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SOME GENERAL COMMENTS ON THE PROBLEM OF STELLAR STABILITY

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Abstract: Some remarks are formulated concerning the influence of nuclear equilibrium on dynamical stability and the implications, for the general stellar stability problem, of neutrino emission by mechanisms related to the direct electron-neutrino interaction. In particular, it is shown that models in which thermonuclear energy generation is balanced mainly by the emission of neutrinos, while secularly stable, would be vibrationally unstable. Finally, it is suggested that the complete differential system being of a higher order than the third, there might exist significant time-scales distinct from the usual three associated with the discussion of stellar stability.

1. When I started getting interested in astrophysics, general texts on the subject were far less numerous than to-day and among those existing at the time, Rosseland's books were fascinating guides for my first steps.

Their well balanced blending of modern physics and physical hydrodynamics had a special charm of its own. The generality of the points of view adopted left one with an impression of dominating the subject and, at the same time, it opened new possibilities of approach or new avenues of research. Here and there, special sections or synthetic comments provided a bird's-eye view of a whole field and, to this day, I remember, with some kind of nostalgia, the enthousiasm aroused by some of the prefaces with their philosophical undertones and their poetical appeal.

When, full of expectations and good intentions, I arrived in Oslo in 1939, Rosseland directed me to the problem of the Cepheids and stellar oscillations which, for all these years, has remained at the centre of my interests and has claimed a good deal of my total activity. Among the first papers that I read on the subject were two from Rosseland [1] himself, opening the series of publications of the University Observatory in Oslo and devoted to oscillating fluid globes and to the stability of gaseous stars. Starting with a striking example concerning the effects of radon on acoustic waves, the latter contains the first derivation, on the basis of the perturbation method, of the coefficient of vibrational stability (or generalized damping coefficient) for radial pulsations.

If the numerical results were vitiated by the lack of information on stellar energy sources and by some of the approximations, on the other hand, and apart from the method which has kept its interest, some of the remarks and comments in this paper especially concerning the behaviour of the external layers and their effects could still be read with profit to-day.

Rosseland continued to be interested in the vibrational stability of the stars and especially in the effects, in this respect, of nuclear energy generation. In fact, just about as I arrived in Oslo, a paper by Rosseland and Randers [2] was coming out of the press in which, for the first time, the effects of phase-delays in energy generation, mentioned but summarily by Eddington, were submitted to a quantitative discussion fixing the pattern followed since then. It seemed particularly fitting to recall this because the present paper will also be essentially concerned with some aspects of the same problem and an attempt at generalizing its approach.

2. To avoid too much algebra, we shall limit ourselves to the case of purely radial perturbations of a hydrostatic model (or quasi-hydrostatic model evolving at a very slow rate) in radiative equilibrium throughout.

In that case, if η represents the relative displacement $(\delta r/r)$, the fundamental linear equations for the Lagrangian perturbations denoted by $\delta \ldots$, are:

a) *Conservation of mass*

$$\frac{\delta \varrho}{\varrho} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^3 \eta) \tag{1}$$

b) *Conservation of momentum*

$$r\frac{\partial^2 \eta}{\partial t^2} = -4\,\eta\frac{1}{\varrho}\frac{\partial p}{\partial r} - \frac{p}{\varrho}\frac{\partial}{\partial r}\left(\frac{\delta p}{p}\right) - \frac{\delta p}{p}\frac{1}{\varrho}\frac{\partial p}{\partial r} \tag{2}$$

in which we have already eliminated the perturbation of the gravitational acceleration.

c) *Conservation of thermal energy*

$$\frac{\partial \delta p}{\partial t} - \frac{\Gamma_1 p}{\varrho}\frac{\partial \delta \varrho}{\partial t} - \frac{1}{N}\frac{\partial \delta N}{\partial t}\frac{6\,\beta(1-\beta)\,p}{(3/2)\,\beta + 12\,(1-\beta)} = -(\Gamma_3 - 1)\,\varrho\left[\delta\frac{dE_s}{dt} + \delta\varepsilon'_\nu + \delta\varepsilon_\nu + \frac{\partial \delta L}{\partial m}\right] \tag{3}$$

with

$$\beta = \frac{p_G}{p_G + p_R} = \frac{p_G}{p} \tag{4}$$

Here, we have kept for the rate of energy generation, its general expression $-(dE_S/dt)$, E_S denoting the subatomic energy. As far as the emission of neutrinos is concerned we have distinguished the losses ε'_ν associated with the neutrinos emitted directly in the β–decay processes occuring in the chain of thermonuclear reactions and the losses ε_ν corresponding to the various mechanisms suggested on the basis of a direct electron-neutrino interaction. The third term in the left-hand member corresponds to the variation of the total number N of particles per unit mass due to nuclear processes. Here, ionization is treated as complete and Γ_1 and Γ_3 are the usual generalized adiabatic coefficients for a mixture of a monatomic gas and radiation.

Equation (3) can also be written

$$\frac{\partial \delta T}{\partial t} - \frac{(\Gamma_3 - 1)\, T}{\varrho} \frac{d\delta\varrho}{dt} + \frac{(3/2)\,\beta}{(3/2)\,\beta + 12\,(1-\beta)} \frac{T}{N} \frac{\partial \delta N}{\partial t} = \tag{5}$$
$$= -\frac{1}{C_v}\left[\delta\left(\frac{dE_S}{dt}\right) + \delta\varepsilon'_\nu + \delta\varepsilon_\nu + \frac{\partial \delta L}{\partial m}\right]$$

where C_v is the generalized specific heat at constant volume for a monatomic gas and radiation.

For the usual thermonuclear reactions, we may write directly

$$\frac{dE_S}{dt} + \varepsilon'_\nu = -\,\varepsilon_N \tag{6}$$

where the rate of energy generation ε_N is a function of ϱ and T. Furthermore, in this case, the variation of N is very small and given by

$$\frac{1}{N}\frac{dN}{dt} = -\,\alpha\,\bar{\mu}\,\frac{\varepsilon_N}{e_N} \to 0 \tag{7}$$

if e_N is the total energy liberated by the transformation of 1 gm of reactant (say H or He^4) and α, a constant of the order of unity ($4\ H \to He^4 : \alpha = 5/4$; $3\ He^4 \to C^{12} : \alpha = 1/6$).

On the other hand, when T and ϱ reach high enough values some kind of statistical equilibrium tends to get established between the different nuclear species and the variations of N are no longer negligible as the equilibrium gets displaced.

3. In the presence of a nuclear statistical equilibrium, E_S itself must be considered as a function of ϱ and T and it is then easier to group the corresponding terms in the left hand members of equations (3) and (5). Developing δN and δE_S in terms of $\delta\varrho$ and δp or $\delta\varrho$ and δT, these equations become:

$$\frac{\partial \delta p}{dt} - \frac{\Gamma_1{}^* p}{\varrho}\frac{\partial \delta\varrho}{dt} = -(\Gamma_3{}^* - 1)\,\varrho\left[\delta\varepsilon_\nu + \frac{\partial \delta L}{\partial m}\right] \tag{8}$$

and

$$\frac{\partial \delta T}{dt} - \frac{(\Gamma_3{}^* - 1)\,T}{\varrho}\frac{\partial \delta\varrho}{dt} = -\frac{1}{C_v{}^*}\left[\delta\varepsilon_\nu + \frac{\partial \delta L}{\partial m}\right] \tag{9}$$

with

$$\Gamma_1{}^* = \frac{16 - 12\,\beta - 1.5\,\beta^2 + 6\,\beta\,(1-\beta)\,N_{\varrho,p} - \beta\,(4 - 3\,\beta)\frac{\varrho}{NkT}\left(\frac{\partial E_S}{\partial \varrho}\right)_p}{12 - 10.5\,\beta - 6\,\beta\,(1-\beta)\,N_{p,\varrho} + \beta\,(4-3\,\beta)\left(\frac{p}{NkT}\right)\left(\frac{\partial E_S}{\partial p}\right)_\varrho} \tag{10}$$

(11) $$\Gamma_3{}^* - 1 = \frac{4 - 3\beta - 1.5\,\beta\, N_{\varrho,T} - (\beta\varrho/NkT)\,(\partial E_S/\partial\varrho)_T}{12 - 10.5\,\beta + 1.5\,\beta\, N_{T,\varrho} + (\beta/Nk)\,(\partial E_S/\partial T)_\varrho}$$

(12) $$C_v{}^* = \frac{3}{2} Nk \left[1 + \frac{8(1-\beta)}{\beta} + N_{T,\varrho} + \frac{2}{3Nk}\left(\frac{\partial E_S}{\partial T}\right)_\varrho\right]$$

$$N_{\varrho,p} = \left(\frac{\partial \log N}{\partial \log \varrho}\right)_p, \quad N_{p,\varrho} = \left(\frac{\partial \log N}{\partial \log p}\right)_\varrho, \quad N_{T,\varrho} = \left(\frac{\partial \log N}{\partial \log T}\right)_\varrho, \text{ etc}\ldots$$

If we write

$$(4 - 3\beta) + \beta N_{T,\varrho} \equiv \Phi$$

the following relations are easily verified

$$\Phi N_{p,\varrho} = N_{T,\varrho}\,;\ \Phi N_{\varrho,p} = (4 - 3\beta)\, N_{\varrho,T} - \beta N_{T,\varrho}$$

(13) $$\Phi\left(\frac{\partial E_S}{\partial \varrho}\right)_p = \Phi\left(\frac{\partial E_S}{\partial \varrho}\right)_T - \left(\frac{\partial E_S}{\partial T}\right)_\varrho \frac{T}{\varrho}\,\beta\,(1 + N_{\varrho,T})$$

$$\Phi\left(\frac{\partial E_S}{\partial p}\right)_\varrho = \left(\frac{\partial E_S}{\partial T}\right)_\varrho \frac{T}{p}, \quad N_{T,\varrho} = -\frac{3}{2} N_{\varrho,T} - \frac{\varrho}{NkT}\left(\frac{\partial E_S}{\partial \varrho}\right)_T$$

which will allow to pass from one pair of independent variables (say p and ϱ) to the other (say ϱ and T).

Passed the iron-peak, $(\partial E_S/\partial T)_\varrho$ and $N_{T,\varrho}$ are positive while $(\partial E_S/\partial \varrho)_T$ and $N_{\varrho,T}$ are negative and since in absolute values they all tend to be large especially the first two, it is obvious that $\Gamma_1{}^*$ and $\Gamma_3{}^*$ may become close to unity. If this is realized in a large enough core the mean Γ's will be smaller than 4/3 and the star will experience a strong dynamical instability.

This is of course well-known from direct energy considerations since it was first advocated by Hoyle [3] in his theory of supernovae. In recent years, many attempts [4] have been made at following in detail the resulting collapse and the subsequent explosion after some mechanism connected with the conservation of angular momentum or the nuclear reactions in the external layers allow the latter to bounce back. Of course, these advanced stages with their strong non-linear effects can only be studied with large computers and even then the introduction of some simplifying ad-hoc assumptions seems generally to be needed. The safest attack might still be through the building up of reasonable stellar models including the effects (especially as regards convection) of the exact run of the Γ^*'s inside them as they approach these very advanced and critical phases of stellar evolution. These models would also provide the necessary information for a critical linear analysis by means of the relevant equations (1) to (9) of the instabilities present and their exact modes of action setting a firm basis for a concerted attack on the complete non-linear problem.

As regards stability, the situation is probably particularly complex in the region where the elements of the iron group are being built as it must correspond to a case somewhat intermediate between that corresponding to thermonuclear reactions proper and a true statistical equilibrium. It is likely that, in these circumstances, an appreciable part of the variation of the energy liberated during a perturbation is in phase not with the perturbations $\delta\varrho$ and δT themselves but rather with their time-derivatives. Let us assume then that, at least for that part of E_S, say E'_S, we can write

$$dE'_S = \chi\, dN$$

where χ can be treated as a constant mean value in a limited range of conditions. Then using the relations (13) and taking into account the fact that χ/kT must still be fairly large with respect to 1, the definitions (10) and (11) yield

$$\Gamma_1{}^* \simeq \Gamma_3{}^* \simeq \frac{5 + 2(\chi/kT)\, N_{T,\,\varrho}}{3 + 2(\chi/kT)\, N_{T,\,\varrho}}$$

Since, in this range, for $dT > 0$, dE_S and dN are negative, χ is positive and $N_{T,\,\varrho}$ negative and we could get all kinds of critical values for the Γ^*'s including values greater than 5/3 if $|(\chi/kT)\, N_{T,\,\varrho}|$ can be smaller than 3/2.

The other part of the energy liberated, in phase with the perturbation, should naturally be included in ε_N and its high sensitivity to ϱ and T coupled with the effects of the distribution of the Γ^*'s, on the run of the pulsational amplitude η through the star might also lead to a rather strong vibrational instability.

Of course, the emission of neutrinos might modify appreciably the results especially if the direct electron-neutrino interaction is definitely confirmed.

4. A detailed investigation of the effects of energy losses through neutrino emission on stellar stability would certainly be very welcome. In this respect, one might perhaps distinguish two main cases depending on whether the neutrino losses are so large as to overwhelm completely any energy generation or, although large, are still compensated at each instant by energy liberated inside the star so that the latter still evolves through a series of quasi-static models.

Let us consider the second case first which can be tackled by the usual method. After eliminating δp and $\delta\varrho$ from (1), (2) and (3), we are left with the following equation

$$r\frac{\partial^3\eta}{\partial t^3} - \frac{1}{\varrho r^3}\left\{\frac{\partial}{\partial r}\left(\Gamma_1 p r^4 \frac{\partial^2\eta}{\partial r\,\partial t}\right) + r^3\frac{\partial\eta}{\partial t}\frac{\partial}{\partial r}\left[(3\Gamma_1 - 4)\, p\right]\right\} = \tag{14}$$

$$= -\frac{1}{\varrho}\frac{\partial}{\partial r}\left[(\Gamma_3 - 1)\,\varrho\left(\delta\varepsilon_N - \delta\varepsilon_\nu - \frac{\partial\delta L}{\partial m}\right)\right]$$

which, with

(15) $$\eta = \xi(r)\, e^{i\sigma t}$$

becomes

$$-\sigma^2 r\,\xi - \frac{1}{\varrho r^3}\left\{\frac{d}{dr}\left(\Gamma_1 p\, r^4 \frac{d\xi}{dr}\right) + r^3 \xi \frac{d}{dr}\left[(3\,\Gamma_1 - 4)\,p\right]\right\} =$$

(16)

$$= -\frac{1}{i\sigma\varrho}\frac{d}{dr}\left[(\Gamma_3 - 1)\,\varrho\left(\delta\varepsilon_N - \delta\varepsilon_\nu - \frac{d\delta L}{dm}\right)\right]$$

For rapid quasi-adiabatic pulsations, the second member of (16) can be treated as a small perturbing term and if ξ_a and σ_a denote one of the eigensolution of the adiabatic problem, the usual procedure yields immediately for the corresponding correction σ' to σ_a

(17) $$\sigma' = -\frac{1}{2\,\sigma_a^2 J_a}\int_0^M \left(\frac{\delta T}{T}\right)_a \left(\delta\varepsilon_N - \delta\varepsilon_\nu - \frac{d\delta L}{dm}\right)_a dm$$

with

$$J_a = \int_0^M r^2\,\xi_a^2\,dm$$

where the subscript a recalls that all quantities have to be evaluated for the adiabatic solution. Depending on whether the coefficient of vibrational stability σ' is positive or negative the oscillation will be damped or amplified and the star will be vibrationally stable or unstable.

As long as ε_ν is small compared to L, one of the fundamental characteristics of the integrand in (17) is that $\delta\varepsilon_N$ (as ε_N itself) is appreciable only in a small central region while the whole star, and especially the external part, contributes to $d\,\delta L/dm$. Furthermore, for the fundamental mode, the only one which we shall consider here, ξ_a, $(\delta\varrho/\varrho)_a$, $(\delta T/T)_a$ increase strongly in absolute value towards the surface (unless the mass becomes very large) so that the stabilizing conduction term $-(d\,\delta L/dm)$ has a large weight in the integral as compared to the distabilizing energy generating term $\delta\varepsilon_N$. The result is that the oscillations of usual stellar models of ordinary masses are strongly damped.

However if $L(r)$ becomes negligibly small compared to ε_ν as in the recent models discussed by Reeves [5], the same is true of δL and the expression of σ' reduces to

(18) $$\sigma' = -\frac{1}{2\,\sigma_a^2 J_a}\int_0^M \left(\frac{\delta T}{T}\right)_a (\delta\varepsilon_N - \delta\varepsilon_\nu)_a\, dm$$

In the relevant ranges of temperatures and densities, we may use for ε_ν formulae of the type

$$\varepsilon_\nu = (\varepsilon_\nu)_0\, \varrho^m\, T^n\,,\; \delta\varepsilon_\nu = \varepsilon_\nu \left(m \frac{\delta\varrho}{\varrho} + n \frac{\delta T}{T}\right) \tag{19}$$

with

$$\begin{array}{lll} m = 0 & , n = 8 & \text{(photoneutrino)} \\ m = -1 & , 23 > n > 11 & \text{(pair-annihilation)} \end{array}$$

depending on the dominant mechanism of neutrino production. On the other hand, as far as the relevant thermonuclear reactions are concerned, if we write

$$\varepsilon_N = (\varepsilon_N)_0\, \varrho^\mu\, T^\nu\,,\; \delta\varepsilon_N = \varepsilon_N \left(\mu \frac{\delta\varrho}{\varrho} + \nu \frac{dT}{T}\right) \tag{20}$$

we have:

Carbon fusion	$\mu = 1$	$30 > \nu > 20$
Neon photodisintegration	$\mu = 0$	$80 > \nu > 33$
Oxygen fusion	$\mu = 1$	$45 > \nu > 35$

With the help of (19) and (20) and noting that here

$$\int_0^M \varepsilon_N\, dm \simeq \int_0^M \varepsilon_\nu\, dm$$

(18) may be written

$$\sigma' = -\frac{\int_0^M \varepsilon_N\, dm}{2\,\sigma_a^2 J_a} (\Gamma_3 - 1) \left\{ \overline{\left(\frac{\delta\varrho}{\varrho}\right)_a^2} [1 + \bar\nu (\Gamma_3 - 1)] - \overline{\overline{\left(\frac{\delta\varrho}{\varrho}\right)_a^2}} [(-1\,, 0) + \bar{\bar{n}} (\Gamma_3 - 1)] \right.$$

where we have denoted by one bar or two bars the averages taken with respect to ε_N or ε_ν respectively. Considering the dependence of ε_N and ε_ν on ϱ and T, the two bar average will effectively extend over a somewhat larger central core however, both ε_N and ε_ν becoming negligible at fairly short distances from the centre $\overline{\overline{|\,\delta\varrho/\varrho\,|}}$ will only be very slightly larger than $\overline{|\,\delta\varrho/\varrho\,|}$. Since, on the contrary $\bar\nu$ is always considerably larger than $\bar{\bar{n}}$, σ' is negative and Reeves's models are vibrationally unstable.

In fact, if $\tau' = 1/\sigma'$ denotes the e-folding time of the amplitude and τ the life-time of the stage associated with one of the thermonuclear reaction considered, one has in a first approximation

$$\frac{\tau'}{\tau} = \frac{2}{9} \frac{(3\,\Gamma_1 - 4)}{(\Gamma_3 - 1)} \frac{GM^2}{R} \frac{1}{aMe_N} \frac{1}{2 + (\Gamma_3 - 1)(\bar\nu - \bar n)}$$

where e_N is the energy liberable per gram of reactant (C^{12}, O^{16}, Ne^{20}) and aM, the total potential fuel which is always a good fraction of the total mass. Using Reeves's figures, one finds that, in general, τ' is still a small fraction of τ (10^{-2} to 10^{-3}) at least for stars which are not too massive. There would thus be ample time for a considerable increase of the amplitude and probably, shedding of material.

However, Reeves's models are very rough and a detailed study on the basis of actual models would be of interest.

In the first case, when ε_ν is largely dominant, we can neglect $\delta\varepsilon_N$ and $d\delta L/dm$ in the second member of (14). But, in these circumstances, Γ_1 and Γ_3 may have to be replaced by Γ_1^* and Γ_3^* which may approach 4/3 and it may be difficult to distinguish between dominant characteristic time-scales. It might be preferable then to keep to the general equation (14) from which $\delta\varepsilon_\nu$ should be eliminated explicitely.

Adopting again a separation of the space and time variables of the type (15) but writing here $s = i\sigma$ and using (1) and (19), we get, from (5),

$$\frac{\delta T}{T} = -\frac{1}{r^2}\frac{d}{dr}(r^3\,\xi)\frac{(\Gamma_3 - 1)\,s\,C_v\,T - m\,\varepsilon_\nu}{s\,C_v\,T + n\,\varepsilon_\nu}$$

and then

$$\frac{\delta\varepsilon_\nu}{\varepsilon_\nu} = -\frac{s}{r^2}\frac{d}{dr}(r^3\,\xi)\frac{m + n(\Gamma_3 - 1)}{s\,C_v\,T + n\,\varepsilon_\nu}$$

If we introduce the latter in (14), we finally get

$$\begin{aligned} s^4\,r\,\xi + 2\,s^3\frac{n\,\varepsilon_\nu}{C_v\,T}r\,\xi + s^2\left[\left(\frac{n\,\varepsilon_\nu}{C_v\,T}\right)^2 r\,\xi - \frac{A}{\varrho r^3}\right] + \\ + s\left[\frac{1}{\varrho}\frac{d}{dr}\left(\frac{B}{C_v\,T}\right) - \frac{2n\,\varepsilon_\nu}{C_v\,T}\frac{A}{\varrho r^3}\right] - \left(\frac{n\,\varepsilon_\nu}{C_v\,T}\right)^2\frac{A}{\varrho r^3} + \\ + \frac{n\,\varepsilon_\nu}{(C_v\,T)^2}\frac{1}{\varrho}\frac{dB}{dr} - \frac{B}{(C_v\,T)^2}\frac{1}{\varrho}\frac{d(n\varepsilon_\nu)}{dr} = 0 \end{aligned} \tag{21}$$

with

$$A = \frac{d}{dr}\left(\Gamma_1\,p\,r^4\frac{d\xi}{dr}\right) + r^3\,\xi\,\frac{d}{dr}[(3\,\Gamma_1 - 4)\,p]$$

$$B = (\Gamma_3 - 1)\,\varrho\,\varepsilon_\nu\frac{1}{r^2}\frac{d}{dr}(r^3\,\xi)\,[m + n\,(\Gamma_3 - 1)]$$

Thus, in this case, the explicit elimination of $(\delta\varrho/\varrho)$ and $(\delta T/T)$ in $\delta\varepsilon_\nu$ raises the order in time of the problem by one unit, introducing the possibility of a new significant time-scale which must be of the order of some average over the star of the cooling time $t_c \simeq C_vT/\varepsilon_\nu$ by neutrino emission. However a detailed discussion of (21) might reveal some unexpected aspects of the stability problem, especially if the Γ's were approaching 4/3.

5. Finally, I would like to attract the attention on a point which has received little attention up to now in the usual treatment of the stellar stability problem. Coming back to usual stellar models with negligible neutrino production, equation (14) becomes

$$r\frac{\partial^3\eta}{\partial t^3}-\frac{1}{\varrho r^3}\left\{\frac{\partial}{\partial r}\left(\Gamma_1 p\, r^4\frac{\partial^2\eta}{\partial r\,\partial t}\right)+r^3\frac{\partial\eta}{\partial t}\frac{\partial}{\partial r}\left[(3\,\Gamma_1-4)\,p\right]\right\}= \tag{22}$$

$$=-\frac{1}{\varrho}\frac{\partial}{\partial r}\left[(\Gamma_3-1)\,\varrho\left(\delta\varepsilon_N-\frac{\partial\delta L}{\partial m}\right)\right]$$

which must be solved with the boundary conditions

$$\delta r=0 \text{ in } r=0\,,\ \delta p=0 \text{ in } r=R\,.$$

The usual procedure, already illustrated to some extent in the first part of section **4** consists essentially in dividing the discussion into three parts:

a) *Dynamical stability*: the terms on the right-hand side are treated as small and negligible and, after separating the time by means of (15), the eigenvalues σ_i^2 of the adiabatic problem are determined. If they are all positive $(\bar{\Gamma}_1 > 4/3)$, the star is said to be dynamically stable.

b) *Vibrational stability*: the evaluation of the corrections due to the small non-adiabatic terms on the right of (22) yields an expression of type (18) where $\delta\varepsilon_\nu$ is replaced by $d\,\delta L/dm$ for the coefficient of vibrational stability.

c) *Secular stability*: a very slow motion determined essentially by the right-hand member of (22) is also possible during which the third order time derivative may be neglected. Substituting s to $i\sigma$, one can also obtain an expression for s which is formally very similar to (18) but which cannot be evaluated by means of the adiabatic solution ξ_a. In fact, we know very little of the adequate solution [6], say ξ_s, except for special cases in which an homology transformation may be used. A systematic study of this aspect of the problem would no doubt be very useful especially in connection with the problem of stellar evolution.

However, in this approach, the problem is really treated as being of the third order in the time as indeed equation (22) seems to suggest at first sight. But actually $\delta\varepsilon_N$ and δL should be explicitly eliminated from (22) by means of their general expressions. In our special case, δL reduces to

$$\delta L(r)=L(r)\left[4\,\eta+(4+u)\,\frac{\delta T}{T}-s\,\frac{\delta\varrho}{\varrho}+\frac{d}{dr}\left(\frac{\delta T}{T}\right)\Big/\frac{1}{T}\frac{dT}{dr}\right] \tag{23}$$

if the opacity is given by

$$\varkappa=\varkappa_0\,\varrho^s\,T^{-u}$$

On the other hand,

$$\varepsilon_N = C\left[\sum_{(i,j)} Q_{ij}\, K_{ij}\, \varrho\, \frac{X_i}{A_i}\frac{X_j}{A_j} + \sum_k Q_k\, \lambda_k \frac{X_k}{A_k}\right]$$

where the sums are to be extended to all the capture reactions (i, j) and the radioactive disintegrations (k) occuring in the special chain of interest. Q_{ij} and Q_k are the energies released in these respective processes; λ_k is independent of the physical conditions while K_{ij} is a function of T only so that

$$\frac{\delta K_{ij}}{K_{ij}} = \nu_{ij}\frac{\delta T}{T}$$

in which ν_{ij} defines the sensitivity of that particular reaction to the temperature. We then have

$$\delta\varepsilon_N = C\Big[\sum_{(i,j)} Q_{ij}\, K_{ij}\, \varrho\, \frac{X_i}{A_i}\frac{X_j}{A_j}\left(\nu_{ij}\frac{\delta T}{T} + \frac{\delta\varrho}{\varrho} + \frac{\delta X_i}{X_i} + \frac{\delta X_j}{X_j}\right) + \sum_k Q_k\, \lambda_k \frac{X_k}{A_k}\frac{\delta X_k}{X_k}\Big] \tag{24}$$

The perturbations of the abundances satisfy general equations of the type

$$\begin{aligned}\frac{1}{A_n}\frac{\partial\,\delta X_n}{\partial t} = &\sum_{(i,j)_n} K_{ij}\frac{X_i}{A_i}\frac{X_j}{A_j}\,\varrho\left(\nu_{ij}\frac{\delta T}{T} + \frac{\delta X_i}{X_i} + \frac{\delta X_j}{X_j} + \frac{\delta\varrho}{\varrho}\right)\\ &- \sum_q K_{nq}\frac{X_n}{A_n}\frac{X_q}{A_q}\left(\nu_{nq}\frac{\delta T}{T} + \frac{\delta X_k}{X_k} + \frac{\delta X_q}{X_q} + \frac{\delta\varrho}{\varrho}\right) - \frac{\lambda_n X_n}{A_n}\frac{\delta X_n}{X_n}\\ &n = 1, 2, \ldots \mathfrak{n}\end{aligned} \tag{25}$$

if the total number of nuclei participating in the reactions is $\mathfrak{n}$.

Thus, to eliminate completely $\delta\varepsilon_N$ and δL from (22) in terms of η we must add to this equation the system formed of the equations (1), (23), (24), (25) and (5) which can be rewritten here

$$\frac{\partial}{\partial t}\left(\frac{\delta T}{T}\right) - (\Gamma_3 - 1)\frac{\partial}{\partial t}\left(\frac{\delta\varrho}{\varrho}\right) = \frac{1}{C_v T}\left(\delta\varepsilon_N - \frac{d\delta L}{dm}\right) \tag{26}$$

Or, one may prefer to go back to the system of the $(6 + \mathfrak{n})$ equations (1), (2), (23), (24), (25), (26) and (3) which altogether is of the $(4 + \mathfrak{n})$ *th* order in the time. Thus one must expect that the solution of the complete problem will introduce extra time-scales distinct from those characterizing the three classical types of stability recalled above. However without a fairly detailed discussion it is difficult to say how significant they may be and with what types of motion they are associated.

Let us note also that even if $\delta\varepsilon_N$ has the simple form (20) and the nuclear kinetic equations are neglected, the explicit elimination of $\delta\varepsilon_N$ and $d\delta L/dm$ leads already to a fourth order equation in s, very much as in the second part of Section **5** although the algebra becomes appreciably more complicated.

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