The Fréchet functional equation for Lie groups

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Abstract

In this paper, we investigate the solutions of Fréchet's functional equation in the context of Lie groups. In particular, we give the explicit right-abelian solutions of this equation for connected Lie groups. We also extend this result to homogeneous spaces and deal with some classical examples.

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1 Introduction

Cauchy's functional equation,

$$f(x+y) = f(x) + f(y) \tag{1}$$

used to play a central role in the mathematical literature [7, 5, 10]. Given a function f defined on the Euclidean space, let us set (see e.g. [3, 8])

$$\Delta_h f(x) = \Delta_h^1 f(x) = f(x+h) - f(x)$$

and

$$\Delta_{h_1,\dots,h_{m+1}}^{m+1} f(x) = \Delta_{h_{m+1}} \Delta_{h_1,\dots,h_m}^{m} f(x) = \Delta_{h_{m+1}} \circ \dots \circ \Delta_{h_1} f(x),$$

for $m \in \mathbf{N}$; if $h_1 = \ldots = h_m$, we will write $\Delta_{h_1}^m f$ instead of $\Delta_{h_1,\dots,h_m}^m f$. Equation (1) naturally leads to the following generalization:

$$\Delta_h^m f(x) = m! f(x). \tag{2}$$

On the other hand, with Cauchy's equation as a starting point, Fréchet proposed in 1909 a functional definition of polynomials by showing that the continuous solutions of the equation

$$\Delta_{h_1,\dots,h_m}^m f(x) = 0 \tag{3}$$

on the Euclidean space are the polynomials of degree at most $m-1$. Later, he proposed this equation as a definition of the "abstract polynomials" of degree at most $m-1$ [6]. From this point of view, equation (2) could serve as a definition for the abstract monomials of order m. In [4], it is shown that equation (3) is equivalent to

$$\Delta_h^m f(x) = 0,$$

in the context of abelian groups.

The idea of defining polynomials in a general setting using Fréchet's equation is quite old (see e.g. [18, 11]), yet still considered nowadays (see e.g. [14, 15, 1, 12] and references therein). It is usually studied in either very general contexts (for abelian [18, 1] or non-abelian groups [11, 15]) or peculiar ones [9, 2, 13, 12]. The goal of this paper is to provide a general framework for studying equation (3) in which explicit solutions can be obtained; we do so by considering the notion of generalized polynomial on Lie groups.

Before going further into the details about the organization of this work, let us introduce the notations that will be used throughout this paper. Given a metrisable space X and a distance d on X, $B_d(x_0, r)$ will stand for the open ball centered at $x_0 \in X$ of radius $r > 0$, that is

$$B_d(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

We use the multiplicative notation for group operation and as usual 1 stands for the identity element. If G is a Lie group, the corresponding Lie algebra is denoted $\mathfrak{g}$, identified with the tangent space T_1G. The exponential map is given by $\exp : \mathfrak{g} \to G$. Given $X \in \mathfrak{g}$, the corresponding left-invariant (resp. right-invariant) vector field is written L_X (resp. R_X). A basis of $\mathfrak{g}$ is $\partial_1, \ldots, \partial_n$ and the corresponding vector fields are respectively given by $L_1, \ldots, L_n$. The dual basis is given by $dx^1, \ldots, dx^n$, so that $dx^i(\partial_j) = \delta^i_j$. Given $x \in G$, the left (resp. right) multiplication by x is denoted by L_x (resp. R_x). Its pullback is simply written L_x^* (resp. R_x^*). In such a general context, we must distinguish left and right difference operators. The left (resp. right) difference operator is defined by

$${}_h\Delta = L_h^* - I \qquad (\text{resp. } \Delta_h = R_h^* - I).$$

Given $m \in \mathbf{N}_0$, the differences of order m are given by

$${}_{h_1,\ldots,h_m}\Delta^m = {}_{h_1}\Delta \circ \cdots \circ {}_{h_m}\Delta \qquad (\text{resp. } \Delta^m_{h_1,\ldots,h_m} = \Delta_{h_m} \circ \cdots \circ \Delta_{h_1}).$$

Moreover, we denote by $\mathrm{Hom}_{\mathrm{Gr}}(G, \mathbf{R})$ the set of smooth group homomorphisms from G to $\mathbf{R}$ and $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{R})$ the set of Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathbf{R}$. Notice that $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{R})$ is just the linear forms vanishing on the commutator subspace

$$[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}([X, Y] : X, Y \in \mathfrak{g}).$$

We will also write $\mathrm{Hom}^{\mathrm{loc}}_{\mathrm{Gr}}(G, \mathbf{R})$ to denote the set of germs of smooth functions satisfying the additive equation in a neighborhood of 1. Given a vector subspace F of E, the annihilator subspace of F, denoted $F^\perp$, consists of the linear forms on E vanishing on F. Finally, $\mathfrak{S}(a_1, \ldots, a_m)$ stands for the group of permutations of $\{a_1, \ldots, a_m\}$.

Let us define a notion a polynomial for a topological group G.

Definition 1. Let G be a topological group; a map $T : G^m \to \mathbf{R}$ is said to be multiadditive of degree m if T is a group homomorphism with respect to any of its variables. It is said to be locally multiadditive if the equalities

$$T(x_1, \ldots, x_j h, \ldots, x_m) = T(x_1, \ldots, x_j, \ldots, x_m) + T(x_1, \ldots, h, \ldots, x_m)$$

hold for any $j \in \{1, \ldots, m\}$ and any $x_1, \ldots, x_m, h$ in a neighborhood of 1. Moreover, T is symmetric if $T(x_{\sigma_1}, \ldots, x_{\sigma_m}) = T(x_1, \ldots, x_m)$ for any σ belonging to $\mathfrak{S}(1, \ldots, m)$.

Definition 2. If $T : G^m \to \mathbf{R}$ is any map, its symmetrization is the map $S(T) : G^m \to \mathbf{R}$ defined by

$$S(T)(x_1, \dots, x_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}(1,\dots,m)} T(x_{\sigma_1}, \dots, x_{\sigma_m}).$$

Definition 3. Let $\mathcal{D}_m : G \to G^m$ be the diagonal map; a local monomial of degree m is a map $f : G \to \mathbf{R}$ that is equal to $T \circ \mathcal{D}_m$ in a neighborhood of 1, where $T : G^m \to \mathbf{R}$ is locally multiadditive and symmetric. A monomial of degree m is the same map f for which the equality holds everywhere. A (local) polynomial is a finite sum of (local) monomials. The degree of a polynomial is the degree of the non-vanishing monomial with the highest degree.

The next definition provides a condition under which we can get an explicit solution for (3).

Definition 4. A function f on G is right-abelian if we have

$$R^*_{h_1 h_2} f = R^*_{h_2 h_1} f,$$

for any h_1, h_2 in G.

Such an assumption is made in [17] and is weaker than the abelian condition given in [16]. It is easy to show that a monomial is right-abelian (see Lemma 2). In the first section, we recover a result obtained in [17] using the notations introduced above, stating that the right-abelian solutions of the Fréchet equation (3) are the polynomials of degree strictly less than m (see Theorem 3 and Theorem 4 for the continuous version originally obtained in [17]). In Section 3, we consider n-dimensional connected Lie groups and show the following result:

Theorem 1. *Let G be a connected Lie group. If $f : G \to \mathbf{R}$ is right abelian, bounded almost everywhere in a neighborhood of 1 and if $\Delta^{m+1}_{h_1,\dots,h_{m+1}} f(x) = 0$ for x and $h_1, \dots, h_{m+1}$ in a neighborhood of 1, then we have the following equality in a neighborhood of 1:*

$$f = \sum_{|\alpha| \le m} a_\alpha f_1^{\alpha_1} \cdots f_k^{\alpha_k}, \tag{4}$$

where the multi-index α runs over $\mathbf{N}^k$ with $k = \dim \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{R}) \le n$, for some $a_\alpha \in \mathbf{R}$ and $f_1, \dots, f_k \in \mathrm{Hom}^{\mathrm{loc}}_{\mathrm{Gr}}(G, \mathbf{R})$. Conversely, any such function is a solution of the local Fréchet equation.

We also consider the case of homogeneous spaces (Proposition 6). The last section is devoted to a few consequences of Theorem 1; some explicit examples are given as well: we consider S^1, the Heisenberg group and $\mathrm{GL}(n, \mathbf{R})$ among others. We also show that if we do not impose the solution f to be right-abelian then it can differ from (4) (see Proposition 7).

2 Fréchet's equation on topological groups

In this section, G denotes an arbitrary topological group; we give a representation theorem for the solutions of Fréchet equation which are right-abelian in this setting. These results have been obtained in [17] with similar proof. They are given here for the sake of completeness, since we are using other notations and slightly different perspectives.

The Fréchet functional equation is the following equation in f,

$$\Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(x) = 0, \tag{5}$$

for $x, h_1, \ldots, h_{m+1} \in G$ and where the solution f has to be a real valued function. We will mainly be interested in the local version of this equation, i.e. in the case where the solutions f are functions defined in a neighborhood of some $x_0 \in G$ and where the variable x belongs to a neighborhood of x_0, the parameters $h_1, \ldots, h_{m+1}$ being in a neighborhood of 1. Such an equation will simply be called the local Fréchet equation at x_0. Let us first reduce the problem to the case where x is in a neighborhood of 1.

Proposition 1. *A function f is a solution of the local Fréchet equation at x_0 if and only if $L^*_{x_0} f$ is a solution of the local Fréchet equation at* 1.

Proof. This is simply due to the fact that the operators $L^*_{x_1}$ and $R^*_{x_2}$ commute for any $x_1, x_2 \in G$. □

Since no ambiguity can arise, the local equation at $x_0 = 1$ will be called the local Fréchet equation. For obvious reasons, we will limit our discussion to the case $x_0 = 1$. Notice that equation (5) concerns right differences only; this problem is equivalent to the one involving left differences.

Lemma 1. *The following formulae hold:*

$$_{h_1,\ldots,h_m}\Delta^m f(x) = \Delta^m_{h_1,\ldots,h_m} R^*_x f(1)$$

and

$$\Delta^m_{h_1,\ldots,h_m} f(x) = {}_{h_1,\ldots,h_m}\Delta^m L^*_x f(1).$$

Proof. The proof is given in [17]. □

The following theorem is inspired by Lemma 2.3 in [17].

Theorem 2. *Let $m \in \mathbf{N}$ and $k \in \{0, \ldots, m+1\}$; the function f is a solution of the Fréchet (resp. local Fréchet) equation if and only if it is a solution of*

$$_{h_{k+1},\ldots,h_{m+1}}\Delta^{m+1-k} \Delta^k_{h_1,\ldots,h_k} f(x) = 0, \tag{6}$$

where $x, h_1, \ldots, h_{m+1}$ are in G (resp. in a neighborhood of 1*).*

Proof. Assume first that f satisfies the Fréchet equation; using the previous result, we get

$$\begin{aligned}
&_{h_{k+1},\ldots,h_{m+1}}\Delta^{m+1-k} \Delta^k_{h_1,\ldots,h_k} f(x) \\
=&\Delta^{m+1-k}_{h_{k+1},\ldots,h_{m+1}} R^*_x \Delta^k_{h_1,\ldots,h_k} f(1) = \Delta^{m+1-k}_{h_{k+1},\ldots,h_{m+1}} (\Delta_x + I) \Delta^k_{h_1,\ldots,h_k} f(1) \\
=&\Delta^{m+2}_{h_1,\ldots,h_k,x,h_{k+1},\ldots,h_{m+1}} f(1) + \Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(1) = 0.
\end{aligned}$$

Conversely, if f is a solution of (6), then

$$\begin{aligned}\Delta^{m+1}_{h_1,\dots,h_{m+1}} f(x) &= \Delta^{m+1-k}_{h_{k+1},\dots,h_{m+1}} \Delta^k_{h_1,\dots,h_k} f(x)\\ &= {}_{h_{k+1},\dots,h_{m+1}}\Delta^{m+1-k} L^*_x \Delta^k_{h_1,\dots,h_k} f(1)\\ &= {}_{h_{k+1},\dots,h_{m+1}}\Delta^{m+1-k} ({}_x\Delta + I) \Delta^k_{h_1,\dots,h_k} f(1)\\ &= {}_{h_{k+1},\dots,h_{m+1},x}\Delta^{m+2-k} \Delta^k_{h_1,\dots,h_k} f(1)\\ &\quad + {}_{h_{k+1},\dots,h_{m+1}}\Delta^{m+1-k} \Delta^k_{h_1,\dots,h_k} f(1)\\ &= 0,\end{aligned}$$

as expected. □

We can now search for a structure theorem for the solution of the Fréchet equation. We first have the following property:

Proposition 2. *If $T : G^m \to \mathbf{R}$ is a multiadditive map, then*

$$\Delta^k_{h_1,\dots,h_k}(T \circ \mathcal{D}_m)(x) = \begin{cases} 0 & \text{if } k > m \\ m!\ S(T)(h_1,\dots,h_m) & \text{if } k = m \end{cases}.$$

Proof. Let us prove this result by induction on m, both formulae being obvious for $m = 1$. Let us consider the first formula: if it is true for $m-1$, let us prove that it also holds for m. We have

$$\begin{aligned}\Delta_{h_1}(T \circ \mathcal{D}_m)(x) &= T(xh_1,\dots,xh_1) - T(x,\dots,x)\\ &= \sum_{l=0}^{m-1} \sum_{i_1<\dots<i_l} (T_{h_1;i_1,\dots,i_l} \circ \mathcal{D}_l)(x),\end{aligned}$$

where $T_{h_1;i_1,\dots,i_l} : G^l \to \mathbf{R}$ is the map defined by

$$\begin{aligned}&T_{h_1;i_1,\dots,i_l}(x_1,\dots,x_l)\\ &= T(h_1,\dots,h_1,\underbrace{x_1}_{i_1},h_1,\dots,h_1,\underbrace{x_2}_{i_2},\dots,\underbrace{x_l}_{i_l},h_1,\dots,h_1),\end{aligned}$$

$T_{h_1;\emptyset}$ being the constant map $T(h_1,\dots,h_1)$. These maps are clearly multiadditive of degree l. By induction, since $l < m$, we get

$$\Delta^k_{u_1,\dots,u_k}(T_{h_1;i_1,\dots,i_l} \circ \mathcal{D}_l)(x) = 0,$$

for $k > l$. We thus have

$$\Delta^k_{h_1,\dots,h_k}(T \circ \mathcal{D}_m)(x) = \sum_{l=0}^{m-1} \sum_{i_1<\dots<i_l} \Delta^{k-1}_{h_2,\dots,h_k}(T_{h_1;i_1,\dots,i_l} \circ \mathcal{D}_l)(x) = 0,$$

for $k > m$.

Let us prove the second formula. To do so, let us suppose that the property is true for the multiadditive maps of order $m-1$ and let T be a multiadditive map of order m. By proceeding in the same manner, we get, using the first formula and the induction hypothesis,

$$\begin{aligned}
\Delta^m_{h_1,\dots,h_m}(T\circ\mathcal{D}_m)(x) &= \sum_{l=0}^{m-1}\sum_{i_1<\cdots<i_l}\Delta^{m-1}_{h_2,\dots,h_k}(T_{h_1;i_1,\dots,i_l}\circ\mathcal{D}_l)(x)\\
&= \sum_{i_1<\dots<i_{m-1}}(m-1)!\ S(T_{h_1;i_1,\dots,i_{m-1}})(h_2,\dots,h_m)\\
&= \sum_{j=2}^{m}\sum_{\sigma\in\mathfrak{S}(2,\dots,m)} T(h_{\sigma_2},\dots,h_{\sigma_j},h_1,h_{\sigma_j+1},\dots,h_{\sigma_m})\\
&\quad+\sum_{\sigma\in\mathfrak{S}(2,\dots,m)} T(h_1,h_{\sigma_2},\dots,h_{\sigma_m})\\
&= \sum_{\mu\in\mathfrak{S}(1,\dots,m)} T(h_{\mu_1},\dots,h_{\mu_m}) = m!\ S(T)(h_1,\dots,h_m),
\end{aligned}$$

which concludes the proof. □

The fact that the monomials (as introduced by Definition 3) are right-abelian is a fundamental property.

Lemma 2. *If f is a monomial (resp. a local monomial), then we have*

$$R^*_{h_1h_2}f = R^*_{h_2h_1}f,$$

for any h_1,h_2 in G (resp. in a neighborhood of 1). This implies that

$$\Delta^k_{h_1,\dots,h_k}f(x)$$

is symmetric with respect to the parameters $h_1,\dots,h_k$ in G (resp. in a neighborhood of 1), for any $k\in\mathbf{N}$.

Proof. Let us proceed once again by induction: let us prove the result for any degree $m>1$, the cases $m=0$ and $m=1$ being trivial. Let $T: G^m\to\mathbf{R}$ be a symmetric multiadditive map such that $f=T\circ\mathcal{D}_m$ and denote by $T_h: G^{m-1}\to\mathbf{R}$ the map defined by $T_h(x_1,\dots,x_{m-1})=T(h,x_1,\dots,x_{m-1})$. We have

$$\begin{aligned}
f(xh_1h_2) &= T(xh_1h_2,\dots,xh_1h_2)\\
&= (T_x\circ\mathcal{D}_{m-1})(xh_1h_2)+(T_{h_1}\circ\mathcal{D}_{m-1})(xh_1h_2)\\
&\quad+(T_{h_2}\circ\mathcal{D}_{m-1})(xh_1h_2)\\
&= f(xh_2h_1),
\end{aligned}$$

where we have used the induction hypothesis on the maps T_h. □

One easily checks that $\Delta^2_{h_1,h_2}f=\Delta^2_{h_2,h_1}f$ for any $h_1,h_2\in G$ if and only if f is right-abelian. Using Theorem 2, if f is right-abelian, we have $\Delta_{h_1,h_2}f(x)=0$ if and only if ${}_{h_1}\Delta\Delta_{h_2}f(x)=0$. As a consequence, we have the following result:

Corollary 1. *Let $m\in\mathbf{N}$ and $k\in\{0,\dots,m+1\}$; a right-abelian function f is a solution of the Fréchet (resp. local Fréchet) equation if and only if it is a solution of*

$$_{h_{\sigma(k+1)},\dots,h_{\sigma(m+1)}}\Delta^{m+1-k}\Delta^k_{h_{\sigma(1)},\dots,h_{\sigma(k)}}f(x)=0, \tag{7}$$

where $\sigma\in\mathfrak{S}\{1,\dots,m+1\}$.

Let us consider the Fréchet equation $\Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(x) = 0$ locally at 1. If $m = 0$ then f is constant in a neighborhood of 1 and if $m = 1$ we have $f = T + C$, where C is a real constant and T is an homomorphism. For larger m, one could expect the solutions to be the local polynomials of degree at most m, i.e.

$$f = \sum_{j=0}^{m} T_j \circ \mathcal{D}_j \tag{8}$$

in a neighborhood of 1, where the maps $T_j : G^j \to \mathbf{R}$ are symmetric and multiadditive. Unfortunately, this is not true in a general setting; a simple counterexample can be obtained with the Heisenberg group (see Section 4). However, in the case $m = 2$, it is easy to show that the right-abelian solutions are of the form (8). The main result of [17] claims that it is indeed the case for any m.

Theorem 3. *Assuming that f is right-abelian, if $\Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(x) = 0$ for $x, h_1, \ldots, h_{m+1}$ in a neighborhood of 1, then there exist symmetric multiadditive maps $T_j : G^j \to \mathbf{R}$ ($j \in \{0, \ldots, m\}$) such that*

$$f = \sum_{j=0}^{m} (T_j \circ \mathcal{D}_j),$$

in a neighborhood of 1. Moreover, these maps $T_0, T_1, \ldots, T_m$ are unique if they are seen as germs of functions at the identity.

Proof. This result can be proved by induction on $m \in \mathbf{N}_0$; assume that the property is satisfied for any $k \in \{0, \ldots, m-1\}$ with $m \geq 2$. It is clear that $\Delta^m_{h_1,\ldots,h_m} f$ is constant for $h_1, \ldots, h_m$ in a neighborhood of 1. Set

$$T_m(h_1, \ldots, h_m) = \frac{1}{m!} \Delta^m_{h_1,\ldots,h_m} f(1).$$

We have

$$\begin{aligned} &\Delta^m_{h_1,\ldots,h_j h'_j,\ldots,h_m} f(1) - \Delta^m_{h_1,\ldots,h_j,\ldots,h_m} f(1) - \Delta^m_{h_1,\ldots,h'_j,\ldots,h_m} f(1) \\ &= \Delta^{m+1}_{h_1,\ldots,h_j,h'_j,\ldots,h_m} f(1) = 0, \end{aligned}$$

which shows that T_m is multiadditive. Moreover, it is symmetric since f is right-abelian.

Finally, set $g = f - T_m \circ \mathcal{D}_m$ to obtain

$$\Delta^m_{h_1,\ldots,h_m} g(x) = \Delta^m_{h_1,\ldots,h_m} f(x) - m! T_m(h_1, \ldots, h_m) = 0,$$

thanks to Proposition 2. The conclusion follows by induction. □

With the same proof, we get the continuous version.

Theorem 4. *If $\Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(x) = 0$ for $x, h_1, \ldots, h_{m+1}$ in a neighborhood of 1 and $f : G \to \mathbf{R}$ is right-abelian and continuous in a neighborhood of 1, then there exist symmetric multiadditive continuous maps $T_j : G^j \to \mathbf{R}$ ($j \in \{0, \ldots, m\}$) such that*

$$f = \sum_{j=0}^{m} (T_j \circ \mathcal{D}_j)$$

in a neighborhood of 1. Moreover, these maps $T_0, T_1, \ldots, T_m$ are unique if they are seen as germs of continuous functions at the identity.

Remark 1. In the case of Lie groups or measurable functions with respect to a Haar measure, it is clear that if the solution of the local Fréchet equation f is smooth or measurable, then the maps $T_0, \dots, T_m$ are also smooth or measurable.

3 The case of Lie groups

Let us now study the Fréchet equation in a Lie group G. Unless otherwise stated, in this section G will stand for a n-dimensional connected Lie group.

For Lie groups, Theorem 4 as an essential consequence.

Proposition 3. *If the right-abelian function f is a smooth solution of the local Fréchet equation on a Lie group G, then $[L_X, L_Y](f)$ vanishes in a neighborhood of 1 for any $X, Y \in \mathfrak{g}$.*

Proof. It is well known that f is a sum of smooth monomials. So, we are reduced to the case $f = T_m \circ \mathcal{D}_m$, where $T_m : G^m \to \mathbf{R}$ is a smooth multiadditive symmetric map. The result being classical for $m = 1$, let us treat the case $m > 1$. We have

$$L_X L_Y(f)(1) = \frac{d}{dt}\frac{d}{ds} T_m(\exp(tX)\exp(sY), \dots, \exp(tX)\exp(sY))|_{s=0,t=0}.$$

Let us show that the last expression is equal to $L_Y L_X f(1)$. This results from the following equality,

$$\begin{aligned} &T_m(\exp(tX)\exp(sY), \dots, \exp(tX)\exp(sY)) \\ &= T_m(\exp(sY)\exp(tX), \dots, \exp(sY)\exp(tX)), \end{aligned}$$

as T_m is multiadditive and these two are homogeneous polynomials of degree m with respect to the variables t and s. We thus have

$$L_X L_Y(f)(1) - L_Y L_X(f)(1) = 0.$$

Since $L_x^* f$ is also a solution of the local equation for x close to 1 and $L_X L_Y$ commutes with pullbacks of left translation by left invariance, we get

$$\begin{aligned} L_X L_Y(f)(x) &= L_x^* L_X L_Y(f)(1) = L_X L_Y(L_x^* f)(1) \\ &= L_Y L_X(L_x^* f)(1) = L_Y L_X(f)(x), \end{aligned}$$

if x is in a neighborhood of 1. □

Let us briefly consider the restricted Fréchet equation on the diagonal:

$$\Delta_h^{m+1} f(x) = 0,$$

where h and x both lie in a neighborhood of 1 (let us recall that $\Delta_h^m = \Delta_{h,\dots,h}^m$). In such a case, we have

$$\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(xh^j).$$

Let us also remember that if $f : \mathbf{R} \to \mathbf{R}$ is smooth in a neighborhood of x_0 and if $\Delta_h^{m+1} f(x)$ is vanishing for any x in a neighborhood U of x_0 and any h in a neighborhood of 0, since we have

$$\frac{d^{m+1}}{dx^{m+1}} f(x) = \lim_{h \to 0} \frac{\Delta_h^{m+1} f(x)}{h^{m+1}} = 0,$$

then f is polynomial of degree lower or equal to m in U. Using this result, we obtain the Taylor series of the solution.

Proposition 4. *If $f : G \to \mathbf{R}$ is smooth in a neighborhood of 1 and if $\Delta_h^{m+1} f(x)$ vanishes for any x and any h both in a neighborhood of 1, then*

$$f(x \exp(X)) = f(x) + \sum_{j=1}^{m} \frac{L_X^j f(x)}{j!},$$

for any X in a neighborhood of 0 and any x in a neighborhood of 1.

Proof. Let us assume that the last equation is true for x and h lying in the neighborhood U of 1 in such a way that exp is a diffeomorphism between a convex neighborhood V of 0 in $\mathfrak{g}$ and U. Given $X \in V$, define

$$\gamma(t) = f(\exp(tX)),$$

for $t \in \mathbf{R}$. Since V is open and convex, U contains $\exp([-\varepsilon, 1+\varepsilon]X)$ for some $\varepsilon > 0$. Given $s, t \in]-\varepsilon, 1+\varepsilon[$, we thus have

$$\begin{aligned}
\Delta_s^{m+1}\gamma(t) &= \sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} f(\exp((t+ks)X)) \\
&= \sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} f(\exp(tX)\exp(sX)^k) \\
&= \Delta_{\exp(sX)}^{m+1} f(\exp(tX)) = 0.
\end{aligned}$$

Therefore, we can write

$$f(\exp(tX)) = \gamma(t) = \sum_{k=0}^{m} a_k t^k,$$

for any $t \in]-\varepsilon, 1+\varepsilon[$, which gives the first coefficient a_0 of the Taylor expansion: $\gamma(0) = f(1)$.

The other coefficients a_k can be obtained by differentiating γ at 0. Let us recall that for any smooth function g on G and any left-invariant vector field X, we have the relation

$$L_X^k g(x \exp(tX)) = \frac{d^k}{dt^k} g(x \exp(tX)),$$

which gives

$$k! a_k = \frac{d^k}{dt^k} \gamma(0) = L_X^k f(1).$$

To conclude, it suffices to consider the translated functions $f \circ L_x$. □

Let us return to the general Fréchet equation.

Proposition 5. *Let G be a connected and simply connected Lie group and let $f : G \to \mathbf{R}$ be a smooth function in a neighborhood of 1 for which $\Delta^{m+1}_{h_1,\dots,h_{m+1}} f(x)$ vanishes for each x and $h_1, \dots, h_{m+1}$ in a neighborhood of 1. Then, in a neighborhood of 1, we have*

$$f = \sum_{|\alpha| \leq m} a_\alpha f_1^{\alpha_1} \cdots f_k^{\alpha_k},$$

where $a_\alpha \in \mathbf{R}$, $k = \dim \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{R}) \leq n$, $f_1, \dots, f_k \in \mathrm{Hom}_{\mathrm{Gr}}(G, \mathbf{R})$ and where the multi-index α runs over $\mathbf{N}^k$.

Proof. Obviously,

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{R}) = \{f \in \mathfrak{g}^* : f|_{[\mathfrak{g},\mathfrak{g}]} = 0\} = [\mathfrak{g}, \mathfrak{g}]^\perp$$

is a subspace of $\mathfrak{g}^*$, which is a finite dimensional vector space. Let us fix a basis $dx^1, \dots, dx^k$ of $[\mathfrak{g}, \mathfrak{g}]^\perp$. Of course, the dual basis $\partial_1, \dots, \partial_k$ viewed as a set of vectors of $\mathfrak{g}$ spans a supplement space of $[\mathfrak{g}, \mathfrak{g}]$, i.e. we have

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \langle \partial_1, \dots, \partial_k \rangle.$$

That being said, let us fix a basis $\partial_{k+1}, \dots, \partial_n$ of $[\mathfrak{g}, \mathfrak{g}]$ and denote the dual basis by $dx^{k+1}, \dots, dx^n$, so that we have

$$\mathfrak{g}^* = [\mathfrak{g}, \mathfrak{g}]^\perp \oplus \langle dx^{k+1}, \dots, dx^n \rangle.$$

By Proposition 4, we get

$$f \circ \exp = f(1) + \sum_{j=1}^{m} \sum_{i_1, \dots, i_j} \frac{L_{i_1 \dots i_j} f(1)}{j!} \, dx^{i_1} \cdots dx^{i_j}.$$

Since $\partial_{k+1}, \dots, \partial_n$ span $[\mathfrak{g}, \mathfrak{g}]$ and $[X, Y](f)$ vanishes in a neighborhood of 1 for all $X, Y \in \mathfrak{g}$ by Proposition 3, we deduce that $L_i f$ vanishes in a neighborhood of 1 for every $i \in \{k+1, \dots, n\}$. Hence, the previous formula can be rewritten

$$f \circ \exp = f(1) + \sum_{j=1}^{m} \sum_{i_1, \dots, i_j \leq k} \frac{L_{i_1 \dots i_j} f(1)}{j!} \, dx^{i_1} \cdots dx^{i_j}.$$

From Lie's theorem, there exists unique maps $f_1, \dots, f_k$ in $\mathrm{Hom}_{\mathrm{Gr}}(G, \mathbf{R})$ such that $dx^1 = d_1 f_1, \dots, dx^k = d_1 f_k$, G being simply connected. Since $f_j \circ \exp = d_1 f$, we get

$$f \circ \exp = f(1) + \sum_{j=1}^{m} \sum_{i_1, \dots, i_j \leq k} L_{i_1 \dots i_j} f(1) \; f_{i_1} \circ \exp \cdots f_{i_j} \circ \exp,$$

which is sufficient to conclude. □

Now, let us weaken the smoothness assumption on f. We first need a classical results transposed in the setting of the Lie groups.

Lemma 3. *Let G be a connected Lie group equipped with a left Haar measure; if $f : G \to \mathbf{R}$ is bounded almost everywhere in a neighborhood of 1 and if $\Delta_h^{m+1} f(x) = 0$ for x and h in a neighborhood of* 1*, then f is bounded in a neighborhood of* 1.

Proof. Let d be a right-invariant distance on G; there exists $\varepsilon > 0$ such that $\Delta_h^{m+1} f(x)$ vanishes for every $x, h \in B_d(1, \varepsilon)$ and a negligible subset N of $B_d(1, \varepsilon)$ for which $|f(x)| \le C$ if $x \in B_d(1, \varepsilon) \setminus N$. Using Fréchet equation, we have

$$|f(x)| = \Big| \sum_{j=0}^{m} (-1)^j \binom{m+1}{j} f(xh^{m-j+1}) \Big| \le 2^{m+1} \sup_{y \in \{xh^j : j \in \{1,\ldots,m+1\}\}} |f(y)|,$$

for $x, h \in B_d(1, \varepsilon)$.

Let us prove that f is bounded on $B_d(1, \frac{\varepsilon}{m+2})$. Let $x \in B_d(1, \frac{\varepsilon}{m+2})$ and set

$$A_j = \{h \in B_d(1, \frac{\varepsilon}{m+2}) : xh^j \in B_d(1, \varepsilon) \setminus N\},$$

for $j \in \{1, \ldots, m+1\}$. The power function $p_j : x \mapsto x^j$ being a diffeomorphism between two neighborhoods of 1, let us denote by r_j the local inverse of p_j; by lowering ε if needed, we may assume that it is a diffeomorphism for all j on $B_d(1, \varepsilon)$. That being said, let us remark that $A_1 \subset (A_1 \setminus A_j) \cup A_j$ and that $A_1 \setminus A_j$ is negligible since

$$A_1 \setminus A_j \subset r_j(x^{-1}N \cap B_d(1, \varepsilon)) \cap B_d\Big(1, \frac{\varepsilon}{m+2}\Big),$$

the set $r_j(x^{-1}N \cap B_d(1, \varepsilon))$ being itself Haar-negligible. We thus have

$$A_1 = \bigcap_{j=1}^{m+1} A_j \cup \bigcup_{j=1}^{m+1} (A_1 \setminus A_j),$$

which implies that A_1 is equal to $\cap_{j=1}^{m+1} A_j$ almost everywhere. Now, $\cap_{j=1}^{m+1} A_j$ is not empty; if it was not the case, A_1 would be negligible, which is impossible since

$$A_1 = \Big(B_d\Big(1, \frac{\varepsilon}{m+2}\Big) \cap x^{-1} B_d(1, \varepsilon)\Big) \setminus x^{-1} N$$

and nonempty open sets are not negligible for the Haar measure on Lie groups. Now, by choosing $h \in \cap_{j=1}^{m+1} A_j$, we get $|f(x)| \le 2^{m+1} C$, which concludes the proof since x is arbitrary in $B_d(1, \frac{\varepsilon}{m+2})$. □

The next lemma can be seen as an adaptation of Theorem 1 from [12] in the case of multiplicative notations.

Lemma 4. *If $f : G \to \mathbf{R}$ is bounded in a neighborhood of* 1 *and if $\Delta_h^{m+1} f(x) = 0$ for x and h in a neighborhood of* 1*, then f is continuous in a neighborhood of* 1.

Proof. Let d be a right-invariant distance on G inducing the same topology of manifold. Choose $\eta > 0$ such that $\Delta_h^{m+1} f(x) = 0$ and $|f(x)|$ is bounded by a constant if $d(x, 1) < \eta$ and $d(h, 1) < \eta$. Next, let $\varepsilon > 0$ be such that $\varepsilon < \eta/(m+1)$ and set $\delta = \eta - (m+1)\varepsilon$. Since $|f|$ is bounded on $B_d(1, \eta)$, the relations

$$d(xh^k, 1) \le d(1, h) + d(h, h^2) + \cdots + d(h^{k-1}, h^k) + d(h^k, xh^k) < \delta + k\varepsilon \le \eta$$

imply the existence of a constant $C > 0$ for which $|\Delta_h^k f(x)| \le C$ for all $k \in \{0, \ldots, m+1\}$, $d(x,1) < \delta$ and $d(h,1) < \varepsilon$.

Given $x \in B_d(1,\delta)$, let us prove that f is continuous at x. Let $r \in \mathbf{N}$ and assume that $d(h,1) < \varepsilon/r$. It is easy to see that

$$f(xh^q) = \sum_{j=0}^{q} \frac{\Delta_h^j f(x)}{j!} (q)_j,$$

where $(q)_j$ denote the falling factorial. The end of the proof is similar to the proof of Theorem 1 in [12]. □

The following result is also an adaptation of Theorem 2 from [12].

Lemma 5. *If $f : G \to \mathbf{R}$ is continuous in a neighborhood of 1 and if $\Delta_h^{m+1} f(x)$ vanishes for any x and any h in a neighborhood of 1, then f is smooth in a neighborhood of 1.*

Proof. Let $X_1, \ldots, X_n$ be a basis of left-invariant vector fields and consider the Lebesgue measure on $\mathfrak{g}$ seen as $\mathbf{R}^n$ using this basis. We need to prove that $f \circ \exp$ is smooth in a neighborhood of 0. First, define H as the Baker-Campbell-Hausdorff series near the origin:

$$H(X,Y) = \log(\exp(X)\exp(Y));$$

we have $H : U^2 \to \mathfrak{g}$, where U is a sufficiently small neighborhood of 0 in $\mathfrak{g}$. Of course, H is smooth (even analytic) in a neighborhood of $(0,0)$. We may assume that $F = f \circ \exp$ is continuous on U and that $\Delta_h^{m+1} f(x)$ vanishes for $x, h \in \exp(U)$. We can also replace the usual differences Δ_h with centered differences δ_h defined by

$$\delta_h f(x) = f(xh) - f(xh^{-1});$$

indeed, the relation

$$\delta_h^m = (R_{h^{-1}}^*)^m \Delta_{h^2}^m$$

implies that that the so-obtained Fréchet equation remains equivalent to the usual one in a sufficiently small neighborhood of 1. Since, at the level of Lie algebras, we have

$$\Delta_Y^{m+1} F(X) = \sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} F(H(X, jY)),$$

we can assume that the Fréchet equation at the level of Lie algebras is satisfied for $X, Y \in U$.

That being done, it is well-known that there exists a function $\Phi \in \mathcal{D}(\mathfrak{g})$ such that $\int \Phi(X)\, DX = 1$ and $F = F|^{\mathfrak{g}} * \Phi$ on a neighborhood of 0 in $\mathfrak{g}$, where $F|^{\mathfrak{g}} : \mathfrak{g} \to \mathbf{R}$ is the extension of F which takes the value 0 outside U (such a construction can be obtained from Remark 2 in [12] for example). Since $F|^{\mathfrak{g}} * \Phi$ is smooth, so is f in a neighborhood of 1. □

Using the universal covering group, Proposition 5 may be generalized to obtain Theorem 1.

Proof of Theorem 1. Let $\mathcal{U}$ denote the universal covering group of G and let U be a neighborhood of 1 for which $\Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(x)$ vanishes if $x, h_1, \ldots, h_{m+1} \in U$ in such a way that the covering map $p : \mathcal{U} \to G$ gives a diffeomorphism between a neighborhood V of 1 in $\mathcal{U}$ and U. Define $\tilde{f} = f \circ p$; since p is a local homomorphism, we have

$$\Delta^{m+1}_{h_1,\ldots,h_{m+1}} \tilde{f}(x) = \Delta^{m+1}_{p(h_1),\ldots,p(h_{m+1})} f(p(x)) = 0,$$

for all $x, h_1, \ldots, h_{m+1} \in V$. Since $\mathcal{U}$ is known to be simply connected, thanks to the previous results, we can write

$$\tilde{f} = \sum_{|\alpha| \leq m} a_\alpha \tilde{f_1}^{\alpha_1} \cdots \tilde{f_k}^{\alpha_k},$$

where $\tilde{f_1}, \ldots, \tilde{f_k} \in \mathrm{Hom}_{\mathrm{Gr}}(G, \mathbf{R})$. Of course, since $\mathcal{U}$ and G are isomorphic as Lie groups in a neighborhood of the identity, they have the same Lie algebra. That being said, it suffices to consider $\tilde{f} \circ (p|_V)^{-1}$ to get the first part of the theorem.

Let us now check that such functions are always solutions of the equation. To do so, we proceed by induction on the order m of the equation. The case $m = 0$ is trivial, since it is a constant function in a neighborhood of 1. Let us take $m \geq 1$; we have to check that the equation is satisfied for a monomial $f = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$. We may assume that $\alpha_k \geq 1$ and gather all the terms f_j with $j < k$ to write $f = g f_k$. We have

$$\Delta_h f(x) = \Delta_h g(x) f_k(x) + g(xh) f_k(h),$$

which shows that the difference operator decreases the total degree $\alpha_1 + \cdots + \alpha_k$. As a consequence, we get $\Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(x) = 0$ locally in a neighborhood of 1. ☐

We can extend this problem to homogeneous spaces as follows. Given $x \in G$ and an element p of a manifold M, let us denote the action of x on p by xp. In this setting, the difference operators is given by

$$\Delta_h f(x) = f(hx) - f(x),$$

for $h \in G$ and $x \in M$. The difference operator of order m is once again naturally given by

$$\Delta^m_{h_1,\ldots,h_m} = \Delta_{h_m} \circ \cdots \circ \Delta_{h_1}.$$

As usual, G_x stands for the isotropy subgroup of G at x. With these notations, we obtain the following generalization of Theorem 1.

Proposition 6. *Let G be a connected Lie group that acts smoothly and transitively on a smooth manifold M; if $f : M \to \mathbf{R}$ is a right-abelian function that is bounded on a neighborhood of x_0 in M such that $\Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(x) = 0$ for x in a neighborhood of x_0 in M and $h_1, \ldots, h_{m+1}$ in a neighborhood of 1 in G, then, for any h in a neighborhood of 1 in G, we have*

$$f(hx_0) = \sum_{|\alpha| \leq m} a_\alpha f_1^{\alpha_1}(h) \cdots f_k^{\alpha_k}(h),$$

with $a_\alpha \in \mathbf{R}$, $k = \dim \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{R})$ and where $f_1, \ldots, f_k \in \mathrm{Hom}^{\mathrm{loc}}_{\mathrm{Gr}}(G, \mathbf{R})$ vanish on a neighborhood of 1 in G_{x_0}.

Proof. Let $\pi : G \to G/G_{x_0}$ be the canonical projection; it is well-known that the map

$$T : [u] \in G/G_{x_0} \mapsto ux_0 \in M$$

is an equivariant diffeomorphism. Let U be an open neighborhood of x_0 in M and V be an open neighborhood of 1 in G such that $\Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(x) = 0$ for all $x \in U$ and all $h_1, \ldots, h_{m+1} \in V$. If we set

$$\tilde{f} = (f \circ T) \circ \pi,$$

$\tilde{f} : G \to \mathbf{R}$ is a map satisfying

$$\Delta^{m+1}_{h_1,\ldots,h_{m+1}} \tilde{f}(u) = \Delta^{m+1}_{h_1,\ldots,h_{m+1}} f(ux_0) = 0,$$

for $h \in V$ and $u \in \pi^{-1}(T^{-1}(U))$. Thanks to Theorem 1, we can write

$$\tilde{f}(h) = \sum_{|\alpha| \leq m} a_\alpha f_1^{\alpha_1}(h) \cdots f_k^{\alpha_k}(h),$$

where $f_1, \ldots, f_k$ belong to $\mathrm{Hom}^{\mathrm{loc}}_{\mathrm{Gr}}(G, \mathbf{R})$, for h in a neighborhood of 1. One may assume that $f_1, \ldots, f_k$ is a basis of $\mathrm{Hom}^{\mathrm{loc}}_{\mathrm{Gr}}(G, \mathbf{R})$, since it is isomorphic to $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{R})$ through the exponential map. We can choose it so that $f_1, ..., f_l$ $(l \leq k)$ is a basis of the subspace

$$\{f \in \mathrm{Hom}^{\mathrm{loc}}_{\mathrm{Gr}}(G, \mathbf{R}) : f|_{G_{x_0}} = 0 \text{ in a neighborhood of } 1\}.$$

Now, as in Proposition 5, let $\partial_1, \ldots, \partial_n$ be a basis of $\mathfrak{g}$ in such a way that $d_1 f_1, \ldots, d_1 f_k$ are projections with respect to the k first components of $X = \sum_{i=1}^n X^i \partial_i$. The solution may be expressed as

$$f(hx_0) = \sum_{|\alpha| \leq m} a_\alpha f_1^{\alpha_1}(h) \cdots f_k^{\alpha_k}(h).$$

In this setting, two elements $h, h' \in G$ in a neighborhood of 1 satisfy $hx_0 = h'x_0$ if and only if one has $h' = hx$ for an element $x \in G_{x_0}$. Therefore, if h is near 1 in G and x is near 1 in G_{x_0}, we must have $f_j(h) = f_j(hx)$ for $j \in \{1, \ldots, l\}$. As a consequence, the following equality must be satisfied:

$$\sum_{|\alpha| \leq m} a_\alpha f_1^{\alpha_1}(h) \cdots f_k^{\alpha_k}(h) = \sum_{|\alpha| \leq m} a_\alpha f_1^{\alpha_1}(hx) \cdots f_k^{\alpha_k}(hx).$$

In other words, we must have

$$\sum_{|\alpha| \leq m} a_\alpha f_1^{\alpha_1}(h) \cdots f_l^{\alpha_l}(h) [f_{l+1}^{\alpha_{l+1}}(h) \cdots f_k^{\alpha_k}(h) - f_{l+1}^{\alpha_{l+1}}(hx) \cdots f_k^{\alpha_k}(hx)] = 0,$$

for h in a neighborhood of 1 in G and x in a neighborhood of 1 in G_{x_0}. Using the exponential map $\exp : \mathfrak{g} \to G$, we get

$$\sum_{|\alpha| \leq m} a_\alpha (X^1)^{\alpha_1} \cdots (X^l)^{\alpha_l}$$
$$[(X^{l+1})^{\alpha_{l+1}} \cdots (X^k)^{\alpha_k} - (X^{l+1} + Y^{l+1})^{\alpha_{l+1}} \cdots (X^k + Y^k)^{\alpha_k}] = 0,$$

for $(X^1, ..., X^k)$ near 0 in $\mathbf{R}^k$ and $(Y^{l+1}, ..., Y^k)$ near 0 in $\mathbf{R}^{k-l}$; this may be seen as a polynomial in $\mathbf{R}^{2k-l}$. Naturally, we have the equality of each homogeneous polynomials

$$\sum_{|\alpha|=d} a_\alpha (X^1)^{\alpha_1} \cdots (X^l)^{\alpha_l}$$
$$[(X^{l+1})^{\alpha_{l+1}} \cdots (X^k)^{\alpha_k} - (X^{l+1} + Y^{l+1})^{\alpha_{l+1}} \cdots (X^k + Y^k)^{\alpha_k}] = 0,$$

in a neighborhood of 0 in $\mathbf{R}^{2k-l}$. Let α be a multi-index such that $|\alpha| = d$ and $\alpha_j \neq 0$ for a $j \geq l+1$; by applying the partial derivative $\partial^{\alpha_1}_{X^1} \cdots \partial^{\alpha_l}_{X^l} \partial^{\alpha_{l+1}}_{Y^{l+1}} \cdots \partial^{\alpha_k}_{Y^k}$ to the equation, we get $a_\alpha = 0$, which leads to the conclusion. □

4 Applications and examples

Let us make some remarks and give some examples. We still consider the right-abelian solutions of the Fréchet equation.

Since $\mathbf{R}$ does not contain any non-trivial compact subgroup, we have the following property.

Corollary 2. *If G is compact, there no global right-abelian solution of the Fréchet equation other than the constants.*

However, there can be local solutions, as attested by the following example.

Example 1. Consider the unit circle S^1. Obviously, $\mathrm{Hom}^{\mathrm{loc}}_{\mathrm{Gr}}(S^1, \mathbf{R})$ is one dimensional and generated by the germ of the argument function at 1, so that a solution is given in a neighborhood of 1 by

$$f(z) = \sum_{j=0}^{m} a_j \mathrm{arg}^j(z).$$

On the other hand, S^3 seen as a Lie group with quaternion product does not have any local solution that is not constant.

Let us now consider the case where G is semisimple. Its commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ being simply $\mathfrak{g}$, we have the following result.

Corollary 3. *If G is semisimple, there is no local right-abelian solution of the Fréchet equation that is not constant.*

Classical examples of semisimple Lie groups are the special linear group, the orthogonal group and the symplectic group.

That begin said, if G is solvable, the derived series must be strictly decreasing and $[\mathfrak{g}, \mathfrak{g}]$ is a proper subalgebra of $\mathfrak{g}$; as a consequence, we get the existence of non-trivial local solution in that case.

Corollary 4. *If G is solvable, there is at least one local solution of Fréchet equation that is not constant.*

Example 2. The Heisenberg group is defined as $G = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ with the group operation given by

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + x' \cdot y).$$

It is easy to check that the dimension of $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbf{R})$ is $2n$ and that it is generated by linear forms that does not depend on the variable t. Therefore, local solutions are given by polynomials that are constant with respect to their last variable t.

Of course, G need not to be solvable to accept non-trivial solutions; an example is given by $\mathrm{GL}(n, \mathbf{R})$.

Example 3. Consider $G = \mathrm{GL}(n, \mathbf{R})$; since $[\mathfrak{gl}(n, \mathbf{R}), \mathfrak{gl}(n, \mathbf{R})] = \mathfrak{sl}(n, \mathbf{R})$, we know that $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{gl}(n, \mathbf{R}), \mathbf{R})$ is one dimensional. It is generated by the trace function. Applying the exponential, we get that local solutions of the equation of order $m+1$ are given by

$$f(M) = \sum_{j=0}^{m} a_j \ln(\det(M))^j,$$

if M is in a neighborhood of the identity matrix.

Finally, let us raise the question of the existence of solutions that are not right-abelian.

Proposition 7. *If G is a nilpotent Lie group with step one, there exists a solution of (3) with $m = 3$ in a neighborhood of 1 that is not right-abelian.*

Proof. Since the group operation is expressed in the Lie algebra by the Baker-Campbell-Hausdorff formula truncated at first order commutators, we have

$$H(X, Y) = X + Y + \frac{1}{2}[X, Y].$$

For convenience, let us write XY instead of $H(X, Y)$. Since $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$, we can find a linear functional f on $\mathfrak{g}$ that does not vanish identically on $[\mathfrak{g}, \mathfrak{g}]$. For such an f, we have

$$\begin{aligned}
\Delta^2_{Y_1,Y_2} f(X) &= f(XY_1Y_2) - f(XY_1) - f(XY_2) + f(X) \\
&= f(X + Y_1 + Y_2 + \frac{1}{2}[Y_1, Y_2] + \frac{1}{2}[X, Y_1 + Y_2 + \frac{1}{2}[Y_1, Y_2]]) \\
&- f(X + Y_1 + \frac{1}{2}[X, Y_1]) - f(X + Y_2 + \frac{1}{2}[X, Y_2]) + f(X) \\
&= \frac{1}{2} f([Y_1, Y_2]),
\end{aligned}$$

so that $\Delta^3_{Y_1,Y_2,Y_3} f(X) = 0$ everywhere on $\mathfrak{g}$. Moreover, we directly get

$$f(XY_1Y_2) = f(X) + f(Y_1) + f(Y_2) + \frac{1}{2} f([Y_1, Y_2]) + \frac{1}{2} f([X, Y_1]) + \frac{1}{2} f([X, Y_2]).$$

Therefore, we have

$$f(XY_1Y_2) - f(XY_2Y_1) = f([Y_1, Y_2]),$$

which does not vanish for all Y_1, Y_2 since f does not vanish identically on the commutator subspace. □

An example of nilpotent group with step one is given by the Heisenberg group.

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