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Sciences

Ultimate periodicity problem for linear numeration systems

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Positional numeration system

A *(positional) numeration system* is a sequence $U = (U_i)_{i \in \mathbb{N}}$ of positive integers s. t.

- U is increasing,
- $U_0 = 1$,
- $C_U = \sup_{i \geq 0} \left\lceil \frac{U_{i+1}}{U_i} \right\rceil$ is finite.

The *alphabet of the numeration* is the set $\Sigma_U = [\![0, C_U - 1]\!]$.
The *greedy U-representation* of a positive integer n is the unique word $\mathrm{rep}_U(n) = w_{\ell-1} \cdots w_0$ over Σ_U s. t.

$$n = \sum_{i=0}^{\ell-1} w_i U_i,\ w_{\ell-1} \neq 0 \text{ and } \forall j \in [\![0, \ell]\!], \sum_{i=0}^{j-1} w_i U_i < U_j.$$

We set $\mathrm{rep}_U(0) = \varepsilon$.
The language $\mathrm{rep}_U(\mathbb{N})$ is the *numeration language*.
A set X is *U-recognizable* if $\mathrm{rep}_U(X)$ is regular.

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The *U-numerical valuation* $\mathrm{val}_U : \mathbb{Z}^* \to \mathbb{N}$ maps a word $w_{\ell-1} \cdots w_0$ to the number $\sum_{i=0}^{\ell-1} w_i U_i$.
If $\mathrm{val}_U(w) = n$, then w is a *U-representation* of n.

Examples

Integer base-b : $1, b, b^2, b^3, \cdots$

- $U_i = b^i \; \forall i \in \mathbb{N}$
- $\Sigma_b = [\![\, 0, b{-}1 \,]\!]$
- $\text{rep}_b(\mathbb{N}) = \{\varepsilon\} \cup (\Sigma_b \setminus \{0\})\Sigma_b^*$

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Fibonacci numeration system : $1, 2, 3, 5, 8, 13, \cdots$

- $F_0 = 1, F_1 = 2$ and $F_{i+2} = F_{i+1} + F_i \; \forall i \in \mathbb{N}$
- $\Sigma_F = [\![0, 1]\!]$
- $\text{rep}_F(\mathbb{N}) = 1\{0, 01\}^* \cup \{\varepsilon\}$

$\text{rep}_F(11) = (10100)_F$ and $\text{val}_F(1001) = 6 = \text{val}_F(111)$

Advantages of regular numeration languages

1. We are able to check with an automaton whether a representation is greedy,
2. the numeration system is linear,
3. ultimately periodic sets are recognizable.

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Integer base b : $U_{i+1} = bU_i$

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Advantages of regular numeration languages

1. We are able to check with an automaton whether a representation is greedy,
2. the numeration system is linear,
3. ultimately periodic sets are recognizable.

Proposition

Let m, r be non-negative integers and let $U = (U_i)_{i\in\mathbb{N}}$ be a linear numeration system. The language

$$\mathrm{val}_U^{-1}(m\,\mathbb{N}+r) = \{w \in \Sigma_U^* : \mathrm{val}_U(w) \in m\,\mathbb{N}+r\}$$

is accepted by a DFA that can be effectively constructed. In particular, if $\mathbb{N}$ is U-recognizable, then any ultimately periodic set is U-recognizable.

Decision problem

Problem

Given a linear numeration system U and a deterministic finite automaton $\mathscr{A}$ whose accepted language is contained in the numeration language $\text{rep}_U(\mathbb{N})$, decide whether the subset X of $\mathbb{N}$ that is recognized by $\mathscr{A}$ is ultimately periodic, i.e. whether or not X is a finite union of arithmetic progressions (along a finite set).

What is known

Integer base

- J. Honkala
- A. Lacroix, N. Rampersad, M. Rigo, E. Vandomme
- B. Boigelot, I. Mainz, V. Marsault, M. Rigo, J. Sakarovitch

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Pisot numeration systems

- First-order logic $\langle \mathbb{N}, +, V_U \rangle$
- X a U-recognizable set, φ a formula describing it

$$(\exists N)(\exists p)(\forall n \geq N)(\varphi(n) \Leftrightarrow \varphi(n+p))$$

- J. Leroux, A. Muchnik

When addition is not recognizable

J. Bell, É. Charlier, A. Fraenkel, M. Rigo

1. $\mathbb{N}$ is U-recognizable,
2. $\lim_{i \to +\infty} (U_{i+1} - U_i) = +\infty$,
3. $\lim_{m \to +\infty} N_U(m) = +\infty$.

$U_{i+3} = 3U_{i+2} + 2U_{i+1} + 3U_i$

Our settings

(H1) $\mathbb{N}$ is U-recognizable,

(H2) $\limsup_{i \to +\infty}(U_{i+1} - U_i) = +\infty$,

(H3) $\exists N \geq 0, \forall i \geq 0, U_{i+1} - U_i \leq U_{i+2} - U_{i+1}$.

Lemma

Let U be a numeration system satisfying (H1), (H2) and (H3). There exists a constant Z such that if w is a greedy U-representation, then for all $z \geq Z$, $10^z w$ is also a greedy U-representation.

Lemma

Let U be a numeration system satisfying (H1), (H2) and (H3). There exists a constant Z such that if w is a greedy U-representation, then for all $z \geq Z$, $10^z w$ is also a greedy U-representation.

$$1, 2, 4, 5, 16, 17, 64, 65, \cdots \qquad U_{i+4} = 5U_{i+2} - 4U_i$$

$$\mathrm{val}_U(1001) = 6 \qquad 1(00)^t 1001$$

$$\cdots, 65, 64, 17, 16, 5, 4, 2, 1$$

Toy examples

Example 1

Consider the numeration system $U_{i+4} = 2U_{i+3} + 2U_{i+2} + 2U_i$ with initial conditions $U_0 = 1, U_1 = 3, U_2 = 9, U_3 = 25$.
The largest root is $\beta \approx 2.804$ and it has also a root $\gamma \approx -1.134$.

Example 2

Consider the numeration system $U_{i+3} = 12U_{i+2} + 6U_{i+1} + 12U_i$ with initial conditions $U_0 = 1, U_1 = 13, U_2 = 163$.

Strategy

- Input : DFA $\mathscr{A}$
- Upper bound on the admissible preperiods and periods
- For each pair (N, p) of possible preperiods and periods, there are at most $2^N 2^p$ corresponding ultimately periodic sets X
- Equality test : $\mathscr{A}_X$ and $\mathscr{A}$

Preperiod

Proposition

Let U be a numeration system satisfying (H1), let X be an ultimately periodic set of non-negative integers and let $\mathscr{A}_X$ be a DFA with $\#Q_X$ states accepting $\mathrm{rep}_U(X)$. Then the preperiod α_X of X is bounded by a computable constant J depending only on the number of states of $\mathscr{A}_X$ and the period π_X of X.

Period

Let $U = (U_i)_{i\in\mathbb{N}}$ satisfying (H1), (H2) and (H3).

$$U_{i+k} = a_{k-1}U_{i+k-1} + \cdots + a_0 U_i$$

Suppose that the minimal automaton $\mathscr{A}_X$ of $\text{rep}_U(X)$ is given. Let π_X be a potential period for X and consider its prime decomposition.

1. Factors that do not divide a_0,
2. factors that divide a_0 but not all the a_j,
3. factors that divide all the a_j.

Factors that do not divide a_0

Proposition

Let $X \subseteq \mathbb{N}$ an ultimately periodic U-recognizable set and let q be a divisor of π_X such that $(q, a_0) = 1$. Then the minimal automaton of $\text{rep}_U(X)$ has at least q states.

Prime factors that divide a_0 but not all the a_j

Take the sequence $U_0 = 1, U_1 = 4, U_2 = 8$ and $U_{i+2} = U_{i+1} + U_i$ for $i \in \mathbb{N}_0$.

$$1, 4, 8, 12, 20, 32, 52, \cdots$$

The sequence $(U_i \bmod 2^\mu)_{i\geq 0}$ has a zero period for $\mu = 1, 2$ because of the particular initial conditions. But the sequence $(U_i \bmod 8)_{i\in\mathbb{N}}$ is given by $1(404)^\omega$.

Theorem

Let p be a prime. The sequence $(U_i \bmod p^\mu)_{i\in\mathbb{N}}$ has a zero period for all $\mu \geq 1$ if and only if all the coefficients $a_0, \ldots, a_{k-1}$ of the linear relation are divisible by p.

Proposition

Let p be a prime not dividing all the coefficients of the recurrence relation and let $\lambda \geq 1$ be the least integer such that $(U_i \bmod p^\lambda)_{i\in\mathbb{N}}$ has a period containing a non-zero element. If $X \subseteq \mathbb{N}$ is an ultimately periodic U-recognizable set with period $\pi_X = p^\mu \cdot r$ where $\mu \geq \lambda$ and r is not divisible by p, then the minimal automaton of $\mathrm{rep}_U(X)$ has at least $p^{\mu-\lambda+1}$ states.

The gcd of the coefficients of the recurrence is 1

Theorem

Let U be a numeration system satisfying (H1), (H2) and (H3) and such that the gcd of the coefficients of the recurrence relation of U is 1. Given a DFA $\mathscr{A}$ accepting a language contained in the numeration language $\mathrm{rep}_U(\mathbb{N})$, it is decidable whether this DFA recognizes an ultimately periodic set.

$$U_{i+5} = 6U_{i+4} + 3U_{i+3} - U_{i+2} + 6U_{i+1} + 3U_i, \forall i \geq 0$$

- $N_U(3^i) \not\to +\infty$
- $\beta = 3 + 2\sqrt{3}$, three roots of modulus 1
- Initial conditions :
  $U_0 = 1, U_1 = 7, U_2 = 45, U_3 = 291, U_5 = 1881$
- Numeration language : set of words over $\{0, \ldots, 6\}$ avoiding $63, 64, 65, 66$
- For all $i \geq 0, U_{i+1} - U_i \geq 5U_i$

Prime factors that divide all the coefficients

$$\boxed{\pi_x = m_X \cdot p_1^{\mu_1} \cdots p_t^{\mu_t}}$$

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Let $j \in [\![1, t]\!], \mu \geq 1$. The sequence $(U_i \bmod p_j^{\mu})_{i \in \mathbb{N}}$ has a zero period. We let $\mathsf{f}_{p_j}(\mu)$ be the integer such that

$$U_{\mathsf{f}_{p_j}(\mu)-1} \not\equiv 0 \pmod{p_j^{\mu}} \quad \text{and} \quad U_i \equiv 0 \pmod{p_j^{\mu}}\ \forall i \geq \mathsf{f}_{p_j}(\mu).$$

Example :

$U_{i+4} = 2U_{i+3} + 2U_{i+2} + 2U_i$, $U_0 = 1, U_1 = 3, U_2 = 9, U_3 = 25$

- $(U_i \bmod 2)_{i \in \mathbb{N}} = 1, 1, 1, 1, 0^{\omega}$ hence $\mathsf{f}_2(1) = 4$
- $(U_i \bmod 4)_{i \in \mathbb{N}} = 1, 3, 1, 3, 2, 0, 2, 2, 0^{\omega}$ hence $\mathsf{f}_2(2) = 8$
- $\mathsf{f}_2(3) = 12, \mathsf{f}_2(4) = 16$

We set

$$F_X = \max_{1 \leq j \leq t} \mathrm{f}_{p_j}(\mu_j).$$

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Example :

$U_{i+3} = 12U_{i+2} + 6U_{i+1} + 12U_i,\ U_0 = 1, U_1 = 13, U_2 = 163$

- $\mathrm{f}_2(1) = 3, \mathrm{f}_2(2) = 5, \mathrm{f}_2(3) = 7$
- $\mathrm{f}_3(1) = 3, \mathrm{f}_3(2) = 6, \mathrm{f}_3(3) = 9$
- $\pi_X = 72 = 2^3 \cdot 3^2,\ F_X = \max(\mathrm{f}_2(3), \mathrm{f}_3(2)) = 7$

$$(U_i \bmod 72)_{i \in \mathbb{N}} = 1, 13, 19, 30, 54, 48, 36, 0^\omega$$

Proposition

Let m, r be non-negative integers and let $U = (U_i)_{i\in\mathbb{N}}$ be a linear numeration system. The language

$$\mathrm{val}_U^{-1}(m\,\mathbb{N}+r) = \{w \in \Sigma_U^* : \mathrm{val}_U(w) \in m\,\mathbb{N}+r\}$$

is accepted by a DFA that can be effectively constructed. In particular, if $\mathbb{N}$ is U-recognizable, then any ultimately periodic set is U-recognizable.

We let γ_m denote the maximum of the number of states of these DFAs for $r \in [\![\, 0, m-1 \,]\!]$.

Theorem

Let $X \subseteq \mathbb{N}$ be an ultimately periodic U-recognizable set with period $\pi_x = m_X \cdot p_1^{\mu_1} \cdots p_t^{\mu_t}$. Assume that $F_X - 1 - \left|\text{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| \geq Z$. Then there is a positive constant C such that the minimal automaton of $0^* \text{rep}_U(X)$ has at least $\frac{C}{\gamma_{m_X}} \log_2\left(\left|\text{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| + 1\right)$ states.

The gcd of the coefficients of the recurrence greater than 1

$$\pi_X = m_X \cdot p_1^{\mu_1} \cdots p_t^{\mu_t},\ t \geq 1$$

$$n_X = F_X - 1 - \left|\text{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| \geq Z$$

Theorem

Let U be a numeration system satisfying (H1), (H2) and (H3), and such that the gcd of the coefficients of the recurrence relation of U is larger than 1. Assume there is a computable positive integer D such that for all ultimately periodic sets X of period $\pi_X = m_X \cdot p_1^{\mu_1} \cdots p_t^{\mu_t}$ with $t \geq 1$, if $\max\{\mu_1, \cdots, \mu_t\} \geq D$, then $n_X \geq Z$. Then, given a DFA $\mathscr{A}$ accepting a language contained in the numeration language $\text{rep}_U(\mathbb{N})$, it is decidable whether this DFA recognizes an ultimately periodic set.

Behaviour of n_X

$$n_X = F_X - 1 - \left| \mathrm{rep}_U \left(\frac{\pi_X}{m_X} - 1 \right) \right|$$

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$$n_X = F_X - 1 - \left|\mathrm{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right|$$

$$F_X = \max_{1 \le j \le t} \mathrm{f}_{p_j}(\mu_j)$$

$$\mathrm{f}_{p_j}(\mu) = M \Leftrightarrow (\nu_{p_j}(U_{M-1}) < \mu \wedge \forall i \ge M, \nu_{p_j}(U_i) \ge \mu)$$

Lemma

Let $j \in [\![1, t]\!]$. Assume that there are $\alpha, \epsilon \in \mathbb{R}_{>0}$ and a non-decreasing function g such that

$$\nu_{p_j}(U_i) < \lfloor \alpha i \rfloor + g(i)$$

for all $i \in \mathbb{N}$ and there exists M such that $g(i) < \epsilon i$ for all $i > M$. Then for large enough μ,

$$\mathrm{f}_{p_j}(\mu) > \frac{\mu}{\alpha + \epsilon}.$$

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$$F_X = \max_{1 \leq j \leq t} \mathrm{f}_{p_j}(\mu_j) > \max_{1 \leq j \leq t} \left(\frac{\mu_j}{\alpha_j + \epsilon_j} \right) \geq \frac{\max\limits_{1 \leq j \leq t} \mu_j}{\max\limits_{1 \leq j \leq t} (\alpha_j + \epsilon_j)}$$

$$\boxed{n_X = F_X - 1 - \left|\mathrm{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right|}$$

$\left|\mathrm{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right|$

Soittola : $\exists u \geq 1, \beta_0, \ldots, \beta_{u-1} \geq 1$, non-zero polynomials $P_0, \ldots, P_{u-1}$ s. t. for $r \in [\![0, u-1]\!]$ and large enough i,

$$U_{ui+r} = P_r(i)\beta_r^i + Q_r(i)$$

where $\frac{Q_r(i)}{\beta_r^i} \to +\infty$ when $i \to +\infty$.

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$$U_{ui+r} = P_r(i)\beta_r^i + Q_r(i)$$

where $\frac{Q_r(i)}{\beta_r^i} \to +\infty$ when $i \to +\infty$.

$$\beta_0 = \cdots = \beta_{u-1} = \beta \qquad \text{and} \qquad \deg(P_0) = \cdots = \deg(P_{u-1}) = d$$

$$U_{ui+r} \sim c_r i^d \beta^i$$

Lemma

If $\beta > 1$, there is a non-negative constant K such that

$$|\operatorname{rep}_U(n)| < u \log_\beta(n) + K$$

for all $n \in \mathbb{N}$.

If $\beta > 1$, then

$$\begin{aligned} \left|\operatorname{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| &\leq u \log_\beta\left(\prod_{j=1}^{t} p_j^{\mu_j}\right) + K \\ &\leq u \left(\max_{1 \leq j \leq t} \mu_j\right) \sum_{j=1}^{t} \log_\beta(p_j) + K \end{aligned}$$

Behaviour of n_X

$$\boxed{n_X = F_X - 1 - \left|\mathrm{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right|}$$

- $F_X \geq \frac{\max\limits_{1\leq j\leq t} \mu_j}{\max\limits_{1\leq j\leq t} (\alpha_j + \epsilon_j)}$
- $\left|\mathrm{rep}_U\left(\frac{\pi_X}{m_X} - 1\right)\right| \leq u\left(\max\limits_{1\leq j\leq t} \mu_j\right) \sum_{j=1}^{t} \log_\beta(p_j) + K$

$$n_X \geq \max_{1\leq j\leq t} \mu_j \left(\frac{1}{\max\limits_{1\leq j\leq t}(\alpha_j + \epsilon_j)} - u\sum_{j=1}^{t} \log_\beta(p_j)\right) - K - 1$$

Some intuition for α_j

$$\boxed{\nu_{p_j}(U_i) < \lfloor \alpha i \rfloor + g(i)}$$

Example 1

Consider the numeration system $U_{i+4} = 2U_{i+3} + 2U_{i+2} + 2U_i$ with initial conditions $U_0 = 1, U_1 = 3, U_2 = 9, U_3 = 25$.

For $41 \leq i \leq 60$, $\nu_2(U_i)$ is

$10, 10, 10, 11, 13, 11, 11, 12, 12, 12, 12, 13, 14, 13, 13, 14, 14, 14, 14, 15.$

Conjecture : $\alpha_1 = \frac{1}{4}$

$$4 = \frac{1}{\alpha_1} > \log_{2.804}(2) \approx 0.672$$

Example 2

Consider the numeration system $U_{i+3} = 12U_{i+2} + 6U_{i+1} + 12U_i$ with initial conditions $U_0 = 1$, $U_1 = 13$, $U_2 = 163$.

For $41 \leq i \leq 60$, $\nu_2(U_i)$ is

$$24, 20, 21, 21, 24, 22, 23, 23, 27, 24, 25, 25, 28, 26, 27, 27, 33, 28, 29, 29$$

and $\nu_3(U_i)$ is

$$13, 14, 14, 14, 15, 15, 15, 16, 17, 16, 17, 17, 17, 18, 18, 18, 19, 20, 19, 20.$$

Conjecture : $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$

$$2 = \frac{1}{\max\{\frac{1}{2}, \frac{1}{3}\}} > \log_{12.554}(2) + \log_{12.554}(3) \approx 0.708$$

Problem to solve

$$\boxed{\nu_{p_j}(U_i) < \lfloor \alpha i \rfloor + g(i)}$$

Example 2

Consider the numeration system $U_{i+3} = 12U_{i+2} + 6U_{i+1} + 12U_i$ with initial conditions $U_0 = 1$, $U_1 = 13$, $U_2 = 163$.

$$\nu_2(U_i) \qquad \text{and} \qquad \nu_3(U_i)$$

Theorem

For all $i \in \mathbb{N}$, we have

$$\nu_3(U_i) = \left\lfloor \frac{i}{3} \right\rfloor + \begin{cases} 0 & \text{if } i \not\equiv 4 \pmod 9 \\ 1 & \text{if } i \equiv 4 \pmod 9. \end{cases}$$

$T_i = U_i/3^{\frac{i-2}{3}}$ for all $i \in \mathbb{N}$
Modulo $9\,\mathbb{Z}[3^{1/3}]$,

$$\begin{array}{ccccccccc} 3^{2/3} & 4\cdot 3^{1/3} & 1 & 7\cdot 3^{2/3} & 3\cdot 3^{1/3} & 1 & 2\cdot 3^{2/3} & 2\cdot 3^{1/3} & 4 \\ 3^{2/3} & 3^{1/3} & 7 & 7\cdot 3^{2/3} & 3\cdot 3^{1/3} & 7 & 8\cdot 3^{2/3} & 5\cdot 3^{1/3} & 1 \\ 3^{2/3} & 7\cdot 3^{1/3} & 4 & 7\cdot 3^{2/3} & 3\cdot 3^{1/3} & 4 & 5\cdot 3^{2/3} & 8\cdot 3^{1/3} & 7. \end{array}$$

and thus the sequence $(\nu_3(T_i))_{i\in\mathbb{N}}$ of 3-adic valuations is

$$\frac{2}{3},\ \frac{1}{3},\ 0,\ \frac{2}{3},\ \frac{4}{3},\ 0,\ \frac{2}{3},\ \frac{1}{3},\ 0,\ \cdots$$

with period 9.

The previous theorem implies $\frac{i-2}{3} \leq \nu_3(U_i) \leq \frac{i+2}{3}$ for all $i \in \mathbb{N}$. In particular, $\nu_3(U_i) < \left\lfloor \frac{i}{3} \right\rfloor + 2$.

What about $\nu_2(U_i)$?

$T_i = U_i / 2^{\frac{i}{2}-1}$ for all $i \in \mathbb{N}$
Modulo $2\,\mathbb{Z}[\sqrt{2}]$,

$$\star, \star, 1, \sqrt{2}, 1, 0, 1, \sqrt{2}, 1, 0, \dots = \star, \star\,(1, \sqrt{2}, 1, 0)^{\omega}.$$

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Theorem

For i large enough such that $i \not\equiv 1 \pmod 4$, we have

$$\nu_2(U_i) = \left\lfloor \frac{i-1}{2} \right\rfloor.$$

p-adic analysis

p-adic valuation $\nu_p(n)$: exponent of the highest power of p dividing n

p-adic absolute value $|n|_p$: $p^{-\nu_p(n)}$

Non-archimedean : $|m+n|_p \leq \max\{|m|_p, |n|_p\}$

$\mathbb{Q}_p$: completion of $\mathbb{Q}$ with respect to the p-adic absolute value

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$\mathbb{Q}_p$: completion of $\mathbb{Q}$ with respect to the p-adic absolute value

Every $\zeta \in \mathbb{Q}_p$ can be written in the form

$$\begin{aligned} \zeta &= d_{-N}p^{-N} + \cdots + d_{-1}p^{-1} + d_0 + d_1 p + d_2 p^2 + \cdots \\ &= \sum_{i \geq -N} d_i p^i, \end{aligned}$$

with $N \in \mathbb{Z}$ and $d_i \in [\![0, p-1]\!]$ for all $i \geq -N$. This representation is unique.

Back to $\nu_2(U_i)$

Construct a piecewise interpolation of U_i to $\mathbb{Z}_2$.

$$\begin{aligned} P(x) &= x^3 - 12x^2 - 6x - 12 \\ &= (x - \beta_1)(x^2 + (\beta_1 - 12)x + (\beta_1^2 - 12\beta_1 - 6)) \end{aligned}$$

$\mathbb{Q}_2(\beta_2)$

$$\begin{aligned} U_i &= c_1\beta_1^i + c_2\beta_2^i + c_3\beta_3^i \\ &= \beta_2^i \left(c_1 \left(\frac{\beta_1}{\beta_2}\right)^i + c_2 + c_3 \left(\frac{\beta_3}{\beta_2}\right)^i \right) \\ &= \beta_2^i \left(c_1 \left(\frac{\beta_1}{\beta_2}\right)^i + f_1(i) \right) \end{aligned}$$

Conjecture ($\star$)

Theorem

Under conjecture ($\star$), for all $i \geq 13$ such that $i \equiv 1 \pmod 4$, we have

$$\nu_2(U_i) = \left\lfloor \frac{i-1}{2} \right\rfloor + \nu_2(i - \zeta).$$

Corollary

Conjecture ($\star$) implies

$$\nu_2(U_i) \leq \frac{i}{2} + \frac{536}{95} \log_2(i)$$

for all $i \geq 10$.

Conclusion

$U_{i+u} = bU_i$

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Proposition

Let $b \geq 2$, $u \geq 1$, $N \geq 0$. Let U be a numeration system $U = (U_i)_{i \in \mathbb{N}}$ such that $U_{i+u} = bU_i$ for all $i \geq N$. If a set is U-recognizable then it is b-recognizable. Moreover, given a DFA accepting $\text{rep}_U(X)$ for some set X, we can compute a DFA accepting $\text{rep}_b(X)$.