Radiation pressure on a multi-level atom: an exact analytical approach

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Nowadays, the mechanical action of laser light is a tool used ubiquitously in cold atom physics, leading to the control of atomic motion with extreme precision. In a recent work, we provided a standardized and exact analytical formalism for computing in the semiclassical regime the radiation force experienced by a two-level atom interacting with an arbitrary number of plane waves with arbitrary intensities, frequencies, phases, and directions [J. Opt. Soc. Am. B **35**, 127-132 (2018)]. Here, we extend this treatment to the multi-level atom case, where degeneracy of the atomic levels is considered and polarization of light enters into play.

I. INTRODUCTION

The mechanical action of laser light on atoms led to a great number of spectacular experiments and achievements in atomic physics during the past decades (see, e.g., Refs. [1–8]). In the semiclassical regime, where the atomic motion is treated classically, the resonant laser radiation acts mechanically as a force on the atomic center-of-mass. Recently, we provided a standardized exact analytical treatment of the mechanical action induced by an arbitrary set of plane waves on a two-level atom [9]. In particular, we showed that the light force always reaches a periodic regime shortly after establishment of the interaction and we provided an exact yet simple expression of all related Fourier components of the force in this regime. The mean net force $\mathbf{F}$ has been shown to be expressible in all cases (coherent and incoherent) in the form $\mathbf{F} = \sum_{j=1}^{N} \mathbf{F}_j$, with

$$\mathbf{F}_j = \frac{\Gamma}{2}\frac{s_j}{1+s}\hbar\mathbf{k}_j \tag{1}$$

the force exerted by the jth plane wave in presence of all other waves. Here, N is the number of plane waves lightening the atom, Γ is the spontaneous de-excitation rate of the upper level of the transition, $\hbar\mathbf{k}_j$ is the jth plane wave photon momentum, and $s = \sum_j s_j$, with s_j a *generalized* saturation parameter

$$s_j = \mathrm{Re}\left[\frac{\Omega_j}{\Gamma/2 - i\delta_j}\sum_{l=1}^{N}\frac{\Omega_l^*}{\Gamma}q_{m_{lj}}\right], \tag{2}$$

where Ω_j and δ_j are the Rabi frequency and the detuning of plane wave j, and $q_{m_{lj}}$ are complex numbers obtained from the solution of an infinite system of equations [9]. In the low-intensity and incoherent regime, we showed that Eq. (2) simplifies to the standard expression of the saturation parameter, i.e.,

$$s_j = \frac{|\Omega_j|^2/2}{\Gamma^2/4 + \delta_j^2}. \tag{3}$$

The question then naturally arises of how this formalism extends to the multi-level atom case, where Zeeman sublevels and arbitrary polarization of light are considered. In this case, a closed form of the mean net force $\mathbf{F}$ is only known in specific situations. For instance, for an atom with degenerate ground and excited states of angular momenta J_g and J_e, respectively, the stationary force exerted by a linearly polarized plane wave is either 0 (if $\Delta J \equiv J_e - J_g = -1$, or $\Delta J = 0$ with integer J_g) or reads

$$\mathbf{F} = \frac{\Gamma}{2}\frac{s}{b+s}\hbar\mathbf{k}, \tag{4}$$

with $\hbar\mathbf{k}$ the plane wave photon momentum, s the standard saturation parameter (3) and b a parameter depending on J_g and J_e [10]. Other specific cases have also been studied in the lin⊥lin and σ^+-σ^- configurations [11–15]. However, no exact analytical extension of Eq. (4) is known for general atomic and laser configurations. In practice, purely numerical approaches are enforced in this case (see, e.g., Refs. [16–22]).

Here, we extend the formalism developed in Ref. [9] to multi-level atoms. We solve the most general case and provide an exact analytical expression for the force exerted by an arbitrary number of plane waves with arbitrary intensities, phases, frequencies, polarizations, and directions acting on the same individual multi-level atom. We provide a generalization of Eqs. (1) and (2) in a matrix formalism.

The paper is organized as follows. In Section II, we provide generalized optical Bloch equations (OBEs) and compute the exact expression of the radiation pressure force in the most general configuration. In Section III, we investigate some specific regimes where interesting simplifications occur and we draw conclusions in Section IV. Finally, two Appendices close this paper, where we detail the effect of a reference frame rotation on the OBEs (Appendix A) and explicit values of specific matrices are given (Appendix B).

II. GENERAL AND EXACT EXPRESSION OF THE RADIATION PRESSURE FORCE

A. Hamiltonian and master equation

We consider an atom with two degenerate levels of energy $E_e \equiv \hbar\omega_e$ and $E_g \equiv \hbar\omega_g$ ($E_e > E_g$), and of total angular momenta J_e and J_g, respectively. The Zeeman sublevels are denoted $|J_e, m_e\rangle$ and $|J_g, m_g\rangle$. We consider an electric dipole transition so as $\Delta J \equiv J_e - J_g \in \{0, \pm 1\}$. We denote the atomic transition angular frequency $\omega_e - \omega_g$ by ω_{eg}. The atom interacts with a classical electromagnetic field $\mathbf{E}(\mathbf{r}, t)$ resulting from the superposition of N arbitrary plane waves: $\mathbf{E}(\mathbf{r}, t) = \sum_{j=1}^{N} \mathbf{E}_j(\mathbf{r}, t)$, with $\mathbf{E}_j(\mathbf{r}, t) = (\mathbf{E}_j/2) e^{i(\omega_j t - \mathbf{k}_j \cdot \mathbf{r} + \varphi_j)} + \text{c.c.}$ Here, ω_j, $\mathbf{k}_j$, and φ_j are the angular frequency, the wave vector and the phase of the jth plane wave, respectively, and $\mathbf{E}_j \equiv E_j \boldsymbol{\epsilon}_j$, with $E_j > 0$ and $\boldsymbol{\epsilon}_j = \sum_q \epsilon_{j,q} \mathbf{e}^q$ the normalized polarisation vector of the corresponding wave written in the upper-index spherical basis $\{\mathbf{e}^q, q = 0, \pm 1\}$ [23]. Nonzero $\epsilon_{j,0}$ and $\epsilon_{j,\pm 1}$ components correspond to so-called π and $\sigma^\pm$ polarization components of radiation, respectively. Accordingly, the vectors $\mathbf{e}^0$, $\mathbf{e}^{\pm 1}$ are also denoted by $\boldsymbol{\pi}$, $\boldsymbol{\sigma}^\pm$, respectively. The quasi-resonance condition is fulfilled for each plane wave: $|\delta_j| \ll \omega_{eg}, \forall j$, where $\delta_j = \omega_j - \omega_{eg}$ is the detuning. We define a weighted mean frequency $\overline{\omega} = \sum_j \kappa_j \omega_j$ of the plane waves and a weighted mean detuning $\overline{\delta} = \sum_j \kappa_j \delta_j = \overline{\omega} - \omega_{eg}$, with $\{\kappa_j\}$ an *a priori* arbitrary set of weighting factors ($\kappa_j \geq 0$ and $\sum_j \kappa_j = 1$).

In the electric-dipole approximation and considering spontaneous emission in the master equation approach [24], the atomic density operator $\hat{\rho}$ obeys

$$\frac{d}{dt}\hat{\rho}(t) = \frac{1}{i\hbar}[\hat{H}(t), \hat{\rho}(t)] + \mathcal{D}(\hat{\rho}(t)) \tag{5}$$

in which $\hat{H}(t) = \hbar\omega_e \hat{P}_e + \hbar\omega_g \hat{P}_g - \hat{\mathbf{D}} \cdot \mathbf{E}(\mathbf{r}, t)$ and

$$\mathcal{D}(\hat{\rho}) = -(\Gamma/2)(\hat{P}_e\hat{\rho} + \hat{\rho}\hat{P}_e) + \Gamma \sum_q (\mathbf{e}^{q*} \cdot \hat{\mathbf{S}}^-)\hat{\rho}(\mathbf{e}^q \cdot \hat{\mathbf{S}}^+). \tag{6}$$

Here, $\hat{P}_k \equiv \sum_{m_k} |J_k, m_k\rangle\langle J_k, m_k|$ $(k = e, g)$, $\hat{\mathbf{D}}$ is the atomic electric dipole operator, $\mathbf{r}$ is the atom position in the electric field, Γ is the spontaneous de-excitation rate of each upper sublevel, and the $\hat{\mathbf{S}}^\pm$ operators are defined according to $\mathbf{e}^q \cdot \hat{\mathbf{S}}^+ |J_g, m_g\rangle = C_{m_g}^{(q)} |J_e, m_g + q\rangle$, $\mathbf{e}^q \cdot \hat{\mathbf{S}}^+ |J_e, m_e\rangle = 0$, and $\mathbf{e}^{q*} \cdot \hat{\mathbf{S}}^- = (\mathbf{e}^q \cdot \hat{\mathbf{S}}^+)^\dagger$, with

$$\mathcal{C}_m^{(q)} \equiv \langle J_g, m; 1, q | J_e, m + q\rangle, \tag{7}$$

where $\langle j_1, m_1; j_2, m_2 | j, m\rangle$ is the Clebsch-Gordan coefficient corresponding to the coupling of $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ into $|j, m\rangle$.

B. Optical Bloch equations

The hermiticity and unit trace of the density operator make all matrix elements $\rho_{(J_k, m_k),(J_l, m_l)} \equiv \langle J_k, m_k|\hat{\rho}|J_l, m_l\rangle$ $(k, l = e, g)$ dependent variables. We consider here the column vector of real and independent variables $\mathbf{x} = (\mathbf{x}_o^T, \mathbf{x}_\xi^T)^T$, with $\mathbf{x}_o$ a column vector of optical coherences and $\mathbf{x}_\xi \equiv (\mathbf{x}_p^T, \mathbf{x}_Z^T)^T$, where $\mathbf{x}_p$ is a column vector of populations and $\mathbf{x}_Z$ a column vector of Zeeman coherences. We defined

$$\mathbf{x}_o = \begin{pmatrix} \mathbf{x}_o^{(-(J_e+J_g))} \\ \vdots \\ \mathbf{x}_o^{(J_e+J_g)} \end{pmatrix}, \text{ with } \mathbf{x}_o^{(\Delta m)} = \begin{pmatrix} u_{o,m_-^{(\Delta m)}}^{(\Delta m)} \\ v_{o,m_-^{(\Delta m)}}^{(\Delta m)} \\ \vdots \\ u_{o,m_+^{(\Delta m)}}^{(\Delta m)} \\ v_{o,m_+^{(\Delta m)}}^{(\Delta m)} \end{pmatrix}, \tag{8}$$

where $u_{o,m}^{(\Delta m)} \equiv \mathrm{Re}(\rho_{(J_g,m),(J_e,m+\Delta m)} e^{-i\overline{\omega}t})$ and $v_{o,m}^{(\Delta m)} \equiv \mathrm{Im}(\rho_{(J_g,m),(J_e,m+\Delta m)} e^{-i\overline{\omega}t})$, with $m = m_-^{(\Delta m)}, \ldots, m_+^{(\Delta m)}$, $m_\pm^{(\Delta m)} = \pm \min(J_g, J_g + \Delta J \mp \Delta m)$, and $\Delta m = -(J_e + J_g), \ldots, J_e + J_g$.

We defined $\mathbf{x}_p \equiv (\mathbf{x}_{p_e}^T, \mathbf{x}_{p_g}^T)^T$, with $(k = e, g)$

$$\mathbf{x}_{p_k} = \begin{pmatrix} w_{k,-J_k+\delta_{k,g}} \\ \vdots \\ w_{k,J_k} \end{pmatrix}, \tag{9}$$

where $w_{k,m_k} = \rho_{(J_k,m_k),(J_k,m_k)} - N_J^{-1}$, with $m_k = -J_k + \delta_{k,g}, \ldots, J_k$ and $N_J \equiv 2(J_e + J_g + 1)$.

We finally defined $\mathbf{x}_Z \equiv (\mathbf{x}_{Z_e}^T, \mathbf{x}_{Z_g}^T)^T$, with $(k = e, g)$

$$\mathbf{x}_{Z_k} = \begin{pmatrix} \mathbf{x}_{Z_k}^{(1)} \\ \vdots \\ \mathbf{x}_{Z_k}^{(2J_k)} \end{pmatrix}, \text{ with } \mathbf{x}_{Z_k}^{(\Delta m)} = \begin{pmatrix} u_{Z_k,-J_k}^{(\Delta m)} \\ v_{Z_k,-J_k}^{(\Delta m)} \\ \vdots \\ u_{Z_k,J_k-\Delta m}^{(\Delta m)} \\ v_{Z_k,J_k-\Delta m}^{(\Delta m)} \end{pmatrix}, \tag{10}$$

where $u_{Z_k,m_k}^{(\Delta m)} = \mathrm{Re}(\rho_{(J_k,m_k),(J_k,m_k+\Delta m)})$ and $v_{Z_k,m_k}^{(\Delta m)} = \mathrm{Im}(\rho_{(J_k,m_k),(J_k,m_k+\Delta m)})$, with $m_k = -J_k, \ldots, J_k - \Delta m$ and $\Delta m = 1, \ldots, 2J_k$.

In the rotating wave approximation (RWA), the time evolution of $\mathbf{x}$ induced by Eq. (5) obeys to the generalized OBEs

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{b}, \tag{11}$$

with a matrix $A(t)$ and a column vector $\mathbf{b}$ as described in the next two subsections.

1. The $A(t)$ matrix

The $A(t)$ matrix reads

$$A(t) = -\Gamma A_0 + \mathrm{Im}\left(\boldsymbol{\Omega}(t) \cdot \mathbf{e}_C\right), \tag{12}$$

where A_0 is a time independent matrix with scalar entries (independent of any reference frame) detailed hereafter, $\boldsymbol{\Omega}(t) = \sum_j \boldsymbol{\Omega}_j(t) \equiv \sum_q \Omega_q(t)\mathbf{e}^q$, with $\boldsymbol{\Omega}_j(t) =$

$\Omega_j e^{i(\omega_j-\overline{\omega})t}\boldsymbol{\epsilon}_j$, where

$$\Omega_j = -\frac{E_j e^{i(-\mathbf{k}_j\cdot\mathbf{r}+\varphi_j)}}{\hbar}\frac{\langle J_e\|\mathbf{D}\|J_g\rangle^*}{\sqrt{2J_e+1}}, \tag{13}$$

in which $\langle J_e\|\mathbf{D}\|J_g\rangle$ denotes the so-called reduced matrix element associated to $\hat{\mathbf{D}}$ and $\mathbf{e}_C$ is the (unnormalized) basis vector of matrices $\mathbf{e}_C = \sum_q C^{(q)}\mathbf{e}_q$, with contravariant components $C^{(q)}$ as described hereafter and where $\mathbf{e}_q = \mathbf{e}^{q*}$ $(q=0,\pm1)$ are the lower-index spherical basis vectors ($\mathbf{e}_C$ is said a basis vector in that it rotates similarly with the spherical basis in case of a basis change, so that $\boldsymbol{\Omega}(t)\cdot\mathbf{e}_C = \sum_q \Omega_q(t)C^{(q)}$ does *not* define a matrix of scalars - see Appendix A). In the quasi-resonance condition, the RWA approximation is fully justified as long as $|\Omega_j|/\omega_j \ll 1, \forall j$ [25].

The A_0 matrix reads $A_0 = \mathrm{diag}(A_{oo}, A_{\xi\xi})$, with matrix blocks $A_{oo} = \oplus^{(\dim\mathbf{x}_o)/2}\Delta(1/2)$ and $A_{\xi\xi} = \mathrm{diag}(A_{pp}, A_{ZZ})$, where

$$\Delta(\alpha) = \begin{pmatrix}\alpha & -\delta/\Gamma\\ \delta/\Gamma & \alpha\end{pmatrix}, \quad \forall\alpha\in\mathbb{C}, \tag{14}$$

and A_{pp} and A_{ZZ} are 2 blocks themselves structured into subblocks according to

$$A_{pp} = \begin{pmatrix}\mathbb{1}_{p_e} & 0\\ A_{p_gp_e} & 0_{p_g}\end{pmatrix} \tag{15}$$

and

$$A_{ZZ} = \begin{pmatrix}\mathbb{1}_{Z_e} & 0\\ A_{Z_gZ_e} & 0_{Z_g}\end{pmatrix}, \tag{16}$$

with $\mathbb{1}_i$ and 0_i the identity and zero matrices of dimension $\dim\mathbf{x}_i\times\dim\mathbf{x}_i$, respectively. The $A_{p_gp_e}$ subblock elements read $(A_{p_gp_e})_{m_gm_e} = -(\mathcal{C}_{m_g}^{(m_e-m_g)})^2$, with $m_k = -J_k+\delta_{k,g},\dots,J_k$ $(k=e,g)$ [here and throughout the paper, we adopt the convention not to index the matrix elements from $(1,1)$ but with indices directly linked to the magnetic sublevels]. The $A_{Z_gZ_e}$ subblock is itself structured into vertically and horizontally ordered subsubblocks $A_{Z_gZ_e}^{(\Delta m_g,\Delta m_e)}$, with respective indices $\Delta m_g = 1,\dots,2J_g$ and $\Delta m_e = 1,\dots,2J_e$. The only *a priori* nonzero of these subsubblocks are for $\Delta m_g = \Delta m_e \equiv \Delta m$, of elements $A_{Z_gZ_e}^{(\Delta m,\Delta m)} = \tilde{A}_{Z_gZ_e}^{(\Delta m,\Delta m)}\otimes\mathbb{1}_2$, with $(\tilde{A}_{Z_gZ_e}^{(\Delta m,\Delta m)})_{m_gm_e} = -\mathcal{C}_{m_g}^{(m_e-m_g)}\mathcal{C}_{m_g+\Delta m}^{(m_e-m_g)}$, where $m_k = -J_k,\dots,J_k-\Delta m$ $(k=e,g)$.

The $C^{(q)}$ matrices read

$$C^{(q)} = \begin{pmatrix}0 & C_{o\xi}^{(q)}\otimes(1,-i)^T\\ C_{\xi o}^{(q)}\otimes(1,-i) & 0\end{pmatrix}, \tag{17}$$

with $C_{o\xi}^{(q)} = (C_{op}^{(q)}, C_{oZ}^{(q)})$ and $C_{\xi o}^{(q)} = ({C_{po}^{(q)}}^T, {C_{Zo}^{(q)}}^T)^T$, where $C_{op}^{(q)} = (C_{op_e}^{(q)}, C_{op_g}^{(q)})$, $C_{po}^{(q)} = ({C_{p_eo}^{(q)}}^T, {C_{p_go}^{(q)}}^T)^T$, $C_{oZ}^{(q)} = (C_{oZ_e}^{(q)}, C_{oZ_g}^{(q)})$, and $C_{Zo}^{(q)} = ({C_{Z_eo}^{(q)}}^T, {C_{Z_go}^{(q)}}^T)^T$. The $C_{op_k}^{(q)}$, $C_{p_ko}^{(q)}$, $C_{oZ_k}^{(q)}$, and $C_{Z_ko}^{(q)}$ $(k=e,g)$ blocks exclusively contain Clebsch-Gordan coefficients and are detailed below. In addition, for $\zeta = \xi, p_e, p_g, p, Z_e, Z_g, Z$, the $C_{o\zeta}^{(q)}$ $[C_{\zeta o}^{(q)}]$ blocks are of dimension $(\dim\mathbf{x}_o/2)\times\dim\mathbf{x}_\zeta$ $[\dim\mathbf{x}_\zeta\times(\dim\mathbf{x}_o/2)]$.

In accordance with Eq. (8), the $C_{op_k}^{(q)}$ blocks are structured into vertically ordered subblocks indexed with $\Delta m = -(J_g+J_e),\dots,J_g+J_e$ and of dimension $(\dim\mathbf{x}_o^{(\Delta m)}/2)\times\dim\mathbf{x}_{p_k}$. The only *a priori* nonzero of these subblocks is for $\Delta m = q$ and we denote it by $\tilde{C}_{op_k}^{(q)}$. Its elements read $(\tilde{C}_{op_k}^{(q)})_{m,m_k} = \mathcal{C}_m^{(q)}(\delta_{m,-J_g} + \tilde{n}_k\delta_{m_k,m+n_kq})/2$, with $n_k = \delta_{e,k}$ and $\tilde{n}_k = 2n_k-1$, where $m = m_-^{(q)},\dots,m_+^{(q)}$ and $m_k = -J_k+\delta_{k,g},\dots,J_k$. In a same way, the $C_{p_ko}^{(q)}$ blocks are structured into horizontally ordered subblocks indexed with $\Delta m = -(J_g+J_e),\dots,J_g+J_e$ and of dimension $\dim\mathbf{x}_{p_k}\times(\dim\mathbf{x}_o^{(\Delta m)}/2)$. Again, the only *a priori* nonzero of these subblocks is for $\Delta m = q$ and it is denoted by $\tilde{C}_{p_ko}^{(q)}$. Its elements read $(\tilde{C}_{p_ko}^{(q)})_{m_k,m} = -\tilde{n}_k\mathcal{C}_m^{(q)}\delta_{m,m_k-n_kq}$, where $m_k = -J_k+\delta_{k,g},\dots,J_k$ and $m = m_-^{(q)},\dots,m_+^{(q)}$.

Finally, $C_{Z_ko}^{(q)} = -{C_{oZ_k}^{(q)}}^T$ and $C_{oZ_k}^{(q)} = \sum_{\epsilon=\pm1} C_{oZ_k,\epsilon}^{(q)}\otimes(1,\epsilon i)$, where, in accordance with Eqs. (8) and (10), $C_{oZ_k,\epsilon}^{(q)}$ is structured into vertically and horizontally ordered subblocks, with respective indices $\Delta m = -(J_g+J_e),\dots,J_g+J_e$ and $\Delta m_k = 1,\dots,2J_k$. These subblocks are of dimension $(\dim\mathbf{x}_o^{(\Delta m)}/2)\times(\dim\mathbf{x}_{Z_k}^{(\Delta m_k)}/2)$. The only *a priori* nonzero of them are for $\Delta m - \epsilon\Delta m_k = q$ and they are denoted by $\tilde{C}_{oZ_k,\epsilon}^{(q)(\Delta m_k)}$. Their elements read $(\tilde{C}_{oZ_k,\epsilon}^{(q)(\Delta m_k)})_{m,m_k} = (\tilde{n}_k/2)\mathcal{C}_{m-\epsilon(n_k-1)\Delta m_k}^{(q)}\delta_{m_k,m+n_kq-(1-\epsilon)\Delta m_k/2}$, where $m = m_-^{(\Delta m)},\dots,m_+^{(\Delta m)}$ and $m_k = -J_k,\dots,J_k-\Delta m_k$.

2. The $\mathbf{b}$ column vector

The $\mathbf{b}$ column vector reads $\mathbf{b} = -\Gamma N_J^{-1}(\mathbf{0}_o^T, \mathbf{b}_\xi^T)^T$, with $\mathbf{0}_o$ the zero column vector of dimension $\dim\mathbf{x}_o$ and $\mathbf{b}_\xi = A_{\xi\xi}\mathbf{u}_\xi$, where $\mathbf{u}_\xi$ is the column vector of dimension $\dim\mathbf{x}_\xi$ having the $2J_e+1$ first components equal to 1 and all others equal to 0.

C. Periodic regime

Because of the time dependence of $A(t)$, the OBEs cannot be solved analytically and require *a priori* numerical integration. However, within the commensurability assumption (all $\omega_j-\overline{\omega}$ commensurable), the $\boldsymbol{\Omega}(t)$ and $A(t)$ quantities are periodic in time with the repetition period $T_c = 2\pi/\omega_c$, where $\omega_c = (\mathrm{LCM}[(\omega_j-\overline{\omega})^{-1}, \forall j:\omega_j\neq\overline{\omega}])^{-1}$, and all $m_j = (\omega_j-\overline{\omega})/\omega_c$ numbers are integer numbers. In particular, if all frequencies ω_j are identical, then $\boldsymbol{\Omega}(t)$

and $A(t)$ are constant in time, i.e., periodic with an arbitrary value of $\omega_c \neq 0$, and all m_j trivially vanish.

Within the commensurability assumption and given initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, the OBEs admit the unique solution (Floquet's theorem; see, e.g., Ref. [26])

$$\mathbf{x}(t) = P_I(t) e^{R(t-t_0)} (\mathbf{x}_0 - \mathbf{x}_{nh}(t_0)) + \mathbf{x}_{nh}(t), \tag{18}$$

where R is a logarithm of the OBEs monodromy matrix divided by T_c [9], $P_I(t)$ is an invertible T_c-periodic matrix equal to $X_I(t)e^{-R(t-t_0)}$ for $t \in [t_0, t_0+T_c]$, with $X_I(t)$ the matriciant of the OBEs [9], and $\mathbf{x}_{nh}(t)$ is an arbitrary particular solution of the nonhomogeneous OBEs.

We denote the maximum value among the real part of all Floquet exponents (the eigenvalues of R) by λ. If $\lambda < 0$, the $e^{R(t-t_0)}$ matrix tends to zero with a characteristic damping time $|\lambda|^{-1}$ and $\mathbf{x}(t) \simeq \mathbf{x}_{nh}(t)$ at long times ($t \gg t_0 + |\lambda|^{-1}$). In addition, the OBEs are ensured to admit a unique T_c-periodic solution in that case [26] that the particular solution $\mathbf{x}_{nh}(t)$ can be set to so that the atomic internal degrees of freedom $\mathbf{x}(t)$ necessarily reach a periodic regime at long times. If $\lambda = 0$, the $e^{R(t-t_0)}$ matrix is not damped. In addition, a unique T_c-periodic particular solution $\mathbf{x}_{nh}(t)$ is ensured to exist if none of the corresponding Floquet exponents imaginary parts is zero [26]; otherwise, a periodic solution may not exist. Finally, the $\lambda > 0$ case can never happen because the solution $\mathbf{x}(t)$ would diverge for $t \to +\infty$, which is unphysical.

When a T_c-periodic regime is reached, the solution of the OBEs can be expressed using the Fourier expansion

$$\mathbf{x}(t) = \sum_{n=-\infty}^{+\infty} \mathbf{x}^{(n)} e^{in\omega_c t}, \tag{19}$$

with $\mathbf{x}^{(n)}$ the corresponding Fourier components. For $\zeta = o, \xi, p, Z, p_e, p_g, Z_e, Z_g$, we denote by $\mathbf{x}_\zeta^{(n)}$ the Fourier components related to the periodic variable $\mathbf{x}_\zeta(t)$. Since $\mathbf{x}(t)$ is real, we have $\mathbf{x}^{(-n)} = \mathbf{x}^{(n)*}$ and, since it is continuous and differentiable, we have further $\sum_n |\mathbf{x}^{(n)}|^2 < \infty$. Inserting Eq. (19) into the OBEs yields an infinite system of equations connecting all Fourier components. We get, $\forall n$,

$$\mathbf{x}_o^{(n)} = A_{oo}^{(n)} \sum_{q,j} \left(\check{C}_{o\xi}^{(q)*} \frac{\Omega_{j,q}^*}{\Gamma} \mathbf{x}_\xi^{(n+m_j)} - \check{C}_{o\xi}^{(q)} \frac{\Omega_{j,q}}{\Gamma} \mathbf{x}_\xi^{(n-m_j)} \right), \tag{20}$$

with $\Omega_{j,q} = \Omega_j \epsilon_{j,q}$, $A_{oo}^{(n)} = -2i\Gamma^2 \tau_n^+ \tau_n^- \mathbb{1}_{\dim \mathbf{x}_o/2} \otimes \Delta(-in\omega_c/\Gamma - 1/2)$ and $\check{C}_{o\xi}^{(q)} = C_{o\xi}^{(q)} \otimes (1,-i)^T$ [see Eq. (17)], where we defined $\tau_n^\pm = 1/[\Gamma + 2i(n\omega_c \pm \bar{\delta})]$, and

$$\mathbf{x}_\xi^{(n)} + \sum_{m\in M_0} \mathcal{W}_{\xi\xi}^{(n,m)} \mathbf{x}_\xi^{(n+m)} = \mathbf{d}_\xi \delta_{n,0}. \tag{21}$$

In the latter, $\delta_{n,0}$ denotes the Kronecker symbol, M_0 is the set of all distinct nonzero integer numbers $m_{lj} \equiv m_l - m_j$ ($j, l = 1, \ldots, N$), and we defined $\mathcal{W}_{\xi\xi}^{(n,m)} = (A_{\xi\xi}^{(n)} + B_{\xi\xi}^{(n,0)})^{-1} B_{\xi\xi}^{(n,m)}$ and $\mathbf{d}_\xi = -N_J^{-1}(A_{\xi\xi} + \tilde{s}_{\xi\xi})^{-1} A_{\xi\xi} \mathbf{u}_\xi$, where we set $A_{\xi\xi}^{(n)} = A_{\xi\xi} + (in\omega_c/\Gamma)\mathbb{1}_\xi$, $B_{\xi\xi}^{(n,m)} = \sum_{j,l:m_{lj}=m} \sum_{q,q'} (\Omega_{j,q}\Omega_{l,q'}^*/\Gamma)(\tau_{n-m_j}^- (C_{\xi\xi})_{q'}^q + \tau_{n+m_l}^+ (C_{\xi\xi})_q^{q'*})$ with $(C_{\xi\xi})_{q'}^q \equiv -C_{\xi o}^{(q)} C_{o\xi}^{(q')*}$, and $\tilde{s}_{\xi\xi} = \sum_j \tilde{s}_{\xi\xi,j}$ with

$$\tilde{s}_{\xi\xi,j} = \mathrm{Re}\left[\sum_{q,q'} \sum_{\substack{l=1 \\ l:m_{lj}=0}}^{N} \frac{\Omega_{j,q}\Omega_{l,q'}^*/\Gamma}{\Gamma/2 - i\delta_j} (C_{\xi\xi})_{q'}^q \right]. \tag{22}$$

We have $\tau_{-n}^\pm = \tau_n^{\mp *}$, $A_{\xi\xi}^{(-n)} = A_{\xi\xi}^{(n)*}$, $B_{\xi\xi}^{(-n,-m)} = B_{\xi\xi}^{(n,m)*}$, and $\mathcal{W}_{\xi\xi}^{(-n,-m)} = \mathcal{W}_{\xi\xi}^{(n,m)*}$.

The system (21) is only defined if all $A_{\xi\xi}^{(n)} + B_{\xi\xi}^{(n,0)}$ matrices are invertible. Otherwise, the OBEs actually do not admit any periodic solution and the following formalism does not apply to these situations. For $\Delta J = -1$ and all waves with the same polarization $\boldsymbol{\epsilon}_j$, we easily get $\det(A_{\xi\xi}^{(0)} + B_{\xi\xi}^{(0,0)}) = 0$ so that no periodic regime exists.

If we define $\mathbf{y} = (\ldots, \mathbf{x}_\xi^{(-1)T}, \mathbf{x}_\xi^{(0)T}, \mathbf{x}_\xi^{(1)T}, \ldots)^T$ the infinite column vector of all $\mathbf{x}_\xi^{(n)}$ components, as well as

$$W = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & W_{\xi\xi}^{(-1,-1)} & W_{\xi\xi}^{(-1,0)} & W_{\xi\xi}^{(-1,1)} & \cdots \\ \cdots & W_{\xi\xi}^{(0,-1)} & W_{\xi\xi}^{(0,0)} & W_{\xi\xi}^{(0,1)} & \cdots \\ \cdots & W_{\xi\xi}^{(1,-1)} & W_{\xi\xi}^{(1,0)} & W_{\xi\xi}^{(1,1)} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{23}$$

the infinite matrix structured into the subblocks $W_{\xi\xi}^{(n,n')} = \sum_{m\in M_0} \mathcal{W}_{\xi\xi}^{(n,m)} \delta_{n',n+m}$ (n, n' ranging from $-\infty$ to $+\infty$), then Eq. (21) yields the complex inhomogeneous infinite system of equations

$$(I + W)\mathbf{y} = \mathbf{c}, \tag{24}$$

with $\mathbf{c} = (\ldots, \mathbf{c}_\xi^{(-1)T}, \mathbf{c}_\xi^{(0)T}, \mathbf{c}_\xi^{(1)T}, \ldots)^T$ the infinite column vector of subblocks $\mathbf{c}_\xi^{(n)} = \mathbf{d}_\xi \delta_{n,0}$ (n ranging from $-\infty$ to $+\infty$) and I the infinite identity matrix. We directly have $W_{\xi\xi}^{(n,n)} = 0$, $\forall n$. The solution of the system (24) is necessarily such as $\mathbf{x}_\xi^{(0)} \neq 0$; otherwise, all other $\mathbf{x}_\xi^{(n)}$ components would solve a homogeneous system of equations and thus vanish, in which case the equation for $n = 0$ could not be satisfied. This allows us to define the matrices $Q_{\xi\xi}^{(n)}$ that map the $\mathbf{x}_\xi^{(0)}$ vector onto $\mathbf{x}_\xi^{(n)}$, $\forall n$: $Q_{\xi\xi}^{(n)} \mathbf{x}_\xi^{(0)} = \mathbf{x}_\xi^{(n)}$. The $Q_{\xi\xi}^{(n)}$ matrices are *a priori* not unique. Such suitable matrices can be given by (see, e.g., Ref. [27]): $Q_{\xi\xi}^{(n)} = 0$ if $\mathbf{x}_\xi^{(n)} = 0$; otherwise $Q_{\xi\xi}^{(n)} = (e^{i\theta} \|\mathbf{x}_\xi^{(n)}\| / \|\mathbf{x}_\xi^{(0)}\|) \mathbb{1}_\xi$ if $\mathbf{x}_\xi^{(n)} / \|\mathbf{x}_\xi^{(n)}\| = e^{i\theta} (\mathbf{x}_\xi^{(0)} / \|\mathbf{x}_\xi^{(0)}\|)$ ($\theta \in [0, 2\pi[$); otherwise $Q_{\xi\xi}^{(n)} = (\|\mathbf{x}_\xi^{(n)}\| / \|\mathbf{x}_\xi^{(0)}\|) U_{\xi\xi}^{(n)}$. In the latter, $U_{\xi\xi}^{(n)}$ is the unitary matrix $e^{i\phi^{(n)}}V_{\xi\xi}^{(n)}$, where $\phi^{(n)}$ is the phase of the complex number $\mathbf{x}_\xi^{(0)^*}\cdot\mathbf{x}_\xi^{(n)}$, conventionally set to 0 if the complex number is zero, and where $V_{\xi\xi}^{(n)}$ is the Householder matrix $V_{\xi\xi}^{(n)} = \mathbb{1}_\xi - (2/\|\mathbf{z}_\xi^{(n)}\|^2)(\mathbf{z}_\xi^{(n)}\mathbf{z}_\xi^{(n)\dagger})$, with $\mathbf{z}_\xi^{(n)} = e^{i\phi^{(n)}}\mathbf{x}_\xi^{(0)} - (\|\mathbf{x}_\xi^{(0)}\|/\|\mathbf{x}_\xi^{(n)}\|)\mathbf{x}_\xi^{(n)}$. In particular, we have in that case $Q_{\xi\xi}^{(0)} = \mathbb{1}_\xi$ and $Q_{\xi\xi}^{(-n)} = Q_{\xi\xi}^{(n)^*}$. Inserting $\mathbf{x}_\xi^{(n)} = Q_{\xi\xi}^{(n)}\mathbf{x}_\xi^{(0)}$ into Eq. (21) for $n = 0$ yields

$$\mathbf{x}_\xi^{(0)} = -\frac{1}{N_J}\frac{A_{\xi\xi}}{A_{\xi\xi}+s_{\xi\xi}}\mathbf{u}_\xi, \tag{25}$$

where $\frac{A_{\xi\xi}}{A_{\xi\xi}+s_{\xi\xi}} \equiv (A_{\xi\xi}+s_{\xi\xi})^{-1}A_{\xi\xi}$, $s_{\xi\xi} = \sum_j s_{\xi\xi,j}$, with

$$s_{\xi\xi,j} = \mathrm{Re}\left[\sum_{q,q'}\sum_{l=1}^{N}\frac{\Omega_{j,q}\Omega_{l,q'}^*/\Gamma}{\Gamma/2 - i\delta_j}(C_{\xi\xi})_{q'}^{q}Q_{\xi\xi}^{(m_{lj})}\right]. \tag{26}$$

Equation (26) is the generalization of Eq. (15) of Ref. [9] to the degenerate two-level atom case.

D. General and exact expression of the radiation force

Proceeding along the same lines as in Ref. [9] (two-level atom case), the total mean power absorbed from all plane waves and the mean net force exerted on the atom can be expressed as $P(t) = \sum_j P_j(t)$ and $\mathbf{F}(t) = \sum_j \mathbf{F}_j(t)$, respectively, with $P_j(t) = R_j(t)\hbar\overline{\omega}$ and $\mathbf{F}_j(t) = R_j(t)\hbar\mathbf{k}_j$, where

$$R_j(t) = \mathrm{Im}\left[\boldsymbol{\Omega}_j(t)\cdot\boldsymbol{\chi}_o(t)\right], \tag{27}$$

with $\boldsymbol{\chi}_o(t) \equiv \sum_q \chi_o^{(q)}(t)\mathbf{e}_q$ the three-dimensional vector of contravariant components

$$\chi_o^{(q)}(t) = -\sum_{m=m_-^{(q)}}^{m_+^{(q)}}\mathcal{C}_m^{(q)}(u_{o,m}^{(q)}(t) - iv_{o,m}^{(q)}(t)). \tag{28}$$

The vector $\boldsymbol{\chi}_o(t)$ is a true polar vector (see Appendix A), so that obviously $\boldsymbol{\Omega}_j(t)\cdot\boldsymbol{\chi}_o(t)$ is a scalar quantity. In the quasi-resonance condition ($\omega_j \simeq \overline{\omega}$, $\forall j$), $R_j(t) \simeq \langle dN/dt\rangle_j(t)$ [9].

Within the commensurability assumption, we have $\omega_j - \overline{\omega} = m_j\omega_c$, $\forall j$. In the periodic regime, $\mathbf{x}(t)$ is in addition T_c-periodic, and thus so are $R_j(t)$, $P_j(t)$, and $\mathbf{F}_j(t)$. In this regime, the Fourier components of $R_j(t) \equiv \sum_{n=-\infty}^{+\infty}R_j^{(n)}e^{in\omega_c t}$ are easily obtained by inserting Eq. (19) into Eq. (27). By using further Eq. (20) and $Q_{\xi\xi}^{(n)}\mathbf{x}_\xi^{(0)} = \mathbf{x}_\xi^{(n)}$ with $\mathbf{x}_\xi^{(0)}$ as in Eq. (25), we get

$$\begin{aligned} R_j^{(n)} &= \frac{\Gamma}{N_J}\mathbf{u}_\xi^T s_{\xi\xi,j}^{(n)}\frac{A_{\xi\xi}}{A_{\xi\xi}+s_{\xi\xi}}\mathbf{u}_\xi \\ &= -\Gamma\mathbf{u}_\xi^T s_{\xi\xi,j}^{(n)}\mathbf{x}_\xi^{(0)}, \end{aligned} \tag{29}$$

with $s_{\xi\xi,j}^{(n)} = \left(\sigma_{\xi\xi,j}^{(n)} + \sigma_{\xi\xi,j}^{(-n)^*}\right)/2$, where

$$\sigma_{\xi\xi,j}^{(n)} = \sum_{q,q'}\sum_{l=1}^{N}\frac{\Omega_{j,q}\Omega_{l,q'}^*/\Gamma}{\Gamma/2 + i(n\omega_c - \delta_j)}(C_{\xi\xi})_{q'}^{q}Q_{\xi\xi}^{(n+m_{lj})}. \tag{30}$$

In particular, the temporal mean value $\overline{R}_j$ of $R_j(t)$ in the periodic regime is given by the Fourier component $R_j^{(0)}$, and observing that $s_{\xi\xi,j}^{(0)} = s_{\xi\xi,j}$ [Eq. (26)], we get

$$\overline{R}_j = \frac{\Gamma}{N_J}\mathbf{u}_\xi^T s_{\xi\xi,j}\frac{A_{\xi\xi}}{A_{\xi\xi}+s_{\xi\xi}}\mathbf{u}_\xi. \tag{31}$$

The Fourier components of the corresponding force $\mathbf{F}_j(t) \equiv \sum_{n=-\infty}^{+\infty}\mathbf{F}_j^{(n)}e^{in\omega_c t}$ in the periodic regime are then given by $\mathbf{F}_j^{(n)} = R_j^{(n)}\hbar\mathbf{k}_j$ and the mean force in this regime thus reads

$$\overline{\mathbf{F}}_j = \frac{\Gamma}{N_J}(\mathbf{u}_\xi^T s_{\xi\xi,j}\frac{A_{\xi\xi}}{A_{\xi\xi}+s_{\xi\xi}}\mathbf{u}_\xi)\hbar\mathbf{k}_j. \tag{32}$$

Equation (32) is a natural extension of Eq. (19) of Ref. [9]. Interestingly, the two-level atom case is also covered within the present formalism. To this aim, it is enough to consider $J_g = J_e = 0$ and to open artificially the forbidden 0-0 transition by forcing $\mathcal{C}_0^{(0)}$ to 1. All equations above then merely simplify to the two-level atom formalism of Ref. [9].

To illustrate our formalism, we computed the stimulated bichromatic force in a standard four traveling-wave configuration [30] for various atomic structures. We considered a detuning $\delta = 10\Gamma$, a Rabi frequency amplitude of $\sqrt{3/2}\delta$, a phase shift of $\pi/2$ for one of the waves, and a pure π polarization ($\boldsymbol{\epsilon}_j = \boldsymbol{\pi}, \forall j$). We show in Fig. 1 the value of the resulting bichromatic force (averaged over the 2π range of the spatially varying relative phase between the opposite waves) acting in the direction of the phased wave on a moving atom as a function of its velocity v for $\Delta J = 1$ with $J_g = 1/2, 1, \ldots, 4$. Figure 1 shows that the π polarization case yields poorer results for the bichromatic force than the ideal two-level atom case (here merely obtained using σ^+ or σ^- light with $\Delta J = 1$).

III. SPECIFIC REGIMES

A. Low-intensity regime

We define the low-intensity regime as the regime where $\sum_j|\Omega_j|/\Gamma \ll 1$ and [31]

$$\sum_{m_2}\sum_{m\in M_0}|(\mathcal{W}_{\xi\xi}^{(n,m)})_{m_1,m_2}| \ll \frac{1}{\sqrt{\dim\mathbf{x}_\xi}}, \quad \forall n\neq 0, \forall m_1. \tag{33}$$

In this regime, we have $s_{\xi\xi,j} \simeq \tilde{s}_{\xi\xi,j}$ and $\overline{R}_j$ can be computed without solving the infinite system (24). Indeed, Eq. (24) can be expressed as a function of $\mathbf{x}_\xi^{(0)}$ according

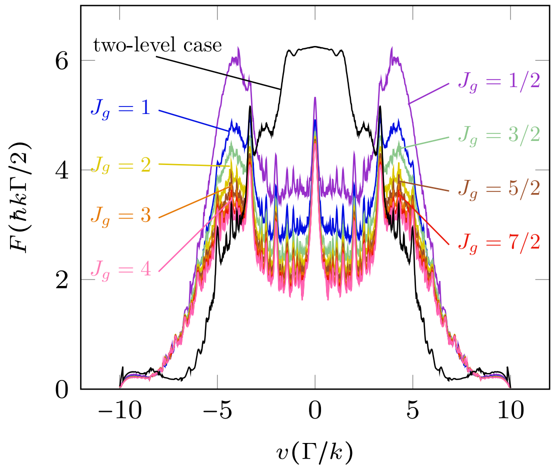

FIG. 1. (Color online) Stimulated bichromatic force F as a function of the atomic velocity v computed via our formalism for a detuning $\delta = 10\Gamma$, a Rabi frequency of $\sqrt{3/2}\delta$, a phase shift of $\pi/2$ for one wave, $\boldsymbol{\epsilon}_j = \boldsymbol{\pi}, \forall j$, and $J_g = 1/2, 1, \ldots, 4$ with $\Delta J = 1$.

to $(\mathbb{1}+W_0)\mathbf{y}_0 = -W_0'\mathbf{x}_\xi^{(0)}$, where $\mathbf{y}_0$ is the $\mathbf{y}$ vector excluding the $\mathbf{x}_\xi^{(0)}$ component, W_0 is the W matrix excluding the $W_{\xi\xi}^{(0,n')}$ and $W_{\xi\xi}^{(n,0)}$ blocks, $\forall n, n'$, and W_0' is the W matrix restricted to the only blocks $W_{\xi\xi}^{(n,0)}$, $\forall n \neq 0$. Under condition (33), we have $(\mathbb{1}+W_0)^{-1} \simeq \mathbb{1}+\sum_{k=1}^{+\infty}(-W_0)^k$. This implies $\mathbf{y}_0 \simeq -W_0'\mathbf{x}_\xi^{(0)}$, so that $\|\mathbf{x}_\xi^{(n)}\| \ll \|\mathbf{x}_\xi^{(0)}\|$ and thus $Q_{\xi\xi}^{(n)} \simeq 0_{\xi\xi}, \forall n \neq 0$, i.e., $s_{\xi\xi,j} \simeq \tilde{s}_{\xi\xi,j}$. If all plane waves have different frequencies, we get

$$s_{\xi\xi,j} \simeq \mathrm{Re}\left[\sum_{q,q'}\left(\frac{\Omega_{j,q}\Omega_{j,q'}^*/\Gamma}{\Gamma/2 - i\delta_j}\right)(C_{\xi\xi})_{q'}^{q}\right]. \qquad (34)$$

If in contrast some plane waves have identical frequencies, coherent effects can be observed. If we are only interested in the incoherent effect of the plane waves, an average $\langle\cdot\rangle_\varphi$ over all phase differences must be performed. For $N = 2$ with $\boldsymbol{\epsilon}_1 = \boldsymbol{\sigma}^+$ and $\boldsymbol{\epsilon}_2 = \boldsymbol{\sigma}^-$, and in the incoherent regime, the statistical delta method [32] at order 0 yields $\overline{R}_j^{\mathrm{inc}} \equiv \langle\overline{R}_j\rangle_\varphi \simeq \Gamma N_J^{-1}\mathbf{u}_\xi^T\langle\tilde{s}_{\xi\xi,j}\rangle_\varphi\left(A_{\xi\xi}+\sum_i\langle\tilde{s}_{\xi\xi,i}\rangle_\varphi\right)^{-1}A_{\xi\xi}\mathbf{u}_\xi$. For such $\boldsymbol{\epsilon}_j$, we have further $\langle\tilde{s}_{\xi\xi,j}\rangle_\varphi = \mathrm{Re}[(|\Omega_j|^2/\Gamma)/(\Gamma/2 - i\delta_j)(C_{\xi\xi})_{q_j}^{q_j}]$, where $q_1 = 1 = -q_2$. Since $(C_{\xi\xi})_q^q$ has the block-diagonal structure (see Appendix B)

$$(C_{\xi\xi})_q^q = \begin{pmatrix}(C_{pp})_q^q & 0\\ 0 & (C_{ZZ})_q^q\end{pmatrix}, \qquad (35)$$

with $(C_{\zeta\zeta})_q^q = -C_{\zeta o}^{(q)}C_{o\zeta}^{(q)^*}$ ($\zeta = p, Z$), it follows that $\langle\tilde{s}_{\xi\xi,j}\rangle_\varphi$ has the same block-diagonal structure. In addition, $A_{\xi\xi}$ behaves similarly and since all $(C_{pp})_q^q$ matrix elements are real numbers, $\overline{R}_j^{\mathrm{inc}}$ simplifies to $\overline{R}_j^{\mathrm{inc}} \simeq \Gamma N_J^{-1}s_j f_j(s_1, s_2)$, with $s_j \equiv (|\Omega_j|^2/2)/(\Gamma^2/4+\delta_j^2)$ and $f_j(s_1,s_2) = \mathbf{u}_p^T(C_{pp})_{q_j}^{q_j}(A_{pp}+\sum_{i=1}^2 s_i(C_{pp})_{q_i}^{q_i})^{-1}A_{pp}\mathbf{u}_p$, where $\mathbf{u}_p$ is the $\dim\mathbf{x}_p$ column vector with the $\dim\mathbf{x}_{p_e}$ first components equal to 1 and all others to 0.

B. Plane waves with same frequency

If all plane waves have the same frequency, the periodic regime is a stationary regime ($R_j^{(n)} = 0, \forall n \neq 0$) and $s_{\xi\xi,j}$ [see Eq. (26)] simplifies to $\mathrm{Re}[\sum_{q,q'}(\underline{s}_j)_q^{q'}(C_{\xi\xi})_{q'}^q]$, where $(\underline{s}_j)_q^{q'}$ is the second-order tensor $(\Omega_{j,q}\Omega_{q'}^*/\Gamma)/(\Gamma/2 - i\delta)$, with $\delta \equiv \delta_j, \forall j$. Here, Ω_q is independent of time and merely identifies to $\sum_j \Omega_{j,q}$.

For $N = 1$, the index $j = 1$ can be omitted and we get $\underline{s}_q^{q'} = s(1+2i\delta/\Gamma)\epsilon_q\epsilon_{q'}^*$ with $s = (|\Omega|^2/2)/(\Gamma^2/4+\delta^2)$ and $\overline{R} = \Gamma N_J^{-1}\mathbf{u}_\xi^T f_{\xi\xi}(A_{\xi\xi}+f_{\xi\xi})^{-1}A_{\xi\xi}\mathbf{u}_\xi$ where we set $f_{\xi\xi} \equiv \mathrm{Re}[\sum_{q,q'}\underline{s}_q^{q'}(C_{\xi\xi})_{q'}^q]$. For a given atomic structure and in contrast to $A_{\xi\xi}$ and $\mathbf{u}_\xi$, the $f_{\xi\xi}$ matrix and $\overline{R}$ depend on s, $\boldsymbol{\epsilon}$, and δ (at constant s): $f_{\xi\xi} \equiv f_{\xi\xi}(s,\boldsymbol{\epsilon},\delta)$ and $\overline{R} \equiv \overline{R}(s,\boldsymbol{\epsilon},\delta)$. In addition, we have $f_{\xi\xi}(s,\boldsymbol{\epsilon},\delta) = f_{\xi\xi}(s,\boldsymbol{\epsilon},0) - (2\delta s/\Gamma)I_{\xi\xi}(\boldsymbol{\epsilon})$ with $I_{\xi\xi}(\boldsymbol{\epsilon}) = \mathrm{Im}[\sum_{q,q'}\epsilon_q\epsilon_{q'}^*(C_{\xi\xi})_{q'}^q]$. Since, for all invertible matrices A and $A-B$, we have $(A-B)^{-1} = A^{-1} + (A-B)^{-1}BA^{-1}$, we get $X_{\xi\xi}(s,\boldsymbol{\epsilon},\delta)^{-1} = X_{\xi\xi}(s,\boldsymbol{\epsilon},0)^{-1} + (2\delta s/\Gamma)X_{\xi\xi}(s,\boldsymbol{\epsilon},\delta)^{-1}I_{\xi\xi}(\boldsymbol{\epsilon})X_{\xi\xi}(s,\boldsymbol{\epsilon},0)^{-1}$, where we set $X_{\xi\xi}(s,\boldsymbol{\epsilon},\delta) \equiv A_{\xi\xi}+f_{\xi\xi}(s,\boldsymbol{\epsilon},\delta)$. It follows that $\overline{R}(s,\boldsymbol{\epsilon},\delta) = \overline{R}(s,\boldsymbol{\epsilon},0) - 2\delta s N_J^{-1}\mathbf{u}_\xi^T A_{\xi\xi}X_{\xi\xi}(s,\boldsymbol{\epsilon},\delta)^{-1}\mathbf{u}_\xi'(s,\boldsymbol{\epsilon})$, where $\mathbf{u}_\xi'(s,\boldsymbol{\epsilon}) \equiv I_{\xi\xi}(\boldsymbol{\epsilon})X_{\xi\xi}(s,\boldsymbol{\epsilon},0)^{-1}A_{\xi\xi}\mathbf{u}_\xi$. The graph of $\|\mathbf{u}_\xi'(s,\boldsymbol{\epsilon})\|^2$ as a function of the most general polarization configuration $\boldsymbol{\epsilon} = \cos(\theta/2)\boldsymbol{\sigma}^+ + e^{i\varphi}\sin(\theta/2)\boldsymbol{\sigma}^-$ ($\theta \in [0,\pi], \varphi \in [0, 2\pi[$) always identifies to 0, whatever s and the atomic structure (J_g, J_e), so that

$$\overline{R}(s,\boldsymbol{\epsilon},\delta) = \overline{R}(s,\boldsymbol{\epsilon},0), \qquad (36)$$

i.e., $\overline{R}$ does not depend on δ at constant s.

C. Plane waves with same pure polarization

If all plane waves have the same pure polarization q ($\boldsymbol{\epsilon}_j = \mathbf{e}^q$, $\forall j$), then each $\mathcal{W}_{\xi\xi}^{(n,m)}$ matrix is block-diagonal with blocks $\mathcal{W}_{pp}^{(n,m)}$ and $\mathcal{W}_{ZZ}^{(n,m)}$ of dimension $\dim\mathbf{x}_p \times \dim\mathbf{x}_p$ and $\dim\mathbf{x}_Z\times\dim\mathbf{x}_Z$, respectively, because so are the $(C_{\xi\xi})_q^q$ matrices, with corresponding blocks $(C_{pp})_q^q$ and $(C_{ZZ})_q^q$. In addition, the $\dim\mathbf{x}_Z$ last components of the $\mathbf{d}_\xi$ vector vanish and its $\dim\mathbf{x}_p$ first components are denoted hereafter by $\mathbf{d}_p$. Equation (21) then yields the decoupled system

$$\begin{cases}\mathbf{x}_p^{(n)} + \sum\limits_{m\in M_0}\mathcal{W}_{pp}^{(n,m)}\mathbf{x}_p^{(n+m)} = \mathbf{d}_p\delta_{n,0},\\ \mathbf{x}_Z^{(n)} + \sum\limits_{m\in M_0}\mathcal{W}_{ZZ}^{(n,m)}\mathbf{x}_Z^{(n+m)} = 0\end{cases} \qquad (37)$$

that can be solved separately for the $\mathbf{x}_p^{(n)}$ and $\mathbf{x}_Z^{(n)}$ Fourier components. The $\mathbf{x}_Z^{(n)}$ components satisfy a homogeneous system with trivial solution $\mathbf{x}_Z^{(n)} = 0, \forall n$. This implies that the $\dim \mathbf{x}_Z$ last components of $\mathbf{z}_\xi^{(n)}$ are zero and then the $Q_{\xi\xi}^{(n)}$ matrix is block-diagonal with blocks $Q_{pp}^{(n)}$ and $Q_{ZZ}^{(n)}$ of dimension $\dim \mathbf{x}_p \times \dim \mathbf{x}_p$ and $\dim \mathbf{x}_Z \times \dim \mathbf{x}_Z$, respectively. As a consequence, $\sigma_{\xi\xi,j}^{(n)}$ are in turn block-diagonal with the same structure, $\forall n$, and so are $s_{\xi\xi,j}^{(n)}$ with corresponding blocks $s_{pp,j}^{(n)}$ and $s_{ZZ,j}^{(n)}$. In particular, Eqs. (25) and (29) yield $\mathbf{x}_p^{(0)} = -N_J^{-1}(A_{pp} + s_{pp})^{-1} A_{pp} \mathbf{u}_p$ and $R_j^{(n)} = \Gamma N_J^{-1} \mathbf{u}_p^T s_{pp,j}^{(n)} (A_{pp} + s_{pp})^{-1} A_{pp} \mathbf{u}_p$, respectively, with $s_{pp} = \sum_j s_{pp,j}$, where $s_{pp,j} \equiv s_{pp,j}^{(0)}$.

In the low-intensity regime and with all plane waves with different frequencies, we get $s_{pp,j} \simeq s_j (C_{pp})_q^q$ and $\overline{R}_j = \Gamma N_J^{-1} s_j \mathbf{u}_p^T (C_{pp})_q^q (A_{pp} + s(C_{pp})_q^q)^{-1} A_{pp} \mathbf{u}_p$, respectively, with $s = \sum_j s_j$ and $s_j = (|\Omega_j|^2/2)/(\Gamma^2/4 + \delta_j^2)$.

D. Plane waves with same frequency and pure polarization

If all lasers have the same angular frequency ω_j and the same pure polarization q, we set $\Omega = \sum_j \Omega_j$ and $\delta \equiv \delta_j, \forall j$, and $\overline{R}_j$ simplifies to

$$\overline{R}_j = \frac{\Gamma}{2} s_j \frac{a_{\Delta J, J_g, q}}{b_{\Delta J, J_g, q} + s}, \qquad (38)$$

with $s_j = \mathrm{Re}[(\Omega_j \Omega^*/\Gamma)/(\Gamma/2 - i\delta)]$ and $s = \sum_j s_j$. For $\Delta J = 1$, or $\Delta J = 0$ with $q = 0$ and half-integer J_g, $a_{\Delta J, J_g, q} = 1$ and

$$b_{\Delta J, J_g, q} = \frac{\det[A_{pp,+}^q] + (-1)^{2J_g} \det[A_{pp,-}^q]}{\det[A_{pp,+}^q] - (-1)^{2J_g} \det[A_{pp,-}^q]}, \qquad (39)$$

with $A_{pp,\pm}^q \equiv A_{pp} \pm (C_{pp})_q^q$. For $\Delta J = 0$ with $q \neq 0$ or integer J_g, $a_{\Delta J, J_g, q} = b_{\Delta J, J_g, q} = 0$. In particular, we have $b_{1, J_g, \pm 1} = 1$. For $\Delta J = 1$, Eq. (38) yields

$$\overline{R}_j = \begin{cases} \dfrac{\Gamma}{2} \dfrac{s_j}{1+s} & \text{for } q = \pm 1, \\ \dfrac{\Gamma}{2} \dfrac{s_j}{b_{1,J_g,0} + s} & \text{for } q = 0. \end{cases} \qquad (40)$$

For $q = \pm 1$, in the periodic regime, the atom is pumped into the $|J_g, m_g = \pm J_g\rangle$ state from which it interacts only with the $|J_e, m_e = \pm J_e\rangle$ state through the laser radiation action. The atom then exactly behaves as a two-level system. For $q = 0$, all populations are nonzero apart from $m_e = \pm J_e$ and the result is more subtle. For $\Delta J = 0$, Eq. (38) yields $\overline{R}_j = 0$ for $q = \pm 1$ and

$$\overline{R}_j = \begin{cases} 0 & \text{if } J_g \text{ is integer,} \\ \dfrac{\Gamma}{2} \dfrac{s_j}{b_{0,J_g,0} + s} & \text{if } J_g \text{ is half-integer,} \end{cases} \qquad (41)$$

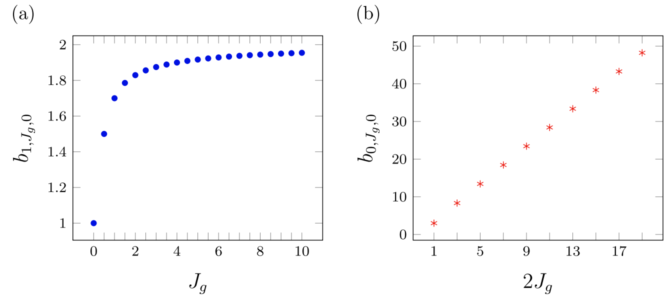

FIG. 2. (Color online) Values of (a) $b_{1,J_g,0}$ and (b) $b_{0,J_g,0}$ as a function of J_g.

for $q = 0$. For $q = \pm 1$, the atom is pumped into the $|J_g, m_g = \pm J_g\rangle$ state on which $\sigma^\pm$-radiation has no effect. For $q = 0$ and integer J_g, since the Clebsch-Gordan coefficient $\mathcal{C}_0^{(0)}$ is zero whatever J_g, the atom is pumped into the $|J_g, m_g = 0\rangle$ state from which π-radiation has no effect. For $q = 0$ and half-integer J_g, all populations are nonzero. We recall that no periodic regime exists for all lasers with same polarization in the $\Delta J = -1$ case [since $\det(A_{\xi\xi}^{(0)} + B_{\xi\xi}^{(0,0)}) = 0$]. Equations (40) and (41) perfectly reproduce the results of Ref. [10] that investigates those specific configurations. We show in Fig. 2 the parameters $b_{1,J_g,0}$ and $b_{0,J_g,0}$ as a function of J_g.

E. $N = 2$ case

For $N = 2$ and $\omega_1 \neq \omega_2$, the system can be tackled in a continued fraction approach. The commensurability assumption implies that $n_2 \kappa_1 = n_1 \kappa_2$, with n_1 and n_2 two positive coprime integer numbers. It follows that $\omega_c = |\omega_1 - \omega_2|/n_s$, with $n_s = n_1 + n_2$, $m_1 = \mathrm{sgn}(\omega_1 - \omega_2) n_2$, $m_2 = \mathrm{sgn}(\omega_2 - \omega_1) n_1$, $m_{12} = \mathrm{sgn}(\omega_1 - \omega_2) n_s$, and $M_0 = \{\pm n_s\}$. The infinite system of Eq. (21) then reads

$$\mathbf{x}_\xi^{(n)} + \mathcal{W}_{\xi\xi}^{(n,n_s)} \mathbf{x}_\xi^{(n+n_s)} + \mathcal{W}_{\xi\xi}^{(n,-n_s)} \mathbf{x}_\xi^{(n-n_s)} = \mathbf{d}_\xi \delta_{n,0}. \qquad (42)$$

This system only couples together the Fourier components $\mathbf{x}_\xi^{(n)}$ with $n = k n_s$ ($k \in \mathbb{Z}$). All other components are totally decoupled from these former and thus vanish, since they satisfy a homogeneous system. Hence, the only *a priori* nonvanishing $Q_{\xi\xi}^{(n)}$ matrices and $R_j^{(n)}$ and $\mathbf{F}_j^{(n)}$ Fourier components are for these specific $n = k n_s$ values, and the periodic regime is rather characterized by the beat period $T_c/n_s = 2\pi/|\omega_1 - \omega_2|$. We then define the matrix $Q_{\xi\xi}^{(n,m)}$ that maps $\mathbf{x}_\xi^{(n)}$ onto $\mathbf{x}_\xi^{(n+m)}, \forall m, n$: $Q_{\xi\xi}^{(n,m)} \mathbf{x}_\xi^{(n)} = \mathbf{x}_\xi^{(n+m)}$. It follows that, for $n = k n_s \neq 0$, Eq. (42) yields $Q_{\xi\xi}^{(n-n_s,n_s)} = -\mathcal{W}_{\xi\xi}^{(n,-n_s)}/(\mathbb{1}_\xi + \mathcal{W}_{\xi\xi}^{(n,n_s)} Q_{\xi\xi}^{(n,n_s)})$ where we adpoted the matrix notation $A/B \equiv B^{-1}A$. Applying recursively this relation for $n = n_s, 2n_s, 3n_s, \ldots$ yields $Q_{\xi\xi}^{(n_s)} = -\mathcal{W}_{\xi\xi}^{(n_s,-n_s)}/[\mathbb{1}_\xi + \mathcal{K}_{k=1}^{\infty}(P_{\xi\xi,k}/\mathbb{1}_\xi)]$, with $P_{\xi\xi,k} = -\mathcal{W}_{\xi\xi}^{(kn_s,n_s)} \mathcal{W}_{\xi\xi}^{((k+1)n_s,-n_s)}$ and $\mathcal{K}_{k=1}^{\infty}(P_{\xi\xi,k}/\mathbb{1}_\xi) \equiv$

$P_{\xi\xi,1}/(\mathbb{1}_\xi+P_{\xi\xi,2}/(\mathbb{1}_\xi+P_{\xi\xi,3}/\ldots))$. For $n=kn_s$ with $k>1$, Eq. (42) implies the recurrence relation $Q_{\xi\xi}^{((k+1)n_s)} = -(Q_{\xi\xi}^{(kn_s)}+\mathcal{W}_{\xi\xi}^{(-kn_s,n_s)*}Q_{\xi\xi}^{((k-1)n_s)})/\mathcal{W}_{\xi\xi}^{(kn_s,n_s)}$ that allows for computing all remaining $Q_{\xi\xi}^{(kn_s)}$ and hence all nonzero Fourier components $R_j^{(kn_s)}$ and $\mathbf{F}_j^{(kn_s)}$.

IV. CONCLUSION

In this paper, we extended the formalism of Ref. [9] to the multi-level atom case, where Zeeman sublevels and arbitrary light polarization are taken into account. In that context, we have provided a general standardized and exact analytical formalism for computing within the usual RWA the mechanical action experienced by a single multi-level atom lightened simultaneously by an arbitrary set of plane waves. By use of a Fourier expansion treatment, we provided an exact analytical expression of all Fourier components $\mathbf{F}_j^{(n)}$ describing the light forces in the periodic regime. In particular, we extended the steady mean force expression (1) into Eq. (32), involving matrix quantities whose dimensions depend on the atomic structure. In addition, we highlighted some simplifications holding in specific regimes. The computation of the Fourier components related to the light forces relies on the solution of an algebraic system of equations and does not require numerical integration of the OBEs with time. Our formalism offers an alternative to purely numerical approaches, where the extraction of the mean force and their Fourier components can be subjected to instabilities, especially when the forces vary very slowly. Our results always converge to the exact values.

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Appendix A: Reference frame rotation

The states $|J_g,m_g\rangle$ and $|J_e,m_e\rangle$ are common eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$, with $\hat{\mathbf{J}}$ the total angular momentum and $\hat{J}_z$ its z component in the considered reference frame S. If S is rotated according to Euler angles α, β, γ to a new configuration $\underline{S}$, the component $\hat{J}_z$ transforms to $\underline{\hat{J}}_z = \hat{R}(\alpha,\beta,\gamma)\hat{J}_z\hat{R}(\alpha,\beta,\gamma)^\dagger$, with the rotation operator $\hat{R}(\alpha,\beta,\gamma)=e^{-i\alpha\hat{J}_z/\hbar}e^{-i\beta\hat{J}_y/\hbar}e^{-i\gamma\hat{J}_z/\hbar}$ [33]. $\hat{\mathbf{J}}^2$ remains unchanged and the common eigenstates of $\hat{\mathbf{J}}^2$ and $\underline{\hat{J}}_z$ read $\underline{|J_k,m_k\rangle} = \hat{R}(\alpha,\beta,\gamma)|J_k,m_k\rangle$ $(k=e,g)$. The elements of the basis transformation matrix $\langle J_k,m_k\underline{|J_k,m_k'\rangle}$ are given by the so-called Wigner functions $\mathcal{D}_{m_k,m_k'}^{(J_k)}(\alpha,\beta,\gamma) = e^{-i(m_k\alpha+m_k'\gamma)}d_{m_k,m_k'}^{(J_k)}(\beta)$, with $d_{m,m'}^{(J)}(\beta) = A_{m,m'}(J)B_{m,m'}(\beta)P_{J-m'}^{(m'-m,m'+m)}(\cos\beta)$, where $A_{m,m'}(J) = \sqrt{[(J+m')!(J-m')!]/[(J+m)!(J-m)!]}$, $B_{m,m'}(\beta) = [\sin(\beta/2)]^{m'-m}[\cos(\beta/2)]^{m'+m}$, and where the $P_n^{(a,b)}(z)$ are the Jacobi polynomials [33]. If $\rho(t)$ and $\underline{\rho}(t)$ denote the density matrices of the atomic state $\hat{\rho}(t)$ in the $\{|J_k,m_k\rangle\}$ and $\{\underline{|J_k,m_k\rangle}\}$ bases, respectively, we get

$$\underline{\rho}(t) = \mathcal{D}(\alpha,\beta,\gamma)^\dagger\rho(t)\mathcal{D}(\alpha,\beta,\gamma), \tag{A1}$$

with $\mathcal{D}(\alpha,\beta,\gamma) = \bigoplus_{k=e,g}\mathcal{D}^{(J_k)}(\alpha,\beta,\gamma)$, where $\mathcal{D}^{(J_k)}(\alpha,\beta,\gamma)$ is the unitary matrix of elements $\mathcal{D}_{m_k,m_k'}^{(J_k)}(\alpha,\beta,\gamma)$. It follows that the associated OBE column vector $\mathbf{x}(t)$ transforms according to

$$\underline{\mathbf{x}}(t) = T(\alpha,\beta,\gamma)\mathbf{x}(t), \tag{A2}$$

with the transformation matrix

$$T(\alpha,\beta,\gamma) = \begin{pmatrix} T_{oo}(\alpha,\beta,\gamma) & 0 \\ 0 & T_{\xi\xi}(\alpha,\beta,\gamma)\end{pmatrix}, \tag{A3}$$

where

$$T_{\xi\xi}(\alpha,\beta,\gamma) = \begin{pmatrix} T_{pp}(\beta) & T_{pZ}(\alpha,\beta) \\ T_{Zp}(\beta,\gamma) & T_{ZZ}(\alpha,\beta,\gamma)\end{pmatrix}, \tag{A4}$$

with blocks $T_{oo}(\alpha,\beta,\gamma)$, $T_{pp}(\beta)$, $T_{pZ}(\alpha,\beta)$, $T_{Zp}(\beta,\gamma)$, and $T_{ZZ}(\alpha,\beta,\gamma)$ as explicitly detailed below. The OBEs (11) in the $\underline{S}$ reference frame then read

$$\dot{\underline{\mathbf{x}}}(t) = \underline{A}(t)\underline{\mathbf{x}}(t) + \underline{\mathbf{b}}, \tag{A5}$$

with

$$\underline{A}(t) = T(\alpha,\beta,\gamma)A(t)T(\alpha,\beta,\gamma)^{-1} \tag{A6}$$

and

$$\underline{\mathbf{b}} = T(\alpha,\beta,\gamma)\mathbf{b}. \tag{A7}$$

Thanks to the orthogonality relations of the Clebsch-Gordan coefficients, to the orthogonality of the $d^{(1)}(\beta)$ matrices of elements $d_{m,m'}^{(1)}(\beta)$, to the transformation law of the spherical components of any three-dimensional space vector $\boldsymbol{v}$, $(\underline{v}_1,\underline{v}_0,\underline{v}_{-1})^T = \mathcal{D}^{(1)}(\alpha,\beta,\gamma)^T(v_1,v_0,v_{-1})^T$, and to the identities $\mathcal{C}_{m_g}^{(q)}d_{m_e,m_g+q}^{(J_e)}(\beta) = \sum_{q'}\mathcal{C}_{m_e-q'}^{(q')}d_{m_e-q',m_g}^{(J_g)}(\beta)d_{q',q}^{(1)}(\beta)$ and $\mathcal{C}_{m_e-q}^{(q)}d_{m_e-q,m_g}^{(J_g)}(\beta) = \sum_{q'}\mathcal{C}_{m_g}^{(q')}d_{m_e,m_g+q'}^{(J_e)}(\beta)d_{q,q'}^{(1)}(\beta)$ (see, e.g., Ref. [34]), which imply, $\forall m_{k_1},m_{k_2} = -J_k,\ldots,J_k$ $(k=e,g)$, $\sum_q\mathcal{C}_{m_{g_1}}^{(q)}d_{m_{e_1},m_{g_1}+q}^{(J_e)}(\beta)\mathcal{C}_{m_{g_2}}^{(q)}d_{m_{e_2},m_{g_2}+q}^{(J_e)}(\beta) =$

$\sum_q \mathcal{C}^{(q)}_{m_{e_1}-q} d^{(J_g)}_{m_{e_1}-q,m_{g_1}}(\beta) \mathcal{C}^{(q)}_{m_{e_2}-q} d^{(J_g)}_{m_{e_2}-q,m_{g_2}}(\beta)$, the explicit calculation of Eqs. (A6) and (A7) yields, as expected,

$$\underline{A}(t) = -\Gamma A_0 + \mathrm{Im}(\mathbf{\Omega}(t)\cdot \underline{\mathbf{e}}_C) \tag{A8}$$

and $\underline{\mathbf{b}} = \mathbf{b}$, where $\underline{\mathbf{e}}_C = \sum_q C^{(q)} \underline{\mathbf{e}}_q$, with $\underline{\mathbf{e}}_q$ the $\underline{S}$ lower-index spherical basis, such that $\mathbf{\Omega}(t)\cdot\underline{\mathbf{e}}_C = \sum_q \underline{\Omega}_q(t) C^{(q)}$.

Similarly, the transformation law $\underline{\mathbf{x}}_o(t) = T_{oo}(\alpha,\beta,\gamma)\mathbf{x}_o(t)$ [see Eq. (A2)] directly yields the standard contravariant transformation law $(\underline{\chi}^{(1)}_o, \underline{\chi}^{(0)}_o, \underline{\chi}^{(-1)}_o)^T = \mathcal{D}^{(1)}(\alpha,\beta,\gamma)^\dagger (\chi^{(1)}_o, \chi^{(0)}_o, \chi^{(-1)}_o)^T$ that proves the vectorial character of $\boldsymbol{\chi}_o(t)$ [see Eq. (27)].

1. The $T_{oo}(\alpha,\beta,\gamma)$ block

In accordance with Eq. (8), the $T_{oo}(\alpha,\beta,\gamma)$ block is structured into vertically and horizontally ordered subblocks $T^{(\Delta m,\Delta m')}_{oo}(\alpha,\beta,\gamma)$, with respective indices Δm and $\Delta m'$ both ranging from $-(J_e + J_g)$ to $J_e + J_g$. The subblocks $T^{(\Delta m,\Delta m')}_{oo}(\alpha,\beta,\gamma)$ read $\tilde{T}^{(\Delta m,\Delta m')}_{oo}(\beta) \otimes U^{(\Delta m,\Delta m')}_{+}(\alpha,\gamma)$, with matrix elements $(\tilde{T}^{(\Delta m,\Delta m')}_{oo}(\beta))_{m,m'} = d^{(J_g)}_{m',m}(\beta) d^{(J_e)}_{m'+\Delta m',m+\Delta m}(\beta)$ ($m = m^{(\Delta m)}_{-},\dots,m^{(\Delta m)}_{+}$ and $m' = m^{(\Delta m')}_{-},\dots,m^{(\Delta m')}_{+}$) and where

$$U^{(\Delta m,\Delta m')}_{\pm}(\alpha,\gamma) = \begin{pmatrix} c^{(\Delta m,\Delta m')}_{\pm}(\alpha,\gamma) & s^{(\Delta m,\Delta m')}_{\pm}(\alpha,\gamma) \\ \mp s^{(\Delta m,\Delta m')}_{\pm}(\alpha,\gamma) & \pm c^{(\Delta m,\Delta m')}_{\pm}(\alpha,\gamma) \end{pmatrix}, \tag{A9}$$

with $c^{(\Delta m,\Delta m')}_{\pm}(\alpha,\gamma) = \cos(\Delta m'\alpha \pm \Delta m\gamma)$ and $s^{(\Delta m,\Delta m')}_{\pm}(\alpha,\gamma) = \sin(\Delta m'\alpha \pm \Delta m\gamma)$.

2. The $T_{pp}(\beta)$ block

The $T_{pp}(\beta)$ block is structured into 4 subblocks (one is zero) as

$$T_{pp}(\beta) = \begin{pmatrix} T_{p_e p_e}(\beta) & 0 \\ T_{p_g p_e}(\beta) & T_{p_g p_g}(\beta) \end{pmatrix}, \tag{A10}$$

with subblock elements $(T_{p_e p_e}(\beta))_{m_e,m'_e} = (d^{(J_e)}_{m'_e,m_e}(\beta))^2$, $(T_{p_g p_e}(\beta))_{m_g,m'_e} = -(d^{(J_g)}_{-J_g,m_g}(\beta))^2$, and $(T_{p_g p_g}(\beta))_{m_g,m'_g} = (d^{(J_g)}_{m'_g,m_g}(\beta))^2 - (d^{(J_g)}_{-J_g,m_g}(\beta))^2$, where $m_k, m'_k = -J_k + \delta_{k,g},\dots,J_k$ ($k = e,g$).

3. The $T_{Zp}(\beta,\gamma)$ block

The $T_{Zp}(\beta,\gamma)$ block is similarly structured into 4 subblocks (among which one is zero) as

$$T_{Zp}(\beta,\gamma) = \begin{pmatrix} T_{Z_e p_e}(\beta,\gamma) & 0 \\ T_{Z_g p_e}(\beta,\gamma) & T_{Z_g p_g}(\beta,\gamma) \end{pmatrix}, \tag{A11}$$

where the subblocks $T_{Z_k p_l}(\beta,\gamma)$ ($k,l = g,e$) are themselves further divided [in accordance with Eq. (10)] into vertically ordered subsubblocks $T^{(\Delta m)}_{Z_k p_l}(\beta,\gamma)$ indexed with $\Delta m = 1,\dots,2J_k$. The subsubblocks $T^{(\Delta m)}_{Z_k p_l}(\beta,\gamma)$ read $\tilde{T}^{(\Delta m)}_{Z_k p_l}(\beta) \otimes (\cos(\Delta m\gamma), -\sin(\Delta m\gamma))^T$, with matrix elements $(\tilde{T}^{(\Delta m)}_{Z_e p_e}(\beta))_{m_e,m'_e} = d^{(J_e)}_{m'_e,m_e}(\beta) d^{(J_e)}_{m'_e,m_e+\Delta m}(\beta)$, $(\tilde{T}^{(\Delta m)}_{Z_g p_e}(\beta))_{m_g,m'_e} = -d^{(J_g)}_{-J_g,m_g}(\beta) d^{(J_g)}_{-J_g,m_g+\Delta m}(\beta)$, and $(\tilde{T}^{(\Delta m)}_{Z_g p_g}(\beta))_{m_g,m'_g} = d^{(J_g)}_{m'_g,m_g}(\beta) d^{(J_g)}_{m'_g,m_g+\Delta m}(\beta) - d^{(J_g)}_{-J_g,m_g}(\beta) d^{(J_g)}_{-J_g,m_g+\Delta m}(\beta)$, where $m_k = -J_k,\dots,J_k - \Delta m$ and $m'_k = -J_k + \delta_{k,g},\dots,J_k$ ($k = e,g$).

4. The $T_{pZ}(\alpha,\beta)$ block

The $T_{pZ}(\alpha,\beta)$ block is structured into 2 diagonal subblocks as

$$T_{pZ}(\alpha,\beta) = \begin{pmatrix} T_{p_e Z_e}(\alpha,\beta) & 0 \\ 0 & T_{p_g Z_g}(\alpha,\beta) \end{pmatrix}, \tag{A12}$$

where the subblocks $T_{p_k Z_k}(\alpha,\beta)$ ($k = e,g$) are themselves further divided [in accordance with Eq. (10)] into horizontally ordered subsubblocks $T^{(\Delta m)}_{p_k Z_k}(\alpha,\beta)$ indexed with $\Delta m = 1,\dots,2J_k$. The subsubblocks $T^{(\Delta m)}_{p_k Z_k}(\alpha,\beta)$ read $\tilde{T}^{(\Delta m)}_{p_k Z_k}(\beta) \otimes (\cos(\Delta m\alpha), \sin(\Delta m\alpha))$, with matrix elements $(\tilde{T}^{(\Delta m)}_{p_k Z_k}(\beta))_{m_k,m'_k} = 2 d^{(J_k)}_{m'_k,m_k}(\beta) d^{(J_k)}_{m'_k+\Delta m,m_k}(\beta)$, where $m_k = -J_k + \delta_{k,g},\dots,J_k$ and $m'_k = -J_k,\dots,J_k - \Delta m$.

5. The $T_{ZZ}(\alpha,\beta,\gamma)$ block

The $T_{ZZ}(\alpha,\beta,\gamma)$ block is similarly structured into 2 diagonal subblocks as

$$T_{ZZ}(\alpha,\beta,\gamma) = \begin{pmatrix} T_{Z_e Z_e}(\alpha,\beta,\gamma) & 0 \\ 0 & T_{Z_g Z_g}(\alpha,\beta,\gamma) \end{pmatrix}, \tag{A13}$$

where the subblocks $T_{Z_k Z_k}(\alpha,\beta,\gamma)$ ($k = e,g$) are themselves divided [in accordance with Eq. (10)] into vertically and horizontally subsubblocks $T^{(\Delta m,\Delta m')}_{Z_k Z_k}(\alpha,\beta,\gamma)$, with respective indices $\Delta m = 1,\dots,2J_k$ and $\Delta m' = 1,\dots,2J'_k$. The subsubblocks $T^{(\Delta m,\Delta m')}_{Z_k Z_k}(\alpha,\beta,\gamma)$ read $\sum_{\epsilon=\pm} \tilde{T}^{(\Delta m,\Delta m')}_{Z_k Z_k,\epsilon}(\beta) \otimes U^{(\Delta m,\Delta m')}_{\epsilon}(\alpha,\gamma)$, with matrix elements $(\tilde{T}^{(\Delta m,\Delta m')}_{Z_k Z_k,+}(\beta))_{m_k,m'_k} = d^{(J_k)}_{m'_k,m_k}(\beta) d^{(J_k)}_{m'_k+\Delta m',m_k+\Delta m}(\beta)$ and $(\tilde{T}^{(\Delta m,\Delta m')}_{Z_k Z_k,-}(\beta))_{m_k,m'_k} = d^{(J_k)}_{m'_k+\Delta m',m_k}(\beta) d^{(J_k)}_{m'_k,m_k+\Delta m}(\beta)$, where $m_k = -J_k,\dots,J_k - \Delta m$ and $m'_k = -J_k,\dots,J_k - \Delta m'$ ($k = e,g$).

Appendix B: Explicit value of the $(C_{\xi\xi})^q_{q'}$ matrices

The matrices $(C_{\xi\xi})^q_{q'} = -C^{(q)}_{\xi o} C^{(q')^*}_{o\xi}$ are structured into 4 blocks as

$$(C_{\xi\xi})^q_{q'} = \begin{pmatrix} (C_{pp})^q_{q'} & (C_{pZ})^q_{q'} \\ (C_{Zp})^q_{q'} & (C_{ZZ})^q_{q'} \end{pmatrix}, \tag{B1}$$

with $(C_{rs})^q_{q'} = -C^{(q)}_{ro} C^{(q')^*}_{os}$ $(r,s = p, Z)$. These blocks are themselves further divided into 4 subblocks as

$$(C_{rs})^q_{q'} = \begin{pmatrix} (C_{r_e s_e})^q_{q'} & (C_{r_e s_g})^q_{q'} \\ (C_{r_g s_e})^q_{q'} & (C_{r_g s_g})^q_{q'} \end{pmatrix}, \tag{B2}$$

where again $(C_{r_k s_l})^q_{q'} = -C^{(q)}_{r_k o} C^{(q')^*}_{o s_l}$ $(k,l = e,g)$. These subblocks are detailed below.

1. The $(C_{p_k p_l})^q_{q'}$ subblocks

For $k,l = e,g$, we have $(C_{p_k p_l})^q_{q'} = (\tilde{C}_{p_k p_l})^q_q \delta_{q,q'}$, with $(\tilde{C}_{p_k p_l})^q_q = -\tilde{C}^{(q)}_{p_k o} \tilde{C}^{(q)}_{o p_l}$. The $(\tilde{C}_{p_k p_l})^q_q$ matrix elements are indexed with the two numbers $m = -J_k + \delta_{k,g}, \dots, J_k$ and $m' = -J_l + \delta_{l,g}, \dots, J_l$. They are *a priori* only nonzero if $m - n_k q \in \{m^{(q)}_-, \dots, m^{(q)}_+\}$, in which case they read explicitly $[(\tilde{C}_{p_k p_l})^q_q]_{m,m'} = \tilde{n}_k (\mathcal{C}^{(q)}_{m - n_k q})^2 (\delta_{m,-J_g + n_k q} + \tilde{n}_l \delta_{m', m + (n_l - n_k) q})/2$. All $(C_{pp})^q_{q'}$ matrix elements are real numbers.

2. The $(C_{p_k Z_l})^q_{q'}$ subblocks

For $k,l = e,g$, we have

$$(C_{p_k Z_l})^q_{q'} = \begin{pmatrix} (\tilde{C}^{(1)}_{p_k Z_l})^q_{q'} & (\tilde{C}^{(2)}_{p_k Z_l})^q_{q'} & 0 \end{pmatrix}. \tag{B3}$$

The 0 block is of dimension $\dim \mathbf{x}_{p_k} \times \sum_{i=3}^{2J_l} \dim \mathbf{x}^{(i)}_{Z_l}$ and, for $j = 1,2$,

$$(\tilde{C}^{(j)}_{p_k Z_l})^q_{q'} = \begin{cases} 0_{\dim \mathbf{x}_{p_k} \times \dim \mathbf{x}^{(j)}_{Z_l}} & \text{if } |\Delta q| \neq j, \\ (\check{C}^{(j)}_{p_k Z_l})^q_{q'} \otimes (1, \operatorname{sgn}(\Delta q) i) & \text{otherwise,} \end{cases} \tag{B4}$$

with $\Delta q \equiv q' - q$ and $(\check{C}^{(j)}_{p_k Z_l})^q_{q'} = -\tilde{C}^{(q)}_{p_k o} \tilde{C}^{(q,|\Delta q|)}_{o Z_l, -\operatorname{sgn}(\Delta q)}$ for $\Delta q \neq 0$ (see Section II). Hence, $(C_{p_k Z_l})^q_q = 0,\ \forall q$. The $(\check{C}^{(j)}_{p_k Z_l})^q_{q'}$ matrix elements are indexed with the two numbers $m = -J_k + \delta_{k,g}, \dots, J_k$ and $m' = -J_l, \dots, J_l - |\Delta q|$. They are *a priori* only nonzero if $m - n_k q \in \{m^{(q)}_-, \dots, m^{(q)}_+\}$, in which case they read explicitly $[(\check{C}^{(j)}_{p_k Z_l})^q_{q'}]_{m,m'} = (\tilde{n}_k \tilde{n}_l \mathcal{C}^{(q)}_{m - n_k q} \mathcal{C}^{(q')}_{m - n_k q + (n_l - 1)\Delta q} \delta_{m', m + (\tilde{n}_l q' - \tilde{n}_k q - j)/2})/2$.

3. The $(C_{Z_k p_l})^q_{q'}$ subblocks

For $k,l = e,g$, we have

$$(C_{Z_k p_l})^q_{q'} = \begin{pmatrix} (\tilde{C}^{(1)}_{Z_k p_l})^q_{q'} \\ (\tilde{C}^{(2)}_{Z_k p_l})^q_{q'} \\ 0 \end{pmatrix}. \tag{B5}$$

The 0 block is of dimension $\sum_{i=3}^{2J_k} \dim \mathbf{x}^{(i)}_{Z_k} \times \dim \mathbf{x}_{p_l}$ and, for $j = 1,2$,

$$(\tilde{C}^{(j)}_{Z_k p_l})^q_{q'} = \begin{cases} 0_{\dim \mathbf{x}^{(j)}_{Z_k} \times \dim \mathbf{x}_{p_l}} & \text{if } |\Delta q| \neq j, \\ (\check{C}^{(j)}_{Z_k p_l})^q_{q'} \otimes (1, \operatorname{sgn}(\Delta q) i)^T & \text{otherwise,} \end{cases} \tag{B6}$$

with $(\check{C}^{(j)}_{Z_k p_l})^q_{q'} = -\tilde{C}^{(|\Delta q|, q')}_{Z_k o, \operatorname{sgn}(\Delta q)} \tilde{C}^{(q')}_{o p_l}$ for $\Delta q \neq 0$, where $\tilde{C}^{(|\Delta q|, q')}_{Z_k o, \epsilon} = -\tilde{C}^{(q', |\Delta q|)^T}_{o Z_k, \epsilon}$ for $\epsilon = \pm 1$ (see Section II). Hence, $(C_{Z_k p_l})^q_q = 0, \forall q$. The $(\check{C}^{(j)}_{Z_k p_l})^q_{q'}$ matrix elements are indexed with the two numbers $m = -J_k, \dots, J_k - |\Delta q|$ and $m' = -J_l + \delta_{l,g}, \dots, J_l$. They are *a priori* only nonzero if $m - n_k q - [1 - \operatorname{sgn}(\Delta q)]\Delta q/2 \in \{m^{(q')}_-, \dots, m^{(q')}_+\}$, in which case they read explicitly $[(\check{C}^{(j)}_{Z_k p_l})^q_{q'}]_{m,m'} = (\tilde{n}_k \mathcal{C}^{(q')}_{-J_g} \mathcal{C}^{(q)}_{-J_g - (n_k - 1)\Delta q} \delta_{m, -J_g + (q' + \tilde{n}_k q - j)/2} + \tilde{n}_k \tilde{n}_l \mathcal{C}^{(q')}_{m' - n_l q'} \mathcal{C}^{(q)}_{m' - n_l q' - (n_k - 1)\Delta q} \delta_{m', m + (\tilde{n}_l q' - \tilde{n}_k q + j)/2})/4$. We note that $(C_{p_k Z_l})^q_{q'} \neq -[(C_{Z_l p_k})^q_{q'}]^T$.

4. The $(C_{Z_k Z_l})^q_{q'}$ subblocks

For $k,l = e,g$, we have

$$\begin{aligned} (C_{Z_k Z_l})^q_{q'} = &\sum_{\epsilon = \pm 1} (\tilde{C}_{Z_k Z_l, \epsilon})^q_{q'} \otimes \begin{pmatrix} 1 & -\epsilon i \\ \epsilon i & 1 \end{pmatrix} \\ &+ (\tilde{C}_{Z_k Z_l})^q_q \otimes \begin{pmatrix} 1 & -qi \\ -qi & -1 \end{pmatrix} \delta_{q', -q}, \end{aligned} \tag{B7}$$

with $(\tilde{C}_{Z_k Z_l, \epsilon})^q_{q'}$ and $(\tilde{C}_{Z_k Z_l})^q_q$ as described below.

If $\operatorname{diag}_p(X_1, \dots, X_n)$ denotes the rectangular matrix whose elements are matrix blocks, with $X_1, \dots, X_n$ the only nonzero such elements exactly located on the p^{th} superdiagonal (or subdiagonal if $p < 0$) of the rectangular matrix, then we have

$$(\tilde{C}_{Z_k Z_l, \epsilon})^q_{q'} = \operatorname{diag}_{-\epsilon \Delta q} \left((\tilde{C}^{(\Delta m_-)}_{Z_k Z_l, \epsilon})^q_{q'}, \dots, (\tilde{C}^{(\Delta m_+)}_{Z_k Z_l, \epsilon})^q_{q'} \right), \tag{B8}$$

where $\Delta m_- = \max[1, 1 + \epsilon \Delta q]$, $\Delta m_+ = \Delta m_- + 2\min(J_e, J_g) - 1$, and, for $\Delta m = \Delta m_-, \dots, \Delta m_+$, $(\tilde{C}^{(\Delta m)}_{Z_k Z_l, \epsilon})^q_{q'}$ is a matrix block of dimension $(\dim \mathbf{x}^{(\Delta m)}_{Z_k}/2) \times (\dim \mathbf{x}^{(\Delta m - \epsilon \Delta q)}_{Z_l}/2)$. These blocks are *a priori* only nonzero for $\Delta m + \epsilon q \le J_e + J_g$, in which case they identify to $-\tilde{C}^{(\Delta m, q + \epsilon \Delta m)}_{Z_k o, \epsilon} \tilde{C}^{(q + \epsilon \Delta m, \Delta m - \epsilon \Delta q)}_{o Z_l, \epsilon}$, with matrix elements $[(\tilde{C}^{(\Delta m)}_{Z_k Z_l, \epsilon})^q_{q'}]_{m,m'} = (\tilde{n}_k \tilde{n}_l / 4)$

$\mathcal{C}^{(q)}_{m-(1+\tilde{n}_k)q/2+(1-\epsilon\tilde{n}_k)\Delta m/2}\mathcal{C}^{(q')}_{m+(1-\epsilon\tilde{n}_l)\Delta m/2-(\tilde{n}_k q-\tilde{n}_l\Delta q+q')/2}\delta_{m',\mu}$, where $\mu = m + (\tilde{n}_l q' - \tilde{n}_k q + \epsilon\Delta q)/2$, $m = -J_k, \ldots, J_k - \Delta m$, and $m' = -J_l, \ldots, J_l - \Delta m + \epsilon\Delta q$.

We also have

$$(\tilde{C}_{Z_k Z_l})^q_q = \begin{cases} 0_{(\dim \mathbf{x}_{Z_k}/2)\times(\dim \mathbf{x}_{Z_l}/2)} & \text{if } q = 0, \\ \begin{pmatrix} (\check{C}_{Z_k Z_l})^q_q & 0 \\ 0 & 0_{d_k\times d_l} \end{pmatrix} & \text{otherwise,} \end{cases} \tag{B9}$$

with, for $k = e, g$, $d_k \equiv \sum_{i=2}^{2J_k} \dim \mathbf{x}^{(i)}_{Z_k}/2 = J_k(2J_k - 1)$ and $(\check{C}_{Z_k Z_l})^q_q = -\tilde{C}^{(1,0)}_{Z_k o,-q}\tilde{C}^{(0,1)}_{o Z_l,q}$. The $(\check{C}_{Z_k Z_l})^q_q$ matrix elements are indexed with the two numbers $m = -J_k, \ldots, J_k - 1$ and $m' = -J_l, \ldots, J_l - 1$. They read explicitly $[(\check{C}_{Z_k Z_l})^q_q]_{m,m'} = (\tilde{n}_k\tilde{n}_l/4)\mathcal{C}^{(q)}_{m+(1-q)/2}\mathcal{C}^{(-q)}_{m'+(1+q)/2}\delta_{m',m-(\tilde{n}_k+\tilde{n}_l)q/2}$.

The $(C_{Z_k Z_l})^q_{q'}$ subblocks are such that $(C_{ZZ})^q_{q'} = [(C_{ZZ})^{q'}_q]^\dagger$ and $(C_{Z_g Z_e})^q_{q'} = [(C_{Z_e Z_g})^q_{q'}]^T$, $\forall q, q'$. In addition, $\sum_q (C_{ZZ})^q_q$ is a real matrix.

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