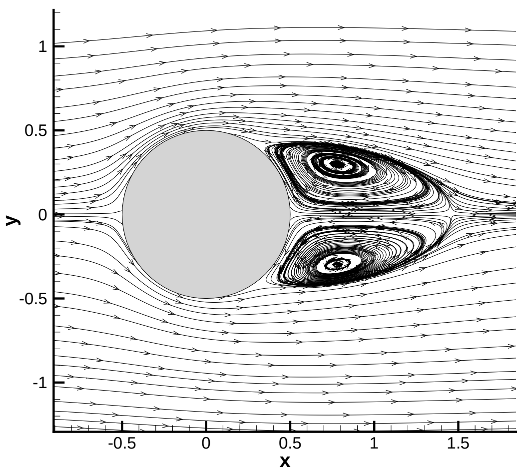


(h) velocity u_y profile.

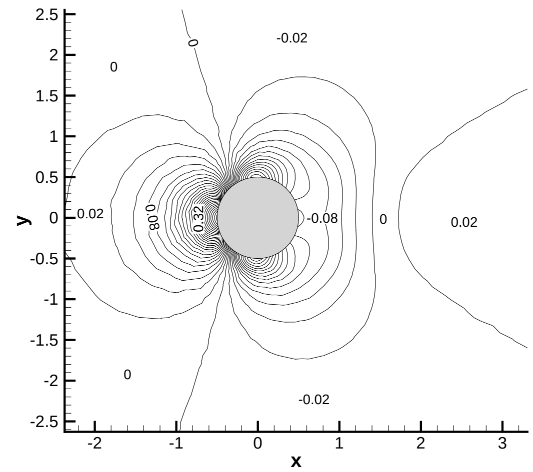


(i) Convergence curves.

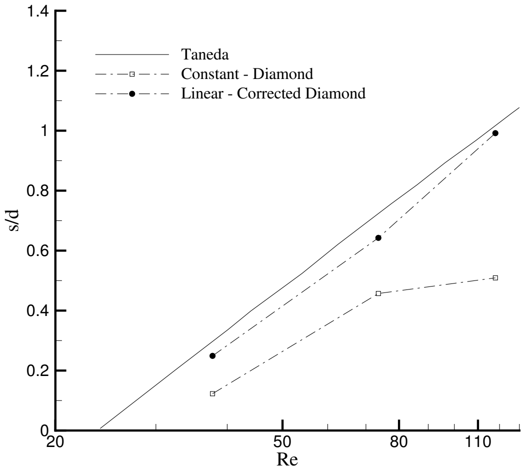
Laminar flow past a sphere $M_0 = 10^{-3}$ on a $3\,10^5$ tetrahedra mesh



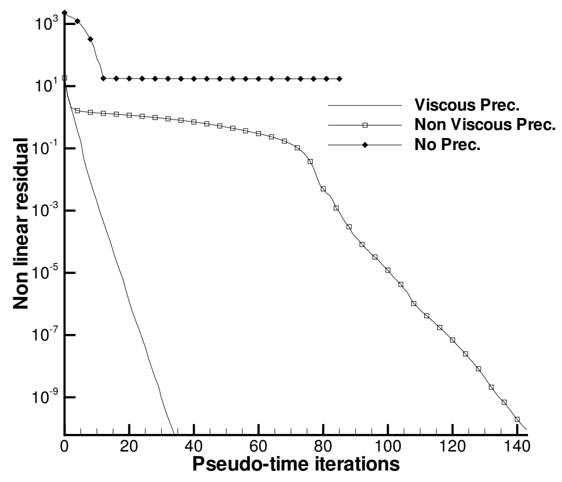
(j) Streamlines $Re = 118$.



(k) Isobar contours $\Delta p = 2\,10^{-2}$.



(l) Comparison with Taneda experiment.



(m) Convergence curves.

Conclusions

- A cell-centered **density-based** finite volume method has been used for the solution of low Mach number flows on **3D unstructured grids**.
- A **fully implicit** pseudo-transient preconditioned Newton-GMRes scheme has been tested.
- Weiss and Smith local preconditioning technique with Liou AUSM+up method have been found to be efficient for viscous (**low cell Reynolds number**) and **low Mach number** flows.

Future developments

- Dual time stepping approach for unsteady flows
- Analysis of boundary conditions treatment
- Implementation of turbulence models