

Characterizing Fundamental, Superharmonic, and Subharmonic Resonances Using Phase Resonance Nonlinear Modes



Martin Volvert and Gaëtan Kerschen

1 Introduction

Modal analysis has been, and continues to be, the dominant dynamical method used in structural design. The goal of modal analysis is to find the vibration modes, resonance frequencies, and damping ratios of the considered system [1]. One key assumption of modal analysis is linearity, which, however, real-world structures violate because they may feature advanced materials, friction, and contact [2]. The theory of nonlinear normal modes (NNMs) was developed to generalize the concept of a vibration mode to nonlinear systems [3]. In direct analogy to a linear mode, Rosenberg defined a NNM as a synchronous vibration of the undamped, unforced system for which all points reach their extreme values or pass through zero simultaneously [4, 5]. This definition is only valid for multi-point, multi-harmonic forcing, which is not always used in practice. The focus of this chapter is on phase resonances for fundamental resonances of harmonically forced, damped systems as well as for superharmonic and subharmonic resonances. For these latter resonances, the phase lag between the harmonic of interest of the displacement and the forcing may not necessarily be equal to $\pi/2$, unlike fundamental resonances. In this context, we propose herein a generalization of phase resonance of nonlinear systems for which the corresponding structural deformation is termed a phase resonance nonlinear mode (PRNM). These PRNMs are applied to the well-known Duffing oscillator.

M. Volvert (✉) · G. Kerschen

Space Structures and Systems Laboratory, Department of Aerospace and Mechanical Engineering, University of Liège, Liège, Belgium

e-mail: m.volvert@uliege.be; g.kerschen@uliege.be

2 Resonances of the Duffing Oscillator

The Duffing oscillator comprises a mass attached to linear and cubic springs and a linear damper. The governing equation of motion of the harmonically forced Duffing oscillator is

$$\ddot{x}(t) + 0.01\dot{x}(t) + x(t) + x^3(t) = f \sin \omega t, \quad (1)$$

where f is the forcing amplitude, whereas ω is the excitation frequency.

Considering the Fourier decomposition of the displacement, where the positive integer ν takes into account the subharmonics of the excitation frequency ω ,

$$x(t) = \frac{c_0}{\sqrt{2}} + \sum_{k=1}^{\infty} \left(s_k \sin \left(k \frac{\omega}{\nu} t \right) + c_k \cos \left(k \frac{\omega}{\nu} t \right) \right) \quad (2)$$

shows that many resonances exist in this simple system. Specifically, each harmonic component of the displacement can trigger a resonance as long as the relation $\frac{k}{\nu}\omega$ corresponds to the frequency of the fundamental resonance of the system. When the ratio $\frac{k}{\nu}$ is lower (greater) than 1, the resonance is said to be subharmonic (superharmonic) and is located after (before) the fundamental resonance. In this chapter, the resonances are divided into four categories, namely:

- Fundamental resonance ($k = 1, \nu = 1$)
- Superharmonic resonance $k:\nu$ ($k > \nu, \nu = 1$)
- Subharmonic resonance $k:\nu$ ($\nu > k, k = 1$)
- Other superharmonic and subharmonic resonances $k:\nu$ ($k > 1, \nu > 1$)

Superharmonic and subharmonic resonances can further be divided into subcategories depending on the parity of k and ν .

The goal of this section is to analyze carefully the resonant response of the Duffing oscillator, as previously achieved in [6]. To this end, the system is analyzed considering four different forcing amplitudes f , i.e., 0.01N, 0.25N, 1N, and 3N. The nonlinear frequency response curves (NFRCs) are depicted in Fig. 1. For a forcing amplitude of 0.01N in Fig. 1a, the only nonlinear effect appearing in the NFRC is the hardening of the fundamental resonance. At 0.25N in Fig. 1b, 3:1 superharmonic and 1:3 subharmonic resonance branches appear before and after the fundamental resonance, respectively. It should be noted that the subharmonic resonance is isolated from the main curve. Additional branches corresponding to 2:1, 4:1, 5:1, and 7:1 superharmonic and 1:2 subharmonic resonances arise in Fig. 1c at 1N. Finally, as the forcing continues to increase, new resonances, for which both k and ν can be different from 1, start to appear, first as isolated singular point solutions and then as growing isolated branches. When the forcing amplitude is 3N, some of these resonances, such as the 7:3, 3:2, 4:3, 7:2, 2:3, 3:4, 5:7, and 3:5 resonances, can be observed in Fig. 1d. A close-up on these specific superharmonic and subharmonic resonances is made in Fig. 1e,f, respectively. The main resonances are examined

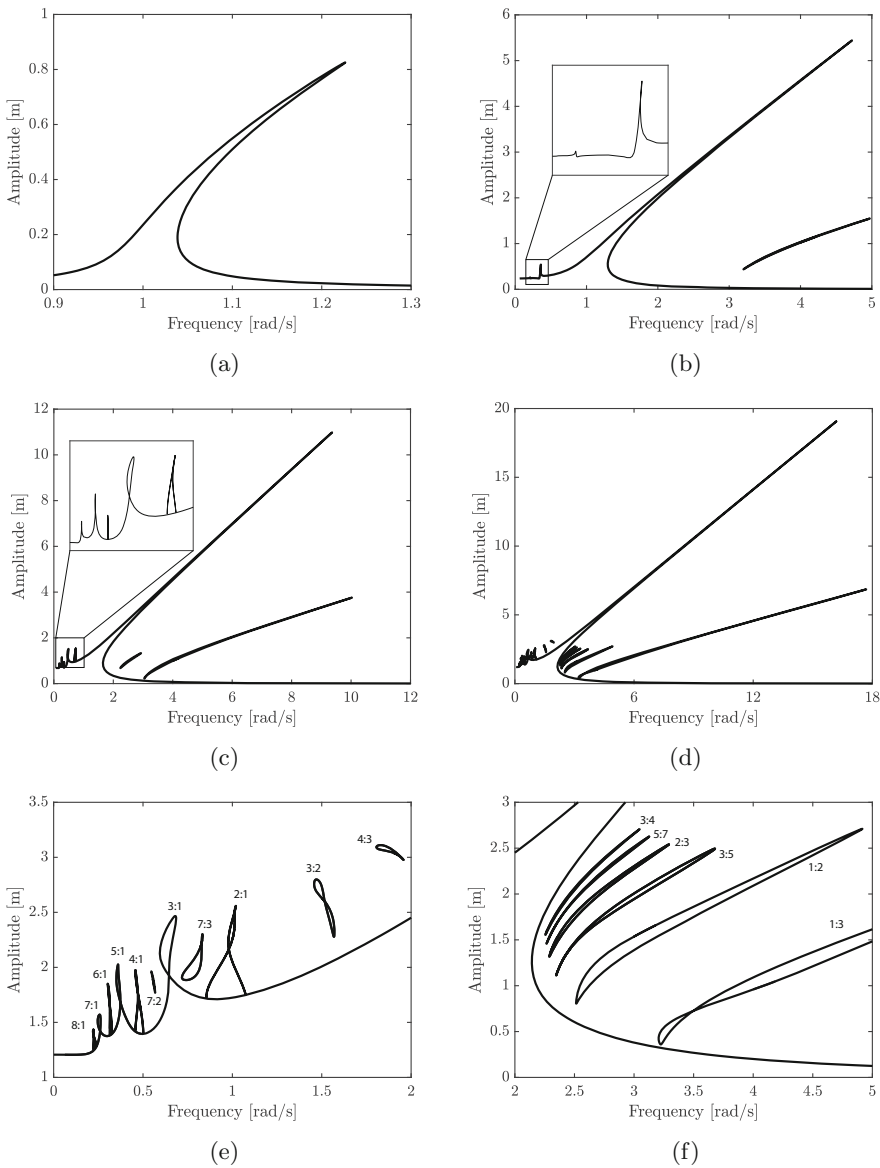


Fig. 1 NFRCs of the Duffing oscillator: (a) $f = 0.01N$, (b) $f = 0.25N$, (c) $f = 1N$, and (d) $f = 3N$, (e) close-up on the superharmonic resonances, (f) close-up on the subharmonic resonances

in greater detail hereafter. Particular attention is devoted to the phase difference between the dominant harmonic component of the displacement and the harmonic excitation.

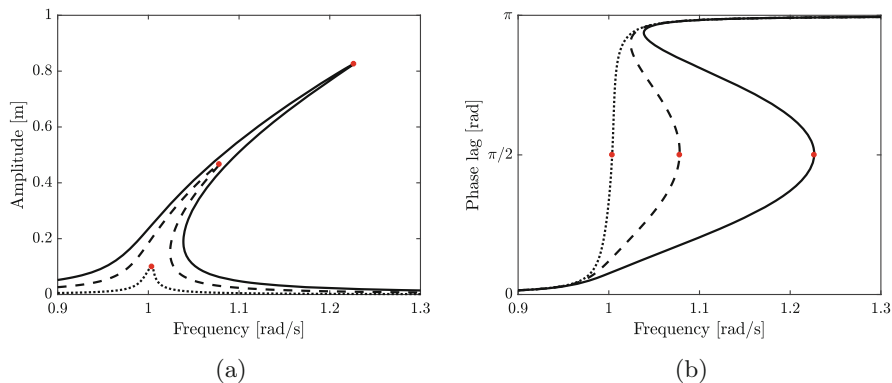


Fig. 2 NFRCs of the fundamental resonance of the Duffing oscillator for $f = 0.001N$ (.....), $f = 0.005N$ (- - -), and $f = 0.01N$ (—): (a) amplitude and (b) phase lag of the first harmonic component. The red dots correspond to phase resonance

2.1 Fundamental Resonance

The amplitude and phase lag of the first harmonic component of the displacement in the neighborhood of the fundamental resonance are displayed in Fig. 2a,b, respectively. The phase lag varies between 0 and π and passes through $\pi/2$ at resonance.

2.2 Superharmonic Resonances ($k > \nu$, $\nu = 1$)

In the case of superharmonic resonances, the ratio $\frac{k}{\nu}$ is greater than one, and the resonance peaks are located before the fundamental resonance. The phase lags of the 3/1 and 2/1 harmonic components of the 3 : 1 and 2 : 1 resonances are depicted in Fig. 3a,b, respectively.

2.2.1 Odd Superharmonic Resonances (k is Odd)

The phase lag of the 3/1 harmonic component of the 3:1 resonance is comprised between 0 and π and, as for the fundamental resonance, passes through $\pi/2$ at resonance. The same observation holds for the 5 : 1 and 7 : 1 superharmonic resonances. These results suggest that phase quadrature between the forcing and the dominant harmonic component exists at resonance for odd superharmonic branches.

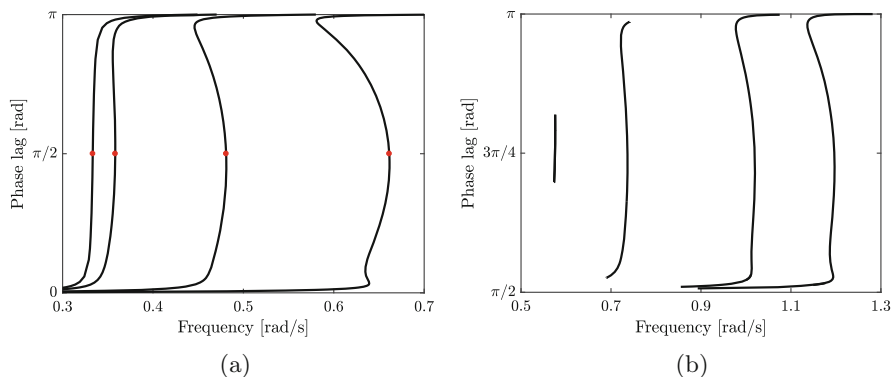


Fig. 3 Phase lags of the 3:1 and 2:1 superharmonic resonances for 4 forcing amplitudes: (a) 3:1 and (b) 2:1. The points where the phase lag is equal to $\pi/2$ are marked by red dots

2.2.2 Even Superharmonic Resonances (k is Even)

The phase lag of the 2/1 harmonic component of the 2:1 resonance is comprised between $\pi/2$ and π and passes through $3\pi/4$ at resonance. The same observation holds for the 4:1, 6:1, and 8:1 resonances. It can be noted that these resonances bifurcate out of the main NFRC.

2.3 Subharmonic Resonances ($\nu > k$, $k = 1$)

The ratio $\frac{k}{\nu}$ is lower than one, and the resonance branches are located beyond the fundamental resonance. The phase lags of the 1/3 and 1/2 harmonic components for the 1 : 3 and 1 : 2 resonances are depicted in Fig. 4a,b, respectively.

2.3.1 Odd Subharmonic Resonances (ν is Odd)

The corresponding phase lag for the 1/3 harmonic component is bounded by $\pi/3$ and $2\pi/3$. Since the branch is isolated, the phase lag is twice equal to $\pi/2$, which happens at the extremities of the isolated branch. For higher-order 1 : ν subharmonic resonances, the phase lag of the $1/\nu^{th}$ harmonic component is located within the interval $[\pi/2 \pm \pi/2\nu]$.

2.3.2 Even Subharmonic Resonances (ν is Even)

As for even superharmonic resonances, the phase lag of even subharmonic resonances is not centered around $\pi/2$. Specifically, for the 1:2 resonance, it is

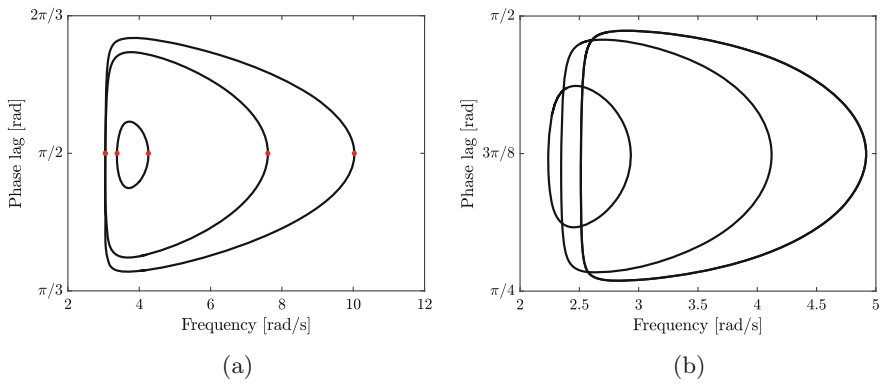


Fig. 4 Phase lags of the 1:3 and 1:2 subharmonic resonances for 3 forcing amplitudes: **(a)** 1:3 and **(b)** 1:2

Table 1 Phase lag of the k/ν harmonic component of the $k : \nu$ resonance

	$k \ \& \ \nu$ are odd	k or ν is even
Phase lag at resonance	$\pi/2$	$3\pi/4\nu$
Phase lag interval	π/ν	$\pi/2\nu$

centered around $3\pi/8$ and comprised between $\pi/4$ and $\pi/2$. For higher-order $1 : \nu$ resonances, the phase lag interval is $\pi/2\nu$ and centered around $3\pi/4\nu$.

2.4 Other Superharmonic and Subharmonic Resonances
($k > 1, \nu > 1$)

Resonances for which neither k nor ν is equal to 1 (see Fig. 1e,f) can also be studied based on the parity of k and ν . Specifically, for the $k : \nu$ resonance, if either k or ν is even, the phase lag of the k/ν harmonic component at resonance is $3\pi/4\nu$ as for even superharmonic and subharmonic resonances, and $\pi/2$ when both k and ν are odd as for odd superharmonic and subharmonic resonances. These results are summarized in Table 1.

3 Phase Resonance Nonlinear Modes

For linear systems, phase resonance takes place when the single-point harmonic forcing and the displacement at the forcing location are in quadrature, i.e., the phase is locked at $\pi/2$ [7]. As illustrated in Fig. 2, this linear definition extends to the fundamental resonances of nonlinear systems.

The results in the previous section allow us to generalize the concept of phase resonance to superharmonic and subharmonic resonances of nonlinear systems. Indeed, they demonstrate that the phase lag can still be used as a robust criterion to track the locus of their resonance peaks, as carried out for fundamental resonances in [8]. The key finding is that phase quadrature between k/ν harmonic component of the $k : \nu$ branch and the forcing is no longer necessarily achieved for such resonances, but depends on the parity of k and ν as indicated in Table 1.

3.1 A New Nonlinear Mode Definition

Considering the unforced linear oscillator

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0, \quad (3)$$

velocity feedback can be considered to drive the system into resonance [9, 10]:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) - \mu\dot{x}(t) = 0, \quad (4)$$

where the feedback term $\mu\dot{x}(t)$ plays the role of *virtual forcing*. Because this virtual forcing and the velocity are collinear, phase quadrature with the displacement $x(t)$, and, hence, phase resonance, is naturally enforced when $\mu = c$.

Phase resonance nonlinear modes (PRNMs) further extend Eq. (4) and take into account superharmonic and subharmonic resonances of nonlinear systems:

The PRNMs of the $k : \nu$ resonance correspond to the periodic responses obtained by feeding back the T -periodic velocity of the harmonic component k/ν shifted by the delay $\nu\alpha/k\omega$ into the autonomous system.

Mathematically, the following equation is to be solved for the Duffing oscillator:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + k_n x^3(t) - \mu\dot{x}_{\frac{k}{\nu}, T}\left(t - \frac{\nu}{k} \frac{\alpha}{\omega}\right) = 0, \quad (5)$$

where ω is the frequency at which the PRNMs are to be calculated, and T is the corresponding period. $\alpha = \pi/2 - \delta$ where δ is the phase lag at resonance given in Table 1. For instance, $\alpha = 0$ for all resonances for which k and ν are odd. The ratio $\frac{\nu}{k}$ in the delay accounts for the fact that the period of the fundamental harmonic component is k/ν times that of the k/ν harmonic component.

Considering the 1:2 subharmonic resonance ($k = 1, \nu = 2$) as an illustrative example, Fig. 5 shows the three steps to calculate the velocity feedback from the original velocity $\dot{x}(t)$ shown in Fig. 5a:

1. Filtering out all the harmonic components of $\dot{x}(t)$ that are different from k/ν to obtain the $\frac{\nu}{k}T$ -periodic signal: $\dot{x}_{\frac{k}{\nu}, \frac{\nu}{k}T}(t)$ (Fig. 5b)
2. Transforming $\dot{x}_{\frac{k}{\nu}, \frac{\nu}{k}T}(t)$ into a T -periodic signal: $\dot{x}_{\frac{k}{\nu}, T}(t)$ (Fig. 5c)

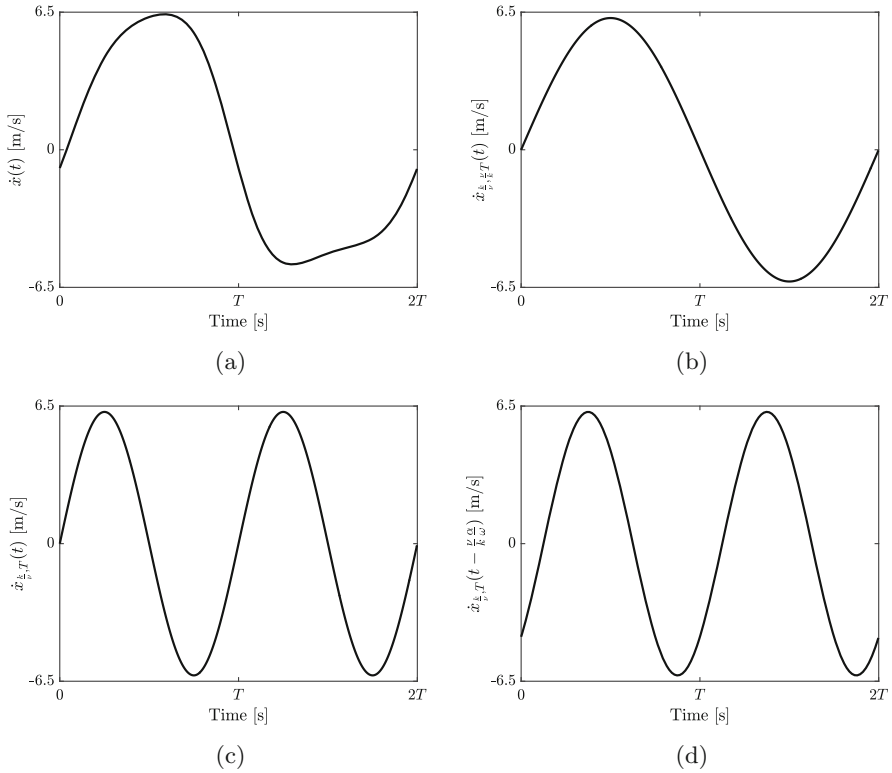


Fig. 5 Calculation of the velocity feedback: (a) original velocity, (b) after step 1 (filtering), (c) after step 2 (T -periodic), and (d) after step 3 (delay)

3. Delaying $\dot{x}_{\frac{1}{k},T}(t)$ by the angle $\frac{\nu}{k}\alpha$, i.e., $\pi/4$ for the 1:2 resonance: $\dot{x}_{\frac{1}{k},T}(t - \frac{\nu}{k} \frac{\alpha}{\omega})$ (Fig. 5d)

For multi-degree-of-freedom systems, the velocity feedback is applied at the degree of freedom where the external forcing is located.

3.2 PRNMs of the Duffing Oscillator

3.2.1 Fundamental Resonance

The PRNM backbone of the fundamental resonance is superposed to the NFRCs of the Duffing oscillator in Fig. 6. As anticipated, the backbone goes exactly through the $\pi/2$ phase lag points in Fig. 6b and traces very closely the locus of the resonance peaks of the different NFRCs in Fig. 6a.

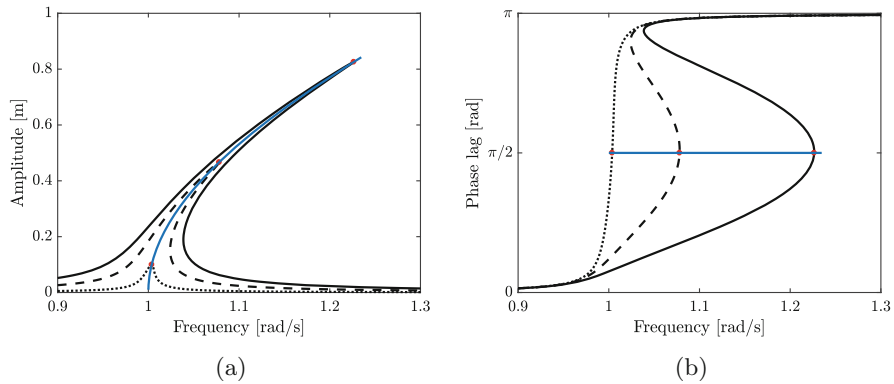


Fig. 6 NFRCs and PRNMs of the fundamental resonance of the Duffing oscillator: (a) amplitude and (b) phase lag. Black: NFRC; blue: PRNM

3.2.2 Superharmonic Resonances

For odd and even superharmonic resonances, the phase lags are $\pi/2$ and $3\pi/4$, respectively. The PRNM backbones corresponding to 3:1 and 2:1 resonances are shown in Fig. 7a,b, respectively. These figures confirm the relevance of the PRNMs for the characterization of superharmonic resonances.

3.2.3 Subharmonic Resonances

The PRNMs of the 1:3 and 1:2 subharmonic resonances are represented in Fig. 8, where the phase lags at resonance are $\pi/2$ and $3\pi/8$, respectively. An important remark is that a critical forcing amplitude is required to activate these resonances. Below this forcing, the isolated resonance branch, and, hence, the PRNM, does not exist.

3.2.4 Other Superharmonic and Subharmonic Resonances

The PRNMs of the remaining superharmonic and subharmonic resonances can also be computed based on the results from Table 1.

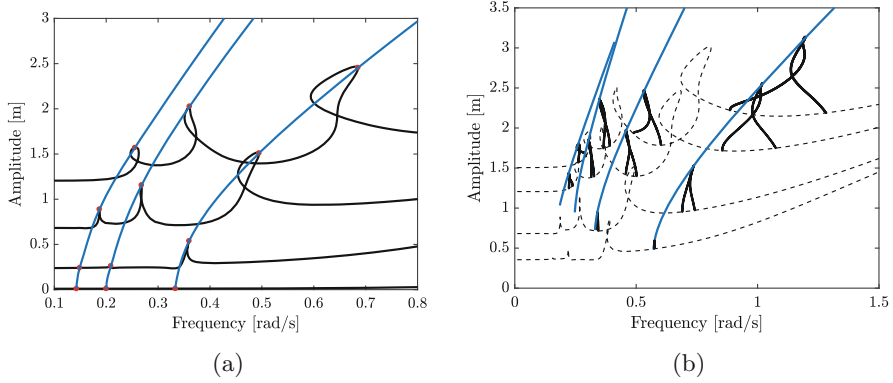


Fig. 7 NFRCs and PRNMs of the 3:1 and 2:1 superharmonic resonances for 4 forcing amplitudes: (a) 3:1 and (b) 2:1. Black: NFRC; blue: PRNM

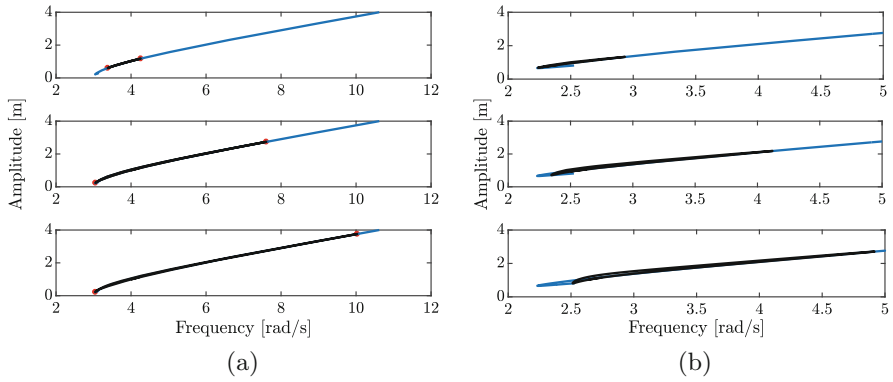


Fig. 8 NFRCs and PRNMs of the 1:3 and 1:2 subharmonic resonances for 3 forcing amplitudes: (a) 1:3 and (b) 1:2. Black: NFRC; blue: PRNM

4 Conclusions

The objective of this chapter was to carry out a detailed study of the phase lags associated with superharmonic and subharmonic resonances of the Duffing oscillator. The study has revealed that phase quadrature still holds for $k : \nu$ resonances when k and ν are both odd. Otherwise, resonance occurs for a phase lag equal to $3\pi/4\nu$. Based on these results, the PRNMs of the $k : \nu$ resonance correspond to the periodic responses obtained by feeding back the delayed velocity of the harmonic component k/ν into the autonomous system at the point where the external forcing is located.

References

1. D.J. Ewins, *Modal Testing: Theory, Practice, and Application*, 2nd edn. (Research Studies Press, Philadelphia, 2000)
2. G. Kerschen, K. Worden, A.F. Vakakis, J.-C. Golinval, Past, present and future of nonlinear system identification in structural dynamics. *Mech. Syst. Sig. Process.* **20**(3), 505–592 (2006)
3. A.F. Vakakis, L.I. Manevitch, Y.V. Mikhlin, V.N. Pilipchuk, A.A. Zevin, *Normal Modes and Localization in Nonlinear Systems* (Wiley, New York, 1996)
4. R.M. Rosenberg, Normal modes of nonlinear dual-mode systems. *J. Appl. Mech.* **27**(2), 263–268 (1960)
5. R.M. Rosenberg, The normal modes of nonlinear n-degree-of-freedom systems. *J. Appl. Mech.* **29**(1), 7–14 (1962)
6. U. Parlitz, W. Lauterborn, Superstructure in the bifurcation set of the Duffing equation. *Phys. Lett.* **107A**, 351–355 (1985)
7. F. De Veubeke, *A Variational Approach to Pure Mode Excitation using Characteristic Phase Lag Theory*. Technical report (1956)
8. L. Renson, A. Gonzalez-Buelga, D.A.W. Barton, S.A. Neild, Robust identification of backbone curves using control-based continuation. *J. Sound Vib.* **367**, 145–158 (2016)
9. M. Krack, Nonlinear modal analysis of nonconservative systems: Extension of the periodic motion concept. *Comput. Struct.* **154**, 59–71 (2015)
10. I. Sokolov, V. Babitsky, Phase control of self-sustained vibration. *J. Sound Vib.* **248**, 725–744 (2001)