Automatic sequences in rational base numeration systems (and even more)

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How it started

Eric Rowland presented our paper

E. Rowland and M. Stipulanti, Avoiding 5/4-powers on the alphabet of nonnegative integers, *Electron. J. Combin.* **27** (2020), Paper 3.42, 39 pp.

at the *One World Combinatorics on Words Seminar.*
He (notably) talked about the word

$$w_{3/2} = 001102100112001103100113001102100114001103 1\cdots ,$$

which is the lexicographically least word avoiding $\frac{3}{2}$-powers on the alphabet of non-negative integers. Then Michel asked:

Is this word $\frac{3}{2}$-regular?

So we started to look at automaticity and regularity in rational bases.

The plan

Define:

- Abstract numeration systems
- Labeled trees and periodic labeled signatures
- Rational base numeration systems
- Automatic sequences

Two main results:

- A version of Cobham's theorem
- "Factor complexity" in trees

Abstract numeration systems

Introduced by Pierre Lecomte and Michel Rigo in 2001, an abstract numeration system (ANS for short) is a triple $S = (L, A, <)$ where

- $(A, <)$ is a totally ordered (finite) alphabet
- L is an infinite language over A.

We say that L is the numeration language.

The map $\mathrm{rep}_S : \mathbb{N} \to L$ is the one-to-one correspondence mapping $n \in \mathbb{N}$ onto the $(n+1)$st word in the radix ordered language L.
This word is called the S-representation of n.
(The S-representation of 0 is the first word in L.)

The inverse map is denoted by $\mathrm{val}_S : L \to \mathbb{N}$. For any word w in L, $\mathrm{val}_S(w)$ is its S-(numerical) value.

Remark: Nothing is assumed on the language L, i.e., it might well be neither regular nor prefix-closed.

Examples of ANS

- Integer base numeration systems:
  Let $k \geq 2$ be an integer. We let $A_k = \{0, 1, \ldots, k-1\}$ and
  $$L_k = \{\varepsilon\} \cup \{1, \ldots, k-1\}\{0, \ldots, k-1\}^*.$$
  Then $S = (L_k, A_k, <)$ with $0 < 1 < \cdots < k-1$ is an ANS.
- Fibonacci (or Zeckendorff) numeration system:

Let $S = (L, \{0, 1\}, <)$ with $0 < 1$ and L containing words avoiding 11.
In radix order: $L = \{\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, \ldots\}$

S-representation	S-value
ε	0
1	1
10	2
100	3
101	4
1000	5

A "non-standard" ANS

$S = (a^*b^*, \{a, b\}, <)$ with $a < b$
$a^*b^* = \{\varepsilon, a, b, aa, ab, bb, aaa, aab, abb, \ldots\}$ in radix order

S-representation	S-value
ε	0
a	1
b	2
aa	3
ab	4
bb	5
aaa	6
aab	7

Position of a word in a^*b^*: $\mathrm{val}_S(a^p b^q) = \frac{(p+q)(p+q+1)}{2} + q$.
For instance, $\mathrm{val}_S(ab) = \frac{2\cdot 3}{2} + 1 = 4$.

From ANS to trees

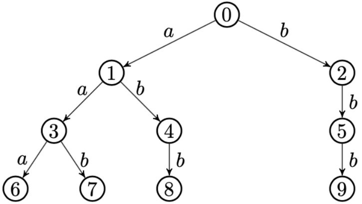
Prefix-closed languages define labeled trees.

Let $S = (L, A, <)$ be an ANS where L is prefix-closed.
We define the tree $T(L)$ as follows.

- The set of nodes of $T(L)$ is L.
- If w and wd are words in L with $d \in A$, then there is an edge from w to wd with label d in $T(L)$.

The children of a node are ordered by the labels of the letters in the ordered alphabet A.
Nodes are enumerated by breadth-first traversal.

Example: $S = (a^*b^*, \{a, b\}, <)$ with $a < b$
$a^*b^* = \{\varepsilon, a, b, aa, ab, bb, aaa, \ldots\}$

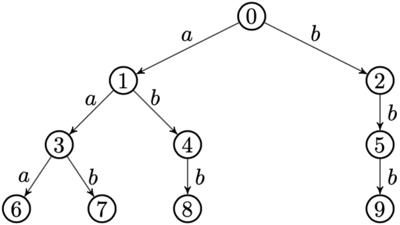


Let T be a labeled tree.

The signature of T is the sequence of the degrees of the nodes visited by the breadth-first traversal of T.

The labeling of T is the sequence of the labels of the edges visited by the breadth-first traversal of T.

Example:



Signature:
$2, 2, 1, 2, 1, 1, 2, 1, 1, 1, 2, \ldots$

Labeling:
$a, b, a, b, b, a, b, b, b, a, b, \ldots$

Remark: Sometimes it is convenient to consider i-trees: the root is assumed to be a child of itself.

Particular ANS: rational bases

Let p and q be two relatively prime integers with $p > q > 1$.

For a word $w = w_\ell w_{\ell-1} \cdots w_0 \in A_p^*$, the value of w in base $\frac{p}{q}$ is the rational number

$$\mathrm{val}_{\frac{p}{q}}(w) = \sum_{i=0}^{\ell} \frac{w_i}{q} \left(\frac{p}{q}\right)^i.$$

Example: $p = 3$, $q = 2$ $\qquad A_3 = \{0, 1, 2\}$

$w \in \{0,1,2\}^*$	$\mathrm{val}_{\frac{3}{2}}(w)$
$\varepsilon, 0$	0
1	$\frac{1}{2} \cdot \left(\frac{3}{2}\right)^0 = \frac{1}{2}$
2	$\frac{2}{2} \cdot \left(\frac{3}{2}\right)^0 = 1$
10	$\frac{1}{2} \cdot \left(\frac{3}{2}\right)^1 + \frac{0}{2} \cdot \left(\frac{3}{2}\right)^0 = \frac{3}{4}$
21	$\frac{2}{2} \cdot \left(\frac{3}{2}\right)^1 + \frac{1}{2} \cdot \left(\frac{3}{2}\right)^0 = 2$

Remark: $\mathrm{val}_{\frac{p}{q}}(w)$ is a not always an integer.

Definition: A word $w \in A_p^*$ is a representation of an integer $n \geq 0$ in base $\frac{p}{q}$ if $\mathrm{val}_{\frac{p}{q}}(w) = n$.

Theorem (Akiyama, Frougny, Sakarovitch, 2008)

Representations in rational bases are unique up to leading zeroes.

In base $\frac{p}{q}$:
$\mathrm{rep}_{\frac{p}{q}}(n)$ denotes the representation of n that does not start with 0.
By convention, the representation of 0 is the empty word ε.
The numeration language is the set

$$L_{\frac{p}{q}} = \left\{ \mathrm{rep}_{\frac{p}{q}}(n) \mid n \geq 0 \right\}.$$

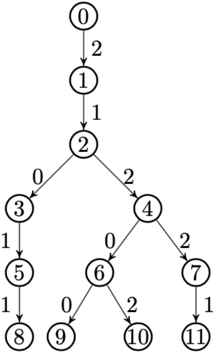
Example: In base $\frac{3}{2}$:

$$L_{\frac{3}{2}} = \{\varepsilon, 2, 21, 210, 212, 2101, 2120, 2122, \ldots\}$$

Properties:

- For all $u, v \in A_p^*$, $\mathrm{val}_{\frac{p}{q}}(uv) = \mathrm{val}_{\frac{p}{q}}(u) \left(\frac{p}{q}\right)^{|v|} + \mathrm{val}_{\frac{p}{q}}(v)$.
- $m < n$ if and only if $\mathrm{rep}_{\frac{p}{q}}(m) < \mathrm{rep}_{\frac{p}{q}}(n)$ for the radix order.
- $L_{\frac{p}{q}} \subseteq A_p^*$ is not regular.
- $L_{\frac{p}{q}} \subseteq A_p^*$ is prefix-closed.

Example: base $\frac{3}{2}$



Signature: $2, 1, 2, 1, 2, 1, \ldots$

(i-tree: if we add an edge of label 0 onto the root)

Labeling: $0, 2, 1, 0, 2, 1, 0, 2, 1, \ldots$

nth node: $\mathrm{rep}_{\frac{3}{2}}(n)$

(breadth-first traversal)

Edges: $n \xrightarrow{a \in A_3} m \iff m = \frac{3}{2} \cdot n + \frac{a}{2}$.

ANS given by periodic labeled signatures

A labeled signature is an infinite sequence $(w_n)_{n\geq 0}$ of finite words providing

- a signature $(|w_n|)_{n\geq 0}$ and
- a consistent labeling (made of the sequence of letters of $(w_n)_{n\geq 0}$)

of a tree.

The canonical breadth-first traversal of this tree produces an ANS.

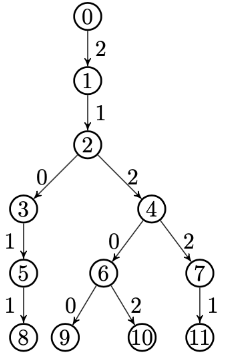
Example:

Labeled signature: $(02, 1)^\omega$

Signature: $\underbrace{|02|}_{=2}, \underbrace{|1|}_{=1}, \underbrace{|02|}_{=2}, \underbrace{|1|}_{=1}, \ldots$

Labeling: $0, 2, 1, 0, 2, 1, 0, 2, 1, \ldots$

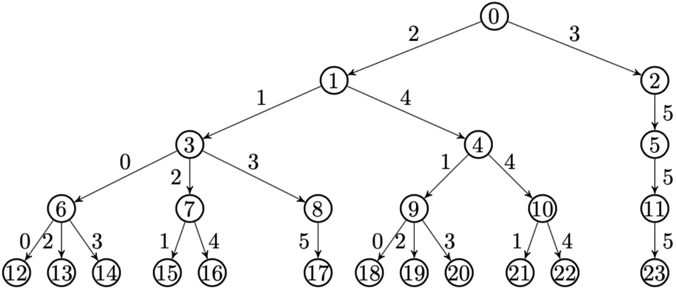
Base $\frac{3}{2}$



Example: Rational bases

p/q	corresp. labeled sign.
$3/2$	$(02, 1)^\omega$
$5/2$	$(024, 13)^\omega$
$7/3$	$(036, 25, 14)^\omega$
$11/4$	$(048, 159, 26(10), 37)^\omega$

Example: i-tree associated with the labeled signature $(023, 14, 5)^\omega$



Automatic sequences

Definition: Let $S = (L, A, <)$ be an ANS and let B be a finite alphabet. An infinite word $\mathbf{x} = x_0 x_1 x_2 \cdots \in B^{\mathbb{N}}$ is S-automatic if there exists a deterministic finite automaton with output (DFAO for short) $\mathcal{A} = (Q, q_0, A, \delta, \mu : Q \to B)$ such that

$$x_n = \mu(\delta(q_0, \mathrm{rep}_S(n))) \quad \forall\, n \geq 0.$$

We read most significant digits first (not a restriction).

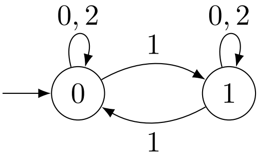
Remark: We talk about...

- k-automatic seq. in the base-k numeration system $(L_k, A_k, <)$
- $\frac{p}{q}$-automatic seq. in the base-$\frac{p}{q}$ numeration system $(L_{\frac{p}{q}}, A_p, <)$.

Toy example: the sum-of-digits in base $\frac{3}{2}$

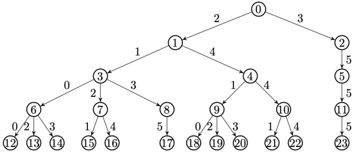
n	0	1	2	3	4	5	6	7
$\mathrm{rep}_{\frac{3}{2}}(n)$	ε	2	21	210	212	2101	2120	2122
$s(n)$	0	2	3	3	5	4	5	7
$t(n) = s(n) \bmod 2$	0	0	1	1	1	0	1	1

The sequence $\mathbf{t}$ is $\frac{3}{2}$-automatic as it is generated by the following DFAO when reading base-$\frac{3}{2}$ representations:



Another example

Periodic labeled signature $(023, 14, 5)^\omega$ producing the i-tree



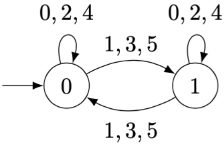
The first few words in the corresponding ANS S:

$\varepsilon, 2, 3, 21, 24, 35, 210, 212,$
$213, 241, 244, 355, \ldots$

The sum-of-digits in S modulo 2

n	0	1	2	3	4	5	6	7
$\text{rep}_S(n)$	ε	2	3	21	24	35	210	212
s.o.d	0	2	3	3	6	8	3	5
s.o.d mod2	0	0	1	1	0	0	1	1

is S-automatic since it is generated by the DFAO



Cobham's theorem

Theorem (Cobham, 1972)

A sequence is k-automatic if and only if it is the image under a coding of a fixed point of a k-uniform morphism.

Many generalizations exist.

Goal: Generalization to our context of S-automatic sequences for ANS built on tree languages with a purely periodic labeled signatures.

Needed: alternate fixed points and block substitutions.

Alternating morphisms

The Kolakoski–Oldenburger word is the unique word $\mathbf{k}$ over $\{1,2\}$ starting with 2 and satisfying $\Delta(\mathbf{k}) = \mathbf{k}$ where Δ is the run-length encoding map

$$\mathbf{k} = 2211212212211\cdots . \qquad \text{[OEIS, A000002]}$$

How to build it?
"Write what you read": each term of $\mathbf{k}$ generates a run of one or two future terms.

2	starting point
$\underline{2}2$	the first letter 2 generates a run of "22"
$22\underline{11}$	the second letter 2 generates a run of "11"
$2211\underline{2}$	the first letter 1 generates a run of "2"
$22112\underline{1}$	the second letter 1 generates a run of "1"
$221121\underline{22}$	and so on and so forth...

It is a well-known and challenging object of study in CoW.
<u>Conjecture</u>: The density of 1 in $\mathbf{k}$ is $\frac{1}{2}$. (Still open.)

Alternate fixed point

Alternative definition (Culik, Karhumäki, Lepistö, 1992): $\mathbf{k} = k_0k_1k_2\cdots$ can be obtained by periodically iterating two morphisms

$$\mathbf{k} = h_0(k_0)h_1(k_1)\cdots h_0(k_{2n})h_1(k_{2n+1})\cdots$$

where

$$h_0 : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 22 \end{cases} \quad \text{and} \quad h_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 11. \end{cases}$$

$\mathbf{k}$	2	2	1	1	2	1	2	2	1
(h_0, h_1)	h_0	h_1	h_0	h_1	h_0	h_1	h_0	h_1	h_0
$(h_0, h_1)(\mathbf{k}) = \mathbf{k}$	22	11	2	1	22	1	22	11	2

We say that $\mathbf{k}$ is an alternate fixed point of (h_0, h_1).

Definition

Let $r \geq 1$ be an integer and let $f_0, \ldots, f_{r-1}$ be morphisms over an alphabet A.
An word $\mathbf{w} = w_0w_1\cdots \in A^\omega$ is an alternate fixed point of $(f_0, \ldots, f_{r-1})$ if $\mathbf{w} = f_0(w_0)f_1(w_1)\cdots f_{r-1}(w_{r-1})f_0(w_r)\cdots f_{i \bmod r}(w_i)\cdots$.

Block substitution

Alternative definition (Dekking, 1980): Since

$$\mathbf{k} = \underbrace{h_0(2)h_1(2)}_{=g(22)}\ \underbrace{h_0(1)h_1(1)}_{=g(11)}\ \underbrace{h_0(2)h_1(1)}_{=g(21)}\ \underbrace{h_0(2)h_1(2)}_{=g(22)}\ \underbrace{h_0(1)h_1(2)}_{=g(12)}\cdots$$
$$= 2211\ \ 21\ \ 221\ \ 2211\ \ 211\cdots$$

$\mathbf{k}$ is also a fixed point of the 2-block substitution

$$g : \begin{cases} 11 \mapsto h_0(1)h_1(1) = 21 \\ 12 \mapsto h_0(1)h_1(2) = 211 \\ 21 \mapsto h_0(2)h_1(1) = 221 \\ 22 \mapsto h_0(2)h_1(2) = 2211. \end{cases}$$

Remark: Lengths of images under g are not all equal.

Definition

An r-block substitution $g : A^r \to A^*$ maps a word $w_0 \cdots w_{rn-1} \in A^*$ to

$$g(w_0 \cdots w_{r-1})g(w_r \cdots w_{2r-1}) \cdots g(w_{r(n-1)} \cdots w_{rn-1}).$$

If the length of the word is not a multiple of r, then the suffix of the word is ignored under the action of g.
An infinite word $\mathbf{w} = w_0 w_1 \cdots \in A^\omega$ is a fixed point of the r-block substitution $g : A^r \to A^*$ if

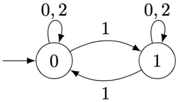
$$\mathbf{w} = g(w_0 \cdots w_{r-1})g(w_r \cdots w_{2r-1}) \cdots .$$

Proposition

If an infinite word over A is an alternate fixed point of $(f_0, \ldots, f_{r-1})$, then it is a fixed point of an r-block substitution.

Proof: For every of length-r word $a_0 \cdots a_{r-1}$, define the r-block substitution $g : A^r \to A^*$ by $g(a_0 \cdots a_{r-1}) = f_0(a_0) \cdots f_{r-1}(a_{r-1})$.

Goal: From a DFAO, build alternating morphisms.



ANS built on the purely periodic labeled signature $(w_0, w_1, \ldots, w_{r-1})^\omega$

Base $\frac{3}{2}$ $\quad (02, 1)^\omega$

Sum-of-digits mod 2

$\mathbf{t} = 0011101111101 \cdots$ $\frac{3}{2}$-automatic

$\mathbf{x}$ automatic produced by a DFA $\mathcal{A} = (Q, q_0, A, \delta)$

$$f_0 : \begin{cases} 0 \mapsto \delta(0,0)\delta(0,2) = 00 \\ 1 \mapsto \delta(1,0)\delta(1,2) = 11 \end{cases}$$

$$f_1 : \begin{cases} 0 \mapsto \delta(0,1) = 1 \\ 1 \mapsto \delta(1,1) = 0 \end{cases}$$

$f_i : Q \to Q^{|w_i|}$:
$q \mapsto \delta(q, w_{i,0}) \cdots \delta(q, w_{i,|w_i|-1})$
$\forall i \in \{0, \ldots, r-1\}$

$f_0(0) = 00$

$f_0(0) f_1(0) = 00\ 1$

$f_0(0) f_1(0) f_0(1) = 00\ 1\ 11$

$f_0(0) f_1(0) f_0(1) f_1(1) f_0(1) = 00\ 1\ 11\ 0\ 11\ 1$

$\mathbf{x}$ is the alternate fixed point of $(f_0, \ldots, f_{r-1})$ starting with q_0

$\mathbf{t}$ alternate fixed point of (f_0, f_1)

Proposition (Rigo, S., 2021)

Let $r \geq 1$ be an integer and let A be a finite alphabet of digits.
Let $w_0, \ldots, w_{r-1}$ be r non-empty words in $\mathsf{inc}(A^*)$.
Consider the language $L(\mathsf{s})$ of the i-tree generated by the purely periodic signature $\mathsf{s} = (w_0, w_1, \ldots, w_{r-1})^\omega$ and the corresponding ANS $S = (L(\mathsf{s}), A, <)$.
Let $\mathcal{A} = (Q, q_0, A, \delta)$ be a DFA.
For $i \in \{0, \ldots, r-1\}$, we define the r morphisms from Q^* to itself by

$$f_i : Q \to Q^{|w_i|}, q \mapsto \delta(q, w_{i,0}) \cdots \delta(q, w_{i,|w_i|-1}),$$

where $w_{i,j}$ denotes the jth letter of w_i.
The alternate fixed point $\mathbf{x} = x_0 x_1 \cdots$ of $(f_0, \ldots, f_{r-1})$ starting with q_0 is the sequence of states reached in $\mathcal{A}$ when reading the words of $L(\mathsf{s})$ in increasing radix order, i.e., for all $n \geq 0$, $x_n = \delta(q_0, \mathrm{rep}_S(n))$.

Alternate fixed points to automatic sequences

Goal: From alternating morphisms, build a DFAO.

$(f_0, \dots, f_{r-1})$ $f_i : A^* \to A^*$ ℓ_i-unif.
f_0 is prolongable on a

ANS S built on the purely periodic labeled signature $(w_0, \dots, w_{r-1})^\omega$

$w_0 = 0 \cdots (\ell_0 - 1)$
$w_1 = \ell_0(\ell_0 + 1) \cdots (\ell_0 + \ell_1 - 1)$
$\vdots$
$w_{r-1} = \left(\sum_{j<r-1} \ell_j\right) \cdots \left(\sum_{j<r} \ell_j - 1\right)$

$\mathcal{A} = (A, a, B, \delta)$ with

$B = \{0, \dots, \sum_{j<r} \ell_j - 1\}$
$\delta(b, i) = [f_{j_i}(b)]_{t_i}$ for $b \in A, i \in B$ such that
$i = \sum_{k \le j-1} \ell_k + t$ with $j \ge 0, 0 \le t < \ell_j$

The alternate fixed point of $(f_0, \dots, f_{r-1})$ starting with a is S-automatic (produced by $\mathcal{A}$)

$$f_0 : \begin{cases} 0 \mapsto 00 \\ 1 \mapsto 11 \end{cases} \quad f_1 : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$$

ANS built on $(\underbrace{01}_{\ell_0=2}, \underbrace{2}_{\ell_1=1})^\omega$

$\mathcal{A} = (\{0,1\}, 0, \{0,1,2\}, \delta)$

Write i in $[0,2)$, $[2,3)$

$0 = 0 + 0 \quad j = 0$

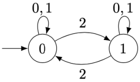
$\delta(0, 0) = [f_0(0)]_0 = 0 \quad \delta(1, 0) = [f_0(1)]_0 = 1$

$1 = 0 + 1 \quad j = 0$

$\delta(0, 1) = [f_0(0)]_1 = 0 \quad \delta(1, 1) = [f_0(1)]_1 = 1$

$2 = 2 + 0 \quad j = 1$

$\delta(0, 2) = [f_1(0)]_0 = 1 \quad \delta(1, 2) = [f_1(1)]_0 = 0$



The alternate fixed point of (f_0, f_1) is automatic. Up to a coding of the ANS, it is equal to $\mathbf{t}$ ($\frac{3}{2}$-automatic).

Proposition (Rigo, S., 2021)

Let $r \geq 1$ be an integer and let A be a finite alphabet.
For all $i \in \{0, \ldots, r-1\}$, let $f_i : A^* \to A^*$ be a ℓ_i-uniform morphism such that f_0 is prolongable on some $a \in A$.
Consider the language $L(\mathsf{s})$ of the i-tree generated by the purely periodic labeled signature

$$\mathsf{s} = \left(0 \cdots (\ell_0 - 1), \ell_0(\ell_0+1) \cdots (\ell_0 + \ell_1 - 1), \ldots, \left(\sum_{j<r-1} \ell_j\right) \cdots \left(\sum_{j<r} \ell_j - 1\right)\right)^\omega$$

and the corresponding ANS $S = (L(\mathsf{s}), B, <)$.
Let $\mathcal{A} = (A, a, B, \delta)$ be the DFA where $B = \{0, \ldots, \sum_{j<r} \ell_j - 1\}$ and its transition function $\delta : A \times B \to A$ is defined as follows: for all $i \in B$, $\exists! j_i \geq 0$ and $\exists! t_i \geq 0$ such that $i = \sum_{k \leq j_i - 1} \ell_k + t_i$ with $t_i < \ell_{j_i}$, so we set $\delta(b, i) = [f_{j_i}(b)]_{t_i}$ $\forall b \in A$.
Then the alternate fixed point $\mathbf{x} = x_0 x_1 \cdots$ of $(f_0, \ldots, f_{r-1})$ starting with a is the sequence of the states reached in $\mathcal{A}$ when reading the words of $L(\mathsf{s})$ by increasing radix order, i.e., for all $n \geq 0$, $x_n = \delta(a, \mathrm{rep}_S(n))$.

Theorem (Rigo, S., 2021)

Let A, B be two finite alphabets. An infinite word over B is the image under a coding $g : A \to B$ of an alternate fixed point of uniform morphisms (not necessarily of the same length) over A if and only if it is S-automatic for an ANS built on a tree language with a purely periodic labeled signature.

Proof: It follows from the previous two propositions.

Particular case in rational bases

Corollary (Rigo, S., 2021)

If a sequence is $\frac{p}{q}$-automatic, then it is the image under a coding of a fixed point of a q-block substitution whose images all have length p.

Example: The sum-of-digits mod 2 $\mathbf{t} = 0011101111101\cdots$ is $\frac{3}{2}$-automatic.

$$f_0 : \begin{cases} 0 \mapsto 00 \\ 1 \mapsto 11 \end{cases} \qquad f_1 : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$$

Then $\mathbf{t}$ is also a fixed point of the 2-block substitution

$$g : \begin{cases} 00 \mapsto f_0(0)f_1(0) = 001 \\ 01 \mapsto f_0(0)f_1(1) = 000 \\ 10 \mapsto f_0(1)f_1(0) = 111 \\ 11 \mapsto f_0(1)f_1(1) = 110. \end{cases}$$

Observe that images under g all have length 3.

Non-morphic sequences

Theorem (Cobham, 1972) rewritten

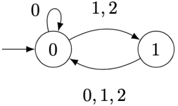
An automatic sequence in an integer base is a morphic word, i.e., the image, under a coding, of a fixed point of a prolongable morphism.

Consider the morphisms $g_0 : 0 \mapsto 01, 1 \mapsto 00$ and $g_1 : 0 \mapsto 1, 1 \mapsto 0$ yielding the 2-block substitution

$$h_2 : 00 \mapsto 011,\ 01 \mapsto 010,\ 10 \mapsto 001,\ 11 \mapsto 000$$

producing the word $\mathbf{F}_2 = 010\,011\,000\,011 \cdots$. It is $\frac{3}{2}$-automatic as it is generated by the following DFAO (built thanks to our proposition):

$$\begin{aligned}
\delta(0,0) &= [g_0(0)]_0 = 0 & \delta(1,0) &= [g_0(1)]_0 = 0\\
\delta(0,1) &= [g_0(0)]_1 = 1 & \delta(1,1) &= [g_0(1)]_1 = 0\\
\delta(0,2) &= [g_1(0)]_0 = 1 & \delta(1,2) &= [g_1(1)]_0 = 0
\end{aligned}$$



Property (Lepistö, 1993)

The word $\mathbf{F}_2$ is neither purely morphic nor morphic.

Decorated trees (not only for Christmas)

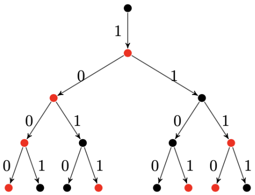
ANS $S = (L, A, <)$ where $L = \{w_0 < w_1 < \cdots\}$ is prefix-closed producing a tree $T(L)$.
We now add an extra information on every node.

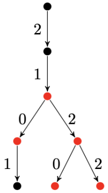
Definition: Let $\mathbf{x} = x_0 x_1 \cdots \in B^{\mathbb{N}}$ where B is a finite alphabet.
A decoration of $T(L)$ by $\mathbf{x}$ is the map $L \to B : w_n \mapsto x_n$.

Example: (0 means black, 1 means red)

Base 2
Decoration:
Thue-Morse
$0110100110010110\cdots$

Base $\frac{3}{2}$
Decoration:
Sum-of-digits modulo 2
$0011101111011011\cdots$



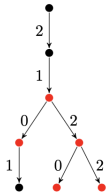


Factors of trees (subtrees)

Definition: Let T be a labeled tree.
The domain $\mathrm{dom}(T)$ of T is the set of labels of paths from the root to its nodes.

Example:

The domain of the tree on the right is $\{\varepsilon, 21, 210, 212, 2101, 2120, 2122\}$.



Definition: Let $w \in L$ and let $h \geq 0$.
We let $T[w, h]$ denote the factor of height h rooted at w of $T(L)$.
The prefix of height h of $T(L)$ is the factor $T[\varepsilon, h]$.

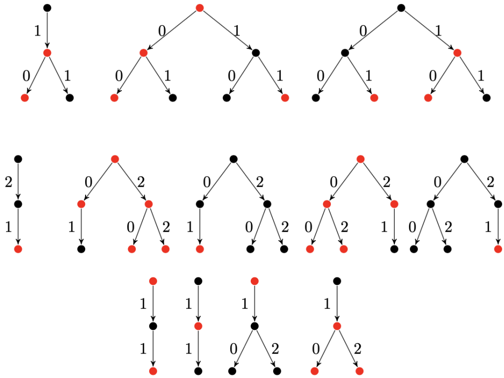
Example: The previous tree is a factor of height 4 of $T(L_{\frac{3}{2}})$. It is its prefix of height 4.

Definition: Two factors $T[w,h]$ and $T[w',h]$ of the same height are equal if they have the same domain and the same decorations.
We let $F_h = \{T[w,h] \mid w \in L\}$ denote the set of factors of height h occurring in $T(L)$.

F_2:

Base 2
Decoration:
Thue-Morse
$0110100110010110\cdots$

Base $\frac{3}{2}$
Decoration:
Sum-of-digits modulo 2
$0011101111011011\cdots$

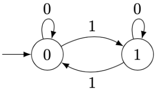


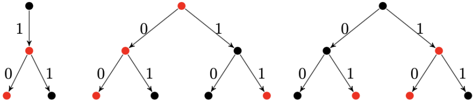
Automatic decorations of trees

Theorem (Rigo, S., 2021)

A sequence $\mathbf{x}$ is k-automatic if and only if, in the labeled tree $T(L_k)$ decorated by $\mathbf{x}$, there exists a height $h \geq 0$ such that $\#F_h = \#F_{h+1}$.

Example: Base 2, decoration being Thue-Morse $0110100110010110\cdots$
Each factor $T[w,h]$ (not the prefix) is determined by $\delta(q_0,w)$: it is a full binary tree, and the decorations are determined by $\tau(\delta(q_0,wu))$ with $u \in A_2^{\leq h}$. Therefore $\#F_h \leq \#Q + 1$.





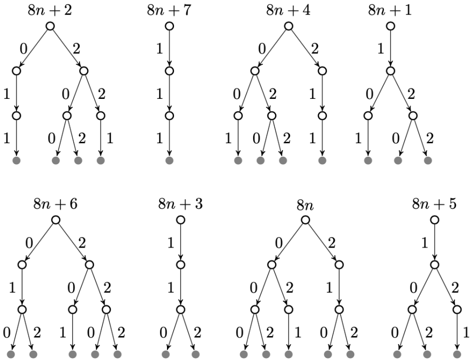
Theorem (Rigo, S., 2021)

Let $S = (L, A, <)$ be an ANS built on a prefix-closed regular language L. A sequence $\mathbf{x}$ is S-automatic if and only if, in the labeled tree $T(L)$ decorated by $\mathbf{x}$, there exists a height $h \geq 0$ such that $\#F_h = \#F_{h+1}$.

The rational bases case

Several extensions: in base $\frac{p}{q}$, except for the height-h prefix, each factor of height h is extended in exactly q ways to a factor of height $h+1$. To the first (leftmost) leaf of a factor of height h are attached children corresponding to one of the q words of the periodic signature.

Example: Base $\frac{3}{2}$ periodic labeled signature $(02,1)^\omega$



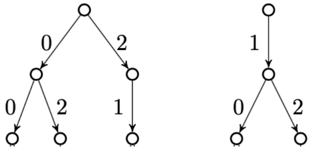
Factors appearing infinitely often

Definition: Let T be a labeled decorated tree and let $h \geq 0$.
We let $F_h^\infty \subseteq F_h$ denote the set of factors of height h occurring infinitely often in T.

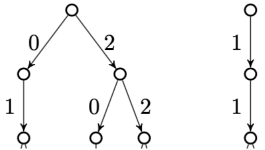
For any suitable letter a in the signature of T, we let $F_{h,a}^\infty \subseteq F_h^\infty$ denote the set of factors of height h occurring infinitely often in T such that the label of the edge between the first node on level $h-1$ and its first child is a. (Otherwise stated, the first word of length h in the domain of the factor ends with a.)

Example: Base $\frac{3}{2}$

$F_{2,0}^\infty$:

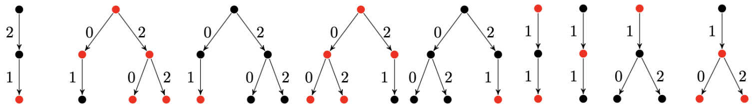


$F_{2,1}^\infty$:



Example: Base $\frac{3}{2}$

Assumptions: Factors of length 1 in $T(L_{\frac{3}{2}})$ can be extended as follows:

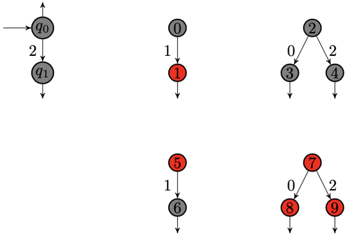


Assumptions:

- The first tree gives the prefix and occurs only once.
- The last eight trees of height 2 occur infinitely often.

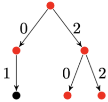
Build an NFA:

- States: $\{T[\varepsilon, 1]\} \cup F_1^{\infty}$
- Initial state: root of $T[\varepsilon, 1]$
- Final states: nodes of $T[\varepsilon, 1]$ and leaves in F_1^{∞}
- Transitions...



Example for transitions on state 7.
The corresponding tree has two extensions:

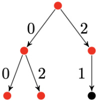
Extension 1



The tree hanging to the child 0 (resp. 2) of the root corresponds to state 5 (resp. 7).

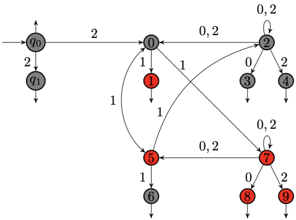
Transitions: $7 \xrightarrow{0} 5$ and $7 \xrightarrow{2} 7$

Extension 2



The tree hanging to the child 0 (resp. 2) of the root corresponds to state 7 (resp. 5).

Transitions: $7 \xrightarrow{0} 7$ and $7 \xrightarrow{2} 5$



Runs in the NFA for $w = 210 \in L_{\frac{3}{2}}$:

$q_0 \xrightarrow{2} q_1$ ✗

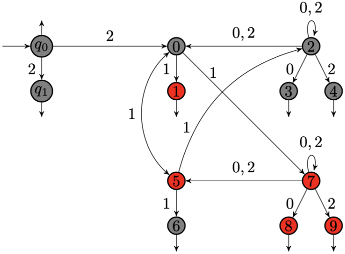
$q_0 \xrightarrow{2} 0 \xrightarrow{1} 1$ ✗

$q_0 \xrightarrow{2} 0 \xrightarrow{1} 5$ ✗

$q_0 \xrightarrow{2} 0 \xrightarrow{1} 7 \xrightarrow{0} 5$ ✗

$q_0 \xrightarrow{2} 0 \xrightarrow{1} 7 \xrightarrow{0} 7$ ✗

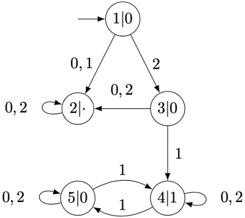
$q_0 \xrightarrow{2} 0 \xrightarrow{1} 7 \xrightarrow{0} 8$ ✓



Determinize the NFA (using the usual subset construction).

No output is set for state 2.

This DFA produces the sum-of-digits modulo 2 in base $\frac{3}{2}$ ($\frac{3}{2}$-automatic).



Theorem (Rigo, S., 2021)

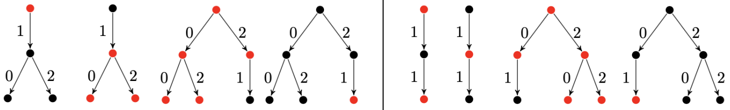
Let the tree $T(L_{\frac{p}{q}})$ be decorated by a sequence $\mathbf{x}$.
Suppose there exists some $h \geq 0$ such that $\#F^{\infty}_{h+1,a} \leq \#F^{\infty}_{h}$ for all $0 \leq j \leq q-1$ and all suitable letters $a \in A_p$.
Then $\mathbf{x}$ is $\frac{p}{q}$-automatic.

Automatic sequences to bounding $\#F^{\infty}_{h,a}$

Example: Base $\frac{3}{2}$

Decoration: the sum-of-digits mod 2 $\mathbf{t} = 001110111110\cdots$, $\frac{3}{2}$-automatic

$F^{\infty}_{2,0}$ $F^{\infty}_{2,1}$



$\#F^{\infty}_{2,0} \leq \#F^{\infty}_{1}$ $\#F^{\infty}_{2,1} \leq \#F^{\infty}_{1}$

Theorem (Rigo, S., 2021)

Let $\mathbf{x}$ be a $\frac{p}{q}$-automatic sequence generated by a DFAO $\mathcal{A} = (Q, q_0, A_p, \delta, \tau : A_p \to B)$ with the following property:

$\exists h \geq 0$ s.t. $\forall q \neq q' \in Q$ and $\forall w \in L_{\frac{p}{q}}$, $\exists u \in w^{-1}L_{\frac{p}{q}}$ with $|u| \leq h$ s. t. $\tau(\delta(q, u)) \neq \tau(\delta(q', u))$.

Then in the tree $T(L_{\frac{p}{q}})$ decorated by $\mathbf{x}$, we have $\#F^{\infty}_{h+1,a} \leq \#F^{\infty}_{h}$ for all $0 \leq j \leq q-1$ and all suitable letters $a \in A_p$.

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