MATHEMATICAL MODELS AND LOCKDOWN

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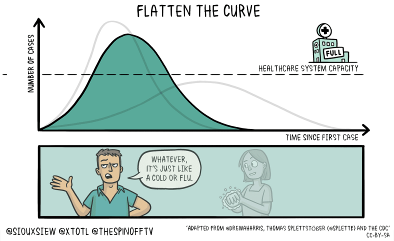
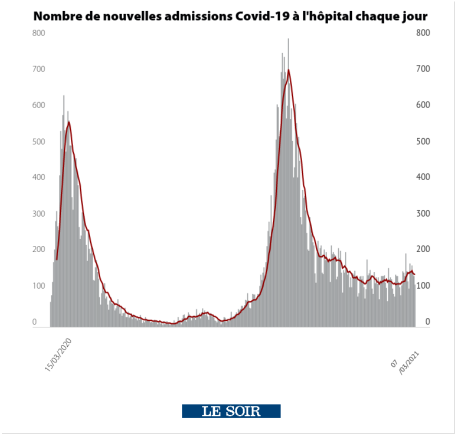
`http://hdl.handle.net/2268/246583`
`https://youtu.be/wJ-K8W321pA`

Belgian Biophysical Society Webinar — 11th March 2021

Disclaimer :

- I'm not at all a specialist of mathematical models, I'm not a statistician ; I do not manipulate data ; but (hopefully) I'm still a mathematician.
- In March 2020, I wanted to understand the sentence "*we have to flatten the curve*".
- Based on available data, we want to predict how the situation will evolve.

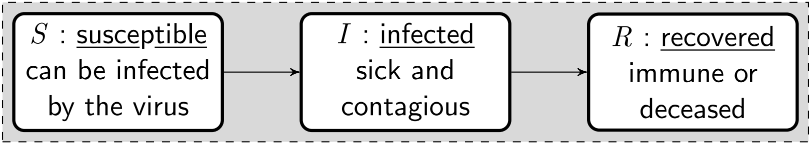




I found the following source useful (easy to Google it) :

- David Smith and Lang Moore, *The SIR Model for Spread of Disease*, Math. Assoc. Amer. 2001.

A “simple” model presented by Kermack and McKendrick in 1927



S : susceptible
can be infected
by the virus

I : infected
sick and
contagious

R : recovered
immune or
deceased

A *recovered* person can still suffer from several long term problems and might even die later on from subsequent complications.

Limitations of the SIR model :

- the incubation period is zero ;
- the period during which a person is contagious coincides with the period during which he or she is ill :

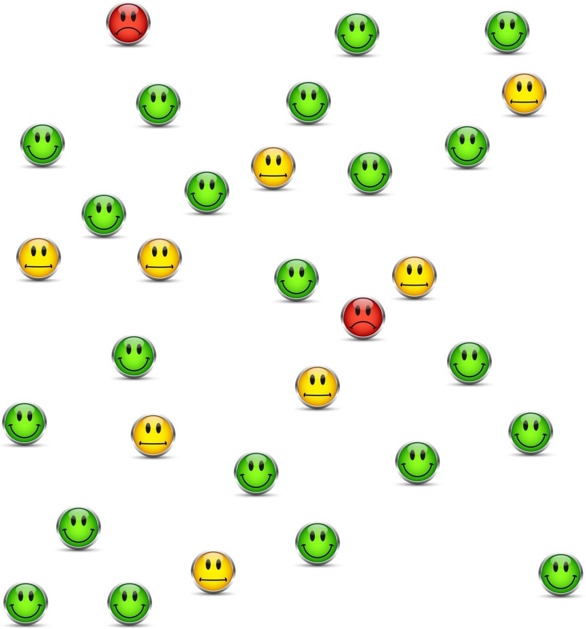
$$\text{infected} = \text{sick} = \text{contagious} = \text{infectious}$$

Remarks :

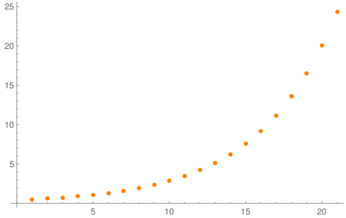
- The category R contains recovered and deceased persons.
- A recovered person cannot get sick again.

$$S \longrightarrow I \longrightarrow R$$

Population : uniform mix of individuals S, I and R



We would like to estimate the evolution $S(t)$, $I(t)$ and $R(t)$ along the time

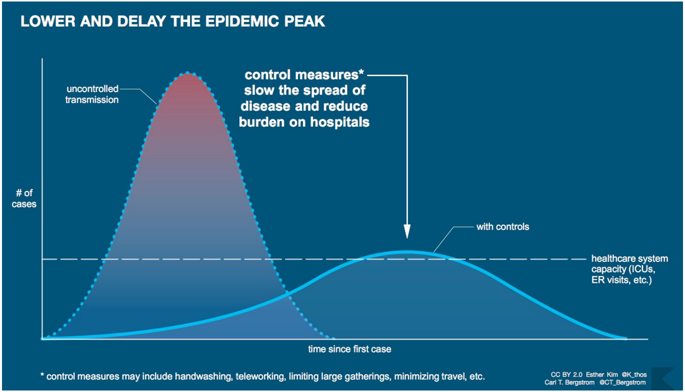


- Not all patients are seriously ill ; in the case of COVID-19 there are also asymptomatic patients.
- A stable proportion of patients will require hospitalization.
- Amongst those, a stable proportion will require ICU.
- The maximum capacity of hospitals is known.
- We never know the “exact” value of $S(t)$, $I(t)$ and $R(t)$.

WHY SHOULD WE ESTIMATE $I(t)$?

We “know” what proportion will require care, e.g. 7%. The goal of policy makers is therefore to maintain

$$\boxed{0,07.I(t) < \text{capacity of hospitals.}}$$



* control measures may include handwashing, teleworking, limiting large gatherings, minimizing travel, etc.

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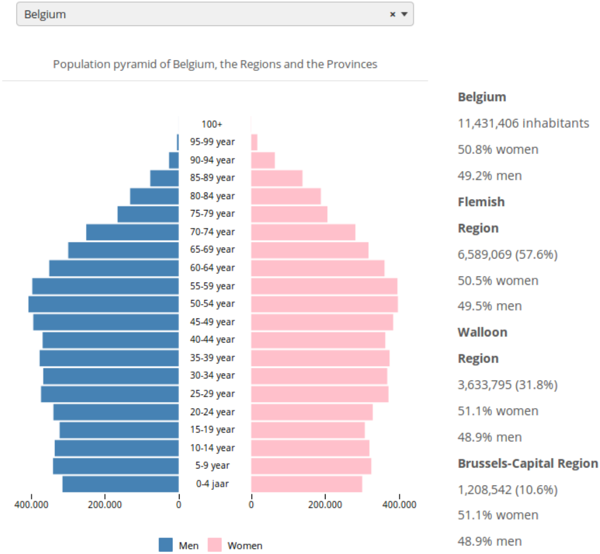
Carl T. Bergstrom @CT_Bergstrom

WHY SHOULD WE ESTIMATE $I(t)$?

population pyramid, cf. `statbel.fgov.be`



Population pyramid of Belgium, the Regions and the Provinces

Belgium

11,431,406 inhabitants

50.8% women

49.2% men

Flemish Region

6,589,069 (57.6%)

50.5% women

49.5% men

Walloon Region

3,633,795 (31.8%)

51.1% women

48.9% men

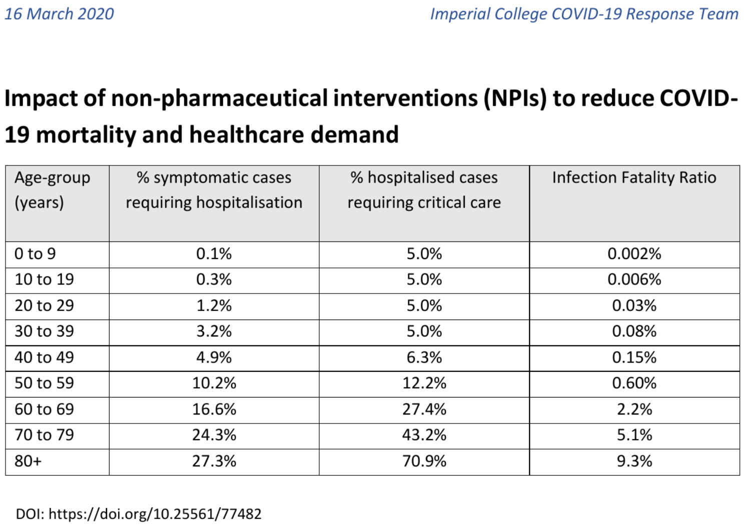
Brussels-Capital Region

1,208,542 (10.6%)

51.1% women

48.9% men

WHY SHOULD WE ESTIMATE $I(t)$?



16 March 2020 *Imperial College COVID-19 Response Team*

Impact of non-pharmaceutical interventions (NPIs) to reduce COVID-19 mortality and healthcare demand

Age-group (years)	% symptomatic cases requiring hospitalisation	% hospitalised cases requiring critical care	Infection Fatality Ratio
0 to 9	0.1%	5.0%	0.002%
10 to 19	0.3%	5.0%	0.006%
20 to 29	1.2%	5.0%	0.03%
30 to 39	3.2%	5.0%	0.08%
40 to 49	4.9%	6.3%	0.15%
50 to 59	10.2%	12.2%	0.60%
60 to 69	16.6%	27.4%	2.2%
70 to 79	24.3%	43.2%	5.1%
80+	27.3%	70.9%	9.3%

DOI: https://doi.org/10.25561/77482

Counting or proportions :

$S(t)$	$I(t)$	$R(t)$	N
160	2	38	200

$$S(t) + I(t) + R(t) = N$$

$s(t)$	$i(t)$	$r(t)$	
80%	1%	19%	100%

$$s(t) = \frac{S(t)}{N} \qquad i(t) = \frac{I(t)}{N} \qquad r(t) = \frac{R(t)}{N}$$

$$s(t) + i(t) + r(t) = 1.$$

EVOLUTION OF S

We place ourselves at the global level of the entire population ; it's a *public health* problem. We look at *average* not at individuals.

β average transmission rate of the disease (*daily and for each infectious person*)

$$\beta = m.p$$

This average transmission rate depends on two parameters

- m : average number of daily contacts
- p : probability to get sick when encountering an infectious person

local ZOOM :

$m = 8$

$p = \frac{1}{2}$

$s(t) = \frac{3}{4}$

An infectious person can only infect a “healthy” person (susceptible to get sick).

He/she will therefore contaminate on average

$$\underbrace{m.p}_{\beta}.s(t) \text{ susceptible persons}$$

on the example : $8.\dfrac{1}{2}.\dfrac{3}{4} = 4.\dfrac{3}{4} = 3$

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Notation, first difference :

$$\Delta I(t) = I(t+1) - I(t)$$

gives the *increment* to move from one day t to the next

Simplification of counting : it is assumed that infectious people meet different people. It's unlikely that someone healthy will meet several infectious persons on the same day.

The total number of contaminations on day t is equal to

$$\underbrace{\beta.s(t)}_{\text{infected by 1}} \times \underbrace{I(t)}_{\text{number of infectious}}$$

First equation of the model :

$$S(t+1) = S(t) - \beta.s(t).I(t)$$

$$\boxed{\underbrace{S(t+1) - S(t)}_{\Delta S(t)} = -\beta.s(t).I(t)}$$

If we express the proportions (we divide by the total number of people)

$$\frac{S(t+1)}{N} = \frac{S(t)}{N} - \beta.s(t).\frac{I(t)}{N}$$

$$\boxed{\Delta s(t) = -\beta.s(t).i(t)}$$

As a reminder, the average transmission rate $\beta = m.p$

$m \downarrow$ with isolation, lock-down, prohibition of large gatherings, ...

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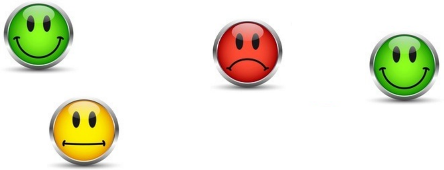

$m = 3$

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$p \downarrow$ with hygiene measures, wearing a mask, social distancing, outdoor activities, ...

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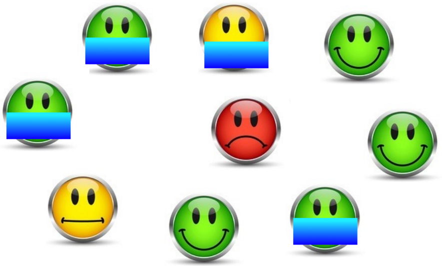
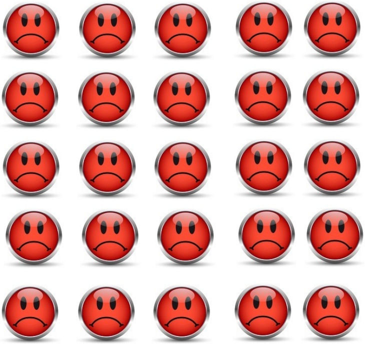


$p = \frac{1}{4}$

EVOLUTION OF R

γ : *the proportion of contaminated people who recover during the course of a day*

Let us say, on average, 5 days are needed to recover.



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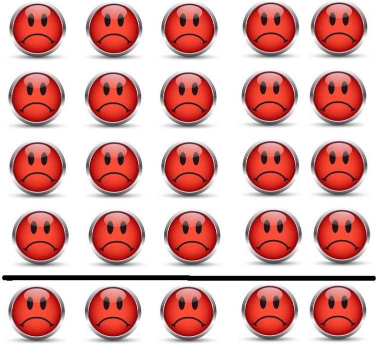

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EVOLUTION OF R

$$S(t+1) = S(t) - \beta.s(t).I(t)$$

Second equation of the model :

$$R(t+1) = R(t) + \gamma.I(t)$$

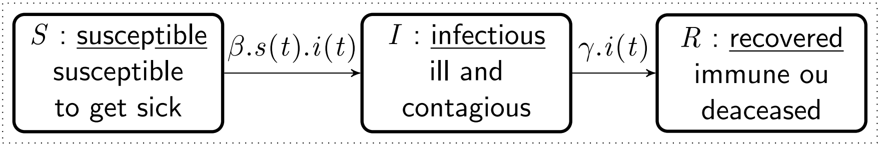
$$\boxed{\underbrace{R(t+1) - R(t)}_{\Delta R(t)} = \gamma.I(t)}$$

If we express the proportions (we divide by the total number of people)

$$\boxed{\Delta r(t) = \gamma.i(t)}$$

THE FULL MODEL

$$\boxed{\Delta s(t) = -\beta.s(t).i(t)} \qquad \boxed{\Delta r(t) = \gamma.i(t)}$$



S : susceptible
susceptible
to get sick

$\beta.s(t).i(t)$

I : infectious
ill and
contagious

$\gamma.i(t)$

R : recovered
immune ou
deaceased

FIGURE – Win and loss within the three categories.

$$\Delta i(t) = i(t+1) - i(t) = \beta.s(t).i(t) - \gamma.i(t)$$

THE FULL MODEL

$\boxed{\Delta s(t) = -\beta.s(t).i(t)}$ $\boxed{\Delta r(t) = \gamma.i(t)}$

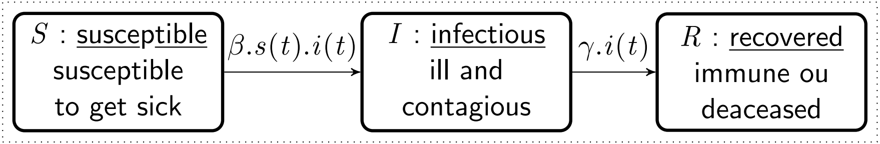


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FIGURE – Win and loss within the three categories.

$$\Delta i(t) = i(t+1) - i(t) = \beta.s(t).i(t) - \gamma.i(t)$$

THE FULL MODEL

We therefore have a system of equations

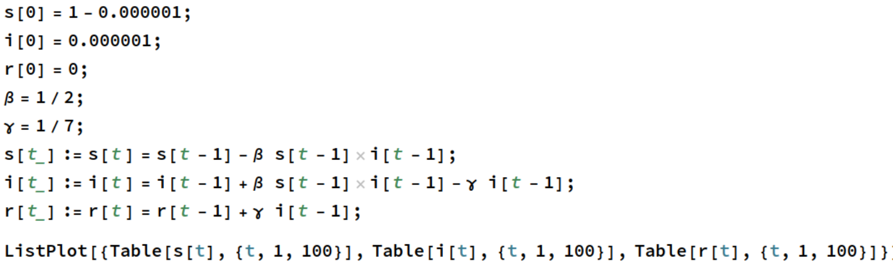
$$\left\{\begin{array}{rcccl}\Delta s(t) & = & s(t+1)-s(t) & = & -\beta.s(t).i(t)\\ \Delta i(t) & = & i(t+1)-i(t) & = & \beta.s(t).i(t)-\gamma.i(t)\\ \Delta r(t) & = & r(t+1)-r(t) & = & \gamma.i(t)\end{array}\right.$$

whose solution depends on initial conditions

If the time interval (instead of a day) tends towards 0, the model is often presented as a system of differential equations

$$\left\{\begin{array}{rcl}s'(t) & = & -\beta.s(t).i(t)\\ i'(t) & = & \beta.s(t).i(t)-\gamma.i(t)\\ r'(t) & = & \gamma.i(t)\end{array}\right.$$

THE FULL MODEL



```
s[0] = 1 - 0.000001;
i[0] = 0.000001;
r[0] = 0;
β = 1 / 2;
γ = 1 / 7;
s[t_] := s[t] = s[t - 1] - β s[t - 1] × i[t - 1];
i[t_] := i[t] = i[t - 1] + β s[t - 1] × i[t - 1] - γ i[t - 1];
r[t_] := r[t] = r[t - 1] + γ i[t - 1];

ListPlot[{Table[s[t], {t, 1, 100}], Table[i[t], {t, 1, 100}], Table[r[t], {t, 1, 100}]}
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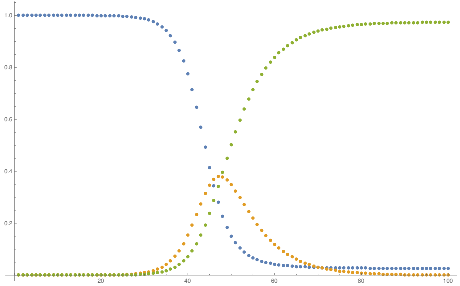
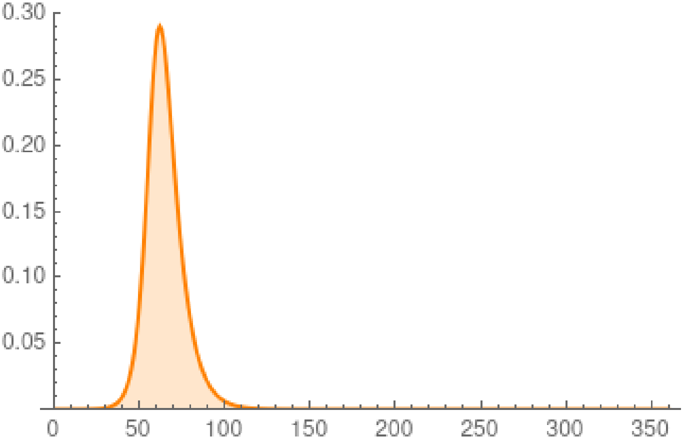


If β evolves ? For instance, between $0,4$ and $0,2$ ($\gamma = 1/7$)



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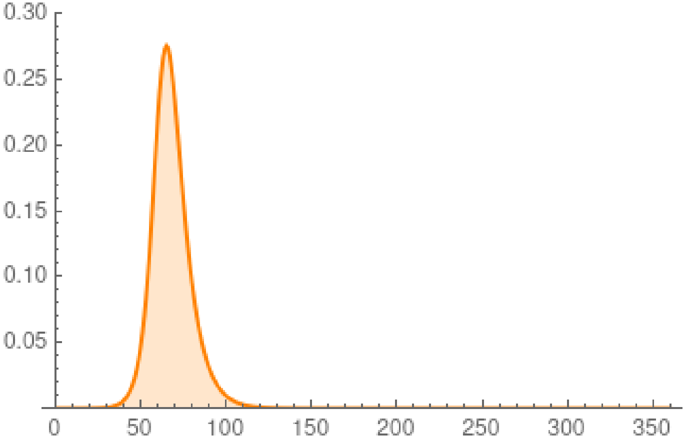

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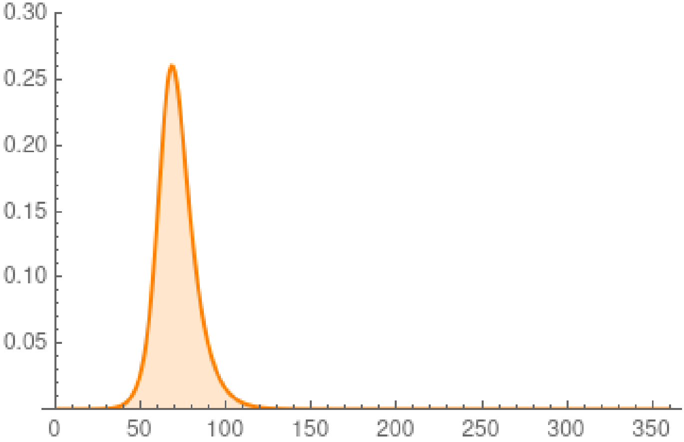

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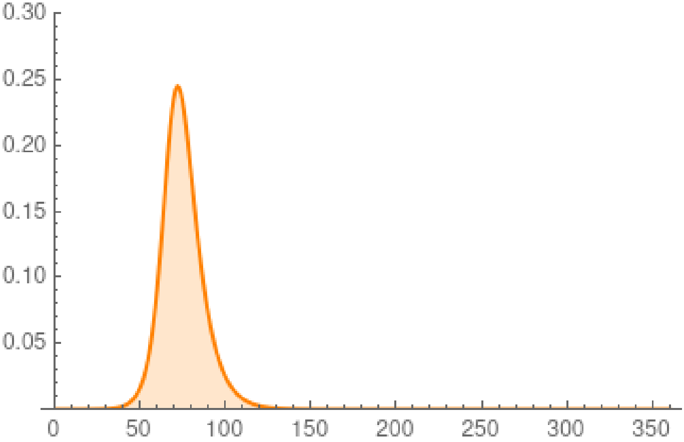

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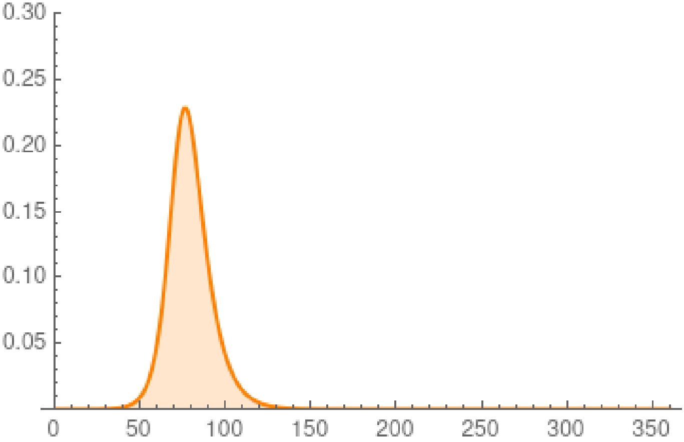

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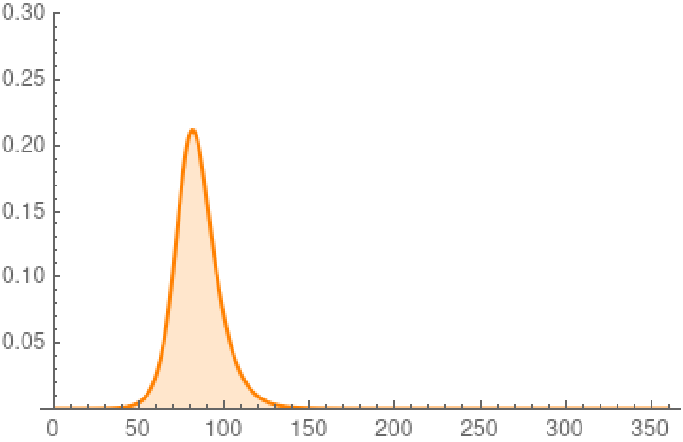

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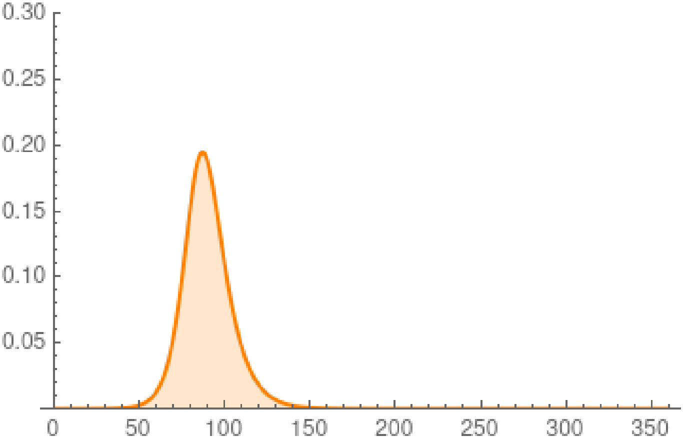

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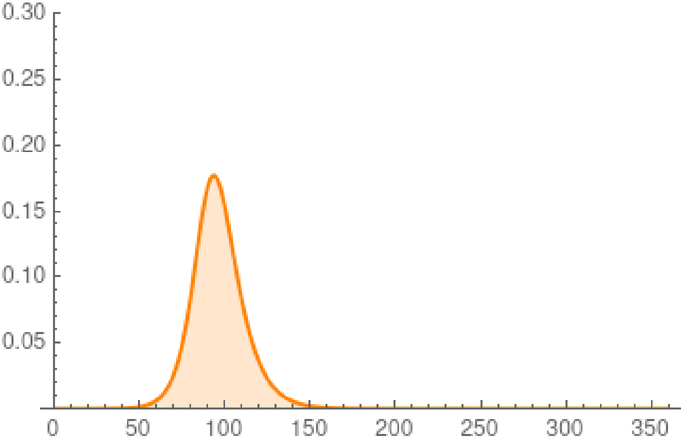

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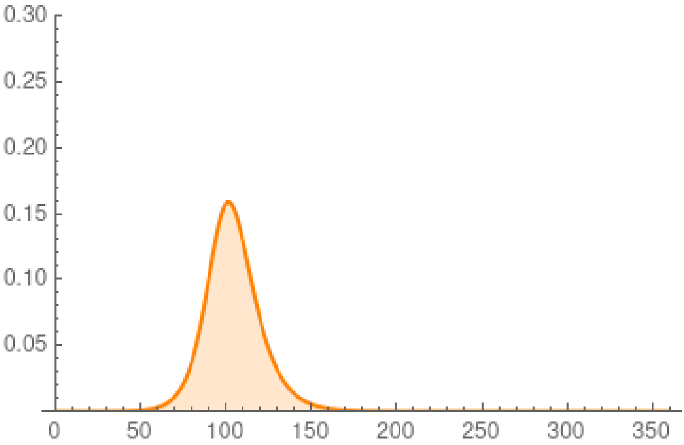

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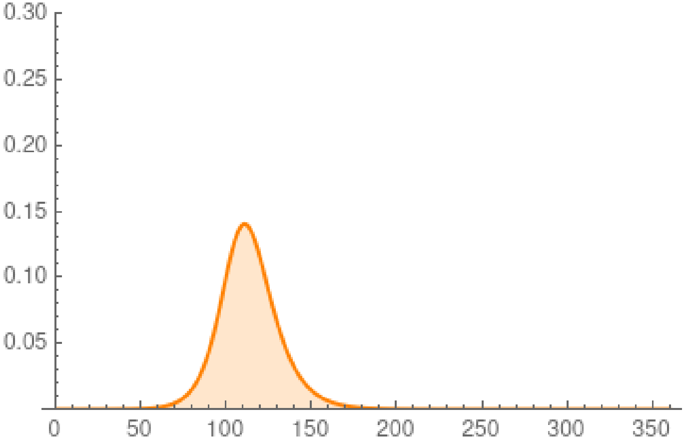

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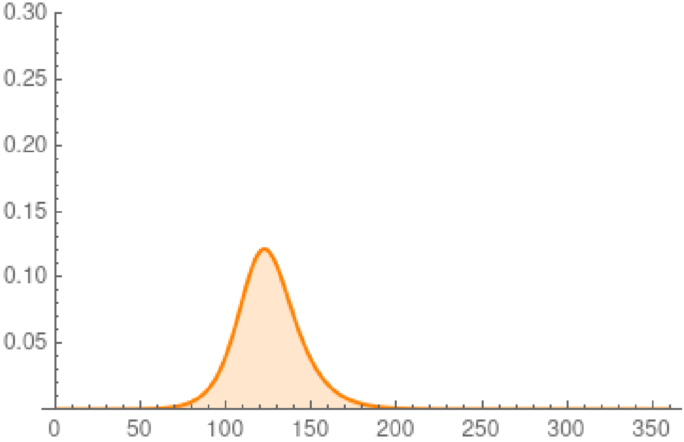

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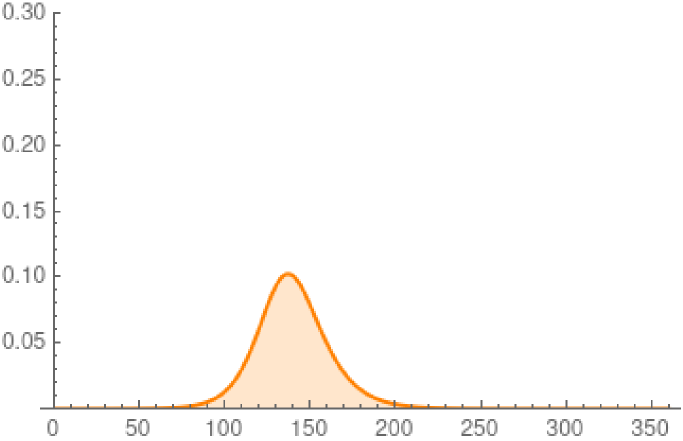

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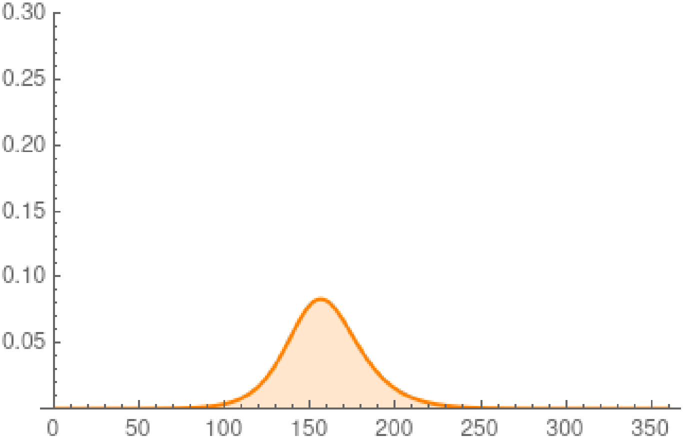

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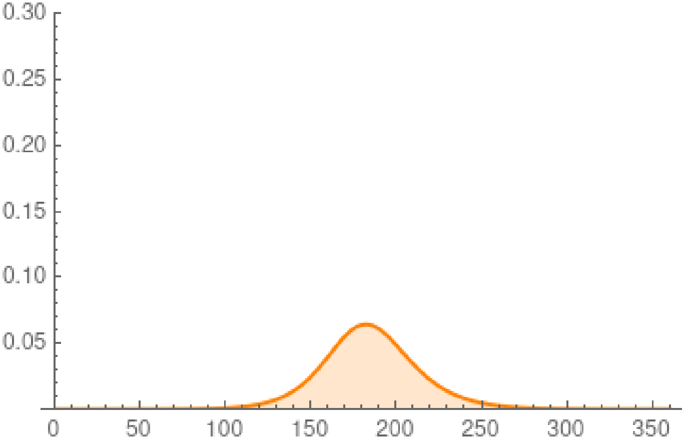

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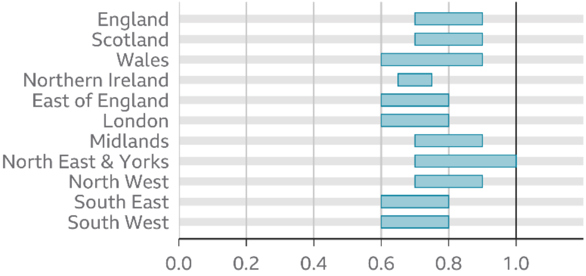
Second (and last) part :

Reproduction rate (or R_0 number) and vaccination

R number estimates in the nations & regions

Range of Covid-19 reproduction numbers



Figures for England published on 5 Mar, Scotland on 4 Mar, Wales on 26 Feb and Northern Ireland on 2 Mar

Source: DHSC, Scottish Government, NI Dept. of Health, Wales TAC

- β average transmission rate (daily, for each infectious person)
- $1/\gamma$ average number of days needed to recover

Reproduction number

$$R_0 := \frac{\beta}{\gamma}$$

average number of transmissions throughout the infectious period for each infectious person

for COVID-19, $R_0 \simeq 2.4$, new Brittish variant 'raises by up to 0.7'. In Belgium, on 9th March 2021, $R_0 \simeq 0.99$.

As we did earlier, if it is multiplied by the proportion of people susceptible to get sick, then we will have the average number

$$\frac{\beta.s(t)}{\gamma}$$

of persons infected by a single infectious person
(over the entire period of time he or she is ill/infectious).

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of persons infected by a single infectious person
(over the entire period of time he or she is ill/infectious).

Recall the third equation

$$\Delta i(t) = \beta.s(t).i(t) - \gamma.i(t)$$

and manipulate

$$\Delta i(t) = \left(\frac{\beta.s(t)}{\gamma} - 1\right).\underbrace{\gamma.i(t)}_{\geq 0}.$$

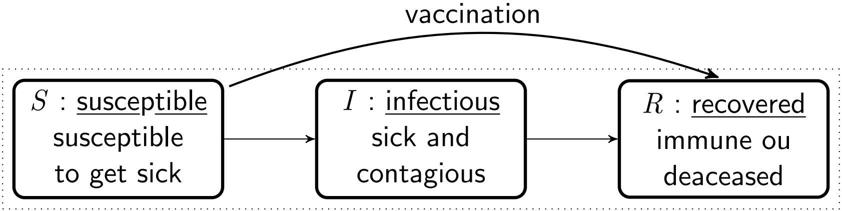
- If $\Delta i(t) > 0$, increase in the proportion of sick people
- If $\Delta i(t) < 0$, decrease in the proportion of sick people

We want that

$$\frac{\beta.s(t)}{\gamma} - 1 < 0$$

Assume that the population is divided into S/R only, we want

$$s(t) < \frac{\gamma}{\beta}, \qquad \text{thus, } r(t) > 1 - \frac{\gamma}{\beta} = 1 - \frac{1}{R_0}$$



If $\beta/\gamma \simeq 2.4$, at least $1-(1/2.4) \simeq 58\%$ vaccination coverage is required

The more contagious a disease is, the greater the vaccination coverage needs to be :

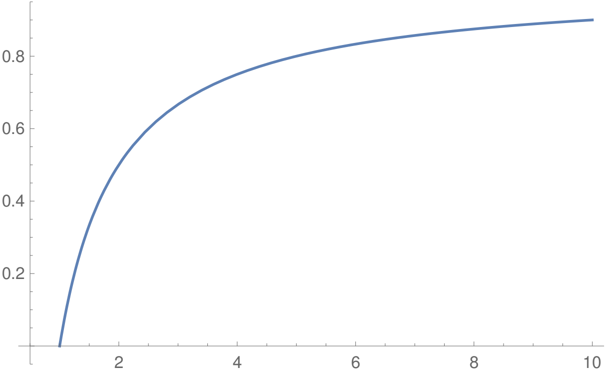


FIGURE – Evolution of vaccination coverage $1 - \gamma/\beta$ depending on the reproduction rate β/γ.

- Much more sophisticated models exist :
  - SEIR, a class of *exposed* individuals in incubation period (infected, but not yet infectious) ;

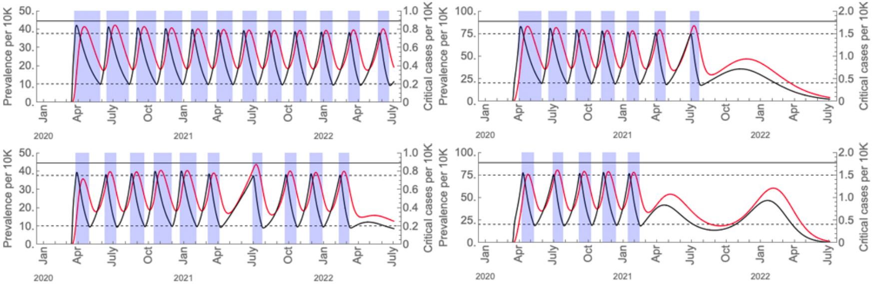
    $$S \longrightarrow E \longrightarrow I \longrightarrow R$$

  - MSEIR, an initial passive immunity exists ;

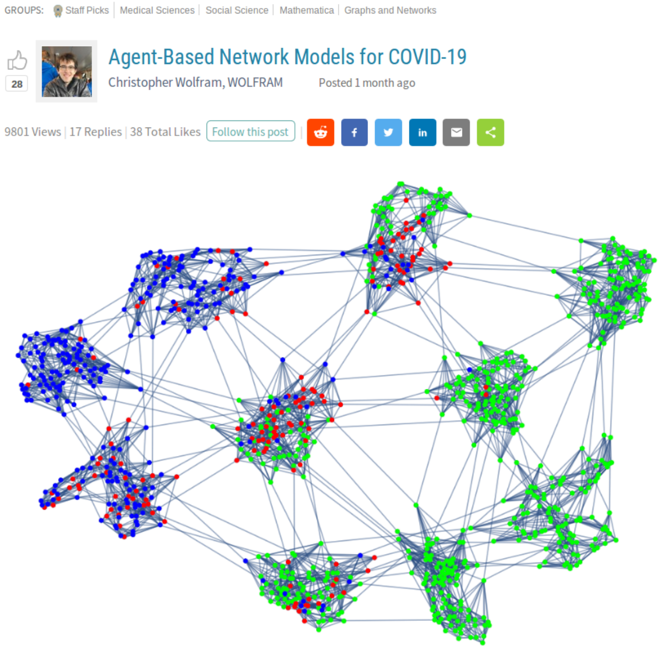
    $$M \longrightarrow S \longrightarrow E \longrightarrow I \longrightarrow R$$

  - MSEIRS, immunity within class R is temporary, return to class S after a certain period of time.
- Taking into account the spatial distribution.
- Having accurate data helps refine predictions ; in particular, estimating $S(t)$ is important but difficult.

What I discovered a year ago — What could be expected ?



intermittent periods of social distancing (Social distancing strategies for curbing the COVID-19 epidemic, Kissler *et al.*)



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Agent-Based Network Models for COVID-19

Christopher Wolfram, WOLFRAM

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