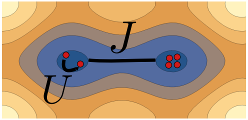
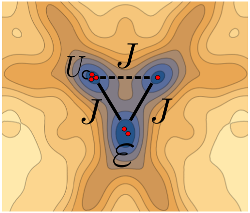
NOON states in a periodically perturbed two-site optical lattice via resonance-assisted tunneling and investigation of the three-site system



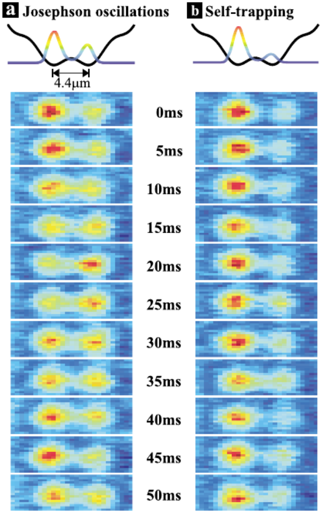


Dresden
13 October 2019 - 14 December 2019

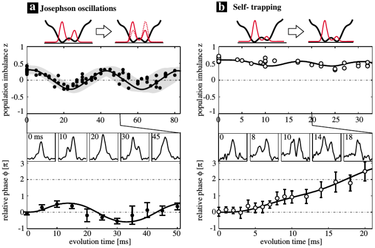
Guillaume Vanhaele

Two-site optical lattice : experimental angle

M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, Phys. Rev. Lett. **95**, 010402 (2005).



$$\psi_{1,2} = \sqrt{N_{1,2}}\, e^{i\theta_{1,2}}$$
$$z = (N_1 - N_2)/N_p$$
$$\phi = \theta_1 - \theta_2$$



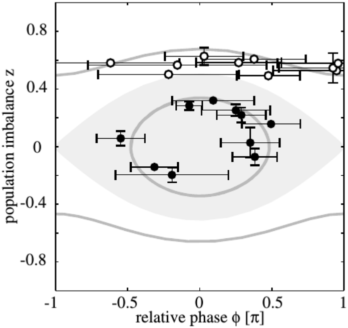
$N_p \simeq 1000$

State of the art : experimental angle

The unperturbed two-site Bose-Hubbard Hamiltonian in the mean-field approximation ($N_p \gg 1$)

$$H_0(\phi, z, t) = \frac{\Lambda}{2}\ z^2 - \sqrt{1 - z^2}\ \cos\phi$$

$\Lambda = N_p U/(2J)$



$$\psi_{1,2} = \sqrt{N_{1,2}}\, e^{i\theta_{1,2}}$$
$$z = (N_1 - N_2)/N_p$$
$$\phi = \theta_1 - \theta_2$$

$$\begin{cases} \dot{z} = -\sqrt{1 - z^2}\sin\phi \\ \dot{\phi} = \Lambda z + \dfrac{z}{\sqrt{1 - z^2}}\cos\phi \end{cases}$$

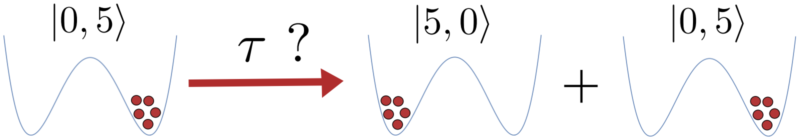
NOON states

$$|\mathrm{NOON}\rangle = \cos\left(\frac{\Omega}{2}t\right)|0, N_p\rangle + i\sin\left(\frac{\Omega}{2}t\right)|N_p, 0\rangle$$

with $\Omega = \dfrac{\epsilon^- - \epsilon^+}{\hbar} = \dfrac{\Delta\epsilon}{\hbar}$.

$$\epsilon^- \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}(|0, N_p\rangle - |N_p, 0\rangle)$$

$$\epsilon^+ \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}(|0, N_p\rangle + |N_p, 0\rangle)$$

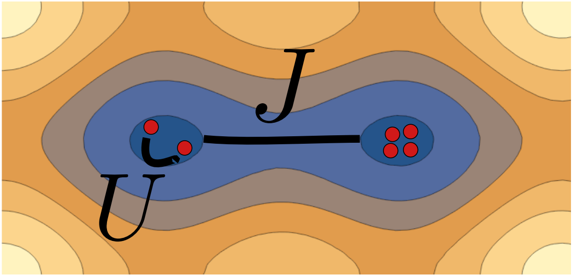


$$\tau = \frac{\pi\hbar}{2|\Delta\epsilon|}$$

Bose-Hubbard model for a two-site optical lattice

$$\hat{H}_0 = -J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{2}(\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2)$$

with J the hopping parameter and U the on-site interaction.



The number of quantum eigenstates (dimension of $\mathcal{H}$) is

$$N_Q = N_p + 1$$

The system is integrable : $[\hat{N}_p, \hat{H}_0] = 0$ and $\hat{H}_0$ is time-independent.

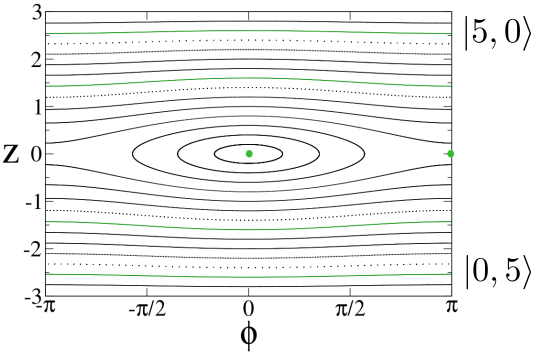
Bose-Hubbard model for a two-site optical lattice : mean-field approximation

$$H_0(\phi, z, t) = \frac{\lambda}{2}\ z^2 - \sqrt{\left(\frac{N_Q}{2}\right)^2 - z^2}\ \cos\phi$$

$\lambda = U/J\,;\ \ N_Q = N_p + 1$ (number of quantum eigenstates)

$z = (N_1 - N_2)/2$ and $\phi = \theta_1 - \theta_2$.

Typical phase space for $N_Q U/J \gg 1$



$N_p = 5$, $U/J = 20$.

Evaluation of τ

The Hamiltonian of one particle trapped in an 1D optical lattice (pendulum Hamiltonian),

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0\frac{\hbar^2 k^2}{m}(1-\cos(kx)).$$

The ground state on site l, $\varphi_l(x) = \varphi(x - l\frac{2\pi}{k})$ is approximated by a Gaussian,

$$\varphi(x) = \frac{1}{\sqrt{\sqrt{\pi}\sigma}}\exp\left(-\frac{x^2}{2\sigma^2}\right).$$

The resolution of the following equation in relation to V_0 enables to give a unit to the energy.

$$\frac{U}{J} = \sqrt{2e\sqrt{V_0}}ka_s\exp\left(2\sqrt{2V_0}\int_{\arccos\eta}^{\pi}\sqrt{\eta-\cos\phi}\,d\phi\right)$$

The time needed to obtain a coherent superposition of the two quasimodes $|5,0\rangle$ and $|0,5\rangle$ reads

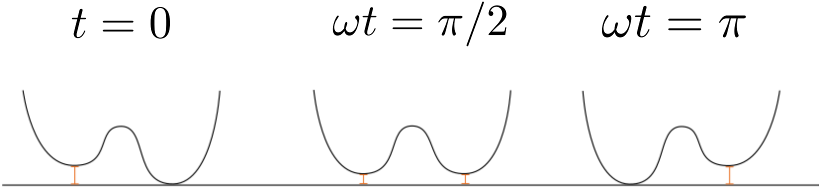
$$\tau_{(5,0)} = \frac{\pi\hbar}{2|\Delta\epsilon_{(5,0)}|} \simeq 2660\text{s}.$$

Life time of a condensate in an optical lattice : a few minutes.

Wave-length of the lasers : $\lambda = 1064$ nm.
s-wave scattering length of Rubidium : $a_s = 5.313$ nm.
Mass of the Rubidium : $m_{\text{Rb}} = 1.443 \times 10^{-25}$ kg.
Recoil energy : $E_R = \hbar^2 k^2/(8m_{Rb}) = 1.34 \times 10^{-30}$ J
with $k = 2\pi/d = 4\pi/\lambda = 2k_\varphi$.

Perturbed Bose-Hubbard model for a two-site optical lattice

$$\hat{H} = -J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{2}(\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2)$$
$$+ \delta \cos(\omega t)(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2).$$



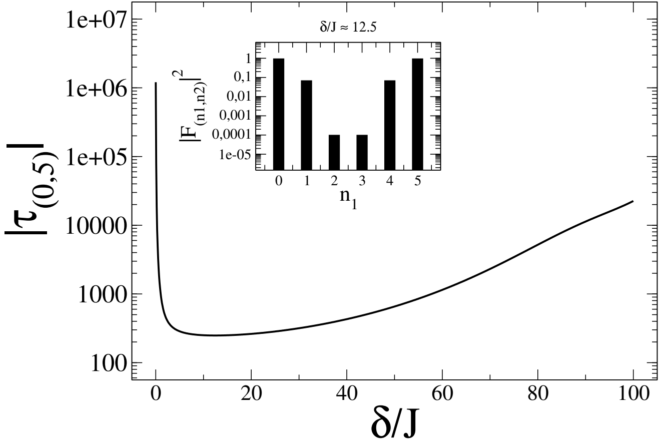
Mean-field approximation :

$$H(\phi, z, t) = \frac{\lambda}{2}\, z^2 - \sqrt{\left(\frac{N_Q}{2}\right)^2 - z^2}\, \cos\phi + \frac{\delta}{J} \cos\left(\frac{\omega}{2J} t\right)\, z$$

with $\lambda = U/J$.

Increase of the tunneling rate by means of resonance-assisted tunneling

$N_p = 5$, $U/J = 20$ and $\omega/J = 80$.

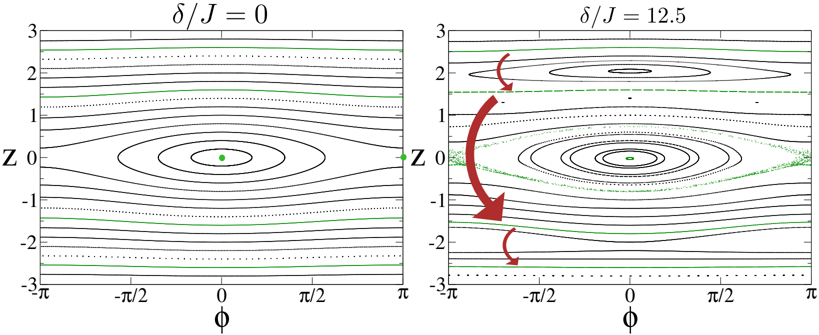


$\tau_{(5,0)}(\delta/J = 0) \simeq 2660\text{s}$

$\tau_{(5,0)}(\delta/J = 12.5) \simeq 0.55\text{s}.$

Increase of the tunneling rate by means of resonance-assisted tunneling

$N_p = 5$, $U/J = 20$ and $\omega/J = 80$.



The r:s=1:1 resonance is symmetrically located ($z_{r:s} = 2$) between the quasimode $|5,0\rangle$ ($z = 2.5$) and the quasimode $|4,1\rangle$ ($z = 1.5$).

$$\begin{cases} \dot{z} = -\sqrt{\left(\frac{N_Q}{2}\right)^2 - z^2}\,\sin\phi \\ \dot{\phi} = \lambda z + \dfrac{z}{\sqrt{\left(\frac{N_Q}{2}\right)^2 - z^2}}\cos\phi + \dfrac{\delta}{J}\cos\left(\dfrac{\omega}{2J}t\right) \end{cases}$$

General method to produce a Schrödinger cat state

KAM theorem :
$\delta \neq 0$ and $\omega \neq 0 \Longrightarrow$ the tori with a rational winding number, $\alpha = \frac{\omega/2J}{\omega_{r:s}} = \frac{r}{s}$, are destroyed. The frequency of the torus located in $z_{r:s}$ reads

$$\omega_{r:s} = \frac{dH_0(z)}{dz}(z = z_{r:s}) \simeq \frac{U}{J} z_{r:s}.$$

The external frequency that must be applied in order to obtain a r:s resonance at $z_{r:s}$ reads

$$\omega = \frac{r}{s}\ 2U z_{r:s}.$$

Maximal connection between $|N_p, 0\rangle$ and $|N_p - r, r\rangle$ when the r:s resonance is symmetrically located between them.

Example :

$\|5, 0\rangle$	$z = 2.5$
1 :1	$z_{r:s} = 2$
$\|4, 1\rangle$	$z = 1.5$

$z = (N_1 - N_2)/2$ is the position of the tori in the phase space.

NOON states and two-site Bose Hubbard system : summary

- The tunneling rate can be increased in the quantum regime.
- A periodic shaking of the two-site optical lattice might produce experimentally a giant entangled state.

$$\tau_{(5,0)}(\delta/J = 0) \simeq 2660\text{s}$$
$$\tau_{(5,0)}(\delta/J = 12.5) \simeq 0.55\text{s}$$

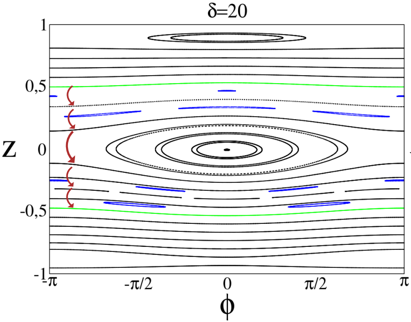
- The resonance-assisted tunneling is a theory suitable to describe the increase of the tunneling rate via the phase space and to choose the appropriate set of parameters.

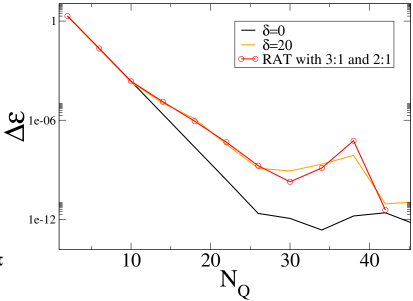
$$\omega = \frac{r}{s}\; 2U z_{r:s}$$

Multi resonance process in the two-mode Bose Hubbard model

In the framework of the resonance-assisted tunneling, the pendulum approximation of a $r{:}s$ nonlinear resonance reads

$$H_{\text{res}}^{(r:s)}(I,\Theta) \simeq \frac{(I - I_{r:s})}{2m_{r:s}} + 2V_{r:s}\cos(r\Theta + \phi_1).$$

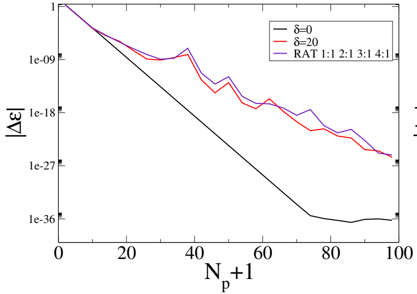


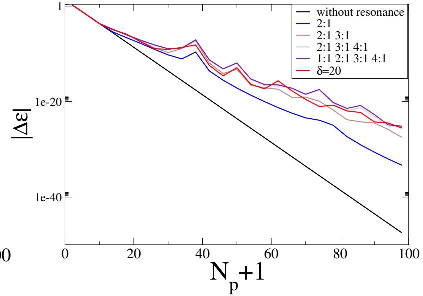


$N_Q U/J = 140$, $\omega/J = 125$
RAT= resonance-assisted tunneling

Multi resonance process in the two-mode Bose Hubbard model

The two-site system reaches the semiclassical regime when $N_Q (= N_p + 1)$ is increased.



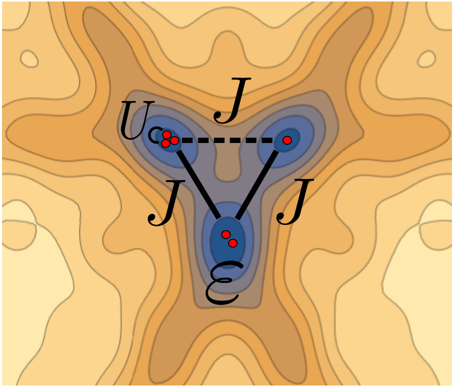


$N_Q U/J = 140$, $\omega = 125$
RAT= resonance-assisted tunneling

N. Mertig, J. Kullig, C. Löbner, A. Bäcker, and R. Ketzmerick, Phys. Rev. E **94**, 062220 (2016).
F. Fritzsch, A. Bäcker, R. Ketzmerick, and N. Mertig, Phys. Rev. E **95**, 020202(R) (2017).

The perturbed three-site Bose Hubbard Hamiltonian

$$\hat{H}(\hat{a}_l^\dagger, \hat{a}_l) = -J\sum_{l=1}^{3}(\hat{a}_l^\dagger \hat{a}_{l+1} + \hat{a}_{l+1}^\dagger \hat{a}_l) + \frac{U}{2}\sum_{l=1}^{3}\hat{a}_l^\dagger \hat{a}_l^\dagger \hat{a}_l \hat{a}_l$$
$$+\,\delta\cos(\omega t)(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) + \varepsilon \hat{a}_3^\dagger \hat{a}_3.$$



$$N_Q = (N_p+1)^2 - \sum_{n_1=1}^{N_p} n_1$$

Polar representation

The mean-field approximation enables to write the ladder operators as a complex number, $< \hat{a}_l > \sim \psi_l$, $< \hat{a}_l^\dagger > \sim \psi_l^*$ and $< \hat{a}_l^\dagger \hat{a}_l > = |\psi_l|^2 - 1/2$ with

$$\psi_l = \sqrt{I_l}\, \mathrm{e}^{i\theta_l} \,.$$

$I_l = N_l + 1/2.$

$$H(\psi_l, \psi_l^*) = -J \sum_{l=1}^{3} (\psi_l^* \psi_{l+1} + \psi_{l+1}^* \psi_l) + \frac{U}{2} \sum_{l=1}^{3} |\psi_l|^4 + \delta \cos(\omega t)(|\psi_1|^2 - |\psi_2|^2) + \varepsilon |\psi_3|^2$$

The Gross-Pitaevskii equations read

$$\begin{cases} i\hbar \dfrac{d\psi_1}{dt} = -J(\psi_2 + \psi_3) + U|\psi_1|^2\psi_1 + \delta \cos(\omega t)\psi_1 \\ i\hbar \dfrac{d\psi_2}{dt} = -J(\psi_3 + \psi_1) + U|\psi_2|^2\psi_2 - \delta \cos(\omega t)\psi_2 \\ i\hbar \dfrac{d\psi_3}{dt} = -J(\psi_1 + \psi_2) + U|\psi_3|^2\psi_3 + \varepsilon\psi_3 \end{cases}$$

The constant of motion,

$$|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 = N_p + 3/2.$$

Action-angle representation

The action-angles variables with one constant of motion read

$$\begin{cases} \alpha_1 = \theta_1 - \theta_3 \\ \alpha_2 = \theta_2 - \theta_3 \\ \alpha_3 = \theta_3 \\ A_1 = I_1 \\ A_2 = I_2 \\ A_3 = I_1 + I_2 + I_3 = N_p + 3/2. \end{cases}$$

The Hamilton equations,

$$\begin{cases} \dot{\alpha}_1 = \partial H/\partial A_1 \\ \dot{\alpha}_2 = \partial H/\partial A_2 \\ \dot{\alpha}_3 = \partial H/\partial A_3 \\ \dot{A}_1 = -\partial H/\partial \alpha_1 \\ \dot{A}_2 = -\partial H/\partial \alpha_2 \\ \dot{A}_3 = -\partial H/\partial \alpha_3 = 0. \end{cases}$$

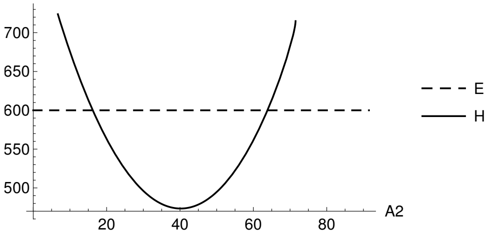
N_l is the number of particle on the site l and N_p is the total number of particles.
$A_1 = N_1 + 1/2$.
$A_2 = N_2 + 1/2$.
$A_3 = N_p + 3/2$.

The Poincaré sections can be built with (α_1, A_1) if $\delta = 0$.

The surface of sections

The perturbation is turned off ($\delta = 0$). A_2 is evaluated by resolving the following equation.

$$H(\alpha_1, \alpha_2 = \beta + 2\pi n, A_1, A_2, A_3 = N_p + 3/2) = E$$



The two criteria to verify in order to draw a point on the surface of section,

$$\alpha_2 = \beta + 2\pi n$$
$$\dot{\alpha}_2 = \frac{\partial H}{\partial A_2} \lesseqgtr 0.$$

The Poincaré sections reads

$$\mathcal{P} = \{(\alpha_1, A_1) \ | \ A_1 + A_2 \leqslant A_3,\ A_1 \geqslant 0,\ A_2 \geqslant 0,\ \alpha_2 = \beta + 2\pi n,$$
$$\dot{\alpha}_2 = \frac{\partial H}{\partial A_2} \lesseqgtr 0,\ H(\alpha_1, \alpha_2 = \beta + 2\pi n, A_1, A_2, A_3 = N_p + 3/2) = E\}.$$

Quadrature representation

The quadrature variables read

$$Q_l = \sqrt{2I_l}\sin\theta_l$$
$$P_l = \sqrt{2I_l}\cos\theta_l.$$

The Hamiltonian in the quadrature representation,

$$H(Q_l, P_l) = -J\sum_l^3 (Q_l Q_{l+1} + P_l P_{l+1}) + \frac{U}{8}\sum_l^3 \left((Q_l^2 + P_l^2)^2 - 8N_l - 1\right)$$
$$+ \frac{\delta}{2}\cos(\omega t)(Q_1^2 + P_1^2 - Q_2^2 - P_2^2) + \frac{1}{2}\varepsilon(Q_3^2 + P_3^2 - 1).$$

The Hamiltonian equations are ($\delta_1 = \delta\cos(\omega t)$, $\delta_2 = -\delta\cos(\omega t)$, $\delta_3 = \varepsilon$ and $l = 1, 2, 3$)

$$\begin{cases} \dot{Q}_l = -J(P_{l-1} + P_{l+1}) + \frac{U}{2}(P_l^3 + P_l Q_l^2) + \delta_l P_l \\ \dot{P}_l = J(Q_{l-1} + Q_{l+1}) - \frac{U}{2}(Q_l^3 + Q_l P_l^2) - \delta_l Q_l. \end{cases}$$

The constant of motion,

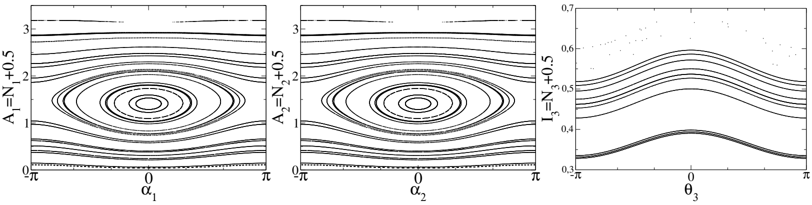
$$\sum_{l=1}^3 \frac{(Q_l^2 + P_l^2)}{2} = N_p + 3/2.$$

Quadrature representation : the surface of sections

It is more stable to verify the sos condition in the quadrature representation.

$$\alpha_2 = \beta + 2\pi n$$

$$\Longleftrightarrow \begin{cases} \dfrac{P_2}{\sqrt{P_2^2 + Q_2^2}} = \dfrac{P_3 \cos\beta - Q_3 \sin\beta}{\sqrt{P_3^2 + Q_3^2}} \\ \dfrac{Q_2}{\sqrt{P_2^2 + Q_2^2}} = \dfrac{Q_3 \cos\beta + P_3 \sin\beta}{\sqrt{P_3^2 + Q_3^2}} \end{cases}$$



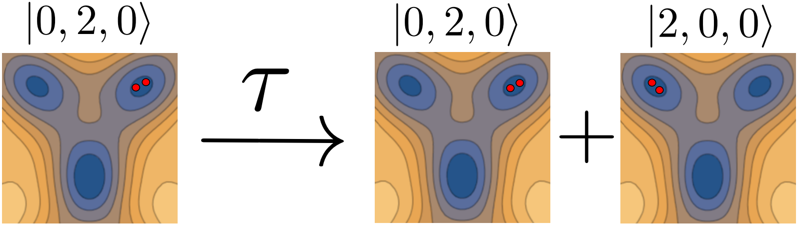
NOON states in a three-site system

$$|\text{NOON}\rangle = \cos(\Omega t)\,|0, N_p - k, k\rangle + i\sin(\Omega t)\,|N_p - k, 0, k\rangle$$

with $\Omega = \dfrac{\epsilon^- - \epsilon^+}{2\hbar} = \dfrac{\Delta\epsilon}{2\hbar}$.

$$\epsilon^- \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}\left(|0, N_p - k, k\rangle - |N_p - k, 0, k\rangle\right)$$

$$\epsilon^+ \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}\left(|0, N_p - k, k\rangle + |N_p - k, 0, k\rangle\right)$$

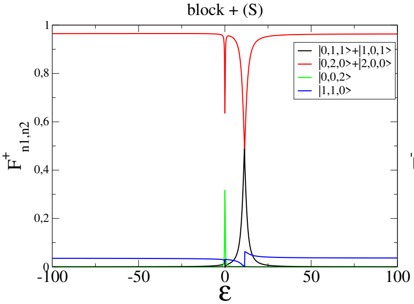


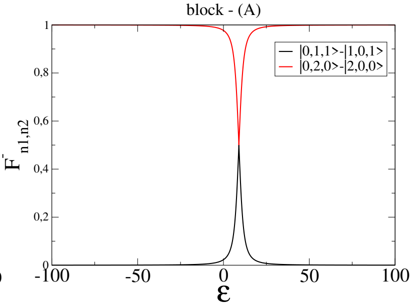
$$\tau = \frac{\pi\hbar}{|\Delta\epsilon|} \qquad\qquad N_Q = (N_p + 1)^2 - \sum_{n_1=1}^{N_p} n_1$$

To choose ε ($\delta = 0$)

We look at the purity of the chosen eigenstate (N_p even).

$$|\psi^+\rangle = \sum_{n_1=0}^{N_p/2-1} \sum_{n_2=n_1+1}^{N_p-n_1} F^+_{n_1,n_2} (|n_1, n_2, N_p - n_1 - n_2\rangle + |n_2, n_1, N_p - n_1 - n_2\rangle) \frac{1}{\sqrt{2}}$$
$$+ \sum_{n_1=0}^{N_p/2} F^+_{n_1} |n_1, n_1, N_p - 2n_1\rangle$$

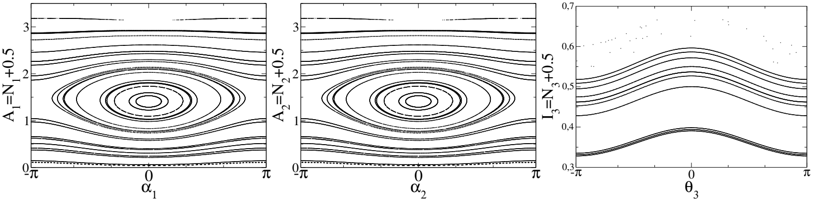




$N_p = 2,\ U = 10,\ J = 1.$
Choosen state : $|\psi^\pm\rangle \simeq \frac{1}{\sqrt{2}}(|0,2,0\rangle \pm |2,0,0\rangle)$

Surface of sections

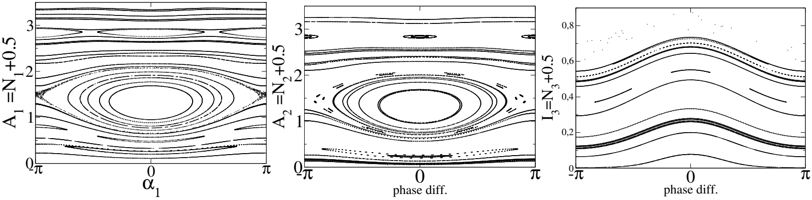
$N_p = 2$, $U = 10$, $J = 1$, $\varepsilon = 100$, $E = 10$ and $\beta = 0$.



The third site is disconnected. It is almost a two-site system.

Surface of sections

$N_p = 2$, $U = 10$, $J = 1$, $\varepsilon = 50$, $E = 10$ and $\beta = 0$.



Tuning ε, δ and ω in order to place nonlinear resonances.
3D phase space slice, arnorld web etc.

S. Lange, M. Richter, F. Onken, A. Bäcker, and R. Ketzmerick, Chaos **24**, 024409 (2014).
Martin Richter, Steffen Lange, Arnd Bäcker, and Roland Ketzmerick, Phys. Rev. E **89**, 022902 (2014).
Markus Firmbach, Felix Fritzsch, Roland Ketzmerick, and Arnd Bäcker, Phys. Rev. E **99**, 042213 (2019).

Increase of the tunneling rate by means of resonance-assisted tunneling

Quantum perturbation theory for a single resonance $r:s$,

$$\Delta E_n \simeq \Delta E_n^{(0)} + \sum_{k=-k_c^-, k\neq 0}^{k_c^+} \left| \mathcal{A}_{n+kr}^{(r:s)} \right|^2 \Delta E_{n+kr}^{(0)}$$

with

$$\mathcal{A}_{n+kr}^{(r:s)} = \prod_{j=\text{sign}(k)}^{k} \frac{\langle n+jr | \hat{H}_{\text{res}}^{(r:s)} | n + (j - \text{sign}(k))r \rangle}{E_n - E_{n+jr} + js\hbar\omega}.$$

The increase of splitting is maximal when the $r:s$ resonance is symmetrically located between the two coupled quasimodes.

$$E_n - E_{n+kr} + ks\hbar\omega \simeq \frac{1}{2m_{r:s}} (I_n - I_{n+kr})(I_n + I_{n+kr} - 2I_{r:s})$$

In our specific case, $I = z$.

Floquet theory

Solutions of Schrödinger equation ($\hat{H}(t) = \hat{H}(t + nT)$),

$$|\psi_\nu(t)\rangle = \mathrm{e}^{-i\epsilon_\nu t}\,|u_\nu(t)\rangle.$$

The Floquet eigenvalue equation,

$$\left(\hat{H}(t) - i\partial_t\right)|u_\nu(t)\rangle = \epsilon_\nu|u_\nu(t)\rangle.$$

The Fourier transformation of the previous equation is evaluated.

$$\sum_{k'=-\infty}^{+\infty}\left(H_{k-k'} + k\omega\delta_{kk'}\right)\tilde{\psi}_{k',\nu} = \epsilon_\nu\tilde{\psi}_{k,\nu}$$

$$\hat{H}(t) = \sum_{k=-\infty}^{+\infty}\hat{H}_k\,\mathrm{e}^{ik\omega t} \qquad\qquad |u_\nu(t)\rangle = \sum_{k=-\infty}^{\infty}\mathrm{e}^{ik\omega t}\,|\tilde{\psi}_{k,\nu}\rangle$$

A basis of solution,

$$\left\{|\psi_\nu(t)\rangle = \mathrm{e}^{-i\epsilon_\nu t}\sum_{k=-\infty}^{+\infty}\mathrm{e}^{ik\omega t}\,|\tilde{\psi}_{k,\nu}\rangle \;\middle|\; 0 \leqslant \epsilon_\nu < \omega\right\}.$$

Discrete symmetry

The permutation operator $\hat{P}$ ($\hat{P}|\psi_{n_1,n_2}\rangle = |\psi_{n_2,n_1}\rangle$) presents the following eigenvalue equation.

$$\hat{P}|\psi_{n_1,n_2}\rangle = \pm|\psi_{n_1,n_2}\rangle$$

An eigenbasis of $\hat{P}$ (for $N_p + 1$ even),

$$\left\{ \frac{1}{\sqrt{2}}(|n_1, N_p - n_1\rangle \pm |N_p - n_1, n_1\rangle) \,\middle|\, n_1 = 0, 1, ..., (N_p+1)/2 - 1 \right\}.$$

Discrete symmetry : $[\hat{H}_0, \hat{P}] = 0$.
Consequence on the unperturbed Hamiltonian ($\delta = 0$),

$$(\hat{H}_0) = \left(\begin{array}{c|c} S & 0 \\ \hline 0 & A \end{array} \right).$$

Consequence on the Floquet matrix ($\delta \neq 0$)

$$(\hat{F}) = \left(\begin{array}{c|c} S_p & 0 \\ \hline 0 & A_p \end{array} \right).$$

The dimension of $(\hat{H}_0)$ is $N_p + 1$ and $(\hat{F})$ infinite.

S. Wimberger, *Nonlinear Dynamics and Quantum Chaos An Introduction* (Springer, 2014).