TOPOLOGY OPTIMIZATION OF CONTINUUM STRUCTURES WITH STRESS CONSTRAINTS

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ABSTRACT

We introduce an extension of topology optimization of continuum structures to deal with local stress criteria. We first consider relevant stress criteria for porous composite materials, initially by studying the stress states of the so-called rank 2 layered materials. Then, an empirical model is proposed for the power law materials (also called SIMP materials). In a second part, solution aspects of topology problems are considered. To deal with the so-called 'singularity' phenomenon of stress constraints in topology design, an ϵ constraint relaxation of the stress constraints is used. We describe the mathematical programming approach that is used to solve the numerical optimization problems. The proposed strategy is applied to illustrative applications.

KEYWORDS

Topology optimization, stress constraints, homogenized stress criteria, constraint relaxation, mathematical programming approach

INTRODUCTION

It is now a well-established methodology to use a material distribution formulation for topology design problems involving continuum structures. Since the introduction of this technique, much research have been devoted to the extension of minimum compliance design to other global criteria (see, e.g. Bendsøe [1] for a survey). Here we present an extension of topology optimization of continuum structures to cater for local stress criteria which are expected to have a non-trivial influence on the resulting designs when there are several load cases (minimum compliance design for single load cases is well capable to handle stress limitations, in accordance with the principle of fully stressed design which holds for discrete structures).

There are two main questions to address when considering stress constraints in continuum topology design. One is the formulation of relevant stress criteria and the other is to devise efficient methods for handling the many constraints in a computational procedure (arising from the local nature of the constraint).

Note that we in the work here have not tried to obtain a consistent mathematical relaxation (in the sense of variational analysis) of the stress constrained 0-1 material-void problem statement, but rather we have pursued the goal of obtaining relevant stress criteria for variable density models which are well-established for the minimum compliance problem.

STRESS CRITERIA FOR POROUS COMPOSITES

For the stress constraints, the developments in this work are devoted to the formulation of relevant stress limits for elastic design, in terms of models which allow for a density description of the structure. Here this includes layered materials and the so-called SIMP method of using a power-law interpolation of material properties. For the former models an evaluation of micro stresses in analytical form allow for the identification of a 'first failure' criterion. From this, relevant formulations for the latter models can also be identified.

In order to exhibit a relevant stress criterion for composite porous materials, we adopt an approach where a relationship between the stresses at the micro-level, the macroscopic stresses and the micro-structural parameters like the density is first established. Then we limit the micro-stress state with a relevant failure criterion. The 'homogenized' macroscopic stress criterion is then the expression of the local criterion in terms of the macroscopic stresses, and it is this criterion which is taken into account in the optimization process.

For convenience we have carried out the scheme outlined above for layered materials. As the effective elastic properties of such materials can be written analytically and because the micro-strains and the micro-stresses in layered materials are constant in each layer, the micro-stress state of layered composites can be determined analytically in terms of the layering parameters and in terms of the macroscopic stresses $<\sigma_{ij}>$ (see Ref. [1, 2])

As an example for a so-called rank-2 layering (with layers aligned with the principal directions of the macroscopic stresses), the von Mises equivalent stress criterion results in the following macroscopic stress criterion

$$
\begin{aligned}
|<\sigma_{11}>/\gamma| &\leq \sigma_l && (1)\\
\sqrt{<\sigma_{11}>^2 + <\sigma_{22}>^2/\mu^2 - <\sigma_{11}><\sigma_{22}>/\mu} &\leq \sigma_l && (2)
\end{aligned}
$$

which guarantees that the stress level is everywhere in the micro-structure under the material elastic limit σ_l. Here μ and γ are the densities of the two orthogonal layers, with a total density of material equal to $\mu + \gamma - \mu\gamma$ (γ is the density of the inner layering of void and material, μ the density of the outer layering). Since there are two levels of layering, one considers the failure in the different layers separately and the overall failure criterion is a 'composite' surface made of two parts. This approach is similar to Hashin's failure criterion [3] of unidirectional composites where the matrix and fibers failure modes are distinguished. The failure criterion (1) for the inner layer looks like a stress limit of a bar of relative cross-section γ. The true local stress increases when the density of this layer decreases while the effective macro-stress is kept constant. By considering uniaxial stress states, one can see that the expression (2) for the outer layer is similar to a Hill's criterion in which the stress limit in the direction orthogonal to the layer is σ_l, while it is reduced to the value $\mu\sigma_l$ in the direction of the layering, due to the relative thickness of the layers. This allows one to understand how the strength domain shrinks and becomes narrower when the porosity increases. This situation is sketched in figure 1.

Apart from models with layered materials, another very convenient technique for practical topology applications is the well known power penalization of stiffness at intermediate densities by a parametrization of rigidity as $<E(\rho)>= \rho^p E^0$ (the so-called SIMP approach). For this case we propose that the allowable stress at intermediate density, like the rigidity,

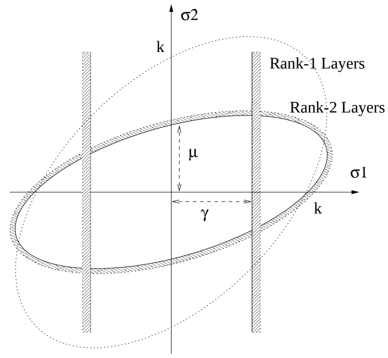

Figure 1: Overall strength domain of a rank 2 layered materials

depends on a power q of the density parameter ρ. The limitation of the von Mises equivalent stress $<\sigma>_{eq}$ then becomes:

$$<\sigma>_{eq} \;\leq\; \rho^q \; \sigma_l \tag{3}$$

where a comparison with the layered model (which is consistent) shows that the natural choice is $q = p$ (see Duysinx and Bendsøe [2] for details).

SOLUTION ASPECTS

A constraint relaxation for stresses

As it was shown for truss structures (cf., Kirsch [4], Cheng and Jiang [5]), a special feature of topology design with stress constraints is the possibility that when a bar area or a element density tends to zero, the stress tend to finite values. As stress constraints must not be considered for zero density, the design space can include degenerated parts and this results in a the so-called 'singularity' of topology design. For the continuum model proposed here, we can likewise demonstrate that, when the density of material tends to zero, the average stresses vanishes while the local stresses (i.e., the constrained stresses) tend to finite values. Due to this 'paradoxical' behaviour at zero density, classical optimization algorithms (based on Kuhn-Tucker conditions) would not be able to remove totally the low density regions and then to reach the true optimal topologies, unless the problem is reformulated.

If $<\|\sigma\|>$ is the homogenized stress criterion defined previously and if ρ is the density parameter, then the constraints are $<\|\sigma\|>\leq \sigma_l$ if $\rho < 0$. To eliminate the zero condition from the constraint, one considers the formulation:

$$\rho \; (<\|\sigma\|>/\sigma_l - 1) \;\leq 0$$

which however, does not remove the algorithmic problems. Nonetheless, one can circumvent the singularity of the design space by rewriting the stress constraints using the ϵ constraint relaxation approach proposed by Cheng and Guo [6]. For our continuum model, one then considers the following stress constraints and side constraints:

$$\rho \; (<\|\sigma\|>/\sigma_l - 1) \leq \epsilon \quad , \quad \epsilon^2 \leq \rho \tag{4}$$

where the ϵ parameter is given. This relaxation (in the sense of mathematical programming) creates continuous point-to-set maps between the parameter ϵ and the relaxed design domains

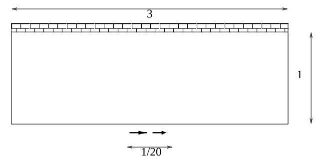

Figure 2: Two bar truss problem

as well as their optimal solutions. This means that the solutions of a sequence of optimization problems with decreasing value of ϵ converges to the original solution of the optimization topology problem. The solution of every relaxed problem is regular and can be attacked with classical mathematical programming algorithms.

A mathematical programming approach of the numerical solution

If one consider the distribution of minimum volume and only stress constraints, the optimal structure is a kind of *fully stressed design*. Thus, the numerical solution of topology optimization problems is characterized by a large number of design variables but also a huge number of active stress constraints. In fact, if the distribution was made only of voids and solids, the percentage of stress constraints being active should be approximately proportional to the ratio of volume used in the structure. The conclusion remains nearly the same with the ϵ-relaxed formulation because relaxation leads to remove the stresses from the active constraint set as soon as the density arrives at its lower bound. Nevertheless, in the designs obtained during the first steps of the optimization process, there exists large zones of intermediate densities and one can have to deal with a huge number of stress constraints.

Our solution procedure is based on a mathematical programming approach using convex approximations and dual solvers and the approach is an extension of the works described in Duysinx [7] and Duysinx et al. [8].

Since the density variables are strictly positive for $\epsilon > 0$, we can treat the stress constraints in a more convenient way:

$$\frac{<\|\sigma\|>}{\sigma_l} - \frac{\epsilon}{\rho} \leq 1 \qquad (5)$$

In this first study of stress constraints, we simply use a CONLIN approximation of the constraints. The experiments showed that the mixed approximations of CONLIN were sufficiently conservative and precise when applied to the statement (5). Solutions of the convex and separable subproblems are realized with a dual method based on a robust second order algorithm designed by Fleury [9]. This algorithm is able to deal with the huge dimensions of the problem and to provide a solution with a reasonable computation time. Important other features to save computational effort are a strategy for selecting potentially dangerous constraints and a careful implementation of sensitivity analysis.

NUMERICAL APPLICATIONS

In the first example, we treat the well known two bar truss topology problem. The two bar truss topology can be recovered by minimizing the volume of material subject to a given bound over the stresses. This problem illustrates clearly the difficulties of topology design with stress constraints. Young's modulus and Poisson's ratio of the material are normalized:

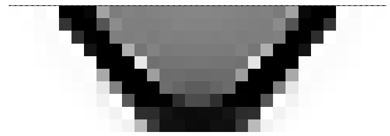

Figure 3: Solution without relaxation

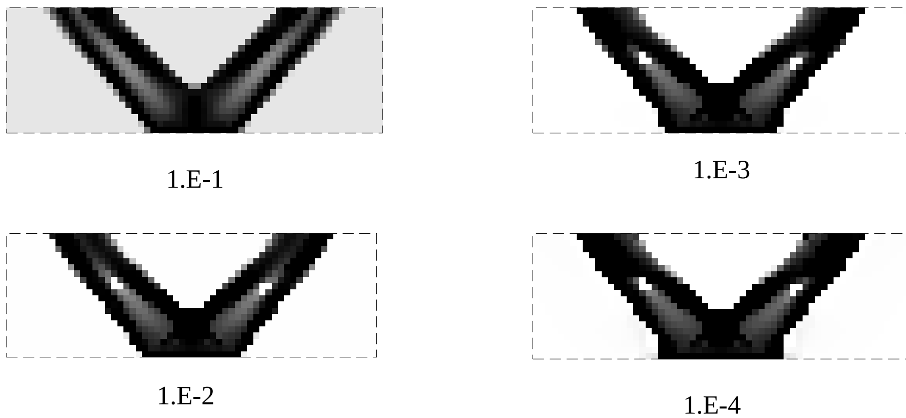

Figure 4: Topology designs with decreasing $\rho_{min} = \epsilon^2$ parameters: 1.e-1,1.e-2, 1.e-3, 1.e-4

$E = 1Nm^2$ and $\nu = 0.3$. Under a distributed shear load of $P = 12N$, the stress is bounded to $\sigma_l = 25.Nm^2$.

If the problem is solved without relaxing the homogenized stress constraints, it is impossible to remove the material membrane between the frame due to the singularity phenomenon. The result is shown in fig.3 on a coarse mesh of 30 by 10 finite elements. To be able to create or delete holes, one has to relax the stress constraints to be able to reach 'singular' designs. Figure 4 shows a sequence of relaxed problems. The problem is solved on a 60 by 20 nine node finite element mesh. The whole optimization process takes 120 iterations and the final area of the structure is 1.010578, *i.e.* around 30 percent of the design area. The effect of the relaxation is shown for example in fig. 5 and in fig. 6. Attention must be paid to the fact that the 2 pictures have not the some grey scale. The relaxed stress criterion is everywhere under the prescribed limit, but in the grey areas, the maximum homogenized von Mises stress is $68Nm^2$. Thus the low density region between the frame is over-stressed compared to the solid in the bars. However, the stress constraints are relaxed so that this region can be removed. During the first design steps the number of active stress constraints is quite large: 1112 potentially dangerous stresses are retained for sensitivity analysis and 648 of them are active in the CONLIN dual optimizer. In the final iterations the number of active constraints is reduced to 446 potentially dangerous stresses and 180 active constraints.

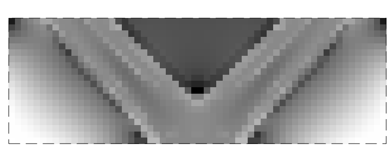

Figure 5: Homogenized von Mises stress for a minimum density of 1.e-1

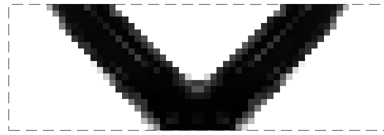

Figure 6: Relaxed stress constraint for a minimum density of 1.e-1

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