Topology Optimization of Continuum Structures with Stress Constraints

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Topology Design with Stress Constraints

Minimum compliance design

$$\begin{aligned} \min_{\rho} \quad & \int \mathbf{f}^{\mathbf{T}} \mathbf{u} \, d\Omega \\ \text{s.t.} \quad & \int \rho \, d\Omega \; \leq \; \bar{V} \\ & \mathbf{K}\mathbf{u} = \mathbf{f} \end{aligned} \tag{1}$$

Stress constraints

$$\begin{aligned} \min_{\rho} \quad & \int \rho \, d\Omega \\ \text{s.t.} \quad & <\|\sigma\|> \; \leq \; \sigma_l \quad \text{if} \quad \rho > 0 \end{aligned} \tag{2}$$

OVERVIEW

RELEVANT STRESS CRITERIA FOR POROUS COMPOSITES

Stresses in layered materials

Stress criterion for rank 2 layered materials

Stress criterion for power law materials

SOLUTION ASPECTS

A constraint relaxation of stresses

A mathematical programming approach of the numerical solution

NUMERICAL APPLICATIONS

RELEVANT STRESS CRITERIA FOR POROUS COMPOSITES

- Establish a relationship between stresses at micro-level, the macroscopic stresses and the micro-structural parameters (density . . .)
- Limit the micro-stress state with a relevant failure criterion
- The 'homogenized' stress criterion is the expression of the local criterion in terms of the macroscopic stresses

Stresses in layered materials

Stresses/strains in each layers are <u>constant</u>

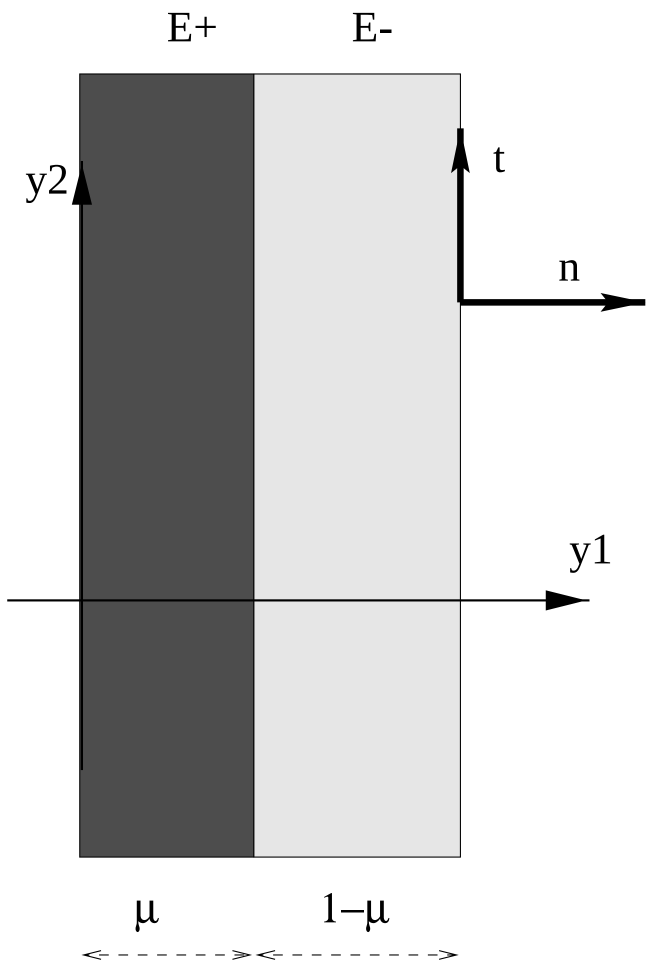
$$\begin{aligned} \sigma_{ij}^{+} &= \bar{\sigma}_{ij} + c_3 t_i t_j \\ \sigma_{ij}^{-} &= \bar{\sigma}_{ij} - \frac{\mu}{1-\mu} c_3 t_i t_j \end{aligned} \qquad (3)$$

with

$$c_3 = \frac{1-\mu}{N(C_{rstu} t_r t_s t_t t_u)} [C_{ijkl}^{-} - C_{ijkl}^{+}] \bar{\sigma}_{kl} \qquad (4)$$

C_{ijkl}^{+}, C_{ijkl}^{-} : compliance tensors of the layers
σ_{ij}^{+}, σ_{ij}^{-} : stresses in the two layers
$\bar{\sigma}_{ij}$: macroscopic stresses
$\mathbf{n}$, $\mathbf{t}$: normal and tangent layer directions
μ : relative thickness of layers of material +
$N(f) = (1-\mu) f^{+} + \mu f^{-}$

Stresses in layered materials



If $\mathbf{n} = (1, 0)$ and $\mathbf{t} = (0, 1)$,

two isotropic materials: (E^+, ν) (strong)

(E^-, ν) (soft)

Stresses in layered materials

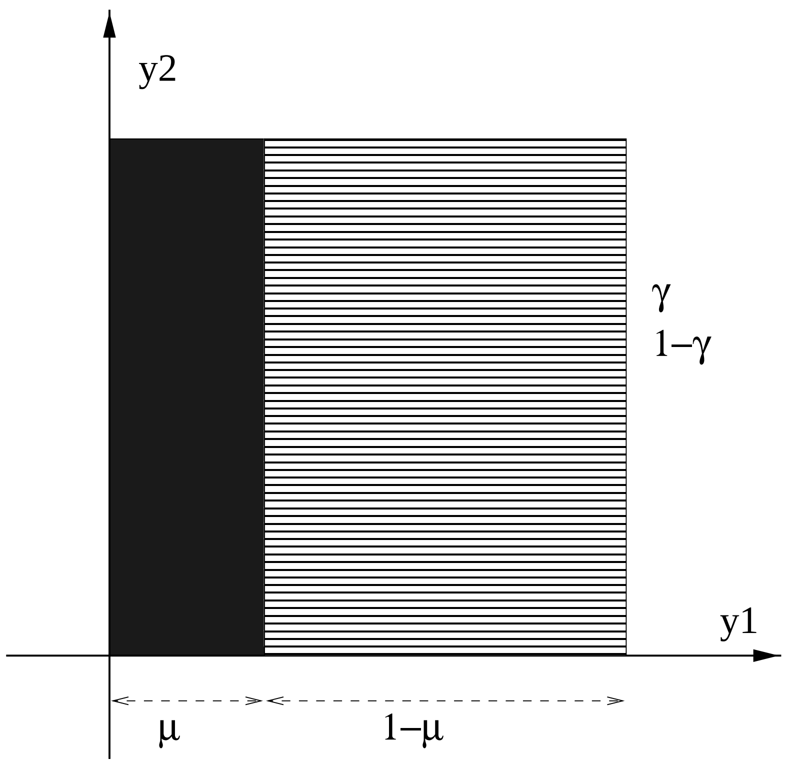
$$
\begin{aligned}
\sigma_{11}^{+} &= \bar{\sigma}_{11} \\
\sigma_{22}^{+} &= \bar{\sigma}_{22} + c_3 \\
\sigma_{12}^{+} &= \bar{\sigma}_{12}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\sigma_{11}^{-} &= \bar{\sigma}_{11} \\
\sigma_{22}^{-} &= \bar{\sigma}_{22} - \frac{\mu}{1-\mu} c_3 \\
\sigma_{12}^{-} &= \bar{\sigma}_{12}
\end{aligned} \tag{6}
$$

with

$$
c_3 = (1-\mu)\frac{E^{+} - E^{-}}{\mu E^{+} + (1-\mu)E^{-}}\left[-\nu\bar{\sigma}_{11} + \bar{\sigma}_{22}\right] \tag{7}
$$

Stresses in porous layered materials of rank 2



Recursive use of previous formula in a top to bottom process in the micro-structure

Stresses in porous layered materials of rank 2

RANK2 LAYERING:

Determine the behaviour of c_3 when $E^- \to 0^+$

$$c_3 \to \frac{1-\mu}{\mu}\,\bar{\sigma}_{22} \tag{8}$$

In the layers of solids:

$$\boxed{\begin{aligned} \sigma_{11}^{L2} = \sigma_{11}^{+} &= \ <\sigma_{11}> \\ \sigma_{22}^{L2} = \sigma_{22}^{+} &= \ <\sigma_{22}>/\mu \\ \sigma_{12}^{L2} = \sigma_{12}^{+} &= \ <\sigma_{12}> \end{aligned} \quad (9)}$$

and in the rank 1 composite layers:

$$\begin{aligned} \sigma_{11}^{-} &= \ <\sigma_{11}> \\ \sigma_{22}^{-} &= 0 \\ \sigma_{12}^{-} &= \ <\sigma_{12}> \end{aligned} \tag{10}$$

Stresses in porous layered materials of rank 2

RANK1 LAYERING:

One more use of recursive formula gives the stresses in the layers of solids

$$\begin{aligned} \sigma_{11}^{+} &= \frac{1}{\gamma}\,\bar{\sigma}_{11} - \frac{1-\gamma}{\gamma}\,\nu\,\bar{\sigma}_{22} \\ \sigma_{22}^{+} &= \bar{\sigma}_{22} \\ \sigma_{12}^{+} &= \bar{\sigma}_{12} \end{aligned} \tag{11}$$

It comes:

$$\begin{aligned} \sigma_{11}^{L1} &= <\sigma_{11}>/\gamma \\ \sigma_{22}^{L1} &= 0 \\ \sigma_{12}^{L1} &= <\sigma_{12}> \end{aligned} \tag{12}$$

Stresses in porous layered materials of rank 2

RANK2 LAYERING:

$$\begin{aligned} \sigma_{11}^{L2} &= <\sigma_{11}> \\ \sigma_{22}^{L2} &= <\sigma_{22}> /\mu \\ \sigma_{12}^{L2} &= <\sigma_{12}> \end{aligned}$$

RANK 1 LAYERING :

$$\begin{aligned} \sigma_{11}^{L1} &= <\sigma_{11}> /\gamma \\ \sigma_{22}^{L1} &= 0 \\ \sigma_{12}^{L1} &= <\sigma_{12}> \end{aligned}$$

REMARK

Since $E^H_{1212} = 0$,

- Rank 2 composite are unable to withstand shear loads
- Assume that layers are aligned with principal directions of macroscopic stresses
- $<\sigma_{12}> = 0$ in layering axes

Homogenized von Mises criterion for rank 2 materials

Von Mises stress in plane stress state

$$\sigma_{eq} = \sqrt{\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2}$$

CRITERION FOR RANK 1 LAYERS

$$\left| <\sigma_{11}>/\gamma \right| \le \sigma_l \qquad (13)$$

CRITERION FOR RANK 2 LAYERS

$$<\sigma_{11}>^2 + \frac{<\sigma_{22}>^2}{\mu^2} - <\sigma_{11}> \frac{<\sigma_{22}>}{\mu} \le \sigma_l^2 \qquad (14)$$

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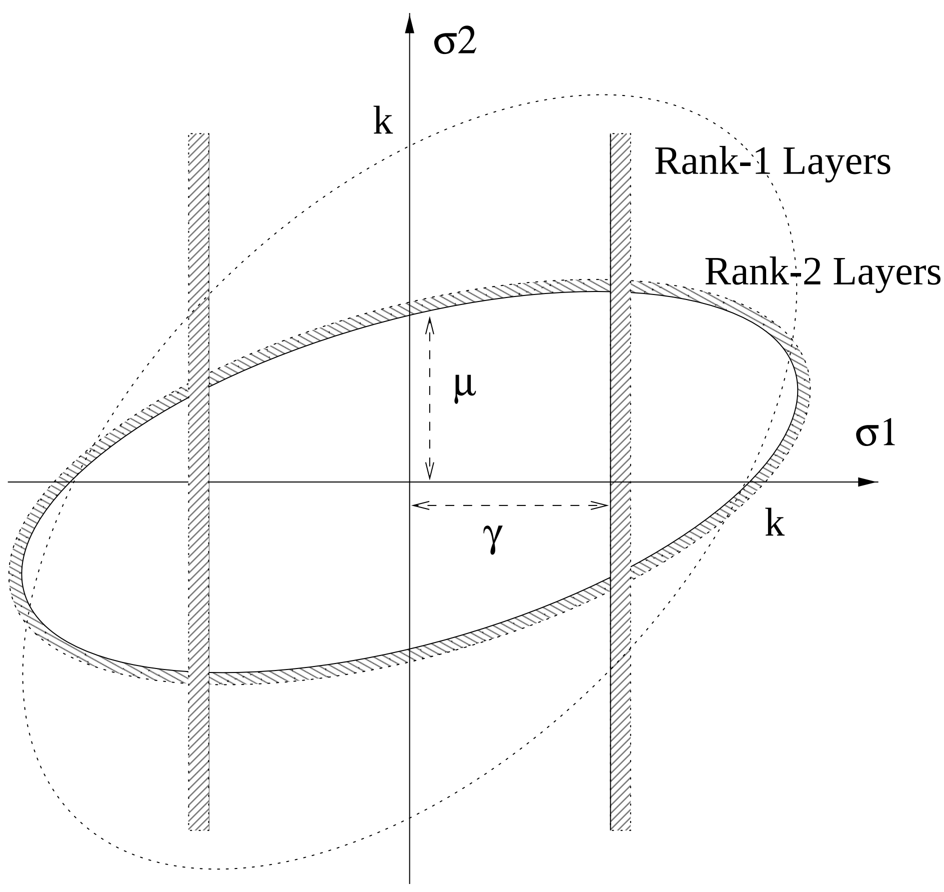
Homogenized von Mises criterion for rank 2 materials



Homogenized von Mises criterion for rank 2 materials

- Failure in the different layers are considered separately
  $\Rightarrow$ overall failure criterion = 'composite' surface
- Similar to Hashin's failure criterion of unidirectional composites
- The failure criterion (13) for the inner layer looks like a stress limit of a fibrous material of relative volume γ
- The failure criterion (14) for the outer layer looks like Hill's criterion with $X_1 = \sigma_l$ and $X_2 = \mu\sigma_l$

Asymptotic behaviour at zero density

MACROSCOPIC STRAINS AND STRESSES

Macroscopic strains $<\varepsilon_{ij}>$ are continuous and keep a finite value

$$\lim_{\mu,\gamma\to 0^+} <\varepsilon_{ij}> = <\varepsilon^0_{ij}> \tag{15}$$

Macroscopic stresses are continuous and vanish at zero density

$$\lim_{\mu,\gamma\to 0^+} <\sigma_{ij}> = E^H_{ijkl} <\varepsilon_{ij}> = 0 \tag{16}$$

Asymptotic behaviour at zero density

LOCAL STRESSES σ_{ij} TEND TO FINITE (NON ZERO) VALUES AT ZERO DENSITY

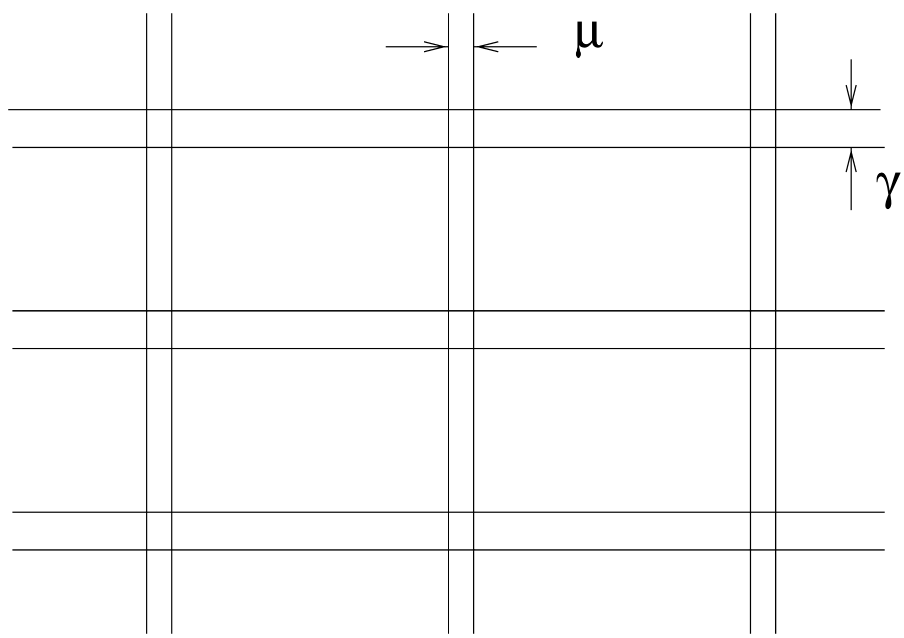
Rank 1:

$$\begin{aligned} \lim_{\mu,\gamma\to 0+} \sigma_{11} &= E <\varepsilon^0_{11}> \\ \lim_{\mu,\gamma\to 0+} \sigma_{22} &= 0 \\ \lim_{\mu,\gamma\to 0+} \sigma_{12} &= 0 \end{aligned} \quad (17)$$

Rank 2:

$$\begin{aligned} \lim_{\mu,\gamma\to 0+} \sigma_{11} &= 0 \\ \lim_{\mu,\gamma\to 0+} \sigma_{22} &= E <\varepsilon^0_{22}> \\ \lim_{\mu,\gamma\to 0+} \sigma_{12} &= 0 \end{aligned} \quad (18)$$

Asymptotic behaviour at zero density



Stress criterion for power law materials

STIFFNESS LAW

$$<E(\mu)>= \rho^p E^0 \tag{19}$$

LOCAL STRESS MODEL

Assume that local stresses are related to macroscopic stresses by a power law of the density

$$\sigma_{ij} = \alpha \frac{<\sigma_{ij}>}{\rho^q} \tag{20}$$

Choose $\alpha = 1$.

Exponent q will be determined later.

Stress criterion for power law materials

STRESS CRITERION

For a local von Mises criterion:

$$<\sigma>_{eq} \;\leq\; \rho^q \;\sigma_l \tag{21}$$

CHOICE OF EXPONENT q

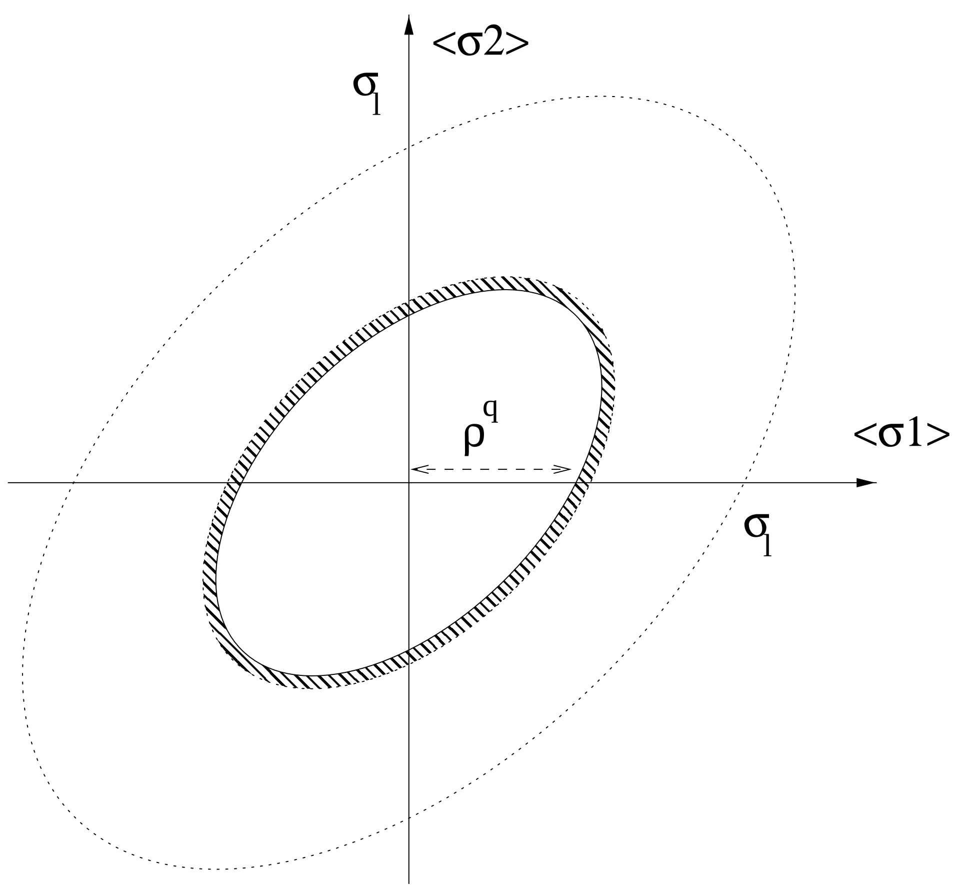
By analogy with the rank 2 materials, local stresses must remain finite and non zero at zero density

$$\lim_{\rho\to 0^+} \sigma_{ij} \;=\; \frac{\rho^p}{\rho^q} E_{ijkl} <\varepsilon^0_{kl}> \neq \; 0 \tag{22}$$

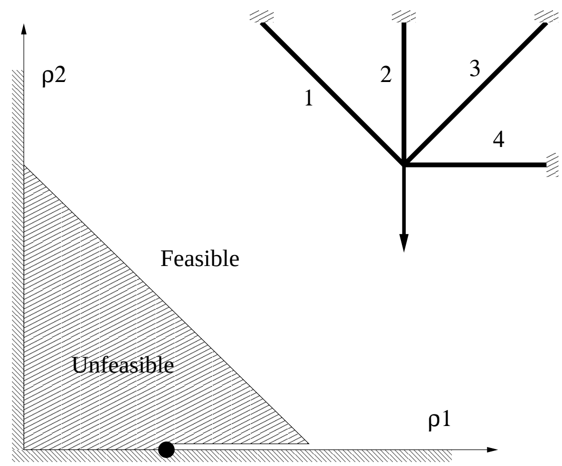
$$\Longleftrightarrow$$

$$\boxed{\mathsf{p} = \mathsf{q}}$$

Stress criterion for power law materials



Singularity Phenomenon with Stress Constraints



$<\|\sigma\|>$: an homogenized stress criterion

ρ : the density parameter

Stress constraints:

$$<\|\sigma\|> \leq \ \sigma_l \ \text{ if } \ \rho < 0$$

Limit value of stresses at zero density

$$\lim_{\rho \to 0^+} <\|\sigma\|> \neq \ 0$$

Singularity Phenomenon with Stress Constraints

- Design domain contains *degenerated* parts
- Optima are generally located in some of these degenerated appendices
- Qualification of constraints (*i.e.* Slatter conditions) are not satisfied in these regions
- Optima are unreachable with algorithms based on Karush-Kuhn-Tucker conditions

Constraint Relaxation of Stresses

- Stress constraints:

$$<\|\sigma\|> \leq \; \sigma_l \quad \text{if} \quad \rho < 0 \tag{23}$$

- Eliminate zero condition:

$$\rho \; (<\|\sigma\|>/\sigma_l - 1) \; \leq 0 \tag{24}$$

- However, it *does not remove the singularity and the algorithmic problems*. To circumvent the singularity of the design space:

- ϵ constraint relaxation (Cheng and Guo, 1997)

$$\begin{aligned} \rho \; (<\|\sigma\|>/\sigma_l - 1) \quad &\leq \quad \epsilon \\ \epsilon^2 \quad &\leq \quad \rho \end{aligned} \tag{25}$$

Constraint Relaxation of Stresses

$$\begin{aligned} \min_{\rho} \quad & \int \rho \; d\Omega \\ \text{s.t.} \quad & \rho \left(<\|\sigma\|>/\sigma_l - 1 \right) \leq \epsilon \\ & \epsilon^2 \leq \rho \end{aligned} \tag{26}$$

- Relaxation in the sense of mathematical programming
- Continuous point-to-set maps between the parameter $\epsilon \to 0$ and the relaxed design domains as well as their optimal solutions
- Solution of relaxed problems is regular and can be attacked with Mathematical Programming algorithms

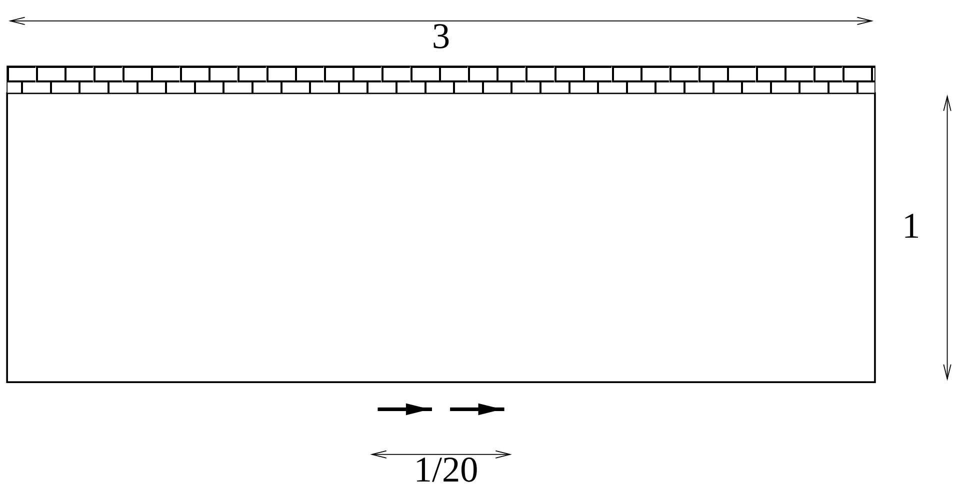
A Mathematical Programming Approach of the Solution

- Large scale optimization problem
  Huge number of design variables and of active constraints
- Problem is a kind of *fully stressed design*. Percentage of active stress constraints proportional to the ratio of volume used
- CONLIN approximation:
  Restriction formulated as:
  $$\frac{<\|\sigma\|>}{\sigma_l} - \frac{\epsilon}{\rho} \leq 1 \tag{27}$$
- Dual algorithms:
  to solve the convex and separable subproblems (CONLIN optimizer from Fleury (1989))

APPLICATIONS

TWO BAR TRUSS TOPOLOGY PROBLEM

Geometry



Material data

Young's modulus: $E = 1Nm^2$

Poisson's ratio : $\nu = 0.3$

Stress limit : $\sigma_l = 25.Nm^2$

Power law : $p = q = 3$

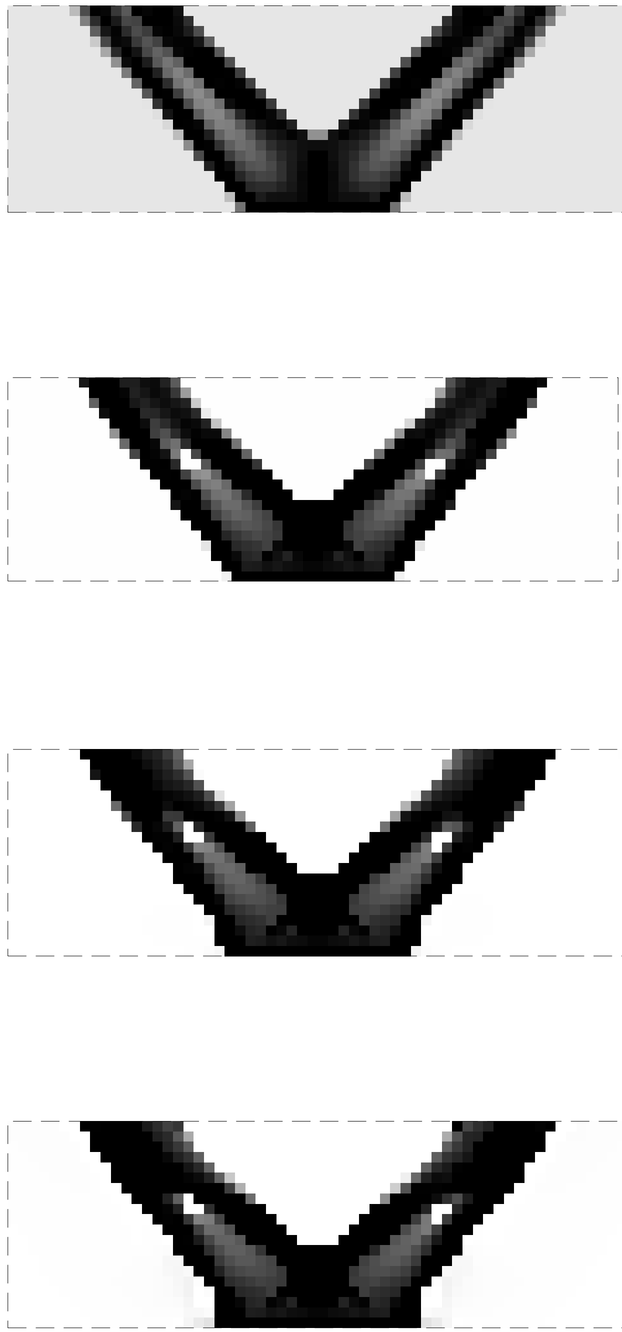
Distributed shear load: $P = 12N$

TWO BAR TRUSS TOPOLOGY PROBLEM

Without constraint relaxation

Relaxed problems

TWO BAR TRUSS TOPOLOGY PROBLEM



Sequence of relaxed problems

TWO BAR TRUSS TOPOLOGY PROBLEM

Relaxed restriction on the stresses

Maximum value: $25.Nm^2$

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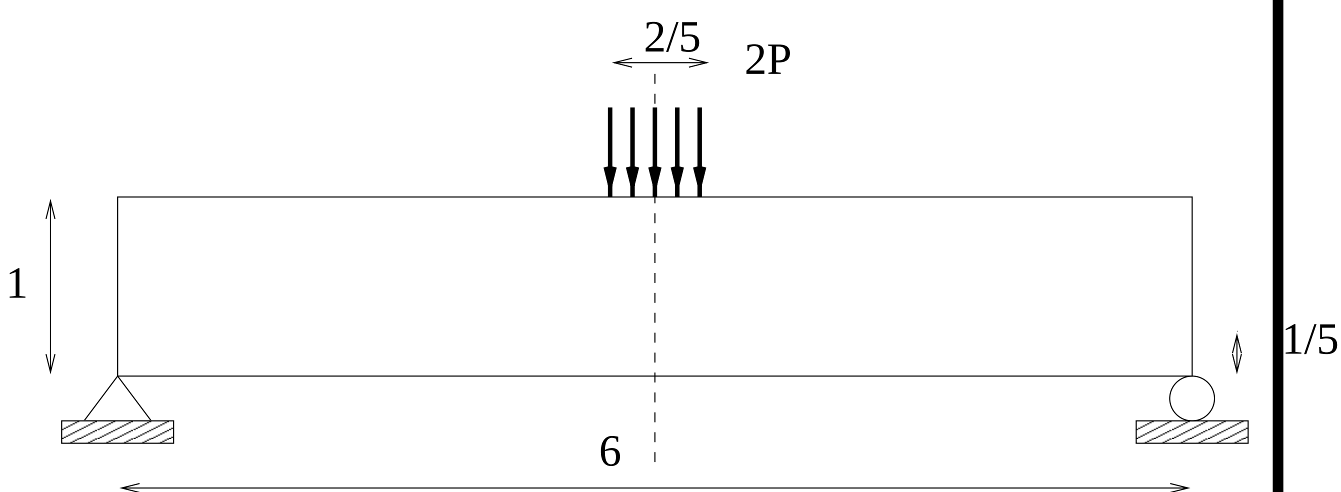
Homogenized stress criterion

Maximum value: $68.Nm^2$

MBB BEAM TOPOLOGY PROBLEM

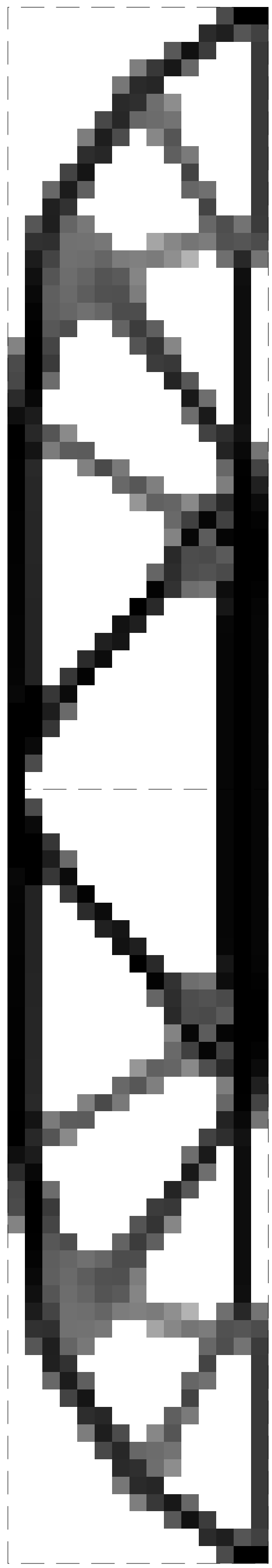
Geometry



Material data

Young's modulus: $E = 1Nm^2$
Poisson's ratio : $\nu = 0.3$
Stress limit : $\sigma_l = 20.Nm^2$
Power law : $p = q = 3$

Distributed load: $P = 1N$



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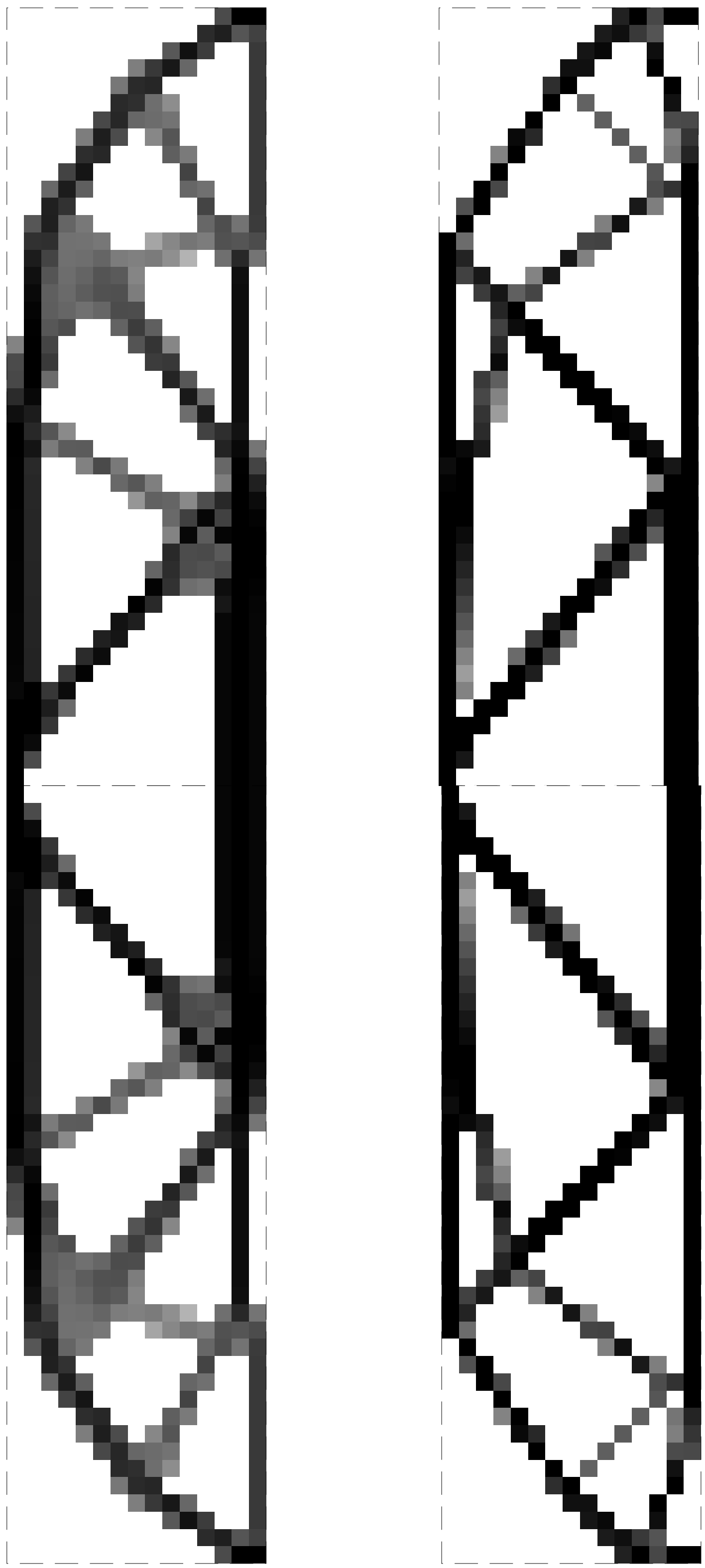
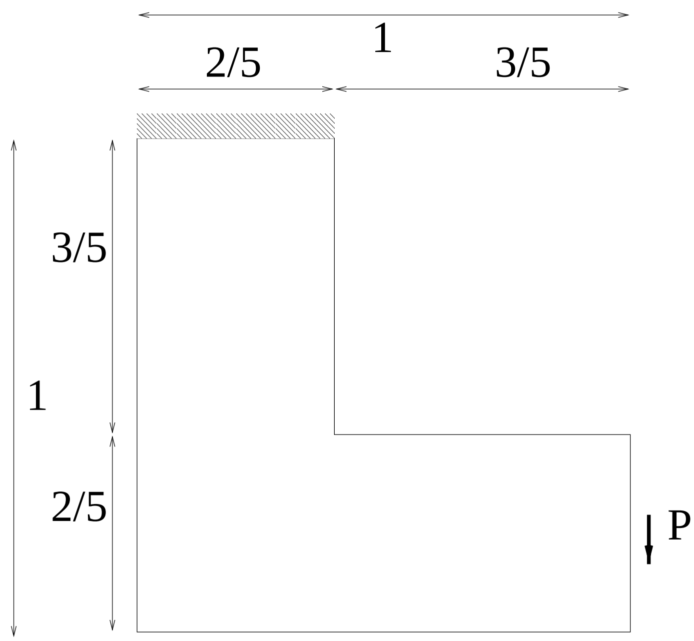


L BEAM TOPOLOGY PROBLEM

Geometry



Material data

Young's modulus: $E = 1 N m^2$

Poisson's ratio : $\nu = 0.3$

Stress limit : $\sigma_l = 30. N m^2$

Power law : $p = q = 3$

Distributed load: $P = 1N$

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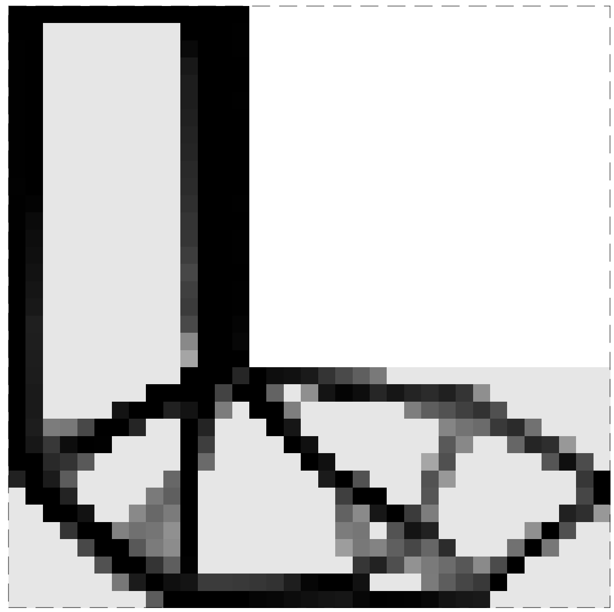
L BEAM TOPOLOGY PROBLEM

Compliance = 235.6

Volume = 0.308

Max stres criterion = 1.0

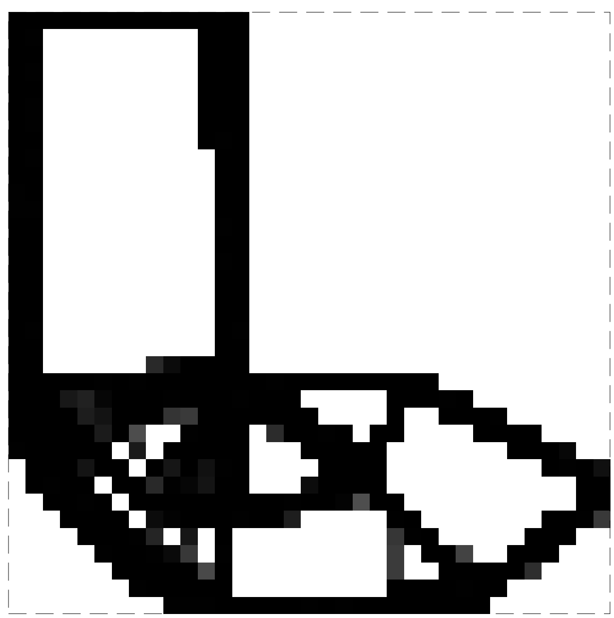


Bounded stress design

Compliance = 206.2

Volume = 0.3

Max stress criterion = 1.83



Minimum compliance design

CONCLUSIONS

- Relevant stress criterion for rank 2 materials
- A criterion for power law materials
- ϵ constraint relaxation of stress constraints to circumvent the singularity phenomenon
- Solution of large optimization problems with a mathematical programming approach

FUTURE WORKS

- Relaxation (in the sense of varitional principles) of topology problems with stress constraints
- *G-closure* problem with stress criteria
- More general topology problems

$$\begin{aligned} \min_{\rho} \quad & \int \rho \, d\Omega \\ \text{s.t.} \quad & <\|\sigma\|> \leq \sigma_l \quad \text{if} \quad \rho > 0 \\ & \int_{\Omega} \mathbf{f}^T \mathbf{u} \, d\Omega \leq \overline{C} \\ & u_j \leq \overline{u}_j \end{aligned} \tag{28}$$

- Improve efficiency of solution algorithms
- Improve capturing of stress concentrations