Selection of approximation schemes in topology optimization

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Abstract

Topology optimization is considered. The formulation is extended to problems including dead loads. The approximation concepts approach is used for the optimization. Numerical tests compare the efficiency of different approximation schemes for different ratios between the applied loads and the structural weight. A proposal for their selection is finally suggested.

Keywords: structural optimization, topology optimization, convex approximations, dead loads
AMS classification:

1 Introduction

Topology optimization is a very general tool from structural optimization techniques. It allows to determine automatically the layout of the structure, that is the optimal distribution of the mechanical properties, in a prescribed design domain for a given amount of material, as illustrated in Figure 1.

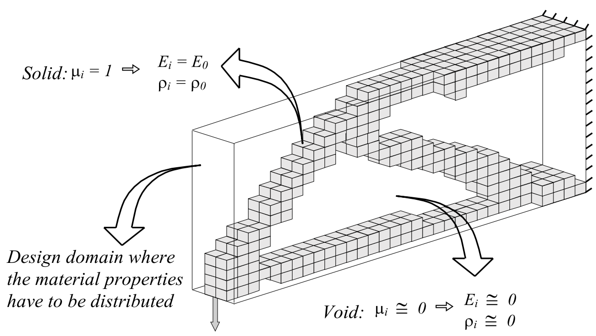

Figure 1: A topology optimization problem

Compliance minimization, that results in the stiffest structure that can be exhibited for a given volume fraction of material at the solution, has been intensively studied in the literature [1]. To solve such large scale problems, optimization methods based on the sequential convex programming [7] proved to be efficient [4]. In this approach, the optimization problem (1.1), which is implicit in terms of the design variables $\mathbf{X} = \{x_i, i = 1 \dots n\}$,

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$$
\begin{array}{lll}
\min\limits_{\mathbf{X}} & g_0(\mathbf{X}) & \\
\text{s.t.:} & g_j(\mathbf{X}) \leq g_j^{max} & j = 1 \dots m \\
 & \underline{x}_i \leq x_i \leq \overline{x}_i & i = 1 \dots n
\end{array}
\tag{1.1}
$$

is replaced by the solution of a sequence of explicit and convex approximated sub-problems that are built based on a variant of a Taylor series expansion of the involved design functions $g_j(\mathbf{X})$:

$$
\begin{array}{lll}
\min\limits_{\mathbf{X}} & \tilde{g}_0^{(k)}(\mathbf{X}) & \\
\text{s.t.:} & \tilde{g}_j^{(k)}(\mathbf{X}) \leq g_j^{max} & j = 1 \dots m \\
 & \underline{x}_i^{(k)} \leq x_i \leq \overline{x}_i^{(k)} & i = 1 \dots n
\end{array}
\tag{1.2}
$$

As illustrated in Figure 2, the optimization problem (1.1) is approximated at the current design point $\mathbf{X}^{(k)}$, where k is the iteration index. This requires to perform a structural and a sensitivity analysis for the computation of the function values and their derivatives. Introducing such a solution procedure allows to decrease the number of structural analyses required to reach the optimum of the problem (1.1). Indeed, the solution of the approximated sub-problem (1.2) is performed on an explicit problem and does no longer require any expensive analysis. A dual approach [7] is used for solving (1.2).

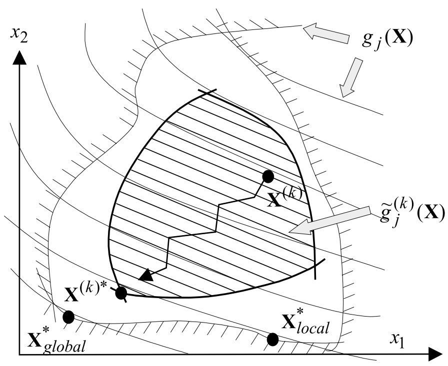

Figure 2: Original optimization problem and its explicit and convex approximation at the current design point $\mathbf{X}^{(k)}$

For most classical topology problems exhibited in the literature, well known approximation schemes, such as the Method of Moving Asymptotes (MMA) derived in [9] and CONLIN [6], proved to be efficient. However, it was recently pointed out that these procedures can fail when dealing with more difficult problems. One of them is the case of structures subjected to their own weight [2]. For such problems, the optimization process diverges mainly because of an unappropriate selection of the approximation scheme of the structural responses in (1.2).
The selection of a right approximation scheme for topology optimization is discussed in this paper. Approximations of the MMA family [3] are compared. The results show that fast solutions can be obtained when appropriate approximations are used.

2 Formulation of a topology optimization problem

The classical formulation of a topology optimization problem consists in maximizing the stiffness of the structure by minimizing the energy of the applied loads, called compliance, for a given volume fraction of the material required at the solution. In this problem, the n design variables are the pseudo-densities $\mu = \{\mu_i, i = 1, ..., n\}$ attached to each finite element of the discretized model:

$$\begin{array}{lll} \min\limits_{\mu} & C = g^T q & \\ \text{s.t.:} & \sum_{i=1}^{n} \mu_i V_i \leq \overline{V} & \\ & \underline{\mu}_i \leq \mu_i \leq \overline{\mu}_i & i = 1 \ldots n \end{array} \tag{2.3}$$

In relation (2.3), $\overline{V}$ is the available amount of material at the solution, V_i is the volume of the i_{th} finite element, g is the vector of nodal loads and q are the nodal displacements. The pseudo-densities μ_i indicate the presence or the absence of material in the i_{th} finite element (see Figure 1), according to relation (2.4) where ρ_o is the density of the base material (e.g. steel) and ρ_i is the effective density:

$$\rho_i = \mu_i \times \rho_o \tag{2.4}$$

When continuous design variables are considered, the pseudo-densities take their values between an upper bound of full density (μ_i=1) and a lower bound close to zero representing the void. For obtaining a 'solid and void' layout at the solution, the intermediate values of those pseudo-densities are penalized by using the following definition of the material stiffness:

$$E_i = {\mu_i}^p E_o \tag{2.5}$$

where E_o is the Young's modulus of the base material. Such a material parameterization, called SIMP (*Simply Isotropic Material with Penalization*) and described in [1], is illustrated in Figure (3) for different values of p ($p = 3$ is used in this article). When dead loads are considered in the problem, the parameterization (2.5) has to be modified for avoiding some undesirable effects for densities close to zero [2]. This is done by selecting $p = 1$ in (2.5) at a given pseudo-density μ_C, as illustrated in Figure 3.

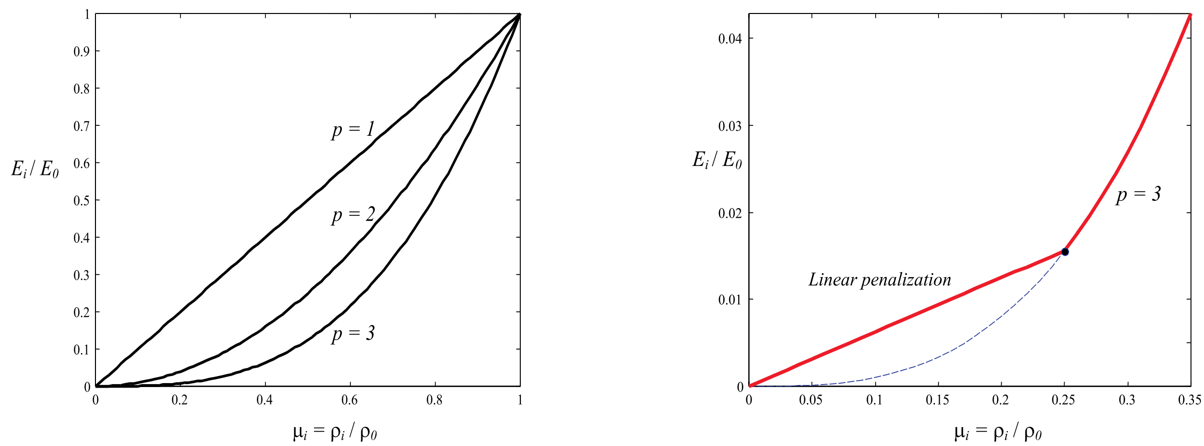

Figure 3: Penalizations of the intermediate densities. $\mu_C = 0.25$ for the modified material parameterization

For avoiding numerical instabilities that often happen in such optimization problems, that is mesh dependency and checkerboard patterns, a filtering technique [8] is used for solving (2.3).

3 Sensitivity analysis

When optimization techniques based on the knowledge of the gradients are used to solve (2.3), a sensitivity analysis has to be performed for determining the behavior of a given structural response according to a design variable change.
As described in [4], the sensitivity of the compliance C is written as:

$$\frac{\partial C}{\partial \mu_i} = 2q^T \frac{\partial g}{\partial \mu_i} - q^T \frac{\partial K}{\partial \mu_i} q \tag{3.6}$$

with the derivative of the stiffness matrix K given by

$$\frac{\partial K}{\partial \mu_i} = p{\mu_i}^{p-1}\overline{K}_i \tag{3.7}$$

and $K_i = \mu_i^p \overline{K}_i$, according to the relation (2.5) and the definition of the stiffness matrix K_i of the i_{th} finite element of the model. For considering dead loads, those definitions are modified according to the linear character of the parameterization for pseudo-densities lower than μ_C (see Figure 3).
In problems including fixed and constant loads, $\frac{\partial g}{\partial \mu_i}$ is null in (3.6), and the sensitivities are always negative: the structural behavior of the compliance is then monotonous. When dead loads are considered, this term is taken into account, and the derivatives of the compliance can pass from positive to negative according to the changing value of the design variables. In this case, the objective function C is non monotonous with respect to the considered design variable μ_i.

4 Approximations of the MMA family

At lot of approximation schemes can be used to approximate the structural responses $g_j(\mu)$, $j = 1, 2$ of (2.3). The approximations of the MMA (Method of Moving Asymptotes) family [3] are considered here and summarized bellow. They are illustrated in Figure 4.

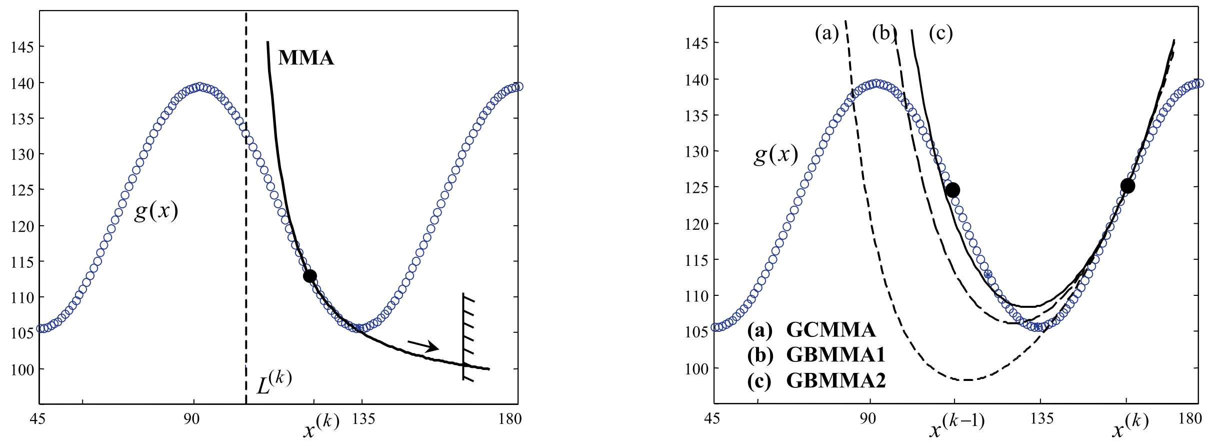

Figure 4: Monotonous and non monotonous approximations of the MMA family

4.1 Non monotonous approximations

In the Globally Convergent version of MMA [10], each function $g_j(\mu)$ is approximated according to (4.8), where $g_j(\mu^{(k)})$ is the function value at the current iteration k. The parameters $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ are computed based on the first order derivatives, on the asymptotes $L_i^{(k)}$ and $U_i^{(k)}$, and on a non monotonic parameter $\rho_j^{(k)}$, heuristically updated at each iteration k. For the approximation to be convex, the parameters $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ in (4.8) have to be positive.

$$\begin{aligned} \tilde{g}_j(\mu) &= g_j(\mu^{(k)}) + \sum_{i=1}^{n} p_{ij}^{(k)} \left(\frac{1}{U_i^{(k)} - \mu_i} - \frac{1}{U_i^{(k)} - \mu_i^{(k)}}\right) \\ &+ \sum_{i=1}^{n} q_{ij}^{(k)} \left(\frac{1}{\mu_i - L_i^{(k)}} - \frac{1}{\mu_i^{(k)} - L_i^{(k)}}\right) \end{aligned} \tag{4.8}$$

In the Gradient Based MMA approximations [3], the gradients from the previous iteration $k-1$ are used in place of $\rho_j^{(k)}$ to build (4.8). For GBMMA1, $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ in (4.8) are analytically computed from:

$$\begin{aligned} \frac{\partial g_j(\mu^{(k)})}{\partial \mu_i} &= \frac{p_{ij}^{(k)}}{(U_i^{(k)} - \mu_i^{(k)})^2} - \frac{q_{ij}^{(k)}}{(\mu_i^{(k)} - L_i^{(k)})^2} \\ \frac{\partial g_j(\mu^{(k-1)})}{\partial \mu_i} &= \frac{p_{ij}^{(k)}}{(U_i^{(k)} - \mu_i^{(k-1)})^2} - \frac{q_{ij}^{(k)}}{(\mu_i^{(k-1)} - L_i^{(k)})^2} \end{aligned} \tag{4.9}$$

In GBMMA2, an estimation of the diagonal second order derivatives (4.10) is used to improve the quality of the approximation (4.8). It is shown in [5] that making such kind of finite difference is the best diagonal quasi Newton update that can be obtained from the second order derivatives when the diagonal assumption is made a priori.

$$\frac{\partial^2 g_j(\mu^{(k)})}{\partial \mu_i^2} \simeq \frac{\dfrac{\partial g_j(\mu^{(k)})}{\partial \mu_i} - \dfrac{\partial g_j(\mu^{(k-1)})}{\partial \mu_i}}{\mu_i^{(k)} - \mu_i^{(k-1)}} \tag{4.10}$$

It was observed on numerical tests that it is interesting to use GBMMA2 when the current design point is in the vicinity of the optimum, that is at the end of the optimization process. Indeed, it make sense that in the final convergence stages, the use of second order information, even if estimated, improves the convergence speed. Based on this, the contribution of a given design variable μ_i in a given design function $g_j(\mu)$ can be approximated by GBMMA2 when the relation (4.11) is verified:

$$\frac{|\mu_i^{(k)} - \mu_i^{(k-1)}|}{\overline{\mu}_i - \underline{\mu}_i} \leq SWITCH \tag{4.11}$$

Otherwise, GBMMA1 is used. This leads to consider the mixed GBMMA1-GBMMA2 approximation, for $SWITCH \in]0,1[$. When $p_{ij}^{(k)}$ and $q_{ij}^{(k)}$ computed by GBMMA1 or GBMMA2 are not positive, GCMMA is selected and the approximation is then convex. The automatic selection of the non monotonous convex approximation based on (4.11) is illustrated in Figure 5.

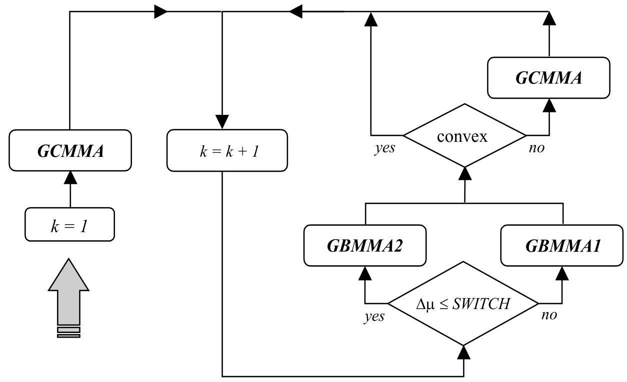

Figure 5: Selection of the non monotonous approximation based on GCMMA, GBMMA1 and GBMMA2

4.2 Monotonous approximations

The monotonous Method of Moving Asymptotes is given in (4.12) and illustrated in Figure 4. A move-limits strategy, given in [9], is used for limiting the range of variation of the design variables along the optimization process.

$$\begin{aligned} \tilde{g}_j(\mu) &= g_j(\mu^{(k)}) + \sum_{+,i}^{n} p_{ij}^{(k)} \left(\frac{1}{U_i^{(k)} - \mu_i} - \frac{1}{U_i^{(k)} - \mu_i^{(k)}}\right) \\ &+ \sum_{-,i}^{n} q_{ij}^{(k)} \left(\frac{1}{\mu_i - L_i^{(k)}} - \frac{1}{\mu_i^{(k)} - L_i^{(k)}}\right) \end{aligned} \tag{4.12}$$

When $L_i^{(k)} = 0$ and $U_i^{(k)} \to \infty$, MMA is reduced to Conlin [6], given in (4.13):

$$\begin{aligned} \tilde{g}_j(\mu) &= g_j(\mu^{(k)}) + \sum_{+,i} \frac{\partial g_j(\mu^{(k)})}{\partial \mu_i}(\mu_i - \mu_i^{(k)}) \\ &- \sum_{-,i} (\mu_i^{(k)})^2 \frac{\partial g_j(\mu^{(k)})}{\partial \mu_i}\left(\frac{1}{\mu_i} - \frac{1}{\mu_i^{(k)}}\right) \end{aligned} \tag{4.13}$$

For the monotonous Conlin approximation, a move-limits strategy (4.14) is used, with a typical value $\Delta\mu = 0.3$:

$$\underline{\mu}_i \leq \; \mu_i^{(k-1)} - \Delta\mu \leq \; \mu_i \leq \; \mu_i^{(k-1)} + \Delta\mu \leq \; \overline{\mu}_i \tag{4.14}$$

5 Performances of the approximation schemes

In the following numerical applications, the optimal solution is supposed to be reached when the maximum variation of the design variables is lower than a user defined precision TOL:

$$\max |\mu_i^{(k)} - \mu_i^{(k-1)}| \leq TOL \tag{5.15}$$

For both presented applications, the volume fraction of material required at the solution is of 100% for problems that are only subjected to their own weight. This value will not influence the resulting topology since, in most of the cases, such topology problems are unconstrained.

5.1 Arch structure

For the first application, the design problem is illustrated in Figure 6, where a non structural mass is placed at the top to load the structure. The topology of the stiffest structure supporting its own weight is looked at. $L = 1m$ is a reference length. Due to symmetry conditions, the half design domain is discretized with 20×20 quadrangular finite elements of 8 degrees of freedom. The mechanical properties of the base material to be distributed in the domain are: $E_o = 1N/m^2$, $\nu = 0.3$ and $\rho_o = 1kg/m^3$. The gravitational acceleration is of $9.81kgm/s^2$.

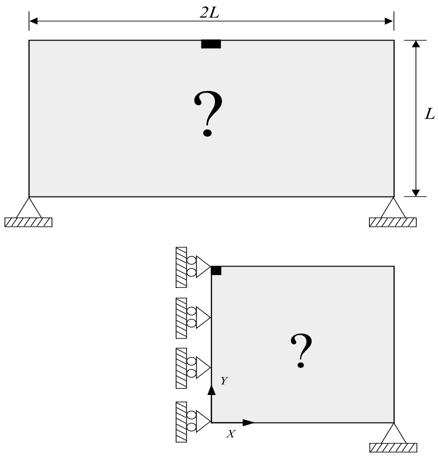

Figure 6: Design domain, supports and non structural mass

When Conlin [6] is used, a topology can not be obtained. In Figure 7, a grey scale is used for representing the emerging structure: black is solid ($\mu_i = \overline{\mu}_i = 1$), white is void ($\mu_i = \underline{\mu}_i = 0.01$), and intermediate grey densities are composite media. Oscillations of the design variables appear during the optimization process: some of them take their values successively at the upper bound and at the lower bound defined in (4.14). The topology changes from one iteration to the other. Such a monotonous approximation scheme is not efficient for solving this non monotonous problem.

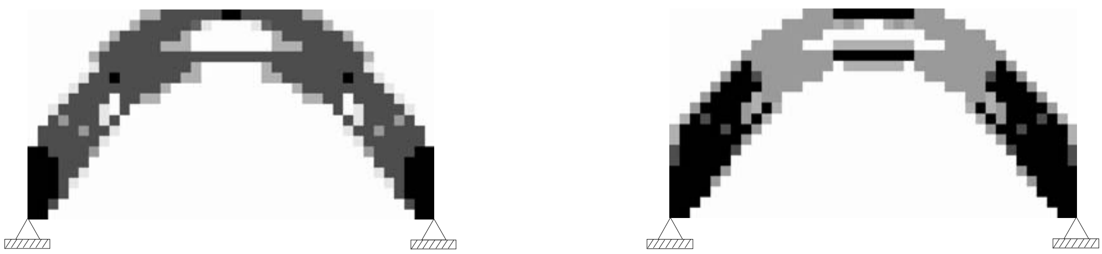

Figure 7: Topologies obtained by Conlin at iterations 199 and 200

For the other approximations described in this paper, a solution can be reached (Figure 8). Although MMA performs monotonous approximations of the design functions, it can find the optimal topology, thanks to a robust move-limits strategy described in [9]. However, the non monotonous approximations are much more efficient, especially when gradients from the previous iteration are used. Results are reported in Table 1 for different values of the precision TOL.

Figure 8: Optimal topology for the arch problem

TOL	MMA	GCMMA	GBMMA1	GBMMA2	GBMMA1-GBMMA2
0.01	130	80	51	73	54
0.001	402	200	98	109	91
0.0001	438	253	130	139	112

Table 1: Number of iterations needed for solving the arch problem. GBMMA1-GBMMA2 is related to Figure 5 ($SWITCH = 0.2$)

5.2 Bridge structure

The design problem is illustrated in Figure 9, where $L = 1m$ is a reference length. Due to symmetry conditions, the half design domain is discretized with 40×20 quadrangular finite elements of 8 degrees of freedom. The mechanical properties of the base material to be distributed in the domain are: $E_o = 100N/m^2$, $\nu = 0.3$ and $\rho_o = 1kg/m^3$. The gravitational acceleration is of $1kgm/s^2$. The obtained volume fraction of material at the solution is of 24% for the problem including only self-weight. For an objective comparison, this limiting value $\overline{V}$ in (2.3) is assigned to the problems including an applied load.

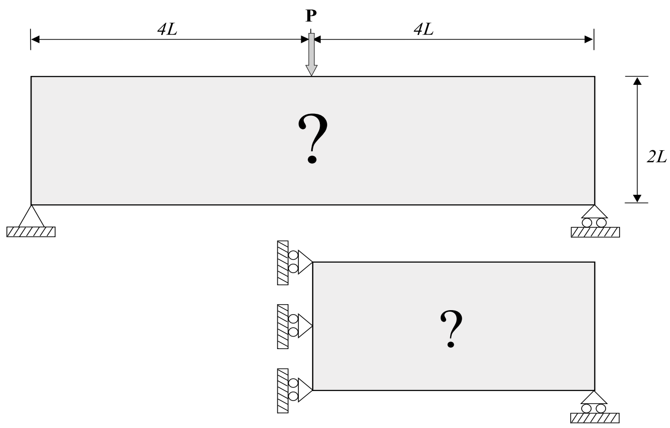

Figure 9: Design domain, supports and applied load

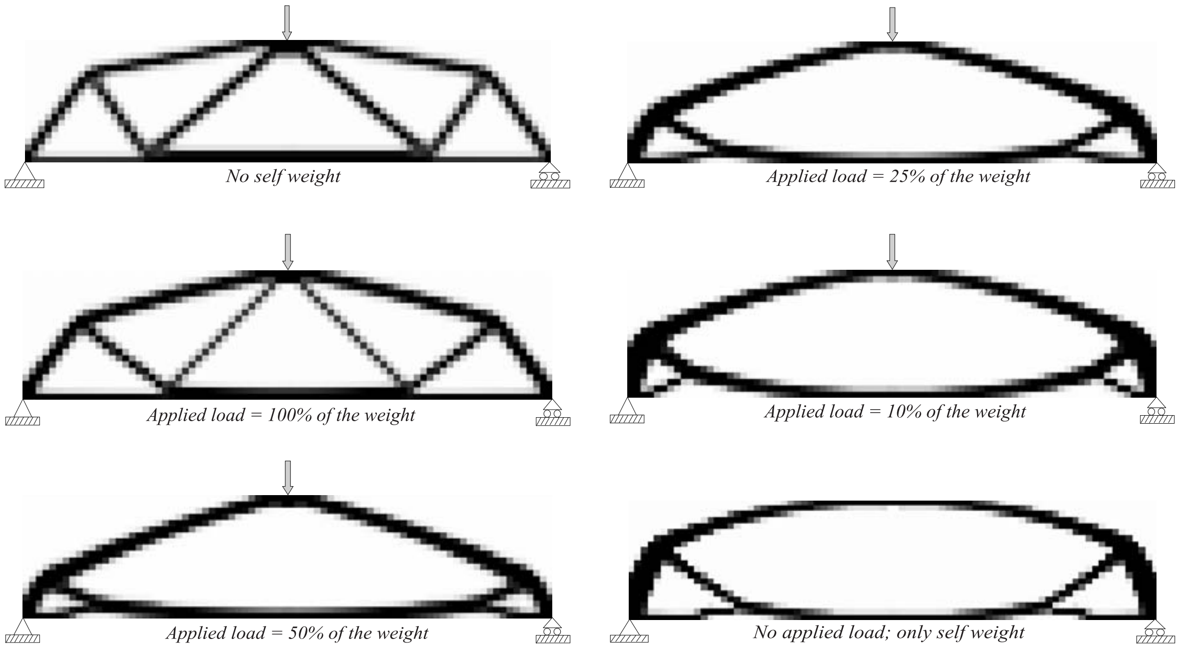

Figure 10: Optimal topologies for different ratios between the applied load and the self-weight

When the self-weight is not taken into account, the amplitude of the load P doesn't have any influence on the resulting optimal topology, shown in Figure 10. In this case, 103 iterations (that is structural analyses) are needed to reach the solution with the monotonous MMA approximation, while 229 steps are required when GCMMA is used. This indicates the too conservative character of the non monotonous GCMMA in solving such a classical topology optimization problem, that is without dead loads.
When the self-weight and the load P are considered simultaneously in the design problem, the resulting topology depends on the ratio between the applied load and the structural weight at the solution, as illustrated in Figure 10. It is seen that when the structural weight becomes preponderant in comparison to the applied load, the stiffeners under the load disappear and the shape of the structure tends to be an arch.
The number of iterations needed to reach the solutions is given in Table 2. Although GCMMA performs non monotonous approximations of the structural functions, it is sometimes not able to reach the optimum in a small number of iterations. This well known behavior [10] results from a too conservative aspect of this approximation (Figure 4). Again, the number of structural analyses needed to get the optimum is decreased when the information from previous iteration is used.
When only dead loads are considered in the design problem, MMA, GCMMA, GBMMA1 and GBMMA2 take 121, 112, 76 and 55 iterations, respectively to find the optimal topology (with $TOL = 0.01$).

6 Conclusions

A comparison of different approximations of the MMA family was performed in the frame of topology design optimization. When the self-weight of the structure is predominant in the problem, a non monotonous approximation is advised. Most of the time, the use of gradients from

ratio load-weight	MMA	GCMMA	GBMMA1-GBMMA2
200%	90	176	101
100%	171	109	80
50%	123	273	106
25%	182	357	117
10%	176	180	120

Table 2: Number of iterations needed for solving the bridge problem. 50% means that the applied load is of 50% of the structural weight at the solution. $TOL = 0.01$ in (5.15). GBMMA1-GBMMA2 is related to Figure 5 ($SWITCH = 0.2$)

the previous iteration will increase the convergence speed. For classical topology optimization (without dead loads), monotonous approximations are reliable for solving the design problem.

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