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Some Comments on Secular Stability Criteria and Applications

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Summary. The general and *formally* exact criterion for secular stability is discussed to point out the interrelation between secular and dynamical stability. Two approximate criteria relating essentially to the stability of stellar nuclear burning are recovered easily from the general criterion.
A general relativistic extension of the rigorous criterion shows that supermassive stars cannot be secularly unstable unless dynamical instability has set in previously. In that case, the secular instability is no more than the consequence of the correlation stressed above and cannot be very significant since the dynamical time-scale becomes soon much shorter than the secular time-scale.

Key words: stellar secular stability – degenerate stars – supermassive stars – general relativity

1. Introduction

Kippenhahn and coworkers (1966, 1970, 1971) and Rakavy and Shaviv (1968) have discussed the secular stability of stars in terms of a generalized specific heat, the sign of which provides, in certain circumstances, a convenient and intuitively satisfying criterion. Of course, as with all simple criteria for secular stability known up to now, this is true only in cases where the perturbation over the star or over some significant part of it has a simple behaviour which can be approximated for instance by an homology transformation

$$\frac{\delta r}{r}=\xi\,,\quad \frac{\delta \varrho}{\varrho}=-3\xi\,,\quad \frac{\delta p}{p}=-4\xi \tag{1}$$

where ξ is constant in space.
But, even in such cases, this kind of criteria might still obscure the real significance of some instabilities. Thus, it seems useful to relate them to some formally exact criterion. For a very slow radial perturbation with a time dependence in e^{st} of a spherical star in equilibrium (the only case considered here), s is given by (Ledoux, 1963, 1965, 1969)

$$s=-\frac{\int_0^M (\Gamma_3-1)\left(\frac{\delta\varrho}{\varrho}\right)^*_s\left(\delta\varepsilon-\frac{d\delta L}{dm}\right)_s dm}{\int_0^R 4\pi\Gamma_1 p r^4\left|\frac{d\xi}{dr}\right|^2_s dr-\int_0^R |\xi|^2_s\frac{d}{dr}[(3\Gamma_1-4)p]4\pi r^3 dr} \tag{2}$$

with fairly conventional notations, δ denoting a lagrangian variation and the subscript s recalling that the corresponding quantity must be evaluated for an eigensolution of the secular stability problem. If ε_N represents the net rate of thermonuclear energy generation (i.e. discounting losses due to the neutrinos produced in these reactions) and ε_ν the rate of all other neutrino losses, $\delta\varepsilon=\delta\varepsilon_N-\delta\varepsilon_\nu$. The variation of the total flux δL should cover all modes of energy transfer: radiation, convection and conduction.
General solutions are often complex (Schwarzschild, 1967; Härm and Schwarzschild, 1972; Aizenman and Perdang, 1971; Gabriel, 1972; Noels, 1972) but here we shall limit the discussion to the real case $\left(\frac{\delta\varrho}{\varrho}\right)=\left(\frac{\delta\varrho}{\varrho}\right)^*$ which, in particular, is always realized if (1) can be used as a solution.
Then we have the obvious correspondence

$$\begin{aligned} &s>0:\ \text{secular instability}\\ &s<0:\ \text{secular stability}\,. \end{aligned} \tag{3}$$

Let us note that (2) can be transformed to various other forms. In particular, let us start from the equation of conservation of energy written in terms of δT and δp

$$\begin{aligned} \delta\varepsilon-\frac{d\delta L}{dm}&=c_p T\left[\frac{d}{dt}\left(\frac{\delta T}{T}\right)-\nabla_a\frac{d}{dt}\left(\frac{\delta p}{p}\right)\right]\\ &=s c_p T\frac{\delta T}{T}\left[1-\nabla_a\left(\frac{\delta p}{p}\right)\Big/\left(\frac{\delta T}{T}\right)\right] \end{aligned} \tag{4}$$

where

$$\nabla_a=\left(\frac{d\ln T}{d\ln p}\right)_s=\frac{\Gamma_2-1}{\Gamma_2}=\frac{\Gamma_3-1}{\Gamma_1} \tag{5}$$

Multiplying Eq. (4) by $(\Gamma_3-1)\left(\frac{\delta\varrho}{\varrho}\right)$ and integrating over the whole mass of the star we get

$$s=-\frac{\int_0^M(\Gamma_3-1)(\delta\varrho/\varrho)\left(\delta\varepsilon-\frac{d\delta L}{dm}\right)dm}{\left[-\int_0^M c_p T\left(\frac{\delta\varrho}{\varrho}\right)(\Gamma_3-1)\left(\frac{\delta T}{T}\right)\cdot[1-V_a(\delta p/p)/(\delta T/T)]\,dm\right]}. \tag{6}$$

We note that the numerators of Eq. (2) and (6) are identical while the denominator of (6) can be reduced easily to that of (2) if we remember that

$$c_p T(\delta T/T-V_a\delta p/p)=\frac{1}{(\Gamma_3-1)}\frac{p}{\varrho}(\delta p/p-\Gamma_1\delta\varrho/\varrho)$$

and eliminate $\delta p/p$ and $\delta\varrho/\varrho$ in terms of ξ by means of the perturbed equations of continuity and of motion (here, of the hydrostatic equation, since we neglect the inertial terms)

$$\frac{\delta\varrho}{\varrho}=-\frac{1}{r^2}\frac{d}{dr}(r^3\xi) \tag{7}$$

$$\frac{d}{dr}(\delta p/p)=-\frac{1}{p}\frac{dp}{dr}(\delta p/p+4\xi) \tag{8}$$

which, in particular, are always satisfied by the homology transformation (1).

The expressions (6) and (2) of s are thus exactly equivalent but they have however their respective advantages. The denominator of (2) is identical to the numerator of the variational expression of the smallest eigenvalue $(\sigma_{a,0})^2$ for radial adiabatic perturbations (i.e. that corresponding to the fundamental mode $\xi_{a,0}$ (Ledoux and Pekeris, 1941))

$$(\sigma_{a,0})^2=\min_u\frac{\int_0^R 4\pi\Gamma_1 pr^4\left(\frac{du}{dr}\right)^2dr-\int_0^R u^2\frac{d}{dr}[(3\Gamma_1-4)p]4\pi r^2dr}{\int_0^R u^2r^2dm(r)}. \tag{9}$$

The minimum is reached when the trial function u which must satisfy the same regularity and boundary conditions as ξ_a becomes identical to $\xi_{a,0}$.

If the star is *dynamically stable* ($\sigma_{a,0}^2>0$), the denominator of (2) will thus be *positive* for any function u and, in particular, for the secular solution ξ_s to be used in (2). In that case, the star will be *secularly stable* or *unstable* depending on whether the numerator of (2) satisfies the first or the second of the following conditions

dyn. stable:

$$\int_0^M(\Gamma_3-1)\left(\frac{\delta\varrho}{\varrho}\right)_s\left(\delta\varepsilon-\frac{d\delta L}{dm}\right)_s dm>0 \tag{10a}$$

: secul stable

$$\int_0^M(\Gamma_3-1)\left(\frac{\delta\varrho}{\varrho}\right)_s\left(\delta\varepsilon-\frac{d\delta L}{dm}\right)_s dm<0 \tag{10b}$$

: secul unstable.

If we consider a sequence of models associated with stellar evolution and all being dynamically stable, then (10b) is a necessary and sufficient condition for secular instability with s going through zero from negative to positive values as the integral in (10) changes sign.

In the case of a dynamically unstable configuration for which the minimum of the numerator of (9) is negative, we cannot tell, in general, from the condition that the denominator of (2) *evaluated for the secular solution* must be larger than this minimum, whether it will be positive or negative. However, close to dynamical instability, the fundamental adiabatic mode $\xi_{a,0}$ tends to the homology transformation (1). If this is also acceptable as a secular mode, the denominator of (2) and the numerator of (9) will have close values and assuming (10a) to remain satisfied all the time, secular instability will appear nearly simultaneously with dynamical instability.

However, this secular instability cannot really be very significant since the corresponding time-scale $\tau_s=1/s$ will be very large compared to the adiabatic time-scale $\tau_a=1/|\sigma_{a,0}|$ associated with the dynamical instability except perhaps extremely close to $\sigma_{a,0}=0$.

But this very small interval around $\sigma_{a,0}=0$, where the usual distinction between the various types of instability breaks down, requires a special and rather delicate investigation.

2. Comparison With Other Approximate Criteria

Both Rakavy and Shaviv (1968) and Kippenhahn *et al.* (1966) were interested in situations where the rate of nuclear energy generation tends to increase strongly in some region upsetting the usual thermal balance between energy production and losses characterizing ordinary static stellar models. Typical examples are, for instance, a star obeying the perfect gas law but sustaining strong neutrino losses, a case which the first authors had especially in mind, or a star where the energy generation occurs in a degenerate region (Mestel, 1952), a case with which the other authors were more directly concerned. Since, in such cases, the problem is mainly one of "stability of nuclear burning", the above authors neglected completely or partially the energy losses and we shall limit our comparison to the same hypothesis. It is possible of course to extend the discussion to include all such losses, but then, if they are really significant, all the complexity of the general problem soon reappears.

Since thermonuclear reactions are usually very sensitive to the temperature T, the question is to know whether a surge of energy generation in some region will result in an increasing or a decreasing temperature taking into account the instantaneous hydrostatic readjustements which lead to conversion of internal energy into gravitational energy or vice-versa. One may then attempt to define a generalized specific heat, say C^*, covering all these energy exchanges. If $C^*>0$, then $\delta T>0$ and the nuclear burning and the star will be secularly unstable while $C^*<0$ implies $\delta T<0$ and the star is secularly (or thermally) stable.

With standard notations for some of Rakavy and Shaviv parameters

$$\alpha=\frac{1}{p}\left(\frac{\partial p}{\partial S}\right)_\varrho=\frac{\varrho T}{p}(\Gamma_3-1),$$

$$\mu=\frac{\varrho}{T}\left(\frac{dT}{d\varrho}\right)_s=\Gamma_3-1, \quad \gamma=\Gamma_1$$

we find that, in the present context, their *sufficient* condition for secular stability, restricted to perturbations of the form (1) is that

$$C_1^{*-1}=C_v^{-1}\left[1-\frac{(\Gamma_3-1)C_vT}{\delta q_0}\frac{\int_0^M(\Gamma_3-1)\delta q_0 dm}{\int_0^M(\Gamma_1-4/3)p/\varrho dm}\right]<0 \tag{11}$$

wherever

$\varepsilon_N>0$ with $(\partial\varepsilon_N/\partial T)>0$.

In (11), δq_0 is a positive quantity (at contraction, $\xi<0$) equal to

$$T\frac{d\delta S}{dt}=\delta\varepsilon-\frac{d\delta L}{dm}$$

if the latter is positive and equal to zero otherwise, the second term in the brackets being evaluated only for the region where $\delta q_0>0$.

In the same conditions, (2) becomes

$$s=\frac{\int_0^M(\Gamma_3-1)\,\delta q_0\,dm}{3\xi\int_0^M(\Gamma_1-4/3)\,p/\varrho\,dm}$$

so that (11) can be written

$$C_1^{*-1}=C_v^{-1}\left[1-\frac{s(\Gamma_3-1)\,C_vT}{\delta q_0}3\xi\right]<0.$$

Since $\xi/\delta q_0<0$, this condition can be satisfied only if $s<0$; in other words, it is indeed a *sufficient condition for secular stability*.

The approach of Kippenhahn *et al.* (1966, 1970) is more direct. Starting from the time-integrated form of (4) and assuming $\delta p/p$ and $\delta T/T$ constant, they define

$$C_2^*=C_p\,[1-\nabla_a(\delta p/p)/(\delta T/T)] \tag{12}$$

in terms of which the criterion of secular instability can be written

$$C_2^*=C_p\,[1-\nabla_a(\delta p/p)/(\delta T/T)]>0. \tag{13}$$

The equation of state yields a relation

$$\frac{\delta\varrho}{\varrho}=\alpha\frac{\delta p}{p}-\delta\frac{\delta T}{T} \tag{14}$$

with

$$\alpha=\left(\frac{\partial\ln\varrho}{\partial\ln p}\right)_T=\frac{C_p}{C_v}\frac{1}{\Gamma_1}$$
$$\delta=-\left(\frac{\partial\ln\varrho}{\partial\ln T}\right)_p=\frac{C_p-C_v}{(\Gamma_3-1)C_v}. \tag{15}$$

Adopting again the homologous perturbation defined by (1) and using (14), the criterion (13) can be written

$$C_2^*=C_p\left(1-\nabla_a\frac{4\delta}{4\alpha-3}\right)>0. \tag{16}$$

Criterion (13) can be generalized by introducing appropriate averages of $\delta p/p$ and $\delta T/T$ (Kippenhahn, 1970), a step which may be essential in special cases as we shall recall in Section 4.

Since they are mainly interested in the stability of nuclear burning, Kippenhahn *et al.* neglect the term $d\delta L/dm$ as well as the dependence of ε on ϱ so that

$$\frac{\delta\varepsilon}{\varepsilon}=\varepsilon_T\frac{\delta T}{T} \tag{17}$$

where

$$\varepsilon_T=\left(\frac{\partial\ln\varepsilon}{\partial\ln T}\right)_\varrho>0.$$

In the same conditions the expression (6) of s becomes

$$s=\frac{\int_0^M(\Gamma_3-1)\,\varepsilon\varepsilon_T dm}{\int_0^M(\Gamma_3-1)\,TC_2^*\,dm}$$

and C_2^* has indeed the same sign as s.

Thus, provided the various conditions and assumptions adopted above remain sufficiently close to the actual situation, there is a straightforward correspondence between the general criterion (2) and those based on the notion of a generalized specific heat. In particular, it is easy to check, either from (2) or from (16), that in the case of a star composed of a perfect gas ($\alpha=\delta=1$) or of a mixture of perfect gas and radiation

$(\alpha=1/\beta, \quad \delta=(4-3\beta)/\beta)$, secular instability can only occur, with the prevalence of energy generation assumed here, after dynamical instability has set in i.e. when some mean value of Γ_1 throughout the configuration is smaller than 4/3.

3. Effects of Non-Relativistic Degeneracy

The analysis of Kippenhahn *et al.* (1966) was mainly directed at this case and, with the approximation (17) for $\delta\varepsilon$, the numerator of (2) (whose denominator anyway is always positive: these configurations are dynamically stable) can be written

$$\int_0^M (\Gamma_3 - 1)(\delta\varrho/\varrho)\,\varepsilon\varepsilon_T(\delta T/T)\,dm$$

and will obviously become negative, implying according to (10b) secular instability, as soon as $\delta T/T$ becomes negative at a compression ($\delta\varrho/\varrho > 0$).
But for the homologous perturbation considered by the authors, we have according to (1) and (14)

$$\frac{\delta T}{T} = \frac{3-4\alpha}{\delta}\xi \tag{18}$$

which is negative at compression, if

$$\alpha < 3/4\,. \tag{19}$$

This may be considered as the real criterion for secular instability in this case, if the homology perturbation applies.
For a partially degenerate gas, the equation of state can be written parametrically (cf. e.g. Gabriel. 1967)

$$p = \frac{\mathscr{R}\varrho T}{\bar\mu}\,\frac{K + D(y)}{1+K}$$

with

$$K = \frac{n_i}{n_e} = \frac{2X + 0.5Y + \sum_{Z>2} X_Z/Z}{1+X},$$

$$\bar\mu = \frac{2}{1+3X+0.5Y}$$

where n_i and n_e represent respectively the number density of ions and of electrons and X, Y and X_Z are respectively the abundances by mass of hydrogen, helium and heavy elements of atomic number Z greater than 2.

$$D(y) = V(\lambda, 3/2)/V(\lambda, 1/2), \qquad y = V(\lambda, 1/2)$$

where

$$V(\lambda, n) = \frac{F_n(\lambda)}{\Gamma(n+1)} = \frac{1}{\Gamma(n+1)}\int_0^\infty \frac{z^n\,dz}{e^{\lambda+z}+1}\,.$$

The degree of degeneracy is fixed by the value of λ or of y which can be estimated in a first approximation from the physical conditions by

$$y = 6.185 \times 10^7 (1+X)\,\varrho\, T^{-3/2},$$

complete non-relativistic degeneracy corresponding to $y \to \infty$.
Using these notations, the definitions (15) of α and δ yield

$$\alpha = \frac{1}{1+x}, \qquad \delta = \frac{1-3/2x}{1+x} \tag{20}$$

with

$$x = \frac{D(y)}{K+D(y)}\,\frac{\partial \ln D(y)}{\partial \ln y} \tag{21}$$

which varies from $x_0 = 0$ (classical gas) to the value $x_\infty = 2/3$ (complete non-relativistic degeneracy with $\delta = 0$).
With (20), condition (19) becomes

$$1 + x > \tfrac{4}{3} \quad \text{or} \quad x > \tfrac{1}{3}\,. \tag{22}$$

The critical value $x = 1/3$ corresponds to a moderate degree of degeneracy corresponding to $\lambda = -3.2$ for which δ is still equal to 3/8 while one has often insisted on the lack of response of the density (or the pressure) to change in the temperature ($\delta = 0$) as the source of the secular instability. Strictly, this of course occurs only for complete degeneracy. But the use of the homology perturbation (1) which, in particular, is incompatible with complete non-relativistic degeneracy ($p \sim \varrho^{5/3}$) may introduce some uncertainties in the exact critical values.

4. Secular Stability of Supermassive Stars

Appenzeller and Kippenhahn (1971) have extended criterion (13) to the supermassive stars considered by Hoyle and Fowler (1963) in view of interpreting the quasars. As the authors point out, the "classical" expression (16) of C_2^* reduces immediately, in this case, to $-(3/2(\mathscr{R}/\bar\mu)$ which would imply that the star is secularly stable.
However, as it is well known (Fowler, 1964; Chandrasekhar, 1964), Γ_1 is already so close to 4/3 due to the predominant radiation pressure in these stars, that they become dynamically unstable on or before reaching the "main sequence" (balance of luminosity and energy production) due to small general relativity corrections if their masses are larger than some critical value, say M_c, of the order of $4.10^5\, M_\odot$. On this basis, the authors conclude that these small corrections to the overall structure of the star may also be important for the secular stability. In that case, it is essential to replace $\delta p/p$ and $\delta T/T$ by appropriate averages in criterion (13) which becomes

$$C_2^* = C_p\,[1 - V_a\overline{(\delta p/p)}/\overline{(\delta T/T)}] \tag{23}$$

and they proceed to compute these averages in a post-newtonian approximation.

Already at this point, if we revert to form (6) of the general criterion and the remarks made there, it is obvious that if secular instability can occur ($s>0$), it is only through the denominator becoming negative since general relativity corrections cannot affect the sign of the numerator which is still determined by the classical approximation. But this cannot occur unless the minimum value of the numerator of (9) itself has already become negative i.e. unless dynamical instability has set in previously. Furthermore, as (1) is still adopted to characterize the secular perturbation and since, in this case ($\Gamma_1 \sim 4/3$), the fundamental adiabatic eigensolution is itself close to (1), one may indeed expect that the setting in of dynamical and that of secular instability will be very close i.e. for critical values of the mass of the same order which is indeed the result of Appenzeller and Kippenhahn. But as we have pointed out above, a secular instability which occurs after dynamical instability has already set in, cannot be very significant since the dynamical time-scale is soon much shorter.

One might perhaps object to the mixed use of relativistic argument and purely classical formulae in our reasoning but these formulae may be extended to cover the general relativity effects. Starting from the general perturbed equations for the small radial motions of a relativistic spherical star (Thorne, 1967; Demaret, 1969) around an equilibrium state, the generalization of (2) (Demaret, unpublished), in the frame of a metric which, with respect to Schwarzschild's coordinates, can be written

$$ds^2 = -e^{\nu(r,t)}c^2dt^2 + e^{\lambda(r,t)}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$

becomes

$$s = -\frac{\int_0^A (\Gamma_3-1)\left(\frac{\delta n}{n}\right)_s e^{-\frac{(\nu+\lambda)}{2}}\left[\delta q_N - \frac{d\delta L}{da} - \delta L\frac{d\nu}{da} - L\frac{d\delta\nu}{da}\right]_s da}{D(\xi_s)} \tag{24}$$

where

$$\begin{aligned} D(\xi_s) = &\int_0^R 4\pi\Gamma_1 p r^4 e^{(\lambda+3\nu)/2}\left(\frac{d\xi_s}{dr}\right) dr \\ &- \int_0^R 4\pi r^3 e^{(\lambda+3\nu)/2}\,\xi_s^2\frac{d}{dr}[(3\Gamma_1-4)p]\,dr \\ &- \int_0^R 4\pi r^3\xi_s^2 e^{(\lambda+3\nu)/2}\left[-\frac{8\pi G}{c^4}e^{\lambda}p(p+\varepsilon)r\right. \\ &\left.+ r/(p+\varepsilon)\left(\frac{dp}{dr}\right)^2 + \frac{3}{2}\Gamma_1 p\left(\frac{d\lambda}{dr}+3\frac{d\nu}{dr}\right)\right] dr\,. \end{aligned} \tag{25}$$

This various symbols have the following meaning:

$a(r)$: number of baryons inside the sphere of radius r is the appropriate lagrangian variable in the relativistic treatment of the problem; A is the total number of baryons in the star.

n: is the number density of baryons, as measured in a reference frame comoving with the matter.

$\xi = \left(\frac{\delta r}{r}\right)e^{-\nu/2}$ is the natural relativistic counterpart of the classical $\xi = \left(\frac{\delta r}{r}\right)$. Note that, as all the coefficients of the variations $\delta n, \xi$, etc., ν and λ here should be considered as independent of time and equal to their value at the same point in the equilibrium state.

G is Newton's gravitational constant.

c is the velocity of light.

$\varepsilon = \varrho_0 c^2 + u$ is the total energy density comprised of the proper mass energy density $\varrho_0 c^2$ and the internal energy density u, both measured in a comoving frame.

q_N is the rate of thermonuclear energy generation per baryon.

$q_N = \varepsilon_N \mu_B$ where ε_N is the rate per unit of proper mass and μ_B is the average rest mass of a baryon.

One has

$$\frac{\delta n}{n} = \frac{\delta\varrho_0}{\varrho_0}$$

and

$$\begin{aligned}\frac{\delta q_N}{q_N} = \frac{\delta\varepsilon_N}{\varepsilon_N} &= (\varepsilon_N)_T\frac{\delta T}{T} + (\varepsilon_N)_{\varrho_0}\frac{\delta\varrho_0}{\varrho_0} \\ &= (\varepsilon_N)_T\frac{\delta T}{T} + (\varepsilon_N)_{\varrho_0}\frac{\delta n}{n}\,.\end{aligned}$$

Let us note that (24) and (25) are quite rigorous expressions except for the neglect in the equilibrium state of an extremely small term in $\partial\lambda/\partial t$ which corresponds to the mass loss associated with the luminosity of the star, a term which has always been neglected anyway.

The study of dynamical stability can also be approached in this case (Chandrasekhar, 1964) by means of a variational principle which yields for the eigenvalue of the fundamental mode

$$(\sigma_{a,0})^2 = \min_u \frac{D(u)}{\int_0^R 4\pi e^{(3\lambda+\nu)/2} r^4\left(\frac{p+\varepsilon}{c^2}\right)u^2 dr} \tag{26}$$

where $D(u)$ is identical to (25) with ξ_s replaced by u. The normalizing integral in the denominator is of course always positive. As in the case of (9) to which (26) reduces for c tending to infinity, the minimum is reached only for $u = \xi_{a,0}$. Thus as long as a supermassive star composed of perfect gas and radiation is dynamically stable, the denominator in (24) is also positive. On the other hand, it is obvious that the small general relativistic corrections in the numerator such as $-\delta L d\nu/da$ and

$- L(d\delta v/da)$ or relativistic factors such as $e^{-(v+\lambda)/2}$ can never change its sign. Thus our previous discussion could be repeated word for word on the basis of these general relativistic expressions leading exactly to the same conclusions.

One knows also (Ledoux, 1945) that a rotation has a stabilizing influence as far as the dynamical stability of an equilibrium configuration is concerned and Roxburgh (1965), Durney and Roxburgh (1967) and Fowler (1966 a, b) have shown that an appropriate uniform or differential rotation can stabilize supermassive stars up to respectively 10^7 or 10^9 $M_\odot$, allowing these heavier stars to reach a state of equilibrium in which the luminosity is balanced by the thermonuclear energy production.

Appenzeller and Kippenhahn (1971) show that rotation has a similar stabilizing influence on secular stability. But again it is natural to conjecture that this only reflects the correlation between dynamical and secular stability stressed above. If rotation restores dynamical stability to supermassive stars in a certain interval of mass, then such stars composed of perfect gas and radiation and for which the numerator in (24) is positive cannot be secularly unstable either.

An absolutely rigorous justification would of course require a generalization of (24) and (26) to the case of rotating stars to show that the denominator of (24) is still formally identical to the numerator of (26). This might not be so easy in the general case although, in principle, there does not seem to be any difficulty. On the other hand, in the classical case, for solid body rotation and as long as $(\xi_{a,0})$ and ξ_s both remain close to the homologous perturbation (1), which seems reasonable for large mass stars, it can be shown that to a high degree of approximation, the denominator of (2) and the numerator of (9) have indeed the same form and the same sign.

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