23. POSSIBLE SOURCES OF INSTABILITY IN STARS

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(1) INTRODUCTION

The most immediate purpose of the study of stellar stability is to discover the sources of incipient instability which must be responsible for the observed variability of a great number of stars.

I will only discuss here the stability of the star as a whole. Local instabilities such as buoyancy due to a super-adiabatic gradient or peculiar magnetic fields [1] will be considered (only) in so far as they might have an influence on the general stability of the star.

Compared to the classical, mainly mechanical questions of stability, a significant difference is that the thermodynamical factors are here of primary importance. Up to now, two lines of attack have been considered. One which has not received a great deal of attention [2] endeavours to formulate a principle of minimum in analogy to the principle of the minimum of potential energy for mechanical systems. In this connexion, and since all considered systems are open, one may wonder whether the principle of the minimum rate of entropy production and the considerable amount of work done in that field during the last few years [3] might not find interesting applications in some aspects of the problem.

The other method more fully worked out by Jeans, Eddington, Rosseland, Cowling and others is based on the theory of infinitesimal perturbations and has led to the recognition of three main types of instability:

(*a*) Secular stability, which has been studied mainly for special types of perturbations corresponding to homologous transformations [4] and which is of interest for very slow changes such as have been usually associated with the notion of stellar evolution.

(*b*) Dynamical stability, which is realized if the star, subjected to a small adiabatic perturbation varying with time as $e^{i\sigma t}$, can oscillate with a finite real frequency σ. Instability here is connected with the occurrence of imaginary frequencies leading to 'explosions'.

(*c*) Vibrational stability, which characterizes the variation with time of the amplitude of the oscillation when deviations from isentropic motion are taken into account.

Here, we are primarily interested in the last two types of instability. Do the observations provide any clue as to the type of instability present in different classes of non-stable stars? Table 1 contains, for some of these stars, three of the characteristics which seem most fundamental in that respect: the period or time separation $\tau_{obs.}$ in days between similar phases (when the phenomenon repeats itself), reduced by the homology factor $\sqrt{\bar{\rho}/\bar{\rho}_\odot}$ where $\bar{\rho}$ is the mean density; the ratio $\Delta L/L$, where ΔL is the maximum deviation from the average (or normal) luminosity L; and the ratio $E/(L_{min.} \times \tau_{obs.})$, where E is the total energy emitted during the outburst.

Nearly all figures given are subject to considerable uncertainties. In particular, the computation of $\bar{\rho}$ rests upon the use of the mass-luminosity relation and the relation between radius, luminosity and spectral type, both of which might show systematic effects when going from population I to population II. However, they are probably sufficient for a qualitative discussion.

Table 1. *Some Fundamental Characteristics of Non-Stable Stars* [5]

Class	$\tau_{obs.}\sqrt{\bar{\rho}/\bar{\rho}_\odot}$	$\Delta L/L$	$\frac{E}{L_{min.} \times \tau_{obs.}}$
Classical cepheids	0·04		1 to $\frac{1}{2}$
Red semi-regular variables	≃0·04		
Long period variables	≃0·07		
RR Lyrae	0·06	$\frac{1}{2}$ (Classical cepheids to Short period variables)	1 to $\frac{1}{2}$
W Virginis stars	0·07–0·15		
Short period variables (in clusters)	≃0·15		
RV Tauri stars	≃0·15–0·30	5–10	1 to $\frac{1}{2}$
SX Herculis stars	≃0·30–0·40		
β Canis Majoris stars	0·04–0·09	Small (β Canis Majoris and SX Phoenicis)	
SX Phoenicis	0·08		
Flare stars	10	5–100	Small ≃0·01
T Tauri stars	?	10–20	
RW Aurigae stars	?		
Z Camelopardalis stars	Large	30–50 (RW Aurigae to U Geminorum)	
U Geminorum stars	500		3 to 5
Recurrent novae	10^4–10^5	Very large (U Geminorum to Classical novae)	1 to $\frac{1}{2}$
Classical novae	Very large		1 to $\frac{1}{2}$?
Super-novae	∞ or extremely large	Extremely large	

The figures in the second column of Table 1 should be compared to the theoretical periods of oscillation. As very complicated modes are unlikely, we can use for this purpose the periods of the fundamental mode of radial pulsation which are given for different models in Table 2.

On this basis, one would be tempted to consider roughly two main groups. In the first one (regular variables and some semi-regular or irregular variables), theoretical and observed periods are of the same order of magnitude and $\Delta L/L$ is of the order of unity. The problem here is essentially that of explaining the appearance and the maintenance of the oscillations: that is, it is a problem of *vibrational instability*.

Table 2. *Theoretical Periods* $\tau_{th}\sqrt{\bar{\rho}/\bar{\rho}_\odot}$ *for Different Models*

	Homogeneous model	Polytrope $n=1\cdot5$ (convective model)	Standard model	Original [6] Epstein model	Modified [7] Epstein model (with large external convection zone)
$\rho_c/\bar{\rho}=$	1	6	54	2×10^6	$1\cdot2\times10^6$
$\tau_{th}\sqrt{\bar{\rho}/\bar{\rho}_\odot}=$	0·1156	0·075	0·039	0·031	0·056

In the second group, the length of the cycle increases extremely rapidly as compared to the theoretical periods which, probably, have no significance for the observed phenomena. At the same time, $\Delta L/L$ becomes very large. It would seem that in those cases, the stars are on the verge of some kind of *dynamical instability* which manifests itself from time to time. On the other hand, if we exclude the flare stars, the last column of Table 1 (which reproduces values quoted by Schatzman [8] in discussing an extension of the Kukarkin-Parenago relation) would rather suggest some kind of underlying unity between all these phenomena. A detailed investigation of this point would be interesting, but we should note that, in these considerations, the phase of minimum luminosity plays a privileged role and this is perhaps not at all justified, especially for the regular variables.

Before discussing the theory, let us recall that the discovery of an instability by the perturbation method does not necessarily mean that the whole star will cease to exist, since the increase of the amplitudes to finite values might very well remove the cause of instability or call in stabilizing factors.

(2) DYNAMICAL INSTABILITY

There is a rather significant difference between radial and non-radial perturbations and we shall discuss them separately.

(a) *Radial perturbation*

It is well known that in this case the formulation of the problem leads to a linear equation of the Sturm-Liouville type where the square of the frequency σ^2 plays the role of the parameter.

The eigenvalues σ_i^2 ($i=0, 1, 2, \ldots$) ordered by increasing values correspond to eigenfunctions ξ_i defining the relative displacements $(\delta r/r)_i$ for successive modes with 0, 1, 2, ... i nodes between the centre and the surface. Thus any instability ($\sigma^2 < 0$) will manifest itself first through the fundamental mode (σ_0, ξ_0), and it is then sufficient to discuss this case. Its frequency is given by [9]

$$\sigma_0{}^2 = -\frac{\int_0^R 4\pi\xi_0 r^3 \frac{d}{dr}[(3\Gamma_1 - 4)P]\,dr}{\int_0^R \xi_0 4\pi\rho r^4\,dr} = \frac{\int_0^R (3\Gamma_1 - 4)\,P\left(-\frac{\delta\rho}{\rho}\right)_0 4\pi r^2\,dr}{\int_0^R \xi_0 4\pi\rho r^4\,dr}, \quad (1)$$

where ξ_0 and $-(\delta\rho/\rho)_0$ are everywhere positive and Γ_1 is a generalized ratio of specific heats for a mixture of radiation and ionized gas, the nuclei of which might be taking part in thermo-nuclear reactions or might even have reached a state of nuclear equilibrium. If Γ_1 is constant, one finds immediately the well-known result that the condition for instability is

$$\Gamma_1 < \tfrac{4}{3}. \quad (2)$$

Of course, strictly speaking, Γ_1 will never be a constant, as the ionization of an electronic shell of an abundant element can lower its value appreciably and even render it smaller than $\frac{4}{3}$ in the region of the star where it takes place.

However, with the accepted predominance of H and He and for normal stars, these layers of low Γ_1 are rather narrow and only occur fairly close to the surface of the star, where the product $P(\delta\rho/\rho)_0$ is small so that they affect very little the appropriate mean $\overline{\Gamma}_1$ (cf. equation (1)), which in this case should replace Γ_1 in equation (2).

A negligible abundance of He and H would be more favourable especially for large masses, where the pressure of radiation would tend to decrease Γ_1 everywhere. However, a detailed discussion of the standard model [10] has failed to reveal any physically significant case of instability for normal dimensions.

Of course, the model chosen might influence the result somewhat since the region of greater weight for $(3\Gamma_1 - 4)$ will displace itself according to the central condensation. However, a rapid comparison between Epstein's model and the standard model did not disclose any important differences. One may then conclude that the chances of dynamical instability towards purely radial perturbations of any aggregate of ordinary matter of stellar dimensions are extremely small.

However, Biermann and Cowling have pointed out [2] that if one goes to a larger and larger radius for a given mass, a configuration devoid of H and He will finally reach a state of dynamical instability for sufficiently large dimensions. A slight admixture of H and He will increase the critical radius and it would be worth while to extend their discussion to the case of large H and He abundances, as it might have interesting cosmological applications in problems such as the formation of stars by condensation of interstellar clouds.

Up to now, we have only considered the effects of ionization on Γ_1, but, of course, the influence of nuclear reactions on these problems could also be discussed through their effects on Γ_1. In fact, thermo-nuclear reactions which lead to a rate of generation of energy ϵ, directly proportional to some powers of the temperature T and density ρ, would add an imaginary part to Γ_1 that would affect the vibrational stability of the star, so that this type of reaction is of no interest here.

On the other hand, in the case of a real nuclear equilibrium, ϵ is proportional to the time derivative of T and ρ, and nuclear reactions contribute a real term to Γ_1. If T and ρ are large enough so that an equilibrium is established between nuclei and elementary particles such that any further increase in T or ρ leads to further dissociation of complex nuclei, Γ_1 could be reduced to a value close to 1 in a large part of the star and this could lead to a violent instability.

For instance, in Hoyle's theory [11] of the formation of heavy elements in stars, the phase of collapse under gravity and the reversal of this into a phase of explosive expansion by means of the centrifugal force is a manifestation of dynamical instability due to this cause.

According to Hoyle, this could explain the origin of super-novae. This very interesting case should be studied anew carefully from the point of view of dynamical instability using quantitative arguments.

(b) Non-radial perturbation

The problem of non-radial perturbations is more complicated, but it presents an interesting possibility of interaction between what we call here dynamical stability and the stability towards convection, which is insured by Schwarzschild's criterion:

$$A=\frac{1}{\rho}\frac{d\rho}{dr}-\frac{1}{\gamma P}\frac{dP}{dr}<0. \tag{3}$$

If equation (3) is violated, it can be shown [12] that non-radial perturbations of sufficiently small wave-lengths will be unstable. On the other hand, the stability of purely radial oscillations (infinite horizontal wave-length) is not affected by the sign of A. What does happen in the interesting case of intermediate wave-lengths? The problem is difficult, but there are a few cases where it can be discussed more or less completely. Let us consider first the case of the homogeneous compressible model which, as Pekeris has shown [13], is amenable to a complete analytical treatment and which is highly unstable towards convection:

$$A = -\frac{1}{\gamma P}\frac{dP}{dr} > 0. \tag{4}$$

Considering non-radial perturbations represented by series of terms of the form

$$f(r)\, P_s^m(\cos\theta)\; e^{\pm im\phi} e^{i\sigma t}, \tag{5}$$

s and m being integers, and $-s < m < s$, Pekeris has shown that as soon as one departs from purely radial perturbations ($s=0$), unstable modes appear. Table 3 summarizes the numerical results obtained by Mme Sauvenier-Goffin [14] for the first few modes. In this Table, s refers to the degree of the spherical harmonics and k to the number of modes of $\delta\rho$ along the radius.

Table 3. *Values of* $\beta = \dfrac{3\sigma^2}{4\pi G\rho}$ *for* $\Gamma_1 = 5/3$

s		$k=0$	$k=1$	$k=2$	$k\to\infty$
0	f	1	12·666	31	$\to\infty$
2	p	8·39	26·23	51·12	$\to\infty$
	g	−0·73	−0·225	−0·117	$\to 0$
3	p	12·0	33·03	61·20	$\to\infty$
	g	−1·0	−0·363	−0·196	$\to 0$
4	p	15·61	39·84	71·28	$\to\infty$
	g	−1·281	−0·502	−0·2806	$\to 0$
0	f	1	12·666	31	$\to\infty$

One notices that for each value of s ($s\neq 0$) and k there are two modes, one with an important vertical displacement δr corresponding to the largest value of σ^2 and another for which the displacement is mainly horizontal; these were called p- and g-oscillations by Cowling. It is through the g-modes that the instability manifests itself, and it increases with s and decreases when k increases: in other words, the most unstable perturbation is that with the smallest horizontal wave-length $\left(\propto \frac{1}{s}\right)$, but the largest vertical wave-length $\left(\propto \frac{1}{k}\right)$. One may expect that this will remain true generally [12] and a recent paper by Skumanich [15] has confirmed it again in the particular case of a polytropic atmosphere. Of course, viscosity and heat conduction might decrease the instability of some of the higher modes or even make them stable [16].

The next simplest case is that of the polytropes discussed by Cowling [17] in neglecting the perturbation of the gravitational potential. Cowling's reasoning can be extended [18] easily to the general case, starting from the basic equations:

$$\frac{d\phi}{dr}+\frac{1}{\Gamma_1 P}\frac{dP}{dr}\phi=\left[\frac{s(s+1)}{\sigma^2}-\frac{r^2\rho}{\Gamma_1 P}\right]y-\frac{s(s+1)U'}{\sigma^2}, \tag{6}$$

$$\frac{dy}{dr}+yA=\frac{1}{\gamma^2}(\sigma^2+Ag)\phi+\frac{\partial U'}{\partial r}, \tag{7}$$

where $\phi=r^2\delta r$, $y=P'/\rho$, P' and U' being the eulerian perturbations of P and U, and $g=-\frac{1}{\rho}\frac{dP}{dr}$. The elimination of U' is difficult, however, and leads to a fourth order differential equation with very complicated coefficients. If, following Cowling, one neglects U', and introduces variables

$$v=r^2\delta r P^{1/\Gamma_1} \quad \text{and} \quad w=y\rho P^{-1/\Gamma_1}=P'P^{-1/\Gamma_1},$$

then equations (6) and (7) can be written:

$$\frac{dv}{dr}=\left(\frac{s(s+1)}{\sigma^2}-\frac{r^2\rho}{\Gamma_1 P}\right)\frac{w}{\rho}P^{2/\Gamma_1}, \tag{8}$$

$$\frac{dw}{dr}=\frac{1}{r^2}(\sigma^2+Ag)vP^{-2/\Gamma_1}\rho. \tag{9}$$

Eliminating w and v successively, one finds

$$\frac{d}{dr}\left[\frac{\rho P^{-2/\Gamma_1}}{\frac{s(s+1)}{\sigma^2}-\frac{r^2\rho}{\Gamma_1 P}}\frac{dv}{dr}\right]=\frac{1}{r^2}(\sigma^2+Ag)\rho P^{-2/\Gamma_1}v, \tag{10}$$

$$\frac{d}{dr}\left[\frac{r^2P^{2/\Gamma_1}}{(\sigma^2+Ag)\rho}\frac{dw}{dr}\right]=\left[\frac{s(s+1)}{\sigma^2}-\frac{r^2\rho}{\Gamma_1 P}\right]\frac{w}{\rho}P^{2/\Gamma_1}. \tag{11}$$

If σ^2 becomes large (p-modes), and one neglects $s(s+1)/\sigma^2$ in comparison to $r^2\rho/\Gamma_1 P$ and Ag in comparison to σ^2, equation (10) becomes

$$\frac{d}{dr}\left(\frac{\Gamma_1 P^{1-2/\Gamma_1}}{r^2}\frac{dv}{dr}\right)+\frac{\sigma^2}{r^2}\rho P^{-2/\Gamma_1}v=0, \tag{12}$$

which is of the Sturm-Liouville type and admits a spectrum of positive eigenvalues σ_p^2 increasing with the order of the mode considered.

In the same way, if σ^2 becomes small (g-modes), equation (11) reduces to

$$\frac{d}{dr}\left(\frac{r^2P^{2/\Gamma_1}}{Ag\rho}\frac{dw}{dr}\right)-\frac{s(s+1)}{\sigma^2}\frac{P^{2/\Gamma_1}}{\rho}w=0, \tag{13}$$

which is again of the Sturm-Liouville type. If A is everywhere negative (condition (3) for thermal stability satisfied everywhere), (13) admits a spectrum of positive eigenvalues σ_g^2, decreasing with the order of the mode considered. In that case, the g-modes as well as the p-modes are all stable.

If A is positive everywhere (condition (3) for thermal stability violated everywhere), the eigenvalues σ_g^2 of (13) will all be negative, and this time, all the g-modes will be unstable.

Multiplying (13) by w and integrating the first term by parts and noting that the integrated part is zero, one gets

$$\sigma_g^2 = -\frac{\int_0^R s(s+1)\, P^{2/\Gamma_1}\rho^{-1}\, w^2 dr}{\int_0^R \frac{r^2 P^{2/\Gamma_1}}{Ag\rho}\left(\frac{dv}{dr}\right)^2 dr}, \tag{14}$$

confirming the fact that if A is of one sign everywhere, σ^2 is of the opposite sign. If A changes sign, one sees that it is the sign of an appropriate mean value of A which determines the stability of the star.

However, the results obtained from these approximate considerations can only be considered as indications. In particular, the passage from (10) and (11) to (12) and (13) involves some delicate points since, close enough to the centre or the surface, the terms neglected become of the same order as the terms kept.

For instance, one could integrate directly (10) multiplied by v and obtain

$$\sigma^2 \int_0^R \frac{\rho v^2}{r^2 P^{2/\Gamma_1}} dr + \int_0^R \frac{gA\rho}{r^2 P^{2/\Gamma_1}} v^2 dr + \int_0^R \frac{\rho\left(\frac{dv}{dr}\right)^2 dr}{\left(\frac{s(s+1)}{\sigma^2} - \frac{\rho r^2}{\Gamma_1 P}\right) P^{2/\Gamma_1}} = 0,$$

or using (8),

$$\sigma^2 \int_0^R \frac{\rho v^2}{r^2 P^{2/\Gamma_1}} dr + \int_0^R \frac{gA\rho}{r^2 P^{2/\Gamma_1}} v^2 dr + \int_0^R \left(\frac{s(s+1)}{\sigma^2} - \frac{r^2\rho}{\Gamma_1 P}\right) \frac{w^2 P^{2/\Gamma_1}}{\rho} dr = 0.$$

If σ^2 is large, one then gets

$$\sigma_p^2 \int_0^R \frac{\rho v^2}{r^2 P^{2/\Gamma_1}} dr = \int_0^R \frac{r^2\rho}{\Gamma_1 P} \frac{w^2}{\rho} P^{2/\Gamma_1} dr - \int_0^R \frac{gA\rho}{r^2 P^{2/\Gamma_1}} v^2 dr, \tag{15}$$

which shows that even for the p-modes, the values of A play a certain role.

For the g-modes (σ^2 small), one would get in the same way,

$$\frac{1}{\sigma_g^2} \int_0^R s(s+1) \frac{w^2}{\rho} P^{2/\Gamma_1} dr = \int_0^R \frac{r^2\rho}{\Gamma_1 P} \frac{w^2}{\rho} P^{2/\Gamma_1} dr - \int_0^R \frac{gA\rho}{r^2 P^{2/\Gamma_1}} v^2 dr, \tag{16}$$

which again shows that even for the *g*-modes, A alone does not really determine the sign of σ^2.

One can easily imagine also that in cases close to instability ($\sigma_g \to \pm 0$), the neglected terms in U' could have important effects.

Any general progress in this problem would certainly be very useful, but even the detailed study of a few special cases would already be very interesting.

The interest of these possibilities of dynamical instability connected with convective instability arises from the fact that the last type of instability must be very common.

Of course, we usually admit that convective equilibrium will replace radiative equilibrium as soon as condition (3) is violated and this is in reasonable agreement with the result mentioned previously, that the most unstable perturbations are those of small horizontal extent.

However, in a region in convective equilibrium, we know that the actual gradient always remains slightly super-adiabatic so that A remains equal to a small positive quantity ϵ. If this situation prevails in the whole star, it is very likely, from our previous discussion, that there will be unstable *g*-oscillations, and it would be very interesting to compute the lowest degree s of the harmonic which can become unstable, let us say, for the first mode.

The same type of problem should also be solved for a star comprising a convection zone and a part in radiative equilibrium. It is possible that many minor and erratic changes could be traced back to this type of instability arising in an external convection zone.

Of course, the situation would be much more favourable to this type of dynamical instability, if A could become large and positive in an important external zone as was once proposed by Biermann [19]. In that case, one might expect a violent instability for fairly low harmonics which would lead to the ejection of material in the form of a few separated jets. Indeed, this seems to occur at least in some novae.

(3) VIBRATIONAL INSTABILITY

Here we want to study the influence of the deviations from isentropic motion. We shall neglect friction for the time being and admit that the oscillation takes place through a true equilibrium state in which

$$\epsilon_0 - \frac{1}{\rho_0} \operatorname{div} \vec{F}_0 = 0, \tag{17}$$

where ϵ_0 is the rate of generation of energy at equilibrium and $\vec{F}_0$, the flux.

In that case, the theory of perturbations [20] shows that the star will be vibrationally unstable towards the k-mode of oscillation if

$$\int_0^M \left(\frac{\delta T}{T}\right)_k \left(\delta\epsilon - \frac{d\delta L(r)}{dm(r)}\right) dm > 0, \tag{18}$$

where $L(r) = 4\pi r^2 F(r)$.

Since for all nuclear reactions of interest, ϵ increases with ρ and T, $\delta\epsilon$ will be of the same sign as δT and the first term of the integral in (18) will always contribute to the instability. As Eddington was the first to point out, however, phase-delays depending on the ratio of the period $(2\pi/\sigma_k)$ to the mean lives of the nuclei may occur in the different reactions considered and some care must be taken in the evaluation of $\delta\epsilon$. One can always write

$$\delta\epsilon = \epsilon_0 \left(\mu_{\text{eff.}} \frac{\delta\rho}{\rho} + \nu_{\text{eff.}} \frac{\delta T}{T}\right), \tag{19}$$

but $\mu_{\text{eff.}}$ and $\nu_{\text{eff.}}$ may be appreciably different from the exponents in $\epsilon_0 = C\rho^\mu T^\nu$ at equilibrium [21]. However, in the current models for main sequence and giant stars, $\mu_{\text{eff.}} \simeq \mu = 1$ and $\nu_{\text{eff.}} \leqslant 20$.

If the heat capacity of the external layers is small, $\delta L(r)$ remains everywhere in phase with δT and for all ordinary opacity laws, the main term in $d[\delta L(r)]/dm$ is proportional to $d(\delta T/T)/dm$ and in a first approximation

$$\frac{\delta T}{T} \frac{d[\delta L(r)]}{dm} \propto \frac{d\left(\frac{\delta T}{T}\right)^2}{dm}. \tag{20}$$

Thus the second part of the integral in (18) will always be negative and contribute to the stability if $(\delta T/T)$ increases in absolute value with $m(r)$. For the fundamental mode of radial oscillations, this is certainly the case for all these models.

Furthermore, the first part of the integral can be limited to the region where the generation of energy takes place, that is, close to the centre where the amplitudes $\delta\rho/\rho$, $\delta T/T$ are small. The second part gets its main contribution from the external layers where the amplitudes are large. As Cowling [22] was the first to show, this makes ordinary stars extremely stable.

With increasing masses, the increasing pressure of radiation lowers the value of Γ_1 and Γ_3, and this reduces the increase of $(\delta r/r)$ from the centre to the surface and vibrational instability becomes possible if $\nu_{\text{eff.}} \simeq 20$, for masses of the order of 100 $\mathbf{m}_\odot$ [23].

There has been very little discussion of the vibrational stability towards higher modes of radial oscillations but, except in very exceptional cases [23], it should even become reinforced since for a given value $(\delta r/r)_0$ at the centre, the value $(\delta r/r)_R$ at the surface increases with the order of the mode. Furthermore, the damping due to friction increases rapidly with the order of the mode.

As far as non-radial oscillations are concerned, $\delta\rho/\rho$ and $\delta T/T$ tend towards zero at the centre, and the tendency to instability due to nuclear reactions would practically disappear altogether in this case. Since the damping due to friction also increases very rapidly here with the degree of the harmonics, it is very probable that the vibrational stability would again be reinforced.

Thus, from now on, we will consider only the fundamental mode of radial pulsation. However, we have seen that for ordinary models, the prospects of vibrational instability are rather poor.

What kind of changes could bring about vibrational instability?

Let us assume first that the calorific capacity of the external layers remains small to avoid any phase difference between δT and $\delta L(r)$. If, furthermore, we keep to models with large central condensations and values of Γ_1 close to $\frac{5}{3}$, it would seem that the only possibility consists in displacing the zone of generation of energy towards the surface. But we are limited by the well-known hydrostatic difficulty of building models with large isothermal cores [23]. Furthermore, since ϵ must be sensitive to ρ and T in order to play a role in condition (18), it seems impossible at the same time to have an appreciable fraction of the luminosity L generated in the external part of gaseous stars.

Changes in the opacity law do not seem to be very promising either [23] in ordinary stars.

In cool stars, surface phenomena which have been invoked repeatedly such as the formation of 'veils' or clouds could play a role, but although they undoubtedly could be responsible at least for light variations, it is difficult to find a natural way of adjusting them to the right phases in order to give the type of instability we are looking for as well as oscillations with a period depending upon the mean density of the star.

According to Schatzman [21], at very high densities some nuclear reactions might have large values of $\mu_{\text{eff.}}$ or $\nu_{\text{eff.}}$. But to realize these conditions towards the centre of a normal star, the central condensation would have to be so large that the increase of $(\delta r/r)$ from $r=0$ to $r=R$ would be enormous, and it is not sure that even those high values of $\mu_{\text{eff.}}$ and $\nu_{\text{eff.}}$ would make such a star unstable.

Of course, the best way would be to reduce the ratio $(\delta r/r)_R/(\delta r/r)_0$. But this requires either a change in the model towards smaller central condensation or an appreciable lowering of Γ_1. Both these occur, for instance, in white dwarfs, and there vibrational instability is difficult to avoid [24] if nuclear reactions take place. But we know that other considerations also tend to exclude them [25]. But for normal gaseous stars, a decrease in Γ_1 seems unlikely and a decrease in central condensation would create new difficulties in explaining the generation of energy at equilibrium.

One might think also of introducing some kind of very strong viscosity in a large external part of the star. Of course this would imply an extra damping proportional to

$$\phi = \tfrac{4}{3}\sigma^2 \mu r^2 \left[\frac{d\left(\frac{\delta r}{r}\right)}{dr} \right]^2, \tag{21}$$

where μ is the coefficient of dynamical viscosity, but at the same time it would tend to make $(\delta r/r)$ a constant in that part. Thus, since the effect of viscosity would decrease the variation of $(\delta r/r)$, the damping due to viscosity itself would decrease as well as that due to the flux, the destabilizing effects of the generation of energy would increase, and one might possibly reach a state of balance.

Of course ordinary viscosity (molecular or radiative) is much too small. Even if one corrects Persico's [26] result by taking into account a large abundance of hydrogen which gives a value of μ that is 200 to 300 times larger, the damping time for the fundamental mode due to viscosity alone remains much too large (of the order of 10^{11} to 10^{13} years, depending on the model). But turbulent viscosity taken proportional to $\rho \bar{c} l$ might be 10^{11} to 10^{12} times larger than ordinary viscosity, and if it would act in a sufficiently large region, a reconsideration of the problem might be worth while.

Finally, one could try to introduce a phase lag between δL and δT through the effects of non-adiabatic processes in the external layers, which must now have an appreciable heat capacity. In that case, the contribution to the integral in (18) of the term

$$\left(\frac{\delta T}{T}\right) \cdot \frac{d[\delta L(r)]}{dm},$$

where $\delta T/T$ is only the adiabatic part of the temperature variation, can be made as small as one wants provided the phase lag between $(\delta T/T)_{\mathrm{ad.}}$ and δL is properly adjusted.

Eddington [27] thought that the convection zone of hydrogen, which is capable of storing energy during the contraction and of liberating it during the expansion, could provide the necessary heat capacity. This point does not seem to have been definitely settled. Recently Zhevakin [28] has considered the effects of the ionization zone of helium and he believes that this effect would raise the capacity sufficiently.

In Eddington's theory, the phase of the displacement δr is affected very little so that the wave remains mainly a standing wave. However, the case, first considered by Schwarzschild [29], where the wave takes a progressive character in the external layers is also very interesting. However, there does not seem to be any discussion of vibrational stability for such a case. For the simple adiabatic progressive wave discussed by Schwarzschild in his first papers on the subject, it is possible to show that the instability with respect to the case of a standing wave is not increased. But it would be interesting to consider more general types of progressive waves.

(4) CONCLUSIONS

Our review of the more classical aspects of the theory did not provide us with many possibilities for instability dynamical or vibrational, and the few cases left open will require probably a good deal of critical study before definite conclusions can be reached.

We have neglected some factors such as rotation or magnetic fields, for instance, which might be important at least in some cases [1, 30], but I doubt that they could contribute to a general instability for normal stars. Schatzman [21, 31] has introduced an interesting idea, according to which a nuclear explosion could be started by an oscillation which becomes vibrationally unstable. But apart from difficulties in following the effects of this explosion, the main problem is still to discover the source of the vibrational instability. It is true, however, that if this idea is used to try to explain a nova outburst, the normal state of the star must be rather peculiar, approaching the white-dwarf stage where vibrational instability might be more common.

One should note also that the perturbation method used here does not necessarily cover all possible cases of instability or periodic variations. In this last respect, non-linear oscillations have received little attention, but one must admit that up to now, the evidence for physical factors capable of maintaining such oscillations seems to be lacking [32].

On the other hand, the purely hydrostatic approach to the problem of the internal structure of stars has revealed that evolutionary sequences of models can, for instance, lead to critical situations such as a maximum convective or isothermal core or very critical conditions for the existence of radiative or convective equilibrium in a part of the star. It would be interesting in these cases to replace the hydrostatic approach by a dynamical one taking into account non-linear terms. This might reveal interesting new possibilities, such as non-static stellar models oscillating periodically around a fictitious position of equilibrium.

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[3] Cf. I. Prigogine, *Etude thermodynamique des phénomènes irréversibles*, Desoer, Liège, 1947.

[4] Cf. for a more general attack of this problem: L. H. Thomas, *M.N.* **91**, 122 (1930), and an interesting discussion in: S. Rosseland, *The Pulsation Theory of Variable Stars*, Oxford, 1949, §§5.12 to 5.14.

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