
# Characterizing fundamental, superharmonic and subharmonic resonances using phase resonance nonlinear modes

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# Nonlinear normal mode definitions

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) + f_{nl}\big(x(t), \dot{x}(t)\big) = f(t)$$

Rosenberg: Periodic solution of the underlying conservative system

$f(t) = C\dot{x}(t) = 0 \quad \& \quad f_{nl}(\dot{x}) = 0$

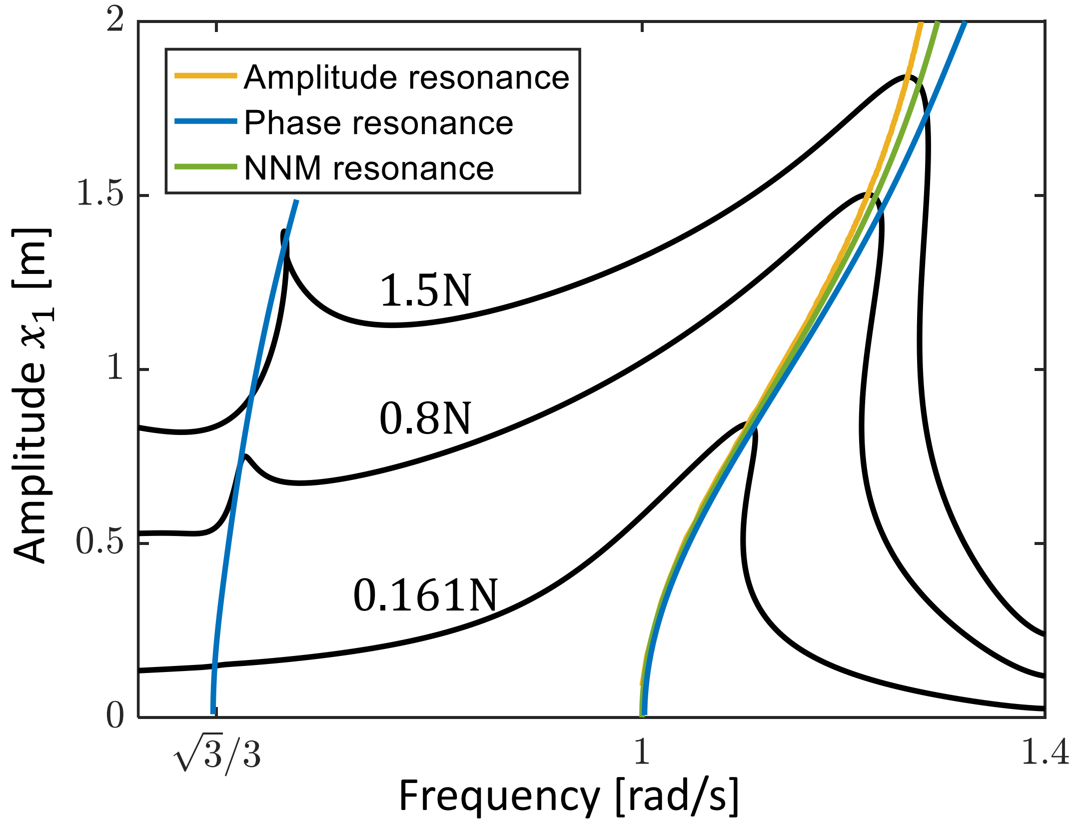
Krack: Extension of the periodic motion concept to nonconservative systems through additional damping

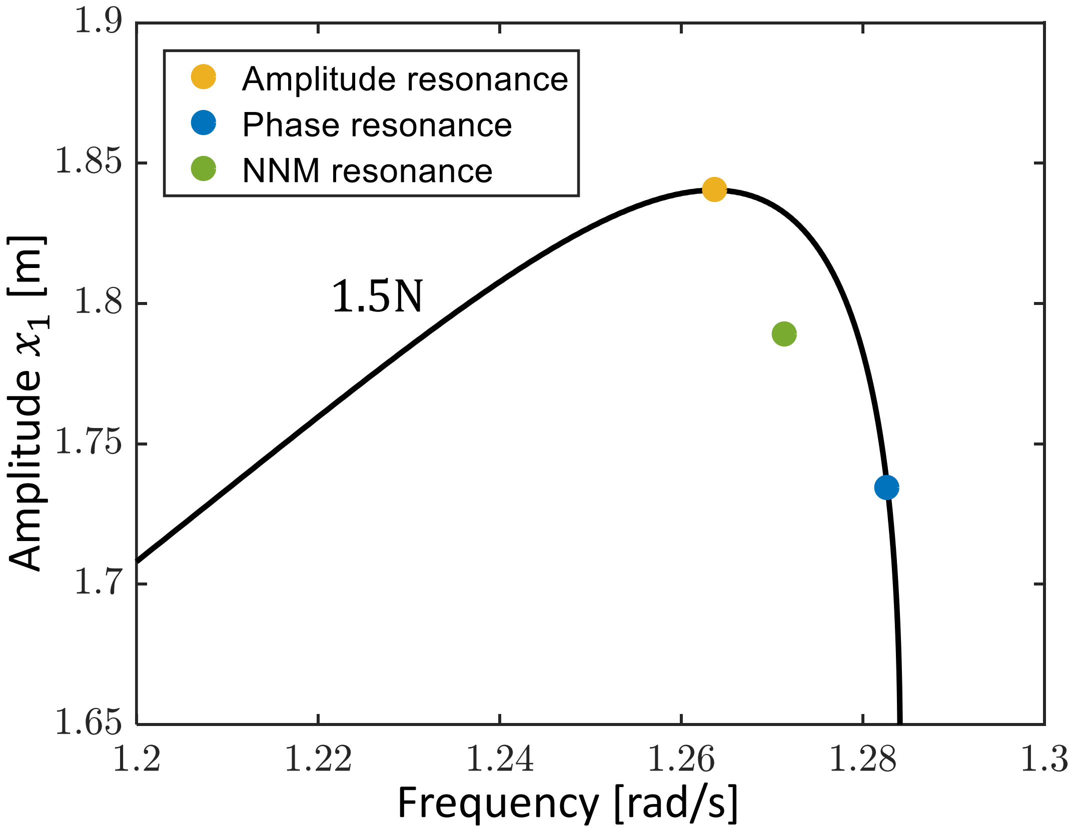
$f(t) = \xi M\dot{x}(t)$

Limitations: Unpractical multi-harmonic, multi-point forcing
Superharmonic and subharmonic resonances not considered

# Motivating example

$$\begin{cases} \ddot{x}_1 + 0.02\dot{x}_1 - 0.01\dot{x}_2 + 2x_1 - x_2 + x_1^3 = f\sin\omega t \\ \ddot{x}_2 + 0.11\dot{x}_2 - 0.01\dot{x}_1 + 2x_2 - x_1 = 0 \end{cases}$$



# $k\colon\nu$ harmonic resonance

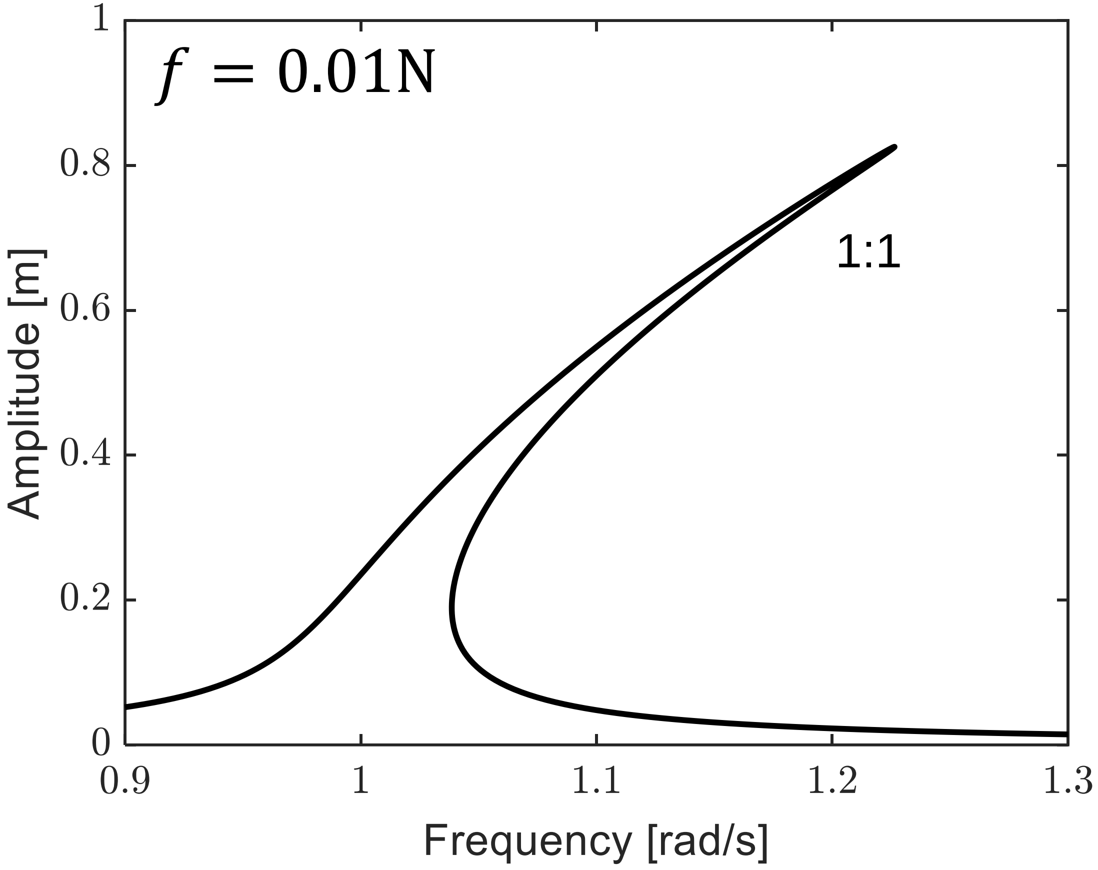
$$x(t) = \frac{c_0}{\sqrt{2}} + \sum_{k=1}^{\infty} \left(c_k \cos k \frac{\omega}{\nu} t + s_k \sin k \frac{\omega}{\nu} t\right)$$

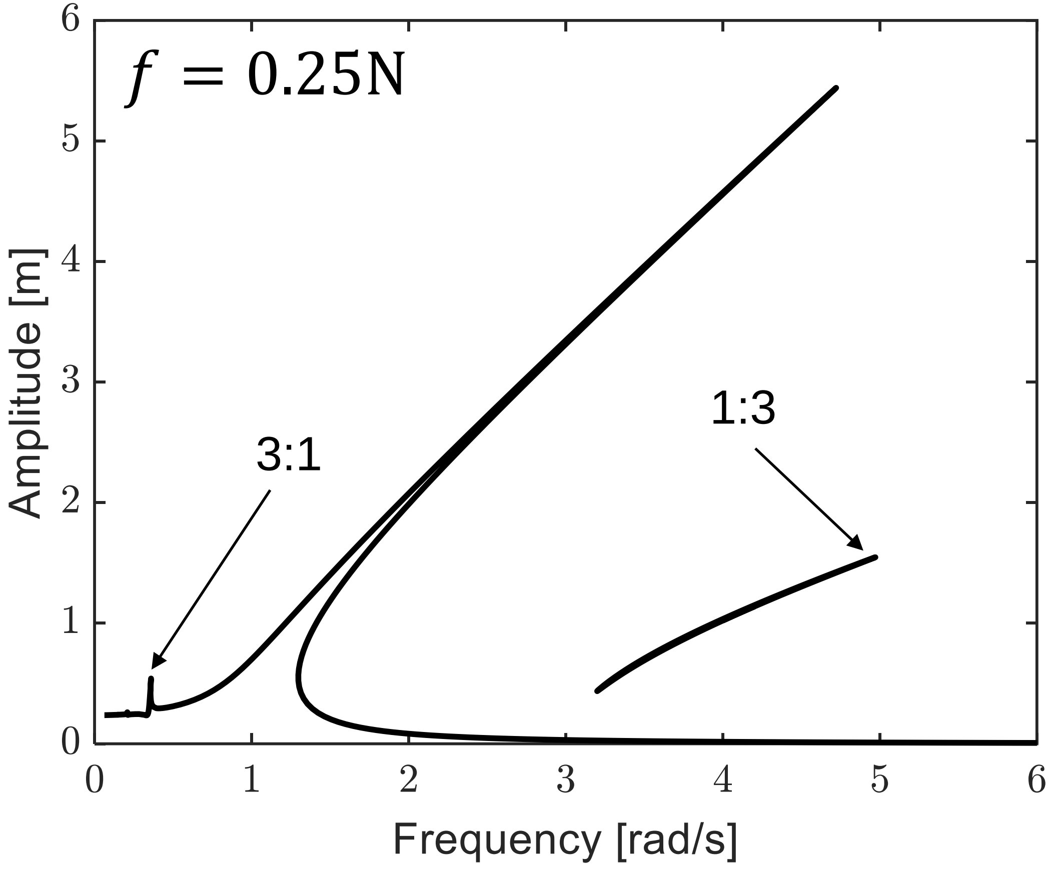
Any harmonic component can trigger a resonance $k\colon\nu$ as long as the relation $k\frac{\omega}{\nu}$ is the frequency of a fundamental resonance of the system

$k = \nu$: fundamendal resonance resonance
$k < \nu$: subharmonic resonance
$k > \nu$: superharmonic resonance

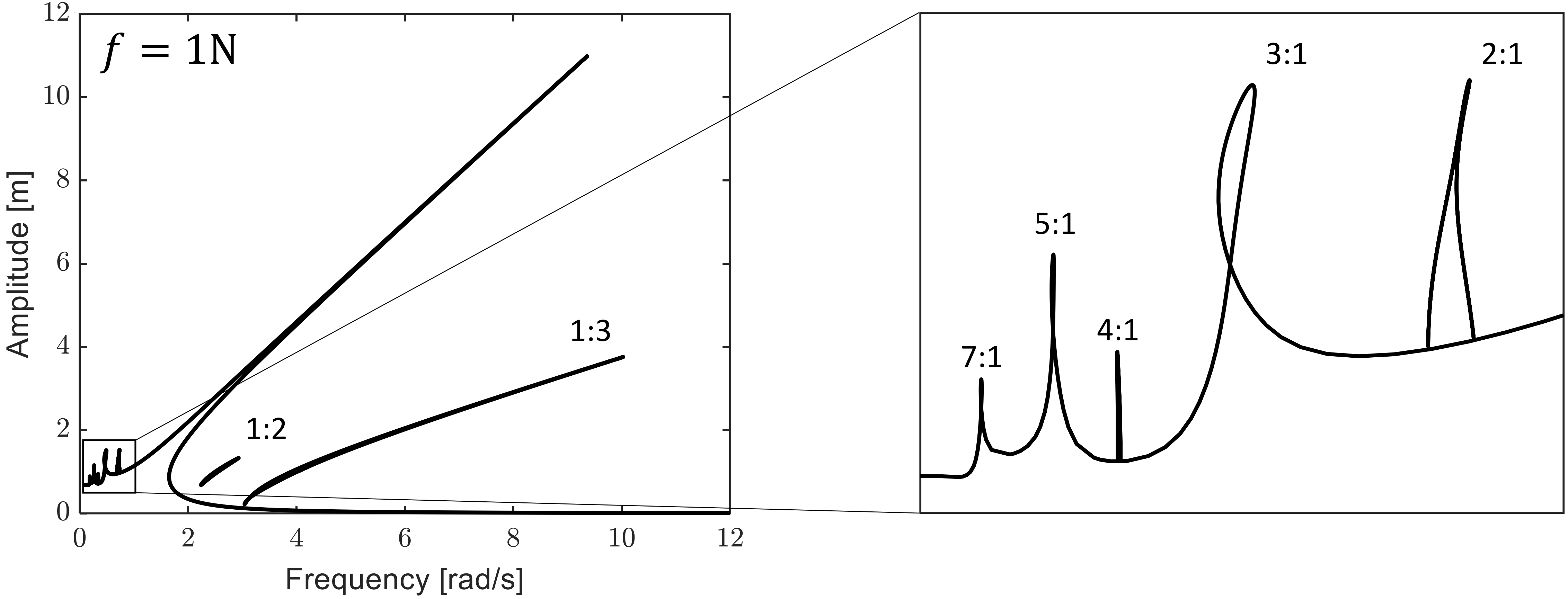
# Illustration on a Duffing oscillator (I)

$$\ddot{x}(t) + 0.01\dot{x}(t) + x(t) + x^3(t) = f \sin \omega t$$

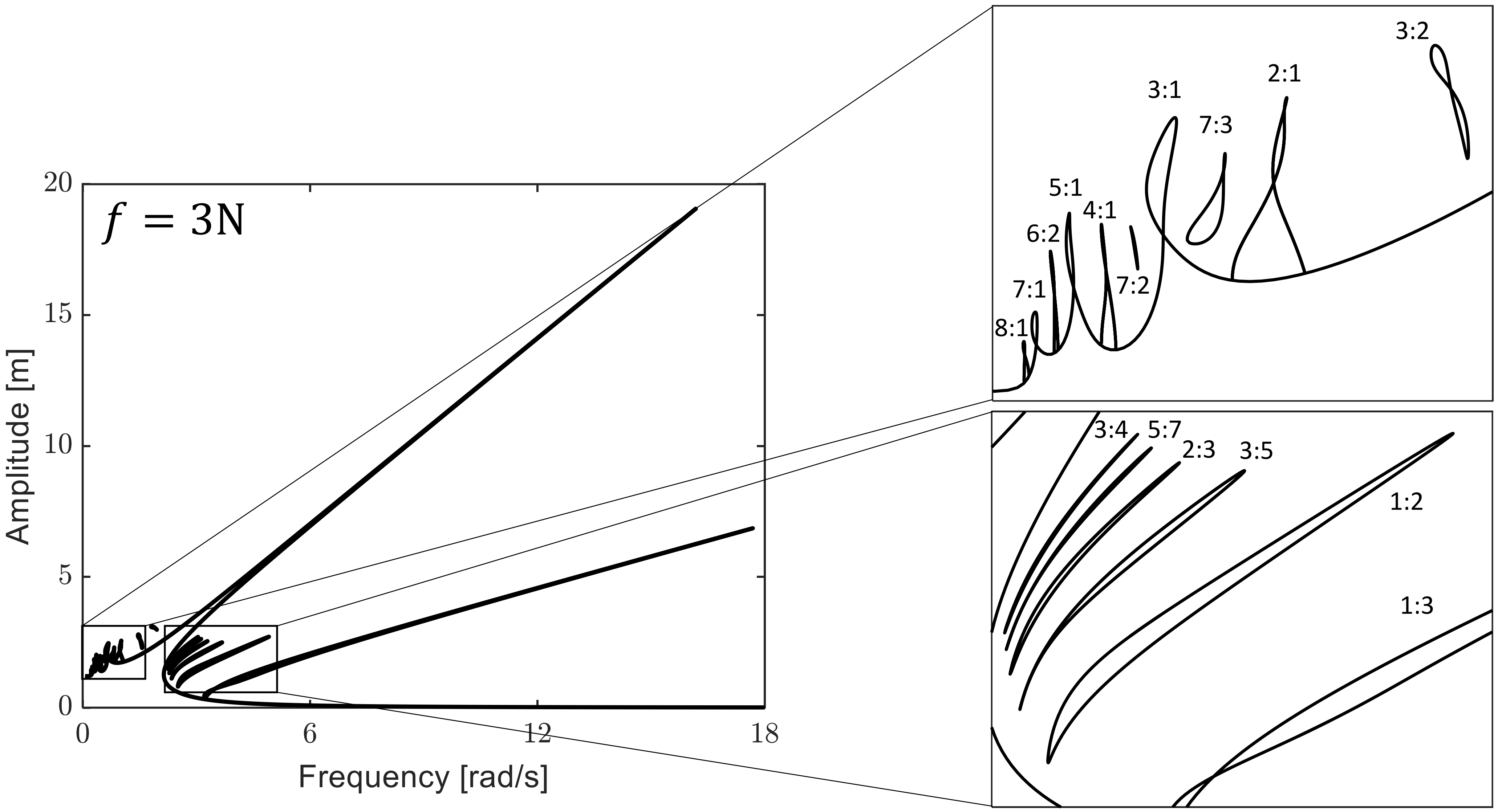


# Illustration on a Duffing oscillator (II)



# Illustration on a Duffing oscillator (III)



# Phase lag of harmonic $k$

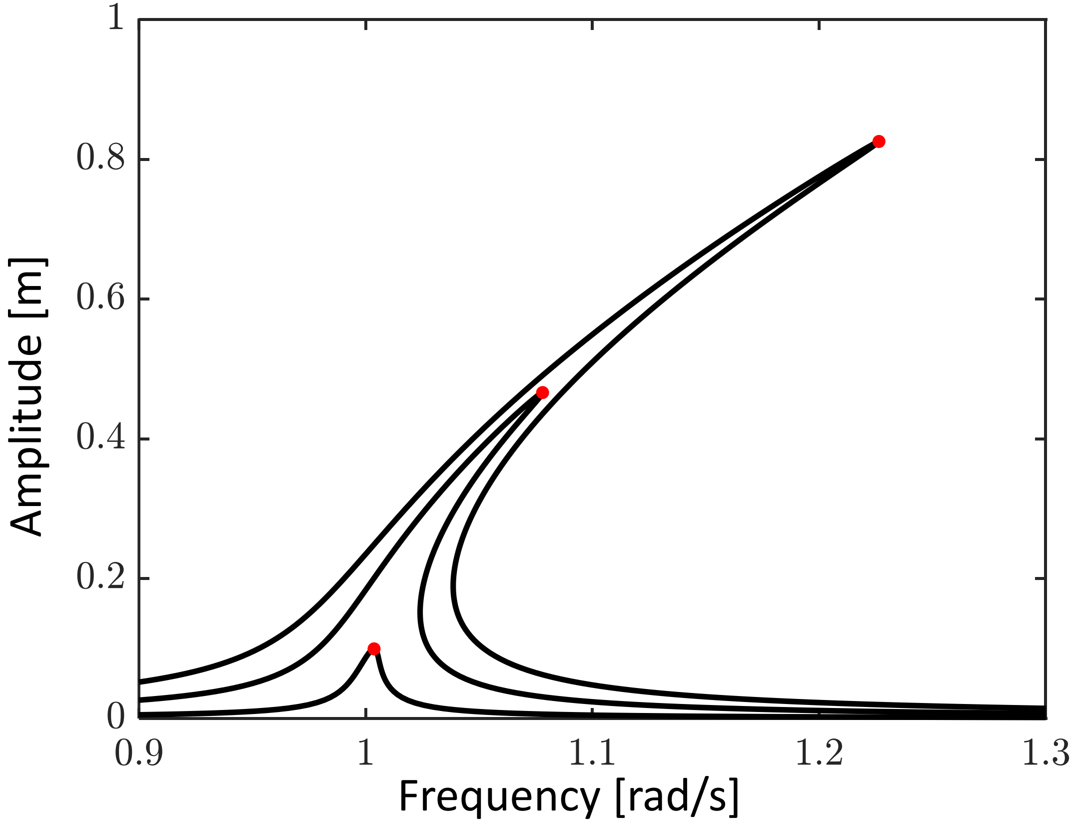
Each harmonic component $k$ can be rewritten

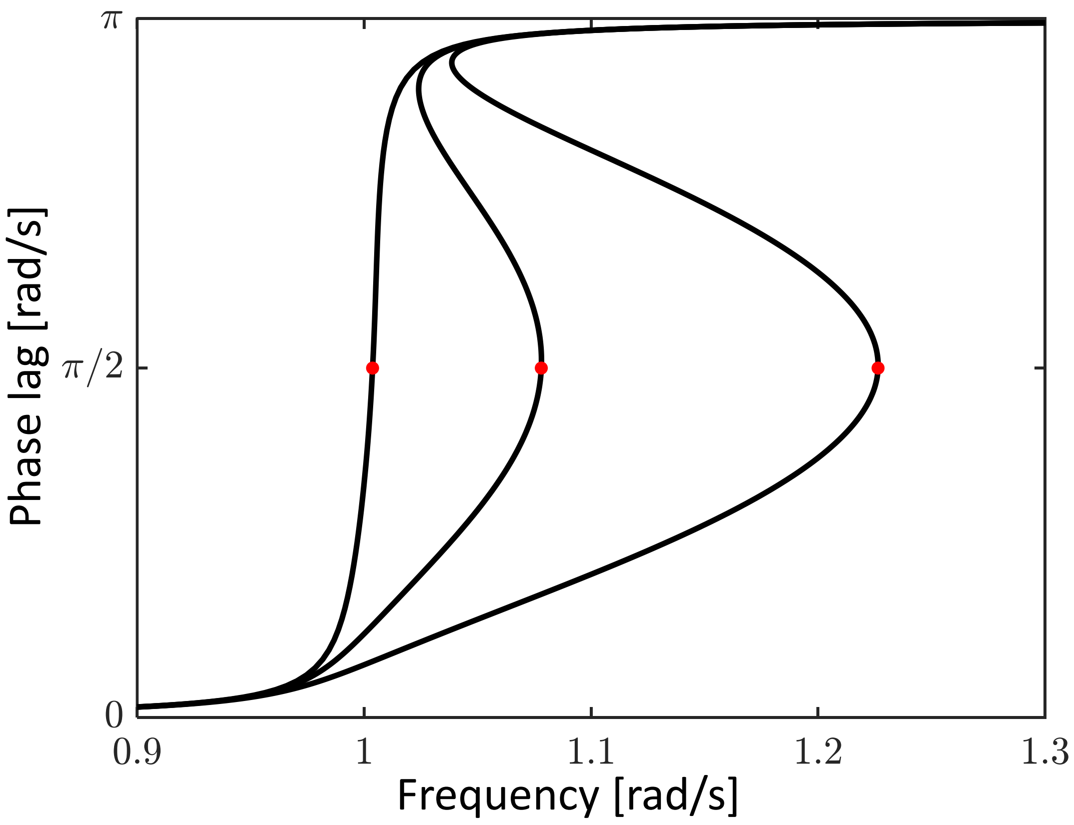
$$x_k(t) = A_k \sin\left(k\frac{\omega}{\nu}t - \phi_k\right)$$

$$A_k = \sqrt{s_k^2 + c_k^2} \qquad \phi_k = \mathrm{atan2}(-c_k, s_k)$$

# Fundamental resonance

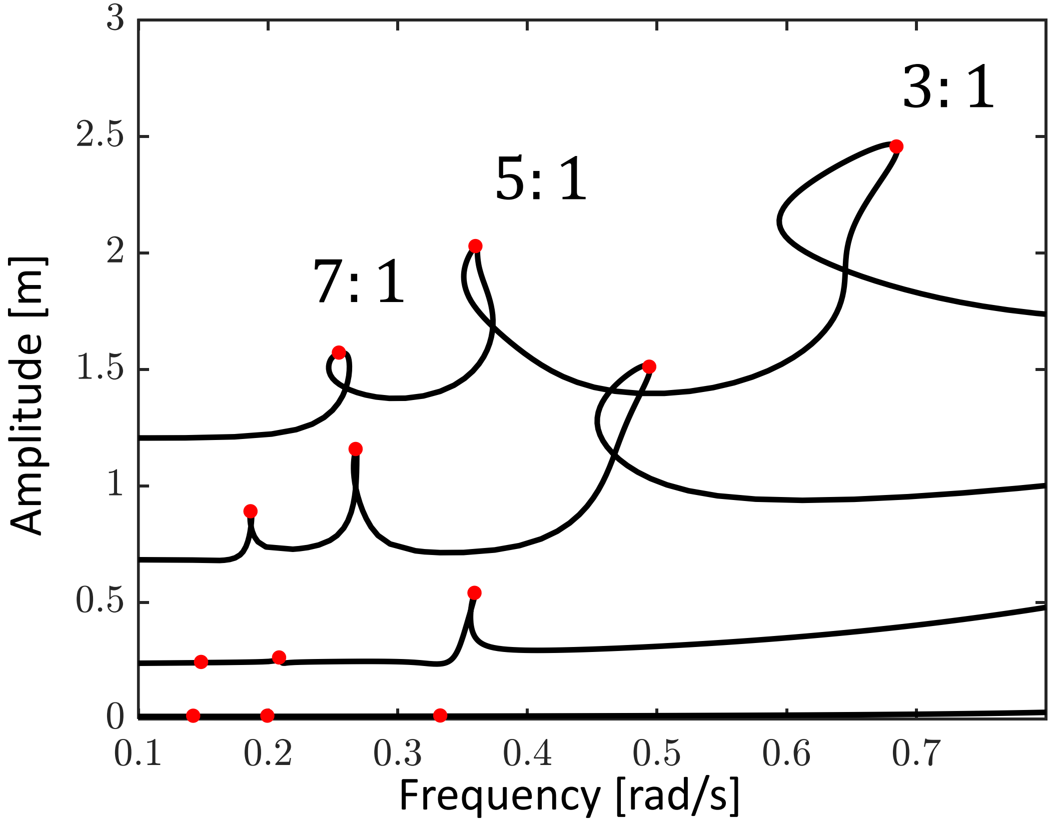
Phase resonance occurs for a phase lag of $\frac{\pi}{2}$ between the harmonic $k = 1$ and the forcing

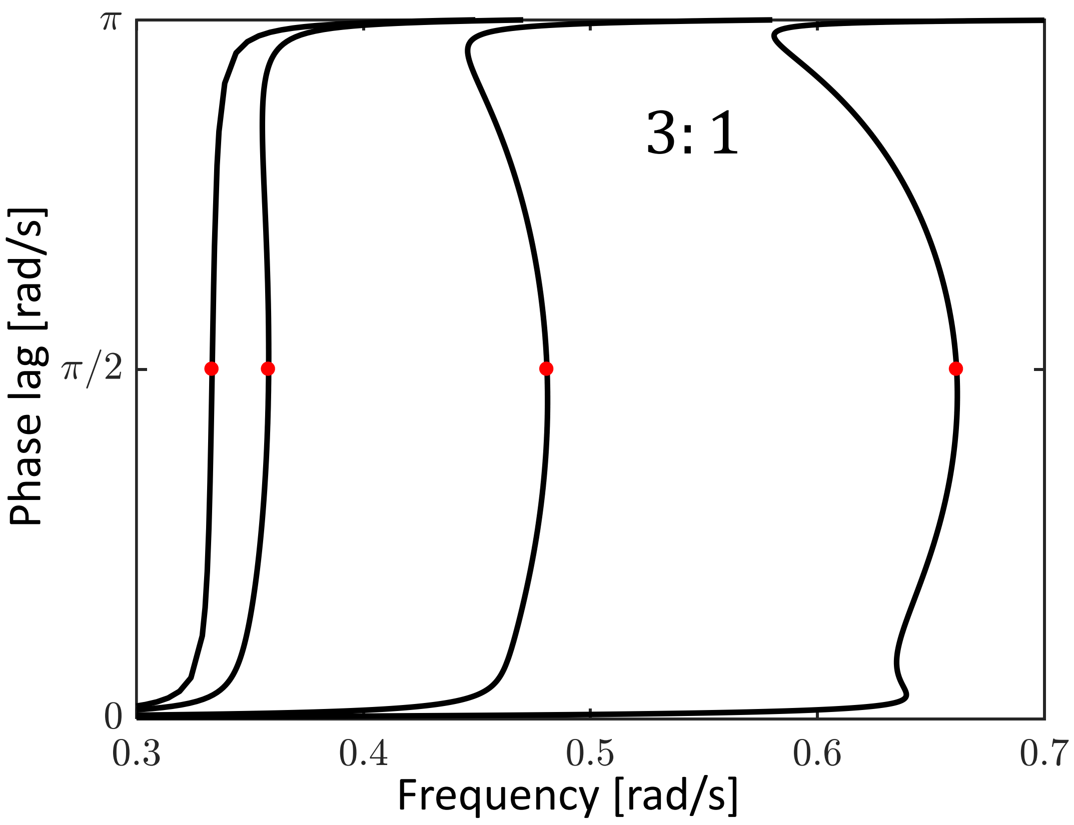


# Odd superharmonic resonance $k$: 1

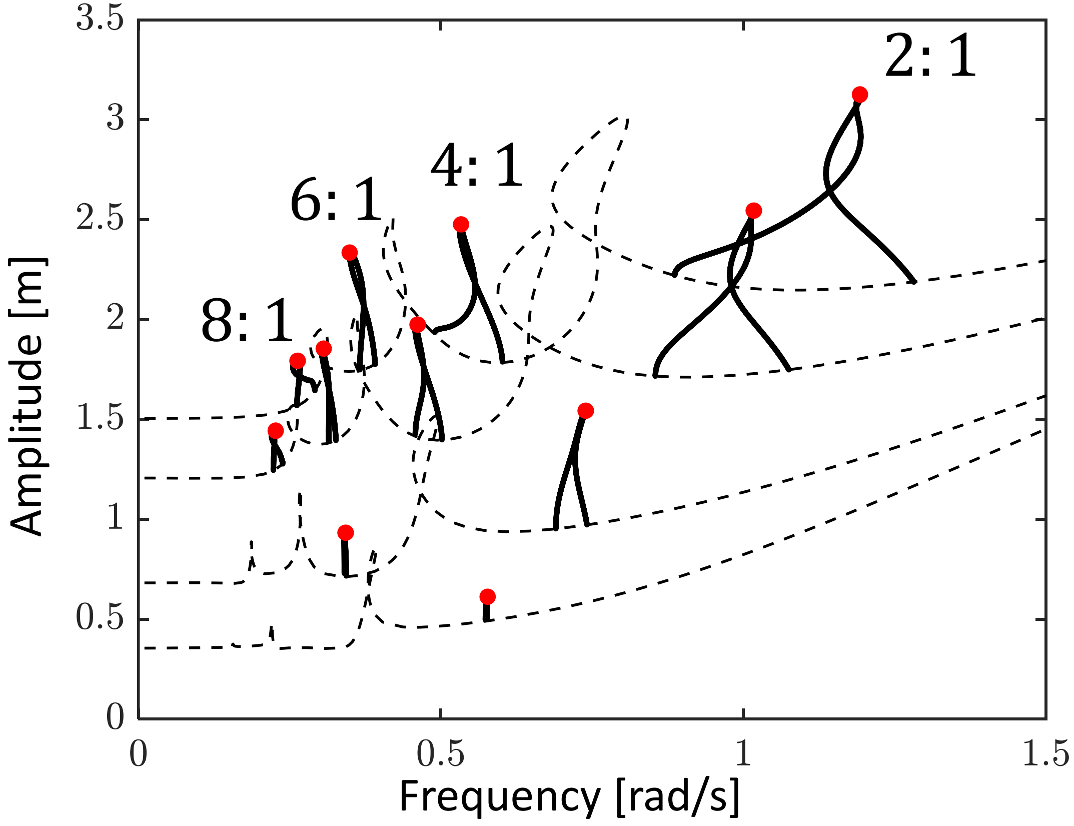
Phase resonance occurs for a phase lag of $\frac{\pi}{2}$ between the harmonic $k$ and the forcing

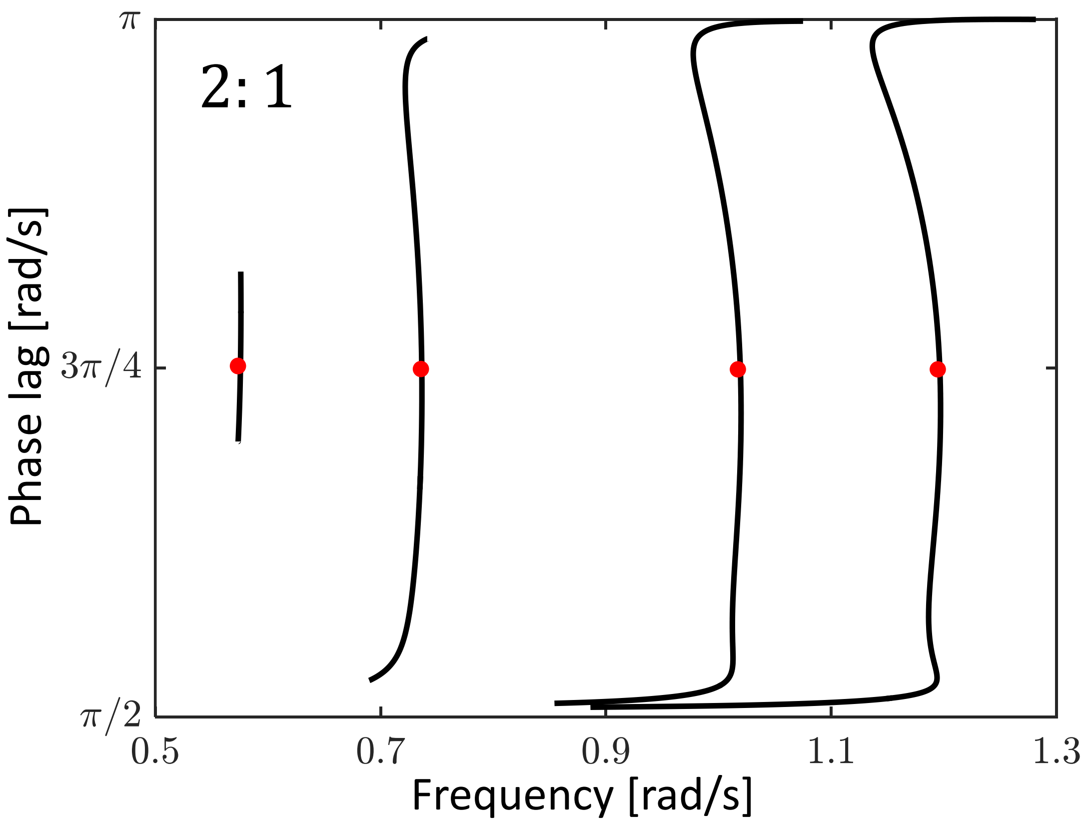


# Even superharmonic resonance $k\!:1$

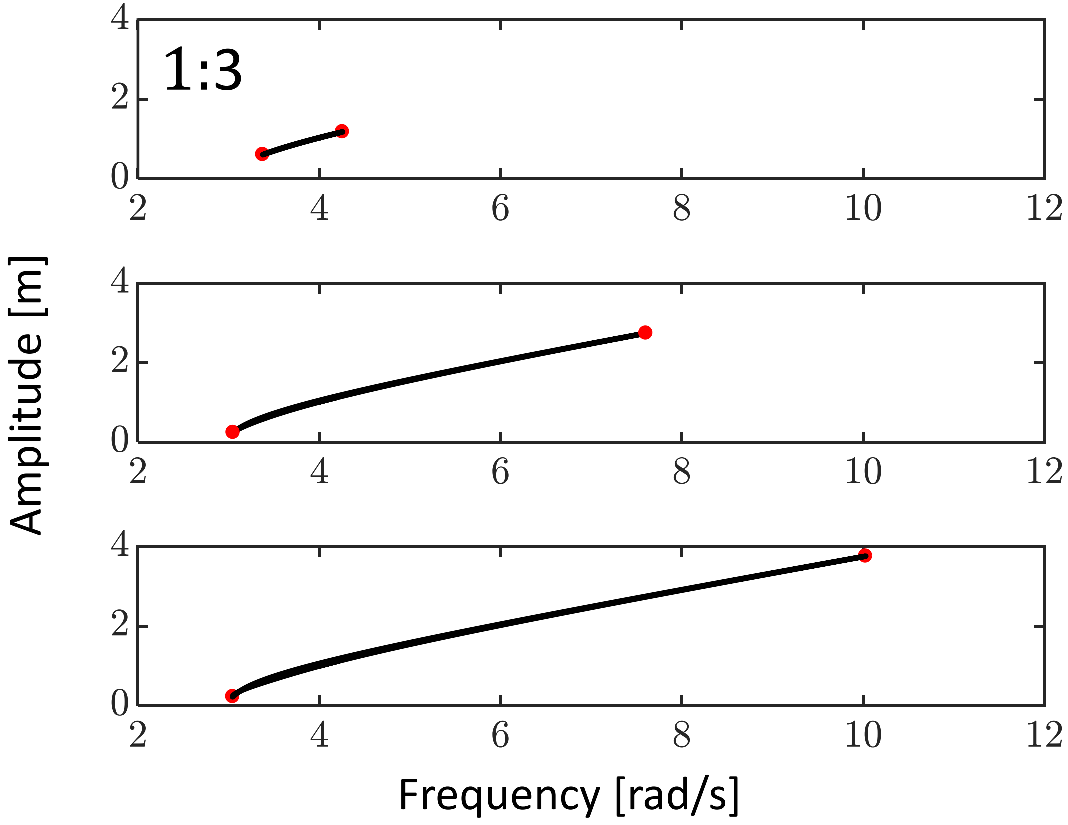
Phase resonance occurs for a phase lag of $\frac{3\pi}{4}$ between the harmonic $k$ and the forcing

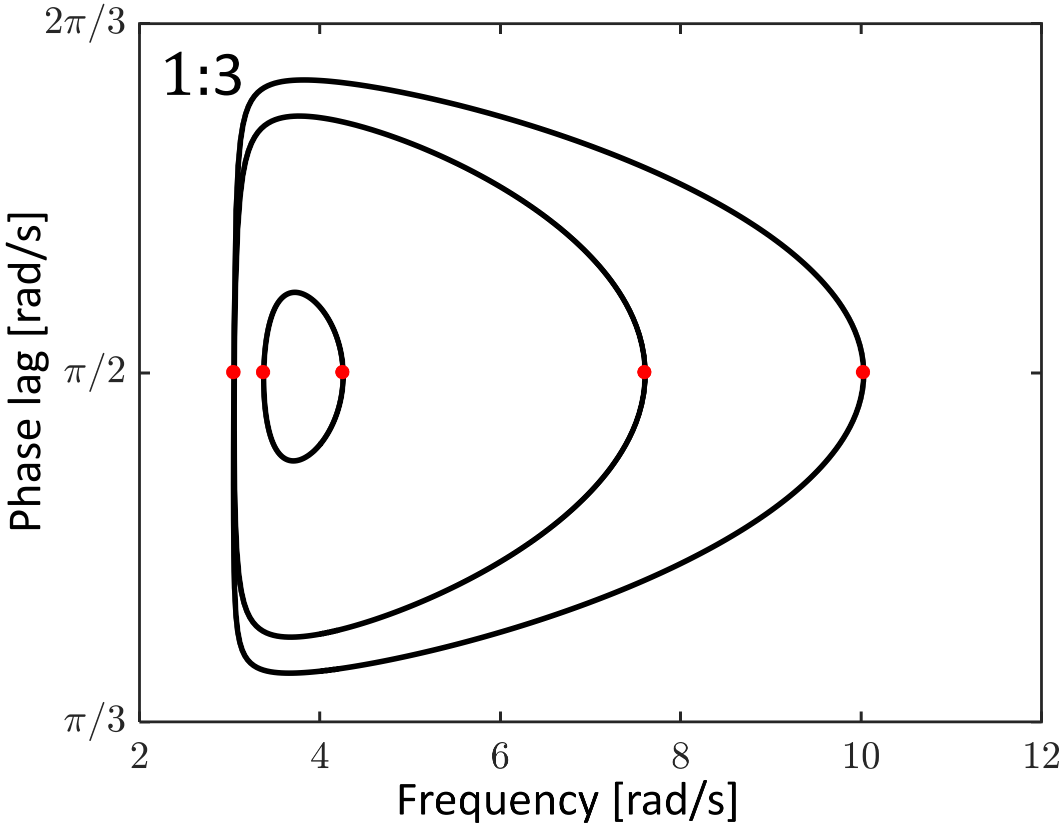


# Odd subharmonic resonance 1: $\nu$

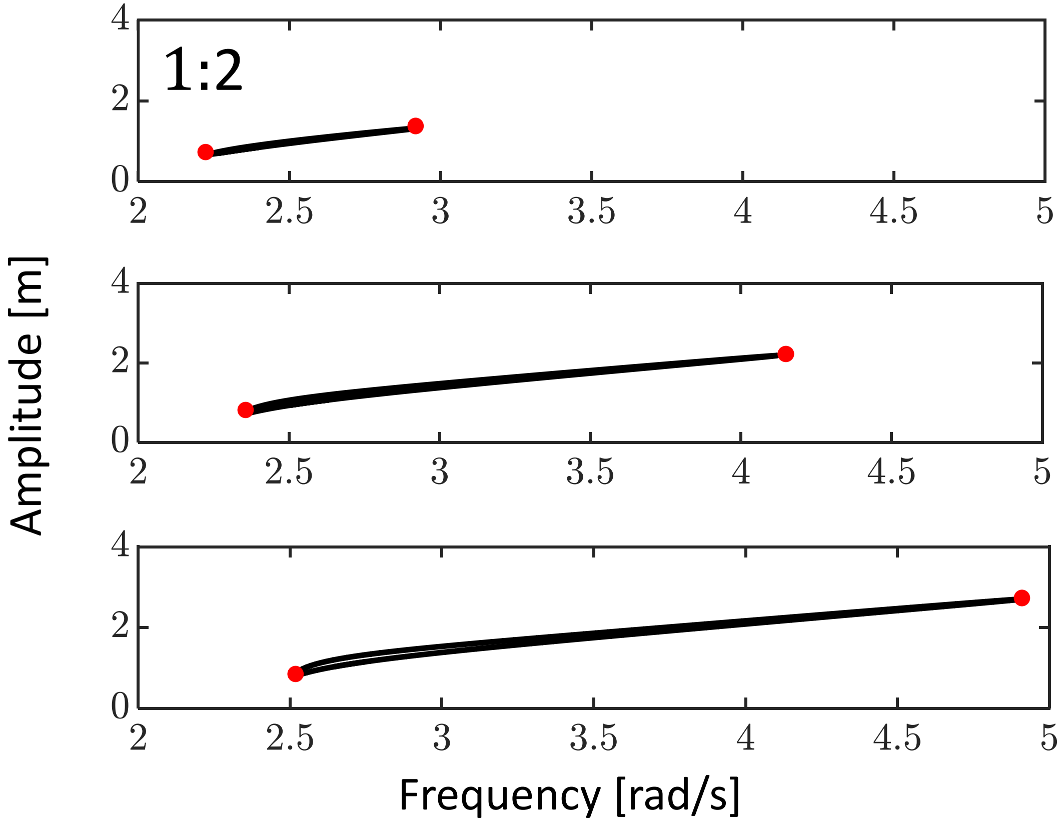
Phase resonance occurs for a phase lag of $\frac{\pi}{2}$ between the harmonic 1 and the forcing, when $\nu > 1$

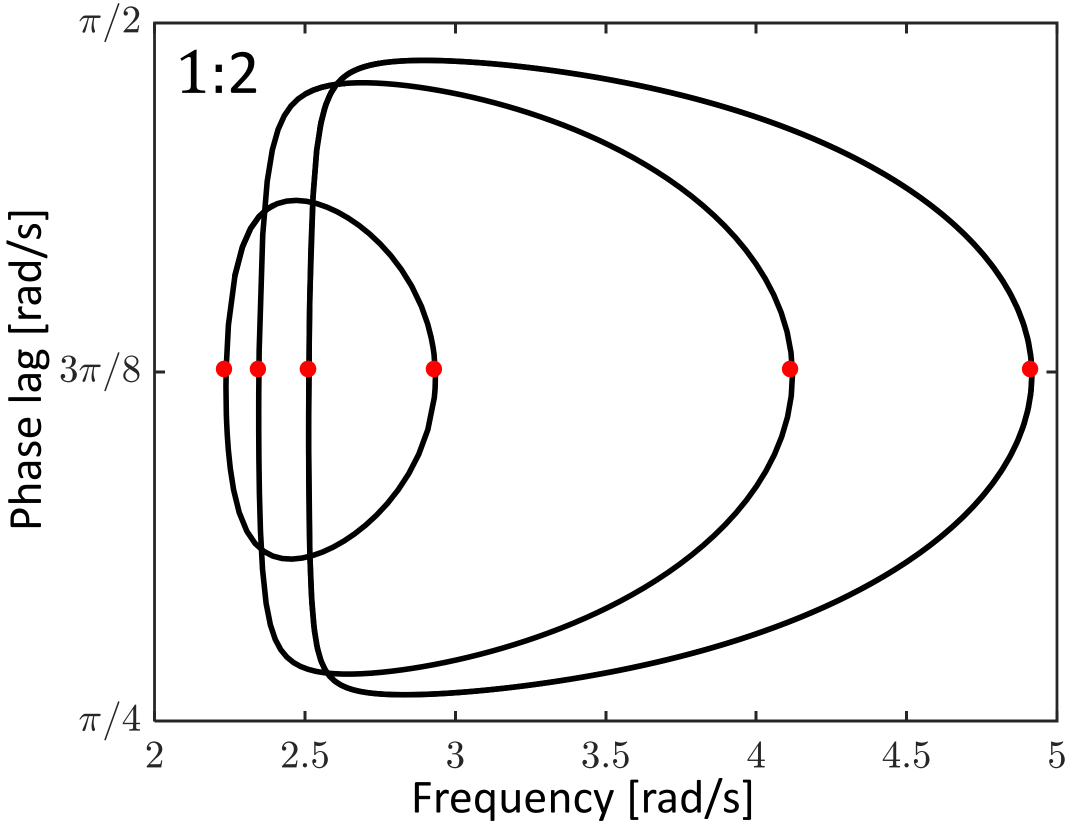


# Even subharmonic resonance $1\!:\!\nu$

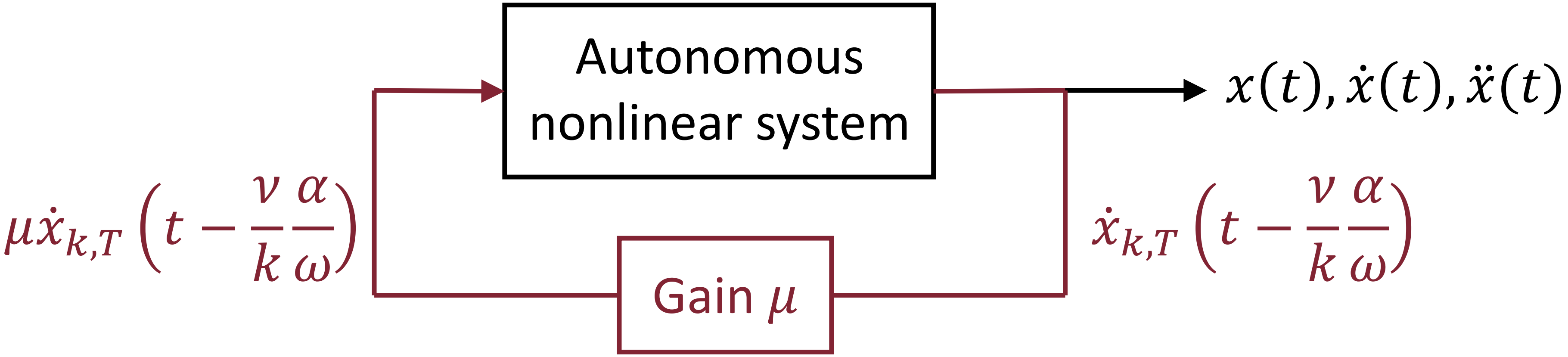
Phase resonance occurs for a phase lag of $\frac{3\pi}{8}$ between the harmonic 1 and the forcing, when $\nu > 1$



# Phase resonance nonlinear mode

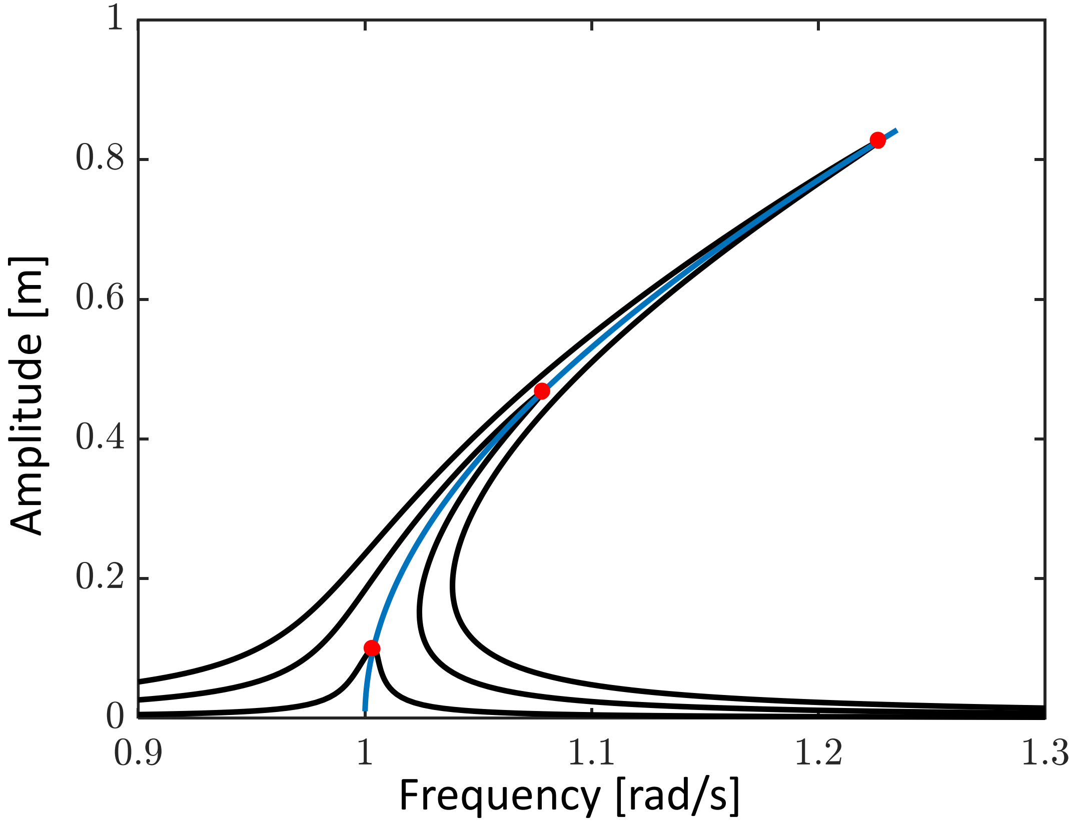
By feeding back the $T$-periodic $k$-th harmonic of the velocity of the considered DOF shifted by a delay $\frac{\nu}{k}\frac{\alpha}{\omega}$, with $\alpha = \frac{\pi}{2} - \delta_k$, the autonomous system is driven into a phase resonance nonlinear mode.

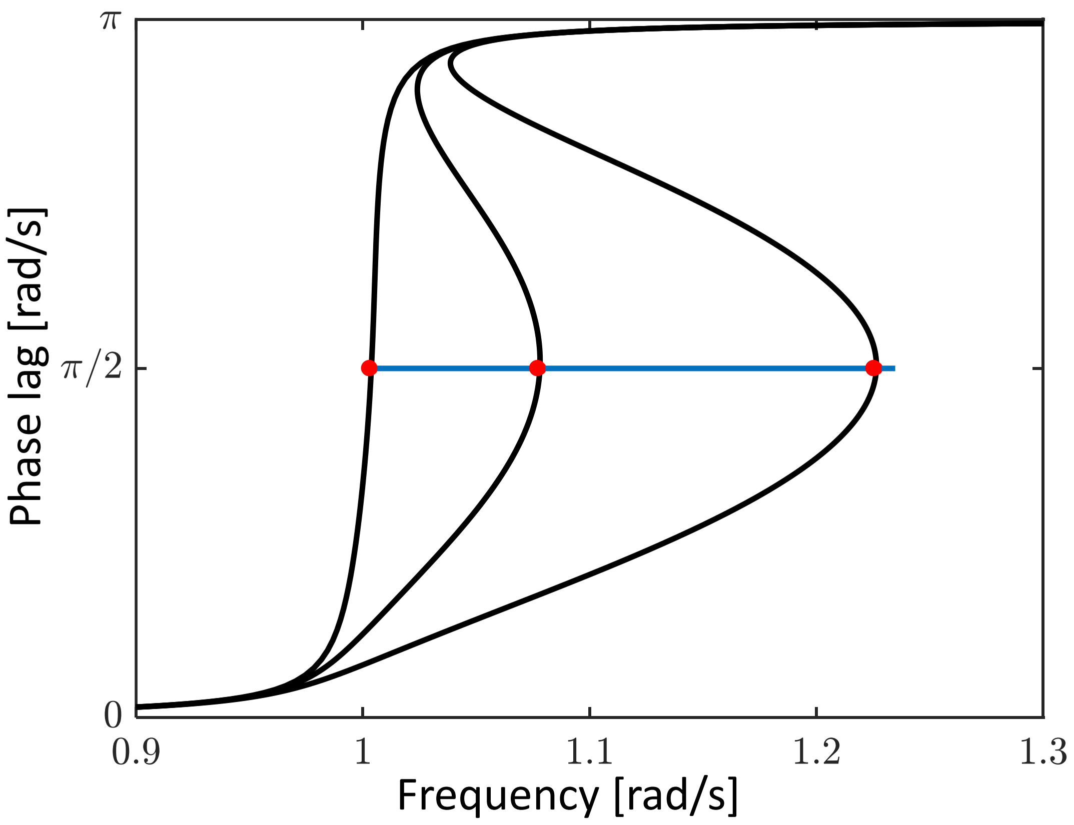



$\delta_k$ is the phase lag at the $k$:$\nu$ resonance $\begin{cases} \delta_k = \frac{\pi}{2} & \text{If } k \text{ and } \nu \text{ are odd} \\ \delta_k = \frac{3\pi}{4\nu} & \text{If either } k \text{ or } \nu \text{ is even} \end{cases}$

# PRNMs of the fundamental resonance

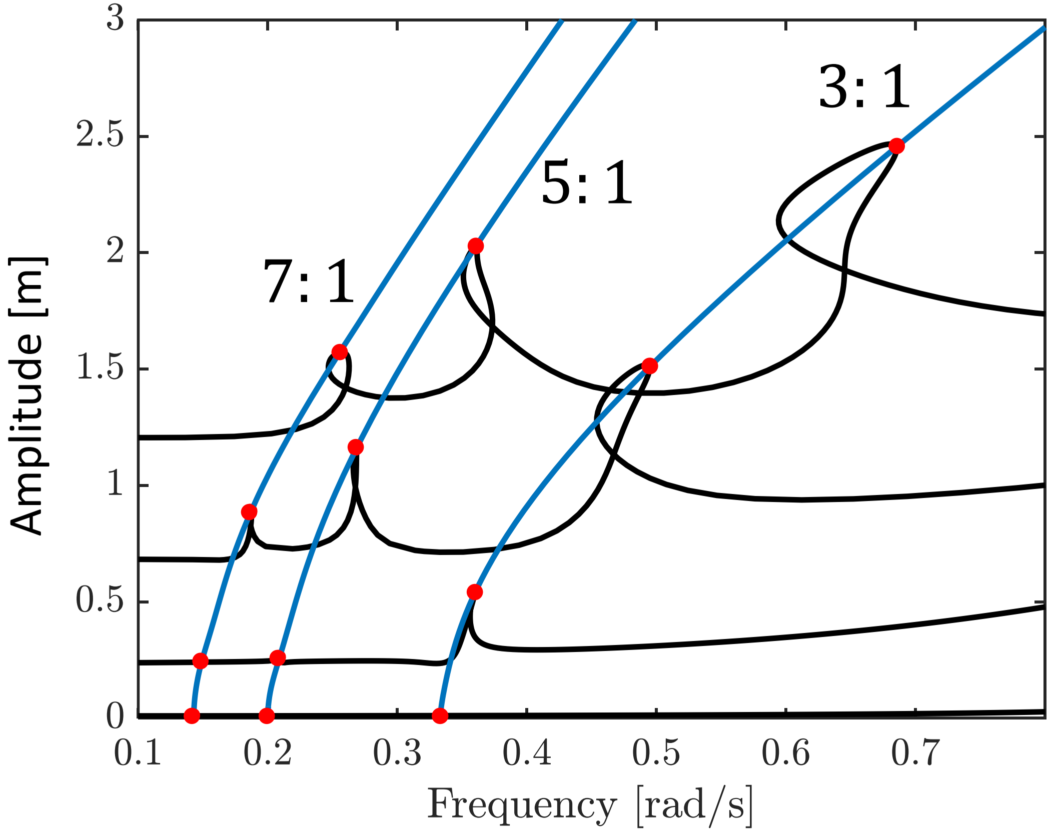
$$\ddot{x}(t) + 0.01\dot{x}(t) + x(t) + x^3(t) - \mu\dot{x}_{1,T}(t) = 0$$

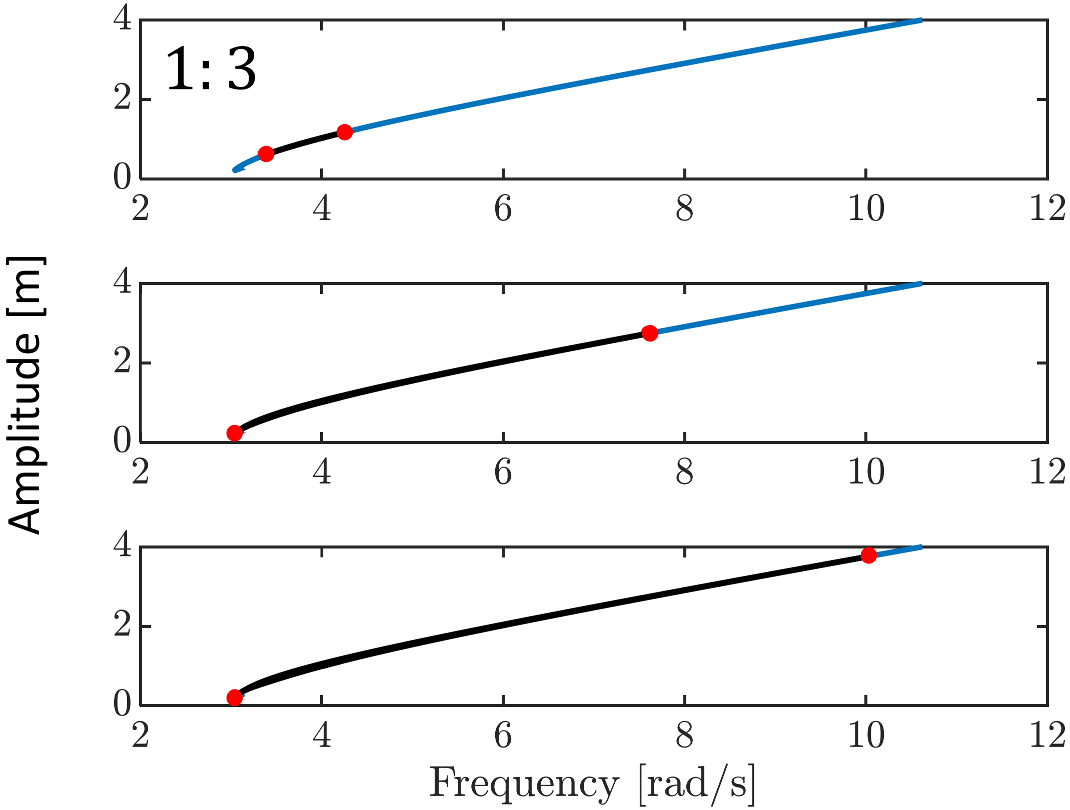


# PRNMs of odd resonances

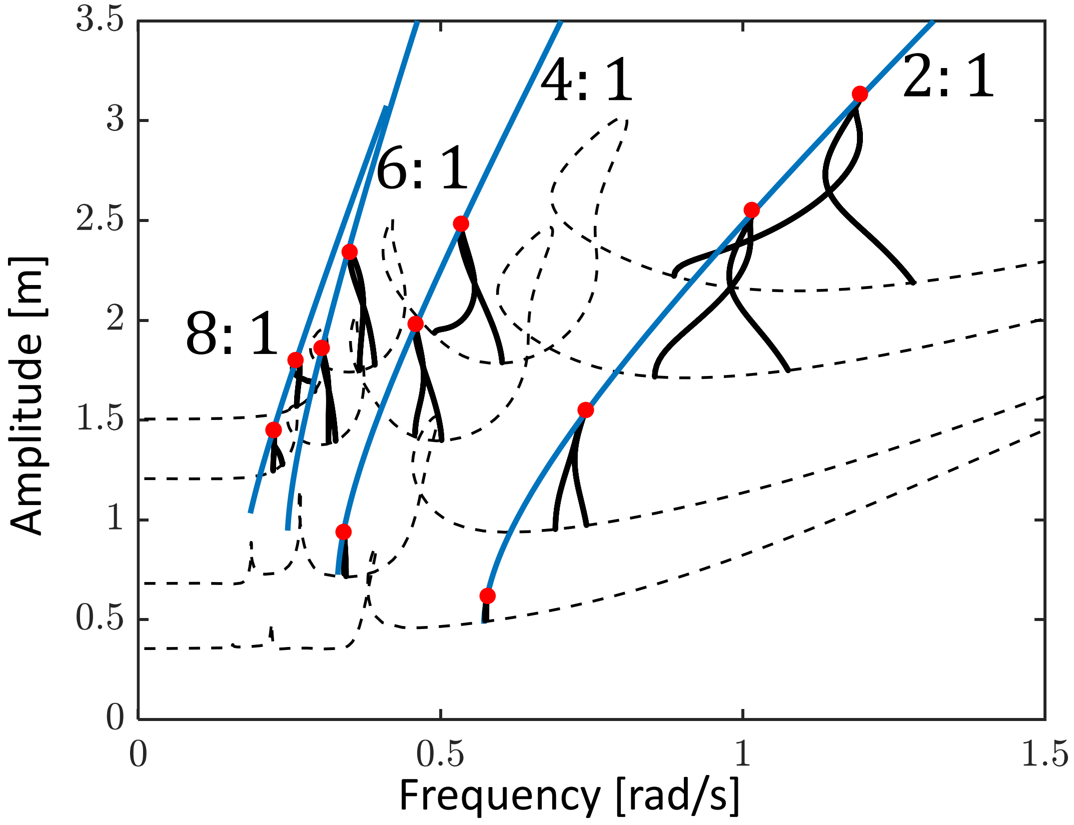
$$\ddot{x}(t) + 0.01\dot{x}(t) + x(t) + x^3(t) - \mu\dot{x}_{k,T}(t) = 0$$

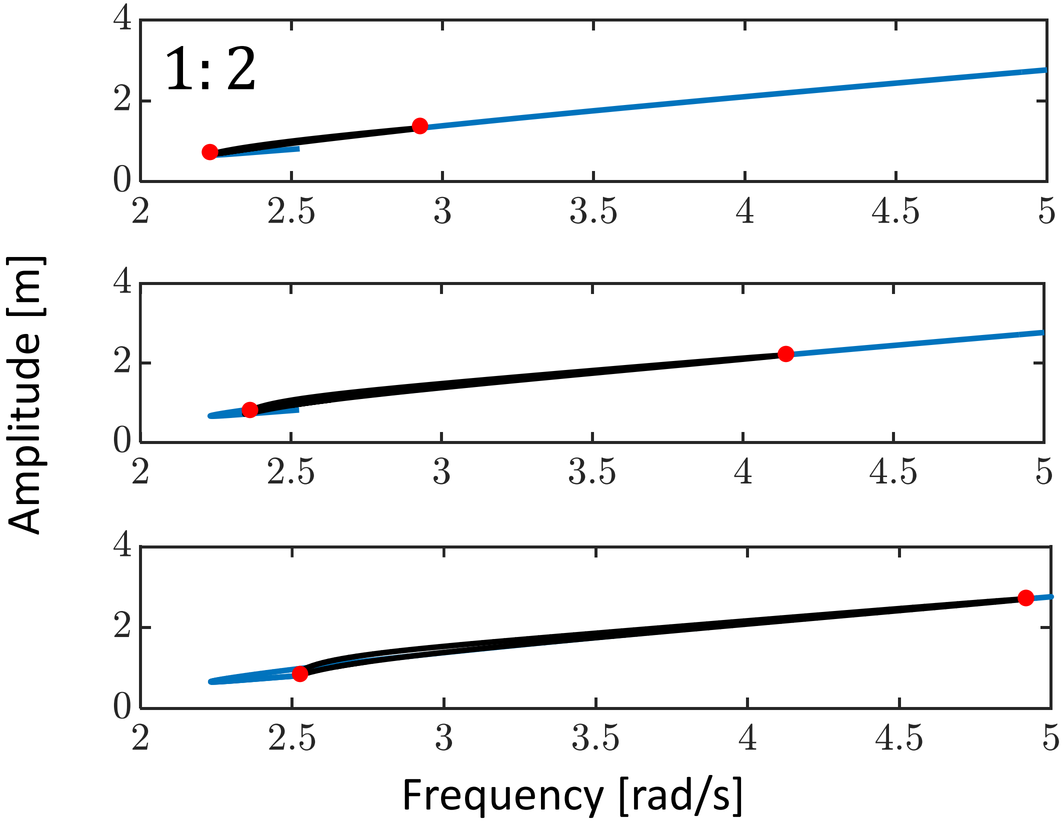


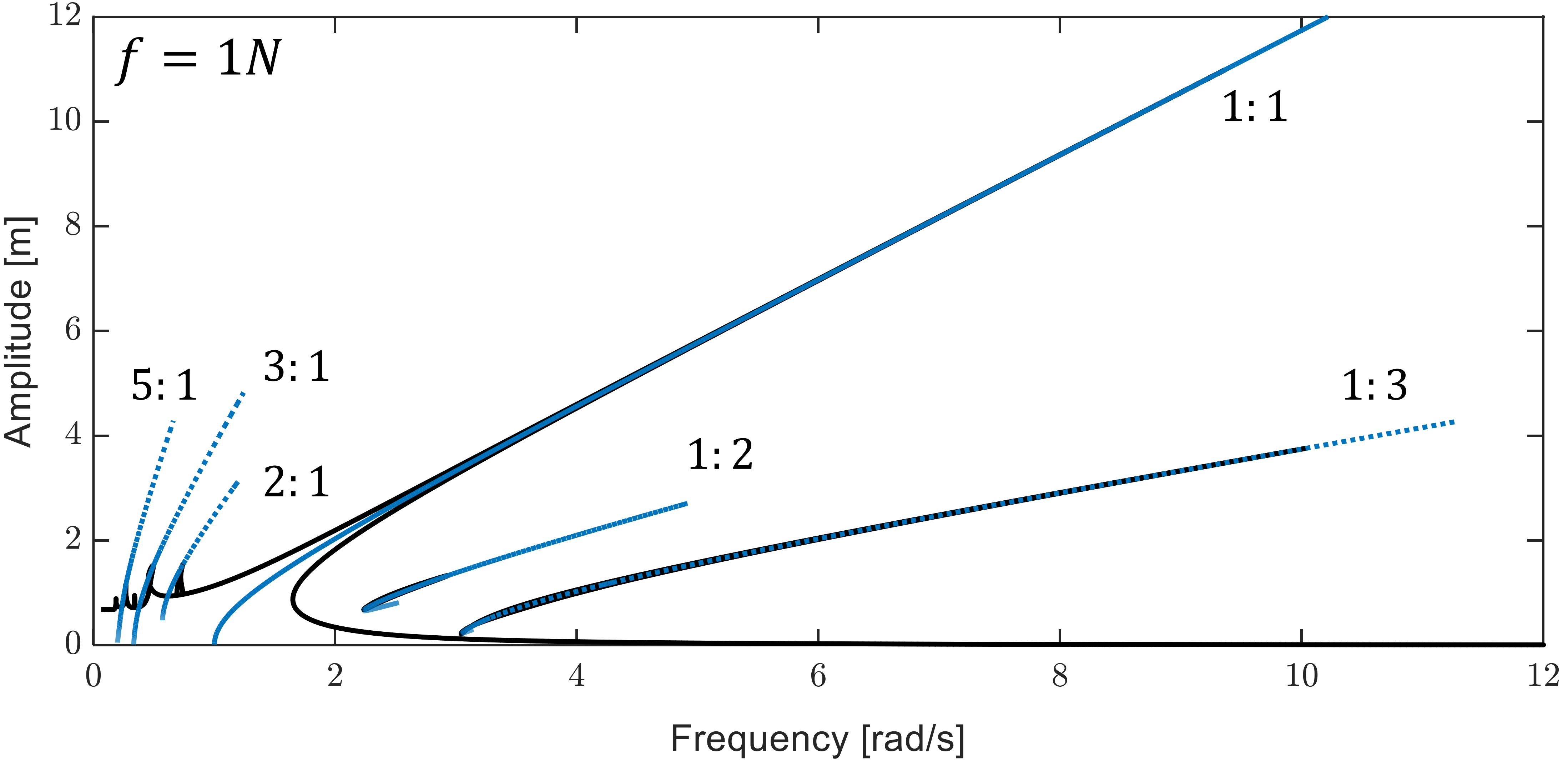
# PRNMs of even resonances

$$\ddot{x}(t) + 0.01\dot{x}(t) + x(t) + x^3(t) - \mu \dot{x}_{k,T}\left(t - \frac{\nu}{k}\frac{\alpha}{\omega}\right) = 0$$



# PRNMs of the Duffing oscillator



# Conclusions and key findings

- The $k{:}\,\nu$ resonance can be enforced by feeding back the $T$-periodic $k$-th harmonic of the velocity of the considered DOF shifted by a delay $\frac{\nu}{k}\frac{\alpha}{\omega}$ into the autonomous system

- The phase lag $\phi_k$ at which the resonance occurs depends on the parity of $k$ and $\nu$ and is either $\frac{\pi}{2}$ or $\frac{3\pi}{4\nu}$

- The exact locus of the phase resonance points, under single-point, harmonic forcing, can be computed thanks to the PRNM equation