Positional Numeration Systems :
Ultimate Periodicity, Complexity and Automatic Sequences

Adeline Massuir

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$$\begin{aligned} 456 &= 4 \times 10^2 + 5 \times 10 + 6 \times 1 \\ &= 1 \times 3^5 + 2 \times 3^4 + 1 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 0 \times 1 \end{aligned}$$

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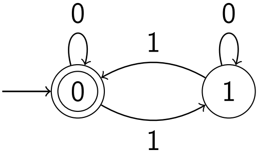
\end{aligned}
$$

$$
\begin{array}{cccccc}
\cdots & 8 & 5 & 3 & 2 & 1 \\
 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

$$8 + 2 = 10$$



recognizability automatic sequences

Part 1

–

Ultimate periodicity problem for linear numeration systems

Problem

Given a linear numeration system U and a deterministic finite automaton $\mathscr{A}$ whose accepted language is contained in the numeration language $\mathrm{rep}_U(\mathbb{N})$,
decide whether the subset X of $\mathbb{N}$ that is recognized by $\mathscr{A}$ is ultimately periodic, i.e. whether or not X is a finite union of arithmetic progressions (along a finite set).

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1994 : Bruyère, Hansel, Michaux, Villemaire
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Syntactic complexity : Lacroix, Rampersad, Rigo, Vandomme (2012)
Logic : Muchnik (2003) – Leroux (2005)
Automata : Marsault, Sakarovitch (2013) – Boigelot, Mainz, Marsault, Rigo (2017) – Marsault (2019)
Morphic : Durand (2013) – Mitrofanov (2013)

Our settings

(H1) $\mathbb{N}$ is U-recognizable,

(H2) $\limsup_{i\to+\infty}(U_{i+1} - U_i) = +\infty$,

(H3) $\exists N \geq 0, \forall i \geq 0, U_{i+1} - U_i \leq U_{i+2} - U_{i+1}$.

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- the numeration system is linear,
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Lemma

Let U be a numeration system satisfying (H1), (H2) and (H3). There exists a constant Z such that if w is a greedy U-representation, then for all $z \geq Z$, $10^z w$ is also a greedy U-representation.

Toy examples

Example 1

Consider the numeration system $U_{i+4} = 2U_{i+3} + 2U_{i+2} + 2U_i$ with initial conditions $U_0 = 1, U_1 = 3, U_2 = 9, U_3 = 25$.
The largest root is $\beta \approx 2.804$ and it has also a root $\gamma \approx -1.134$.

Example 2

Consider the numeration system $U_{i+3} = 12U_{i+2} + 6U_{i+1} + 12U_i$ with initial conditions $U_0 = 1, U_1 = 13, U_2 = 163$.

Strategy

- Input : DFA $\mathscr{A}$
- Upper bound on the admissible preperiods and periods
- For each pair (N, p) of possible preperiods and periods, there are at most $2^N 2^p$ corresponding ultimately periodic sets X
- Equality test : $\mathscr{A}_X$ and $\mathscr{A}$

Let $U = (U_i)_{i\in\mathbb{N}}$ satisfying (H1), (H2) and (H3).

$$U_{i+k} = a_{k-1}U_{i+k-1} + \cdots + a_0 U_i, \; a_0 \neq 0$$

Suppose that the minimal automaton $\mathscr{A}_X$ of $\mathrm{rep}_U(X)$ is given. Let π_X be a potential period for X and consider its prime decomposition.

1. Factors that do not divide a_0,
2. factors that divide a_0 but not all the a_j,
3. factors that divide all the a_j.

The gcd of the coefficients of the recurrence is 1

Proposition

Let $X \subseteq \mathbb{N}$ an ultimately periodic U-recognizable set and let q be a divisor of π_X such that $(q, a_0) = 1$. Then the minimal automaton of $\text{rep}_U(X)$ has at least q states.

Proposition

Let p be a prime not dividing all the coefficients of the recurrence relation and let $\lambda \geq 1$ be the least integer such that $(U_i \bmod p^\lambda)_{i \in \mathbb{N}}$ has a period containing a non-zero element.
If $X \subseteq \mathbb{N}$ is an ultimately periodic U-recognizable set with period $\pi_X = p^\mu \cdot r$ where $\mu \geq \lambda$ and r is not divisible by p, then the minimal automaton of $\text{rep}_U(X)$ has at least $p^{\mu-\lambda+1}$ states.

Theorem

Let U be a numeration system satisfying (H1), (H2) and (H3) and such that the gcd of the coefficients of the recurrence relation of U is 1.
Given a DFA $\mathscr{A}$ accepting a language contained in the numeration language $\mathrm{rep}_U(\mathbb{N})$, it is decidable whether this DFA recognizes an ultimately periodic set.

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Given a DFA $\mathscr{A}$ accepting a language contained in the numeration language $\mathrm{rep}_U(\mathbb{N})$, it is decidable whether this DFA recognizes an ultimately periodic set.

$$U_{i+5} = 6U_{i+4} + 3U_{i+3} - U_{i+2} + 6U_{i+1} + 3U_i, \forall i \geq 0$$

with $U_0 = 1, U_1 = 7, U_2 = 45, U_3 = 291, U_5 = 1881$

Prime factors that divide all the coefficients

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Let F_X be the length of the preperiod of $(U_i \pmod{\frac{\pi_X}{m_X}})_{i\in\mathbb{N}}$.

Theorem

Let $X \subseteq \mathbb{N}$ be an ultimately periodic U-recognizable set with period $\pi_X = m_X \cdot p_1^{\mu_1} \cdots p_t^{\mu_t}$. Assume that $F_X - 1 - \left|\mathrm{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| \geq Z$. Then there is a positive constant C such that the minimal automaton of $0^* \mathrm{rep}_U(X)$ has at least $\frac{C}{\gamma_{m_X}} \log_2\left(\left|\mathrm{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| + 1\right)$ states.

Lemma

Let U be a numeration system satisfying (H1), (H2) and (H3). There exists a constant Z such that if w is a greedy U-representation, then for all $z \geq Z$, $10^z w$ is also a greedy U-representation.

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Lemma

Assume that for all $j \in [\![1, t]\!]$, there are $\alpha_j, \epsilon_j \in \mathbb{R}_{>0}$ and a non-decreasing function g_j such that

$$\nu_{p_j}(U_i) < \lfloor \alpha_j i \rfloor + g_j(i)$$

for all $i \in \mathbb{N}$ and there exists M_j such that $g_j(i) < \epsilon_j i$ for all $i > M_j$. Then for large enough $\mu_1, \ldots, \mu_t$,

$$F_X > \frac{\max\limits_{1 \le j \le t} \mu_j}{\max\limits_{1 \le j \le t} (\alpha_j + \epsilon_j)}.$$

$$\boxed{F_X - 1 - \left|\operatorname{rep}_U\left(\tfrac{\pi_X}{m_X} - 1\right)\right|}$$

Lemma

If $\beta > 1$, there is a non-negative constant K such that

$$|\operatorname{rep}_U(n)| < u \log_\beta(n) + K$$

for all $n \in \mathbb{N}$. In particular,

$$\left|\operatorname{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| \leq u \left(\max_{1 \leq j \leq t} \mu_j\right) \sum_{j=1}^{t} \log_\beta(p_j) + K$$

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for all $i \in \mathbb{N}$ and there exists M_j such that $g_j(i) < \epsilon_j i$ for all $i > M_j$. Assume also that $\beta > 1$. Then

$$\begin{aligned} F_X - 1 - \left|\operatorname{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| \\ \geq \max_{1 \leq j \leq t} \mu_j \left(\frac{1}{\max\limits_{1 \leq j \leq t}(\alpha_j + \epsilon_j)} - u \sum_{j=1}^{t} \log_\beta(p_j)\right) - K - 1 \end{aligned}$$

Search for the α_j

Example 2

$$U_{i+3} = 12U_{i+2} + 6U_{i+1} + 12U_i,\ U_0 = 1, U_1 = 13, U_2 = 163.$$

Proposition

For all $i \in \mathbb{N}$, one has

$$\nu_3(U_i) < \left\lfloor \frac{i}{3} \right\rfloor + 2.$$

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Proposition

Under Conjecture $(\star)$, one has

$$\nu_2(U_i) \leq \frac{i}{2} + \frac{536}{95} \log_2(i)$$

for all $i \geq 10$.

Part 2

–

Minimal automaton for multiplying and translating the Thue-Morse set

Theorem (Alexeev, 2004)

The state complexity of the language $0^* \operatorname{rep}_b(m\,\mathbb{N})$ is

$$\min_{N\geq 0}\left\{\frac{m}{\gcd(m,b^N)}+\sum_{n=0}^{N-1}\frac{b^n}{\gcd(b^n,m)}\right\}$$

2011 : Charlier, Rampersad, Rigo, Waxweiler

Thue-Morse set

Definition

The Thue-Morse set is the set

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}\,.$$

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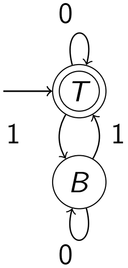

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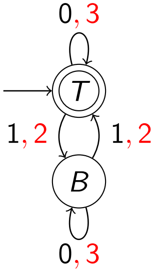

The result

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}$$

Theorem

Let $p, m \in \mathbb{N}_{\geq 1}$ and $r \in [\![0, m-1]\!]$. The state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T} + r)$ is

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

The method

Automaton	Accepted language
$\mathscr{A}_{\mathscr{T},2^p}$	$(0,0)^* \,\mathsf{rep}_{2^p}\,(\mathscr{T} \times \mathbb{N})$

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$\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,r,2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\{(t, mt+r) : t \in \mathscr{T}\})$

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$\Pi\left(\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,r,2^p}\right)$	$0^* \operatorname{rep}_{2^p}(m\mathscr{T} + r)$

The automaton $\mathscr{A}_{\mathscr{T},2^p}$

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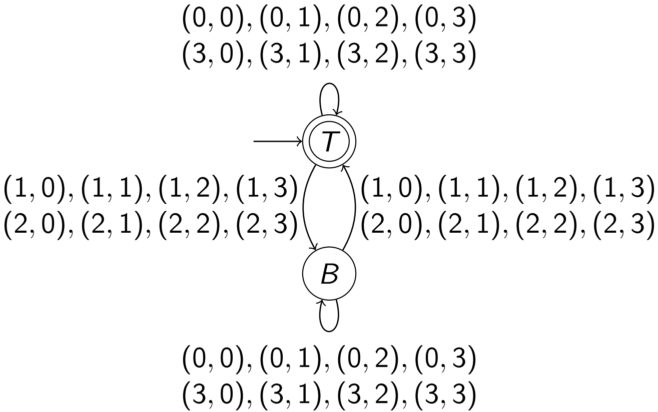

The automaton $\mathscr{A}_{m,r,b}$

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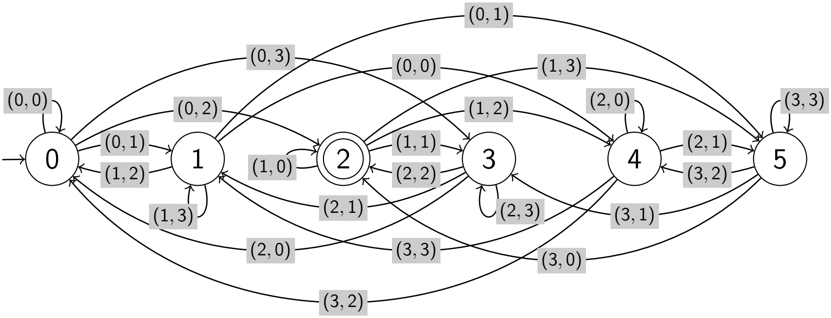
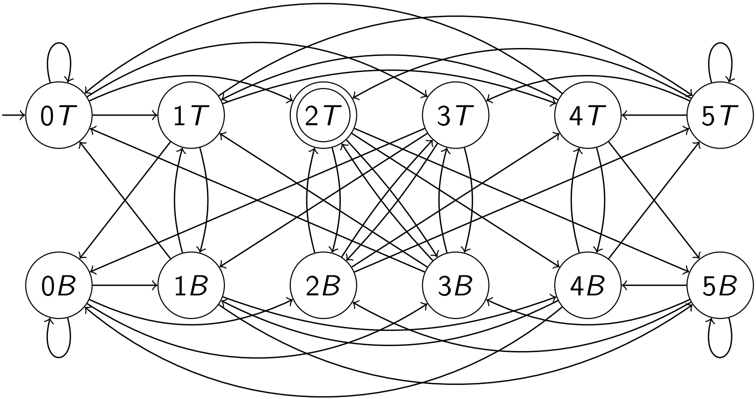


$$\delta_{m,b}(i,(d,e)) = j \Leftrightarrow bi + e = md + j$$

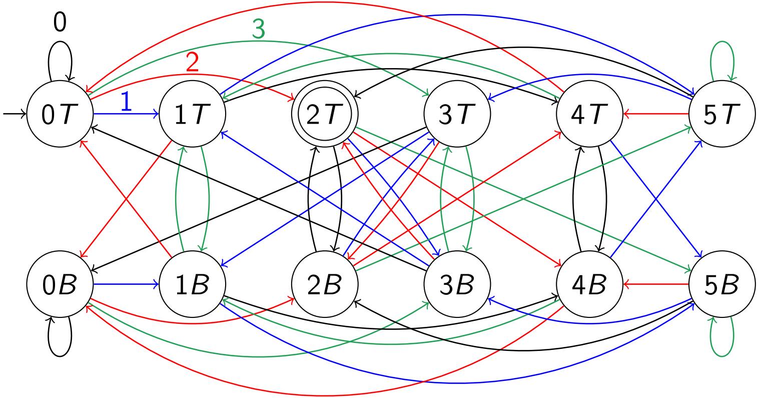
The product automaton $\mathscr{A}_{m,r,2^p} \times \mathscr{A}_{\mathscr{T},2^p}$

$(0,0)^* \left\{\mathrm{rep}_{2^p}(t, mt + r) : t \in \mathscr{T}\right\}$



The projected automaton $\Pi\left(\mathscr{A}_{m,r,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$

$$0^* \operatorname{rep}_{2^p}(m\mathscr{T} + r) = 0^* \{\operatorname{rep}_{2^p}(mt + r) : t \in \mathscr{T}\}$$



Proposition

The automaton $\Pi\left(\mathscr{A}_{m,r,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$ is

- complete,
- deterministic,
- accessible,
- coaccessible.

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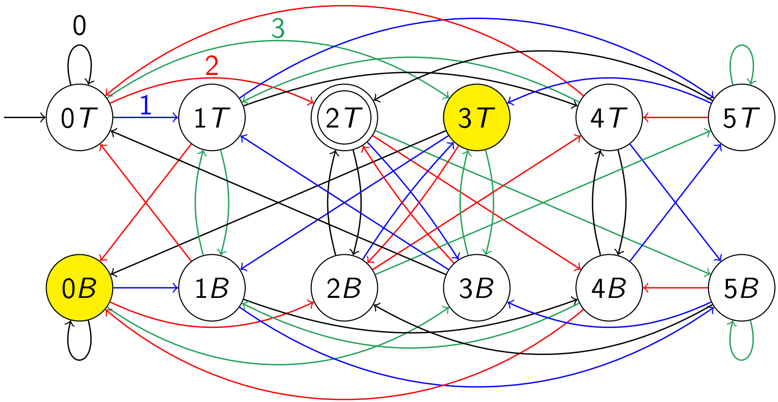
Proposition

In the automaton $\Pi\left(\mathscr{A}_{m,r,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$, the states (i, T) and (i, B) are disjoined for all $i \in [\![\, 0, m-1 \,]\!]$.

The automaton $\Pi\left(\mathscr{A}_{6,2,4} \times \mathscr{A}_{\mathscr{T},4}\right)$



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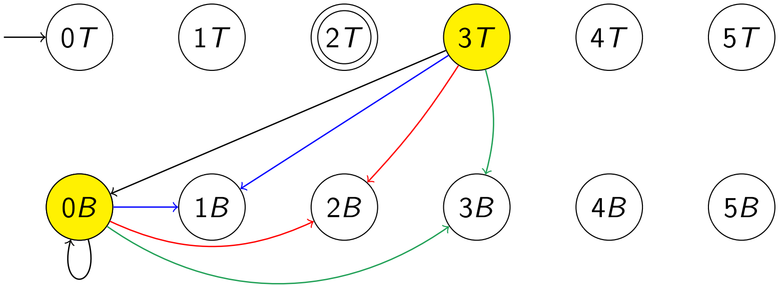
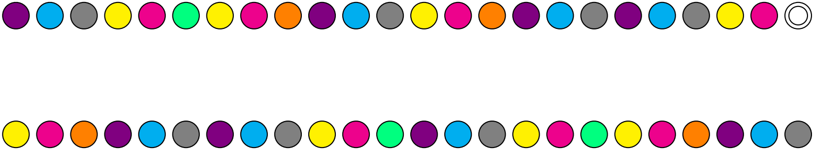


The classes of the automaton $\Pi(\mathscr{A}_{24,23,4} \times \mathscr{A}_{\mathscr{T},4})$

$\mathrm{rep}_4(23) = 113$



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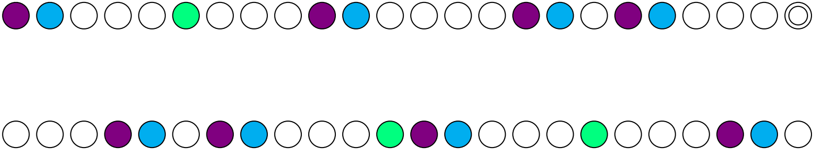

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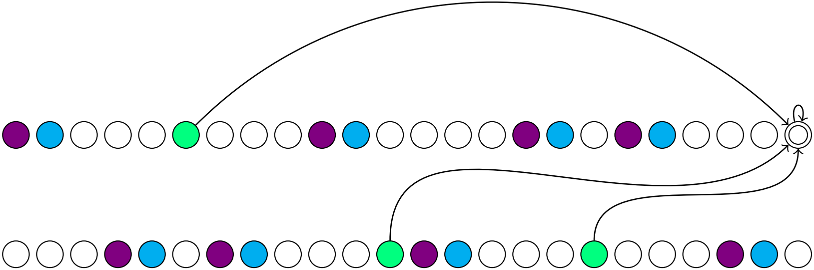

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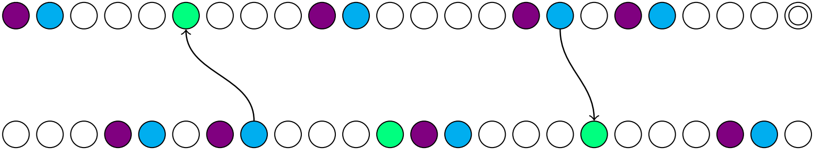

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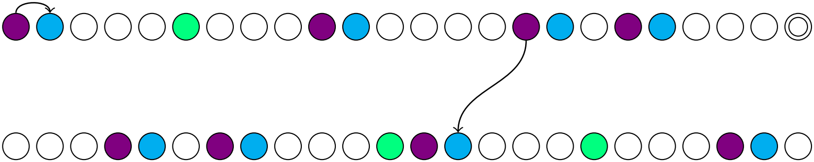

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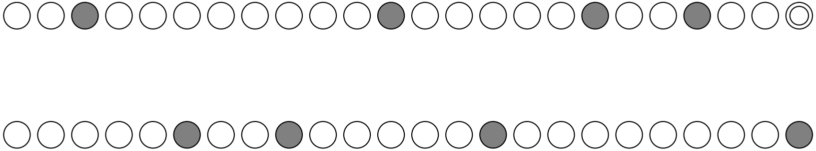

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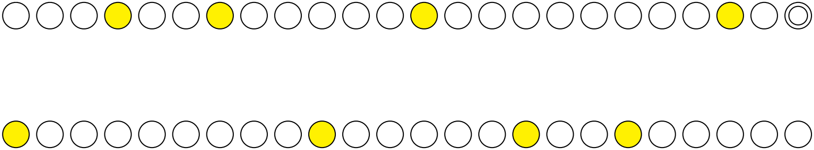

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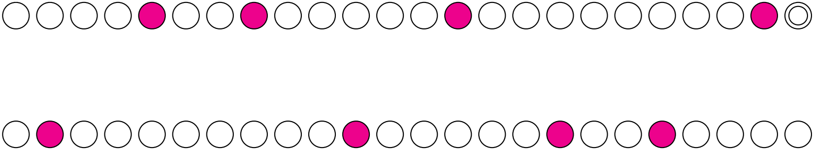

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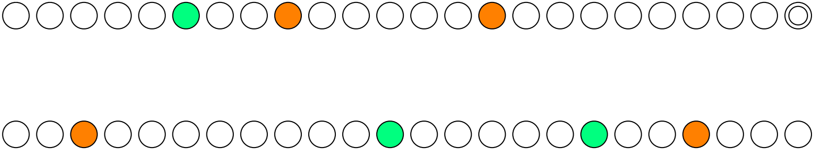

Definition

Let $N = \max\left\{\left\lceil \frac{z}{p} \right\rceil, |\operatorname{rep}_{2^p}(r)|\right\}$. For $\alpha \in [\![0, N]\!]$,

$$C'_\alpha = \begin{cases} \left\{\left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + \ell \frac{m}{2^{\alpha p}}, T_\ell\right) : 0 \leq \ell \leq 2^{\alpha p} - 1\right\} & \text{if } \alpha \leq \frac{z}{p} \\ \left\{\left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + \ell k, T_\ell\right) : 0 \leq \ell \leq 2^{z} - 1\right\} & \text{else.} \end{cases}$$

and $C_\alpha = C'_\alpha \setminus \bigcup_{\beta=0}^{\alpha-1} C'_\beta$.

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and $C_\alpha = C'_\alpha \setminus \bigcup_{\beta=0}^{\alpha-1} C'_\beta$.

For $(j, X) \in ([\![0, k-1]\!] \times \{T, B\}) \setminus \{(0, T)\}$, we define

$$D'_{(j,X)} = \{(j + k\ell, X_\ell) : 0 \leq \ell \leq 2^z - 1\}$$

and

$$D_{(j,X)} = D'_{(j,X)} \setminus \bigcup_{\alpha=0}^{N} C_\alpha.$$

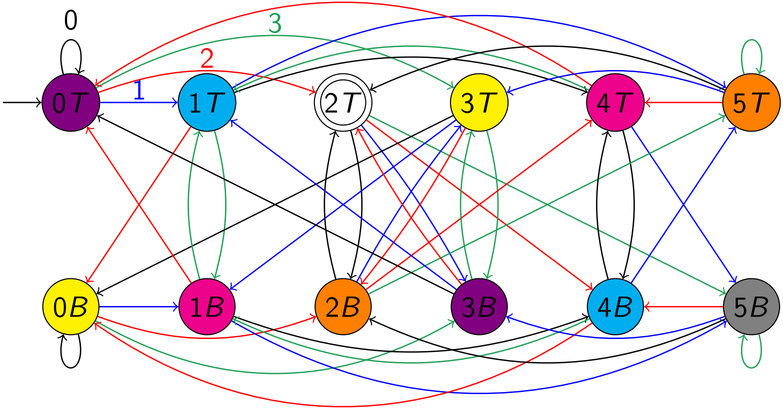
Theorem

Let $p, m \in \mathbb{N}_{\geq 1}$ and $r \in [\![0, m-1]\!]$. The state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T} + r)$ is equal to

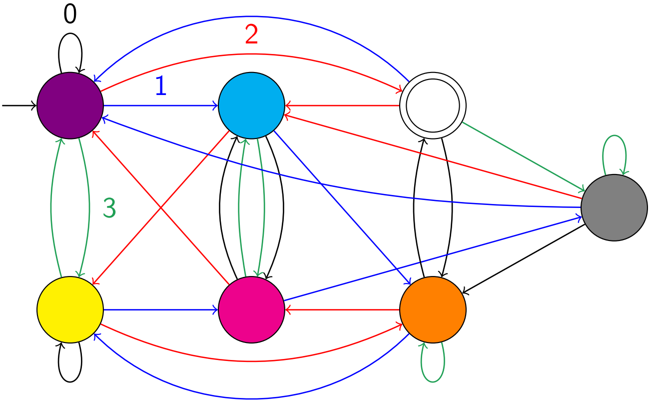
$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

Back to $6\mathscr{T}+2$



Back to $6\mathscr{T}+2$



$2 \times 3 + \lceil \frac{1}{2} \rceil = 7$

Part 3

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Automatic sequences based on Parry or Bertrand numeration systems

Integer base systems $\subsetneq$ Pisot numeration systems

$U_{i+1}=2U_i, U_0=1$

$F_{i+2}=F_{i+1}+F_i, F_0=1, F_1=2$

$\subsetneq$ Parry numeration systems

$U_{i+4}=3U_{i+3}+2U_{i+2}+3U_i,$

$U_0=1, U_1=4, U_2=15, U_3=54$

$\subsetneq$ Bertrand numeration systems

$B_{i+1}=3B_i+1, B_0=1$

Factor complexity

Definition

The *factor complexity function* $p_{\mathbf{x}}(n)$ of an infinite word $\mathbf{x}$ counts the number of factors of length n in $\mathbf{x}$.

Proposition (Cobham, 1972)

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Theorem

The factor complexity function of a Parry-automatic sequence is sublinear.

Theorem

There exists a Bertrand-automatic sequence with superlinear factor complexity.

Closure properties

Integer base systems $\subsetneq$ Pisot n. s. $\subsetneq$ Parry n. s. $\subsetneq$ Bertrand n. s.

Proposition (Bruyère, Hansel, 1997)

The image of a Pisot-automatic sequence under a substitution of constant length is a Pisot-automatic sequence.

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Theorem

There is a Parry numeration system U such that the class of U-automatic sequences is not closed under taking image by a uniform morphism.

$U_{i+4} = 3U_{i+3} + 2U_{i+2} + 3U_i$, $U_0 = 1$, $U_1 = 4$, $U_2 = 15$, $U_3 = 54$
Consider the characteristic sequence $\mathbf{x}$ of the set $\{U_i : i \in \mathbb{N}\}$:

$$\mathbf{x} = 0100100000000001000000000000 \cdots$$

$\mu : 0 \mapsto 0^t, 1 \mapsto 10^{t-1}, t \geq 4$

Proposition (Bruyère, Hansel, 1997)

Pisot-automatic sequences are closed under periodic deletion.

Theorem

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Theorem

There exists a Parry numeration system U such that the class of U-automatic sequences is not closed under periodic deletion.

$\{U_i/2 : i \in \mathbb{N}, U_i \in 2\,\mathbb{N}\}$

$$\mathbf{y} = 0010000000000000000000000001 \cdots$$

Perspectives

Ultimate periodicity

- Can we weaken our hypotheses ?
- Who are the numeration systems not satisfying

$$F_X - 1 - \left|\mathrm{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| \geq Z?$$

$U_{i+2} = 4U_i : 1, 3, 4, 6, 8, 12, \cdots$

- Is there a strategy working for both integer bases and other positional numeration systems ?
- What about the time complexity ?

Thue-Morse

- Replacing $\mathscr{T}$ by a b-recognizable set X : $mX + r$?
- What about positional numeration system which are not an integer base ?
- LSDF

Theorem (Pansiot, 1984)

Let $\mathbf{x}$ be a purely morphic word. Then one of the following holds :

- $p_{\mathbf{x}}(n) = \Theta(1)$,
- $p_{\mathbf{x}}(n) = \Theta(n)$,
- $p_{\mathbf{x}}(n) = \Theta(n \log \log n)$,
- $p_{\mathbf{x}}(n) = \Theta(n \log n)$,
- $p_{\mathbf{x}}(n) = \Theta(n^2)$.