DYNAMICAL BEHAVIOR OF ALTERNATE BASE EXPANSIONS

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Abstract. We generalize the greedy and lazy β-transformations for a real base β to the setting of alternate bases $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$, which were recently introduced by the first and second authors as a particular case of Cantor bases. As in the real base case, these new transformations, denoted $T_{\boldsymbol{\beta}}$ and $L_{\boldsymbol{\beta}}$ respectively, can be iterated in order to generate the digits of the greedy and lazy $\boldsymbol{\beta}$-expansions of real numbers. The aim of this paper is to describe the measure theoretical dynamical behaviors of $T_{\boldsymbol{\beta}}$ and $L_{\boldsymbol{\beta}}$. We first prove the existence of a unique absolutely continuous (with respect to an extended Lebesgue measure, called the p-Lebesgue measure) $T_{\boldsymbol{\beta}}$-invariant measure. We then show that this unique measure is in fact equivalent to the p-Lebesgue measure and that the corresponding dynamical system is ergodic and has entropy $\frac{1}{p}\log(\beta_{p-1}\cdots\beta_0)$. We give an explicit expression of the density function of this invariant measure and compute the frequencies of letters in the greedy $\boldsymbol{\beta}$-expansions. The dynamical properties of $L_{\boldsymbol{\beta}}$ are obtained by showing that the lazy dynamical system is isomorphic to the greedy one. We also provide an isomorphism with a suitable extension of the β-shift. Finally, we show that the $\boldsymbol{\beta}$-expansions can be seen as $(\beta_{p-1}\cdots\beta_0)$-representations over general digit sets and we compare both frameworks.

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1. Introduction

A representation of a non-negative real number x in a real base $\beta > 1$ is an infinite sequence $a_0a_1a_2\cdots$ of non-negative integers such that $x = \sum_{i=0}^{\infty} \frac{a_i}{\beta^{i+1}}$. These representations were first considered by Rényi [23] and Parry [21] for points x in the unit interval with digits a_n belonging to the set $\{0, 1, \cdots, \lceil\beta\rceil - 1\}$. Typically each point in $[0, 1)$ has uncountably many representations [25]. The largest in the lexicographic order is called the greedy expansion and the smallest is called the lazy expansion. An interesting feature of these extreme cases is that they can be generated dynamically by iterating the so-called greedy β-transformation T_β and lazy β-transformation L_β respectively (see Section 2.2 for definitions). The dynamical properties of T_β and L_β are now well understood since the seminal works of Rényi and Parry; for example, see [11]. Pedicini [22] extended the definition of real base representations by considering digits a_i belonging to some fixed finite

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set of reals Δ. In the last fifteen years, generalizations of classical results such as characterizations of greedy and lazy expansions and the properties of their underlying dynamical systems have been obtained; see for example [2, 7, 16]. To distinguish the general digit set from the classical case, we refer to the resulting representations as (β, Δ)-representations.

In a recent work, the first two authors introduced the notion of expansions of real numbers in a real Cantor base [5]. One starts with an infinite sequence $\boldsymbol{\beta} = (\beta_n)_{n\geq 0}$ of real bases greater than 1 and satisfying $\prod_{n=0}^{\infty} \beta_n = \infty$, and representations of a non-negative real number x are infinite sequences $a_0a_1a_2\cdots$ of non-negative integers such that $x = \sum_{n=0}^{+\infty} \frac{a_n}{\beta_n\cdots\beta_0}$. In this initial work, generalizations of several combinatorial results of real base representations were obtained, such as Parry's criterion for greedy β-expansions [5, Theorem 26] or Bertrand-Mathis characterization of sofic β-shifts [5, Theorem 48]. The latter result was obtained for periodic Cantor bases, which are called alternate bases and are central in the present paper.

Representations involving more than one base have recently gained momentum as shown by the five simultaneous and independent works [4, 5, 18, 20, 27]. In particular, these papers all present a generalization of Parry's theorem to their respective frameworks. But so far, all the research was concentrated on the symbolic properties of these representations.

The aim of this paper is to study the measure theoretical dynamical behaviors of the greedy and lazy expansions in a periodic Cantor base $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots)$, which we refer to as an alternate base. This is done by introducing two new transformations, the alternate greedy transformation $T_{\boldsymbol{\beta}}$ and the alternate lazy transformation $L_{\boldsymbol{\beta}}$, iterations of which generate the greedy and lazy alternate base expansions respectively. We find for each transformation a natural invariant ergodic measure absolutely continuous with respect to an appropriate generalization of the Lebesgue measure and calculate its measure theoretical entropy (Theorems 4.12 and 5.3). Using tools from ergodic theory, we are able to exhibit some statistical properties of these expansions, such as the frequency of digits in the greedy expansion of a typical point (Proposition 4.18). Furthermore, we show that the dynamical system underlying the greedy expansion is measure theoretically isomorphic to the dynamical system underlying the lazy expansion (Proposition 5.1) as well as to the dynamical system underlying a natural generalization of the so-called β-shift (Proposition 6.2); as a consequence, the three transformations have the same dynamical behavior. Another interesting property of the alternate base expansions is that when every p-terms are written as one fraction, then one is able to rewrite the involved series in the form $x = \sum_{n=0}^{+\infty} \frac{d_n}{(\beta_{p-1}\cdots\beta_0)^n}$, with d_n belonging to some fixed digit set $\Delta_{\boldsymbol{\beta}}$ of real numbers, see formula (13). This algebraic operation transforms the alternate base expansion to a $(\beta_{p-1}\cdots\beta_0, \Delta_{\boldsymbol{\beta}})$-representation. We give a sufficient condition for this transformed representation to be greedy or lazy (Theorem 7.6).

The article is organized as follows. In Section 2, we provide the necessary background on measure theory and on expansions of real numbers in a real base. In Section 3, we introduce the greedy and lazy alternate base expansions and define the associated transformations $T_{\boldsymbol{\beta}}$ and $L_{\boldsymbol{\beta}}$. Section 4 is concerned with the dynamical properties of the greedy transformation. We first prove the existence of a unique absolutely continuous (with respect to a generalization of the Lebesgue measure, which is defined in (8) and called the p-Lebesgue measure) $T_{\boldsymbol{\beta}}$-invariant measure and then prove that this measure is equivalent to the p-Lebesgue measure and that the corresponding dynamical system is ergodic. We then express the density function of this measure and compute the frequencies of letters in the greedy $\boldsymbol{\beta}$-expansions. In Section 5 and 6, we prove that the greedy dynamical system is isomorphic to the lazy one, as well as to a suitable extension of the β-shift. In Section 7, we show that the $\boldsymbol{\beta}$-expansions can be seen as $(\beta_{p-1}\cdots\beta_0)$-representations over general digit sets and we compare both frameworks.

2. Preliminaries

2.1. **Measure preserving dynamical systems.** In this subsection we summerize the ergodic properties that will be used throughout this paper, for more detail we refer the reader to [3, 10, 13, 15, 26].

A *probability space* is a triplet $(X,\mathcal{F},\mu)$ where X is a set, $\mathcal{F}$ is a σ-algebra over X and μ is a measure on $\mathcal{F}$ such that $\mu(X)=1$. For a measurable transformation $T\colon X\to X$ and a measure μ on $\mathcal{F}$, the measure μ is *T-invariant*, or equivalently, the transformation $T\colon X\to X$ is *measure preserving with respect to* μ, if for all $B\in\mathcal{F}$, $\mu(T^{-1}(B))=\mu(B)$. A *(measure preserving) dynamical system* is a quadruple $(X,\mathcal{F},\mu,T)$ where $(X,\mathcal{F},\mu)$ is a probability space and $T\colon X\to X$ is a measure preserving transformation with respect to μ. A dynamical system $(X,\mathcal{F},\mu,T)$ is *ergodic* if all $B\in\mathcal{F}$ such that $T^{-1}(B)=B$ satisfy $\mu(B)\in\{0,1\}$, and it is *exact* if $\bigcap_{n=0}^{\infty}\{T^{-n}(B):B\in\mathcal{F}\}$ only contains sets of measure 0 or 1. Clearly, any exact dynamical system is ergodic. Two dynamical systems $(X,\mathcal{F}_X,\mu_X,T_X)$ and $(Y,\mathcal{F}_Y,\mu_Y,T_Y)$ are *(measure preservingly) isomorphic* if there exist $M\in\mathcal{F}_X$ and $N\in\mathcal{F}_Y$ with $\mu_X(M)=\mu_Y(N)=0$ and $T_X(X\setminus M)\subset X\setminus M$, $T_Y(Y\setminus N)\subset Y\setminus N$, and if there exists a bijective map $\psi\colon X\setminus M\to Y\setminus N$ which is bimeasurable with respect to the σ-algebras $\mathcal{F}_X\cap(X\setminus M)$ and $\mathcal{F}_Y\cap(Y\setminus N)$ and such that for all $B\in\mathcal{F}_{\mathcal{Y}}\cap(Y\setminus N)$, $\mu_Y(B)=\mu_X(\psi^{-1}(B))$, and finally, such that for all $x\in X\setminus M$, $\psi(T_X(x))=T_Y(\psi(x))$. Here and throughout the paper, for a subset A of X, the notation $\mathcal{F}\cap A$ designates the σ-algebra $\{B\cap A:B\in\mathcal{F}\}$ over A.

With any given dynamical system $(X,\mathcal{F},\mu,T)$, one associates a non-negative real number $h_\mu(T)$, called the measure theoretical entropy of T, that measures the average amount of information gained by each application of T. Moreover, the entropy is an isomorphic invariant, in the sense that isomorphic systems have the same entropy. Formally, the *measure theoretical entropy* is defined as

$$h_\mu(T)=\sup_\alpha\lim_{n\to\infty}\frac{1}{n}H_\mu\Big(\bigvee_{i=0}^{n-1}T^{-i}(\alpha)\Big),$$

where α denotes a finite (measurable) partition of X, $\bigvee_{i=0}^{n-1}T^{-i}(\alpha)$ is the refined partition consisting of all sets of the form $A_{i_0}\cap T^{-1}(A_{i_1})\cap\cdots\cap T^{-(n-1)}(A_{i_{n-1}})$ with $A_{i_j}\in\alpha$, and

$$H_\mu\Big(\bigvee_{i=0}^{n-1}T^{-i}\alpha\Big)=-\sum_{D\in\bigvee_{i=0}^{n-1}T^{-i}\alpha}\mu(D)\log\big(\mu(D)\big).$$

Given a dynamical system $(X,\mathcal{F},\mu,T)$ and $A\in\mathcal{F}$ with $\mu(A)>0$, one can restrict the dynamics to the sub-probability space $(A,\mathcal{F}\cap A,\mu_A)$ where $\mu_A(C)=\frac{\mu(C)}{\mu(A)}$ for $C\in\mathcal{F}\cap A$. This is done by defining for $x\in A$, the first return time $r(x)=\inf\{n\geq 1:T^n(x)\in A\}$. By the classical Poincaré Recurrence Theorem, $r(x)$ is finite for μ_A-almost all $x\in A$. We then define $T_A\colon A\to A$ by setting $T_A(x)=T^{r(x)}(x)$. This function is almost everywhere defined, but by throwing away a set of measure zero one can assume with no less of generality that $r(x)$ is finite on A. The *induced dynamical system* $(A,\mathcal{F}\cap A,\mu_A,T_A)$ inherits many nice properties of the original system. For example T_A is measure preserving with respect to μ_A. If the original system is ergodic, then the induced system is also ergodic. The converse holds true if $\mu(\bigcup_{n=0}^{\infty}T^{-n}(A))=1$. A famous result of Abramov [1] relates the entropy of the original system with the entropy of the induced system. To be more precise, the theorem states that if $(X,\mathcal{F},\mu,T)$ is measure preserving and ergodic, then $h_\mu(T)=\mu(A)h_{\mu_A}(T_A)$.

For two measures μ and ν on the same σ-algebra $\mathcal{F}$, we say that μ is *absolutely continuous with respect to* ν if for all $B \in \mathcal{F}$, $\nu(B) = 0$ implies $\mu(B) = 0$, and we say that μ and ν are *equivalent* if they are absolutely continuous with respect to each other. In what follows, we will be concerned by the Borel σ-algebras $\mathcal{B}(A)$, where $A \subset \mathbb{R}$. In particular, a measure on $\mathcal{B}(A)$ is *absolutely continuous* if it is absolutely continuous with respect to the Lebesgue measure λ restricted to $\mathcal{B}(A)$. The Radon-Nikodym theorem states that μ and ν are two probability measures such that μ is absolutely continuous with respect to ν, then there exists a ν-integrable map $f\colon X \mapsto [0, +\infty)$ such that for all $B \in \mathcal{F}$, $\mu(B) = \int_B f\, d\nu$. Moreover, the map f is ν-almost everywhere unique. Such a map is called the *density function of the measure* μ *with respect to the measure* ν and is usually denoted $\frac{d\mu}{d\nu}$.

2.2. **Real base expansions.** Let β be a real number greater than 1. A *β-representation* of a non-negative real number x is an infinite sequence $a_0a_1a_2\cdots$ over $\mathbb{N}$ such that $x = \sum_{i=0}^{\infty} \frac{a_i}{\beta^{i+1}}$. For $x \in [0,1)$, a particular β-representation of x, called the *greedy β-expansion of* x, is obtained by using the *greedy algorithm*. If the first N digits of the β-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest integer such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} \leq x.$$

Note that, by definition of the greedy algorithm, the β-expansion of a real number $x \in [0,1)$ is written over the restricted alphabet $[\![0, \lceil\beta\rceil - 1]\!]$. Here and throughout the text, for $i, j \in \mathbb{Z}$, the notation $[\![i, j]\!]$ designates the interval of integers $\{i, \ldots, j\}$. The greedy β-expansion can also be obtained by iterating the *greedy β-transformation*

$$T_\beta\colon [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor$$

by setting $a_n = \lfloor \beta T_\beta^n(x) \rfloor$ for all $n \in \mathbb{N}$.

Example 2.1. In this example and throughout the paper, φ designates the golden ratio, i.e., $\varphi = \frac{1+\sqrt{5}}{2}$. The transformation T_{φ^2} is depicted in Figure 1.

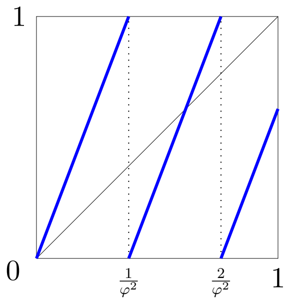

FIGURE 1. The transformation T_{φ^2}.

Real base expansions have been studied through various points of view. We refer the reader to [19, Chapter 7] for a survey on their combinatorial properties and [10] for a survey on their dynamical properties. A fundamental dynamical result is the following. This summarizes results from [21, 23, 24].

Theorem 2.2. *There exists a unique T_β-invariant absolutely continuous probability measure μ_β on $\mathcal{B}([0,1))$. Furthermore, the measure μ_β is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$ and the dynamical system $([0,1), \mathcal{B}([0,1)), \mu_\beta, T_\beta)$ is ergodic and has entropy $\log(\beta)$.*

Remark 2.3. It follows from Theorem 2.2 that T_β is *non-singular with respect to the Lebesgue measure*, i.e., for all $B \in \mathcal{B}([0,1))$, $\lambda(B) = 0$ if and only if $\lambda(T_\beta^{-1}(B)) = 0$.

In what follows, we let

$$x_\beta = \frac{\lceil \beta \rceil - 1}{\beta - 1}.$$

This value corresponds to the greatest real number that has a β-representation over the alphabet $[\![0, \lceil\beta\rceil - 1]\!]$. Clearly, we have $x_\beta \geq 1$. The *extended greedy β-transformation*, denoted T_β^{ext}, is defined in [11] as

$$T_\beta^{\mathrm{ext}} \colon [0, x_\beta) \to [0, x_\beta),\ x \mapsto \begin{cases} \beta x - \lfloor \beta x \rfloor & \text{if } x \in [0,1) \\ \beta x - (\lceil\beta\rceil - 1) & \text{if } x \in [1, x_\beta). \end{cases}$$

Note that for all $x \in \left[\frac{\lceil\beta\rceil-1}{\beta}, \frac{\lceil\beta\rceil}{\beta}\right)$, the two cases of the definition coincide since $\lfloor \beta x \rfloor = \lceil\beta\rceil - 1$. The extended β-transformation restricted to the interval $[0,1)$ gives back the classical greedy β-transformation defined above. Moreover, for all $x \in [0, x_\beta)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $(T_\beta^{\mathrm{ext}})^n(x) \in [0,1)$.

Example 2.4. We continue Example 2.1. The extended greedy transformation $T_{\varphi^2}^{\mathrm{ext}}$ is depicted in Figure 2.

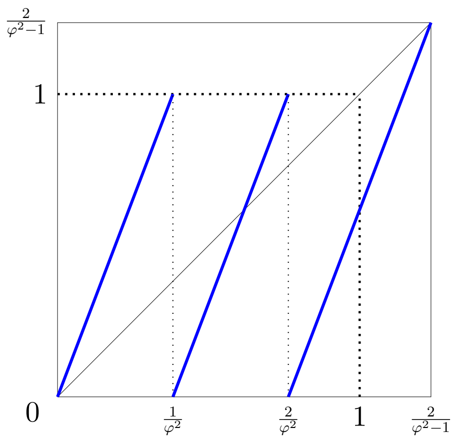

FIGURE 2. The extended transformation $T_{\varphi^2}^{\mathrm{ext}}$.

In the greedy algorithm, each digit is chosen as the largest possible among $0, 1, \ldots, \lceil\beta\rceil - 1$ at the considered position. At the other extreme, the *lazy algorithm* picks the least possible digit at each step [12]: if the first N digits of the expansion of a real number $x \in (0, x_\beta]$ are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least element in $[\![0, \lceil\beta\rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \sum_{n=N+1}^{\infty} \frac{\lceil\beta\rceil - 1}{\beta^{n+1}} \geq x,$$

or equivalently,

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \frac{x_\beta}{\beta^{N+1}} \geq x.$$

The so-obtained β-representation is called the *lazy β-expansion* of x. The *lazy β-transformation* dynamically generating the lazy β-expansion is the transformation L_β defined as follows [10]:

$$L_\beta\colon (0,x_\beta]\to(0,x_\beta],\ x\mapsto\begin{cases}\beta x & \text{if } x\in(0,x_\beta-1]\\ \beta x-\lceil\beta x-x_\beta\rceil & \text{if } x\in(x_\beta-1,x_\beta].\end{cases}$$

Observe that for all $x\in\left(\frac{x_\beta-1}{\beta},\frac{x_\beta}{\beta}\right]$, the two cases of the definition coincide since $\lceil\beta x-x_\beta\rceil=0$. Moreover, since $L_\beta\big((x_\beta-1,x_\beta]\big)=(x_\beta-1,x_\beta]$, the lazy transformation L_β can be restricted to the length-one interval $(x_\beta-1,x_\beta]$. Also note that for all $x\in(0,x_\beta]$, there exists $N\in\mathbb{N}$ such that for all $n\geq N$, $L_\beta^n(x)\in(x_\beta-1,x_\beta]$. Furthermore, for all $x\in(x_\beta-1,x_\beta]$ and $n\in\mathbb{N}$, we have $a_n=\lceil\beta L_\beta^n(x)-x_\beta\rceil$.

Example 2.5. The lazy transformation L_{φ^2} is depicted in Figure 3.

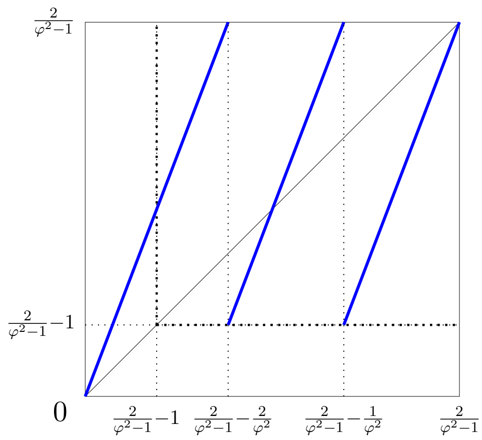

FIGURE 3. The transformation L_{φ^2}.

It is proven in [11] that there is an isomorphism between the greedy and the lazy β-transformations. As a direct consequence of this property, an analogue of Theorem 2.2 is obtained for the lazy transformation restricted to the interval $(x_\beta-1,x_\beta]$.

3. Alternate base expansions

Let p be a positive integer and $\boldsymbol{\beta}=(\beta_0,\ldots,\beta_{p-1})$ be a p-tuple of real numbers greater than 1. Such a p-tuple $\boldsymbol{\beta}$ is called an *alternate base* and p is called its *length*. A *$\boldsymbol{\beta}$-representation* of a non-negative real number x is an infinite sequence $a_0a_1a_2\cdots$ over $\mathbb{N}$ such that

$$(1)\qquad x=\frac{a_0}{\beta_0}+\frac{a_1}{\beta_1\beta_0}+\cdots+\frac{a_{p-1}}{\beta_{p-1}\cdots\beta_0}+\frac{a_p}{\beta_0(\beta_{p-1}\cdots\beta_0)}+\frac{a_{p+1}}{\beta_1\beta_0(\beta_{p-1}\cdots\beta_0)}+\cdots+\frac{a_{2p-1}}{(\beta_{p-1}\cdots\beta_0)^2}+\cdots$$

We use the convention that for all $n\in\mathbb{Z}$, $\beta_n=\beta_{n \bmod p}$ and $\boldsymbol{\beta}^{(n)}=(\beta_n,\ldots,\beta_{n+p-1})$. Therefore, the equality (1) can be rewritten as:

$$x=\sum_{n=0}^{+\infty}\frac{a_n}{\prod_{k=0}^{n}\beta_k}.$$

The alternate bases are particular cases of Cantor real bases, which were introduced and studied in [5].

In this paper, our aim is to study the dynamics behind some distinguished representation in alternate bases, namely the greedy and lazy $\boldsymbol{\beta}$-expansions. Firstly, we recall the notion of greedy $\boldsymbol{\beta}$-expansions defined in [5] and we introduce the greedy $\boldsymbol{\beta}$-transformation dynamically generating the digits of the greedy $\boldsymbol{\beta}$-expansions. Secondly, we introduce the notion of lazy $\boldsymbol{\beta}$-expansions and the corresponding lazy $\boldsymbol{\beta}$-transformation.

3.1. **The greedy $\boldsymbol{\beta}$-expansion.** For $x \in [0,1)$, a distinguished $\boldsymbol{\beta}$-representation, called the *greedy $\boldsymbol{\beta}$-expansion* of x, is obtained from the *greedy algorithm*. If the first N digits of the greedy $\boldsymbol{\beta}$-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest integer such that

$$\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} \leq x.$$

Note that, by the definition of the greedy algorithm, for every $n \in \mathbb{N}$, the n-th digit of the $\boldsymbol{\beta}$-expansion of a real number $x \in [0,1)$ belongs to the restricted alphabet $[\![0, \lceil \beta_n \rceil - 1]\!]$. The greedy $\boldsymbol{\beta}$-expansion can also be obtained by alternating the β_i-transformations: for all $x \in [0,1)$ and $n \in \mathbb{N}$, $a_n = \lfloor \beta_n \big(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)\big) \rfloor$. The greedy $\boldsymbol{\beta}$-expansion of x is denoted $d_{\boldsymbol{\beta}}(x)$. In particular, if $p = 1$ then it corresponds to the usual greedy β-expansion as defined in Section 2.2.

Example 3.1. Consider the alternate base $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$ already studied in [5]. The greedy $\boldsymbol{\beta}$-expansions are obtained by alternating the transformations $T_{\frac{1+\sqrt{13}}{2}}$ and $T_{\frac{5+\sqrt{13}}{6}}$, which are both depicted in Figure 4. Moreover, in Figure 5 we see the computation of the first five digits of the greedy $\boldsymbol{\beta}$-expansion of $\frac{1+\sqrt{5}}{5}$.

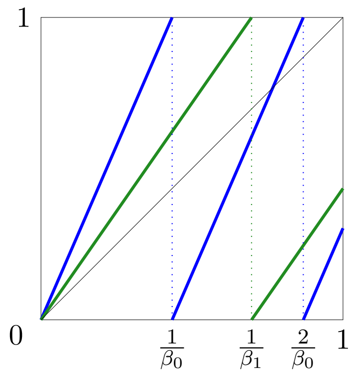

FIGURE 4. The transformations $T_{\frac{1+\sqrt{13}}{2}}$ (blue) and $T_{\frac{5+\sqrt{13}}{6}}$ (green).

We now define the *greedy $\boldsymbol{\beta}$-transformation* by

$$T_{\boldsymbol{\beta}} \colon [\![0, p-1]\!] \times [0,1) \to [\![0, p-1]\!] \times [0,1), \ (i,x) \mapsto \big((i+1) \bmod p, T_{\beta_i}(x)\big). \tag{2}$$

The greedy $\boldsymbol{\beta}$-transformation generates the digits of the greedy $\boldsymbol{\beta}$-expansions as follows. For all $x \in [0,1)$ and $n \in \mathbb{N}$, the digit a_n of $d_{\boldsymbol{\beta}}(x)$ is equal to $\lfloor \beta_n \pi_2 \big(T_{\boldsymbol{\beta}}^n(0,x)\big) \rfloor$ where

$$\pi_2 \colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \ (n,x) \mapsto x.$$

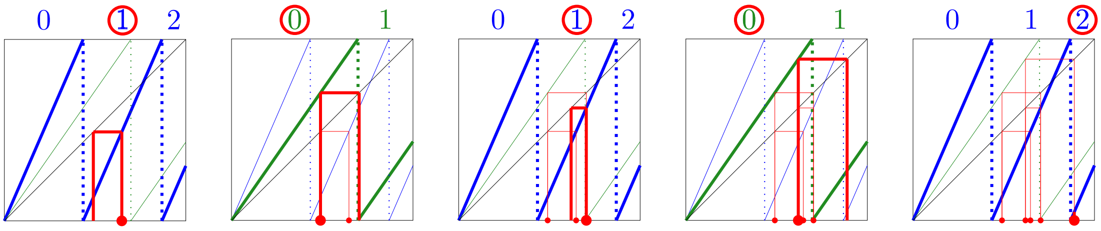

FIGURE 5. The first five digits of the greedy $\boldsymbol{\beta}$-expansion of $\frac{1+\sqrt{5}}{5}$ are 10102 for $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

As in Section 2.2, the greedy $\boldsymbol{\beta}$-transformation can be extended to an interval of real numbers bigger than $[0, 1)$. To do so, we define

$$x_{\boldsymbol{\beta}} = \sum_{n=0}^{\infty} \frac{\lceil \beta_n \rceil - 1}{\prod_{k=0}^{n} \beta_k}. \tag{3}$$

It can be easily seen that $1 \leq x_{\boldsymbol{\beta}} < \infty$. This value corresponds to the greatest real number that has a $\boldsymbol{\beta}$-representation $a_0 a_1 a_2 \cdots$ such that each digit a_n belongs to the alphabet $[\![0, \lceil \beta_n \rceil - 1]\!]$, that is, $x_{\boldsymbol{\beta}}$ is the real number having $(\lceil \beta_0 \rceil - 1)(\lceil \beta_1 \rceil - 1) \cdots$ as a $\boldsymbol{\beta}$-representation. Similarly, for all $n \in \mathbb{Z}$, the largest number that has a $\boldsymbol{\beta}^{(n)}$-representation $a_0 a_1 a_2 \cdots$ such that each digit a_m belongs to the alphabet $[\![0, \lceil \beta_{n+m} \rceil - 1]\!]$ is given by

$$x_{\boldsymbol{\beta}^{(n)}} = \sum_{m=0}^{\infty} \frac{\lceil \beta_{n+m} \rceil - 1}{\prod_{k=0}^{m} \beta_{n+k}}.$$

Hence, for all $n \in \mathbb{Z}$, we get

$$x_{\boldsymbol{\beta}^{(n)}} = \frac{x_{\boldsymbol{\beta}^{(n+1)}} + \lceil \beta_n \rceil - 1}{\beta_n}. \tag{4}$$

We define the *extended greedy $\boldsymbol{\beta}$-transformation*, denoted $T_{\boldsymbol{\beta}}^{\text{ext}}$, by

$$T_{\boldsymbol{\beta}}^{\text{ext}} \colon \bigcup_{i=0}^{p-1} \left(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\right) \to \bigcup_{i=0}^{p-1} \left(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\right), \tag{5}$$
$$(i, x) \mapsto \begin{cases} \left((i+1) \bmod p, \beta_i x - \lfloor \beta_i x \rfloor\right) & \text{if } x \in [0, 1) \\ \left((i+1) \bmod p, \beta_i x - (\lceil \beta_i \rceil - 1)\right) & \text{if } x \in [1, x_{\boldsymbol{\beta}^{(i)}}). \end{cases}$$

The greedy $\boldsymbol{\beta}$-expansion of $x \in [0, x_{\boldsymbol{\beta}})$ is obtained by alternating the p maps

$$\pi_2 \circ T_{\boldsymbol{\beta}}^{\text{ext}} \circ \delta_i\big|_{[0, x_{\boldsymbol{\beta}^{(i)}})} \colon [0, x_{\boldsymbol{\beta}^{(i)}}) \to [0, x_{\boldsymbol{\beta}^{(i+1)}})$$

for $i \in [\![0, p-1]\!]$, where

$$\delta_i \colon \mathbb{R} \to \{i\} \times \mathbb{R}, \ x \mapsto (i, x).$$

Proposition 3.2. *For all $x \in [0, x_{\boldsymbol{\beta}})$ and $n \in \mathbb{N}$, we have*

$$\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^n \circ \delta_0(x) = \beta_{n-1} \cdots \beta_0 x - \sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k$$

where $(c_0, \ldots, c_{n-1})$ is the lexicographically greatest n-tuple in $\prod_{k=0}^{n-1} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k}{\beta_{n-1} \cdots \beta_0} \leq x$.

Proof. We proceed by induction on n. The base case $n = 0$ is immediate: both members of the equality are equal to x. Now, suppose that the result is satisfied for some $n \in \mathbb{N}$. Let $x \in [0, x_{\boldsymbol{\beta}})$. Let $(c_0, \ldots, c_{n-1})$ is the lexicographically greatest n-tuple in $\prod_{k=0}^{n-1} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k}{\beta_{n-1} \cdots \beta_0} \le x$. Then it is easily seen that for all $m < n$, $(c_0, \ldots, c_m)$ is the lexicographically greatest $(m+1)$-tuple in $\prod_{k=0}^{m} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{m} \beta_m \cdots \beta_{k+1} c_k}{\beta_m \cdots \beta_0} \le x$. Now, set $y = \pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^n \circ \delta_0(x)$. Then $y \in [0, x_{\boldsymbol{\beta}^{(n)}})$ and by induction hypothesis, we obtain that $y = \beta_{n-1} \cdots \beta_0 x - \sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k$. Then, by setting

$$c_n = \begin{cases} \lfloor \beta_n y \rfloor & \text{if } y \in [0,1) \\ \lceil \beta_n \rceil - 1 & \text{if } y \in [1, x_{\boldsymbol{\beta}^{(n)}}) \end{cases}$$

we obtain that $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^{n+1} \circ \delta_0(x) = \beta_n \cdots \beta_0 x - \sum_{k=0}^{n} \beta_n \cdots \beta_{k+1} c_k$. In order to conclude, we have to show that

a) $\frac{\sum_{k=0}^{n} \beta_n \cdots \beta_{k+1} c_k}{\beta_n \cdots \beta_0} \le x$

b) $(c_0, \ldots, c_n)$ is the lexicographically greatest $(n+1)$-tuple in $\prod_{k=0}^{n} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that a) holds.

By definition of c_n, we have $c_n \le \beta_n y$. Therefore,

$$\sum_{k=0}^{n} \beta_n \cdots \beta_{k+1} c_k = \beta_n \sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k + c_n = \beta_n(\beta_{n-1} \cdots \beta_0 x - y) + c_n \le \beta_n \cdots \beta_0 x.$$

This shows that a) holds.

Let us show b) by contradiction. Suppose that there exists $(c'_0, \ldots, c'_n) \in \prod_{k=0}^{n} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $(c'_0, \ldots, c'_n) >_{\mathrm{lex}} (c_0, \ldots, c_n)$ and $\frac{\sum_{k=0}^{n} \beta_n \cdots \beta_{k+1} c'_k}{\beta_n \cdots \beta_0} \le x$. Then there exists $m \le n$ such that $c'_0 = c_0, \ldots, c'_{m-1} = c_{m-1}$ and $c'_m \ge c_m + 1$. We again consider two cases. First, suppose that $m < n$. Since $(c'_0, \ldots, c'_m) >_{\mathrm{lex}} (c_0, \ldots, c_m)$, we get $\frac{\sum_{k=0}^{m} \beta_m \cdots \beta_{k+1} c'_k}{\beta_m \cdots \beta_0} > x$. But then

$$\sum_{k=0}^{n} \beta_n \cdots \beta_{k+1} c'_k \ge \beta_n \cdots \beta_{m+1} \sum_{k=0}^{m} \beta_m \cdots \beta_{k+1} c'_k > \beta_n \cdots \beta_0 x,$$

a contradiction. Second, suppose that $m = n$. Then

$$\beta_n \cdots \beta_0 x \ge \sum_{k=0}^{n} \beta_n \cdots \beta_{k+1} c'_k \ge \sum_{k=0}^{n-1} \beta_n \cdots \beta_{k+1} c_k + c_n + 1,$$

hence $\beta_n y \ge c_n + 1$. If $y \in [0,1)$ then $c_n + 1 = \lfloor \beta_n y \rfloor + 1 > \beta_n y$, a contradiction. Otherwise, $y \in [1, x_{\boldsymbol{\beta}^{(n)}})$ and $c_n + 1 = \lceil \beta_n \rceil$. But then $c'_n \ge \lceil \beta_n \rceil$, which is impossible since $c'_n \in [\![0, \lceil \beta_n \rceil - 1]\!]$. This shows b) and ends the proof. □

The restriction of the extended greedy $\boldsymbol{\beta}$-transformation to the domain $[\![0, p-1]\!] \times [0,1)$ gives back the greedy $\boldsymbol{\beta}$-transformation initially defined in (2). Moreover, the subspace $[\![0, p-1]\!] \times [0,1)$ is an attractor of $T_{\boldsymbol{\beta}}^{\mathrm{ext}}$ in the sense given by the following proposition.

Proposition 3.3. *For each $(i, x) \in \bigcup_{i=0}^{p-1} \left(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\right)$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $(T_{\boldsymbol{\beta}}^{\mathrm{ext}})^n(i, x) \in [\![0, p-1]\!] \times [0,1)$.*

Proof. Let $(i, x) \in \bigcup_{i=0}^{p-1} \left(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\right)$. On the one hand, if $(T_{\boldsymbol{\beta}}^{\mathrm{ext}})^N(i, x) \in [\![0, p-1]\!] \times [0,1)$ for some $N \in \mathbb{N}$, then clearly $(T_{\boldsymbol{\beta}}^{\mathrm{ext}})^n(i, x) \in [\![0, p-1]\!] \times [0,1)$ for all $n \ge N$. On the other hand, if $(T_{\boldsymbol{\beta}}^{\mathrm{ext}})^n(i, x) \notin [\![0, p-1]\!] \times [0,1)$ for all $n \in \mathbb{N}$, then we would get that $x = x_{\boldsymbol{\beta}}^{(i)}$ since at each step n, the greedy algorithm would pick the maximal digit $\lceil \beta_{i+n} \rceil - 1$. □

Example 3.4. Let $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$ be the alternate base of Example 3.1. The maps $\pi_2 \circ T_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})} : [0, x_{\boldsymbol{\beta}}) \to [0, x_{\boldsymbol{\beta}^{(1)}})$ and $\pi_2 \circ T_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \delta_1\big|_{[0,x_{\boldsymbol{\beta}^{(1)}})} : [0, x_{\boldsymbol{\beta}^{(1)}}) \to [0, x_{\boldsymbol{\beta}})$ are depicted in Figure 6.

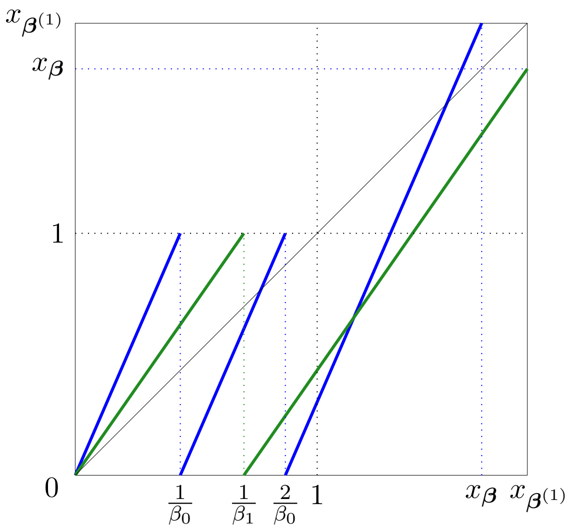

FIGURE 6. The maps $\pi_2 \circ T_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ (blue) and $\pi_2 \circ T_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \delta_1\big|_{[0,x_{\boldsymbol{\beta}^{(1)}})}$ (green) with $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

3.2. **The lazy $\boldsymbol{\beta}$-expansion.** As in the real base case, in the greedy $\boldsymbol{\beta}$-expansion, each digit is chosen as the largest possible at the considered position. Here, we define and study the other extreme $\boldsymbol{\beta}$-representation, called the *lazy $\boldsymbol{\beta}$-expansion*, taking the least possible digit at each step. For $x \in [0, x_{\boldsymbol{\beta}})$, if the first N digits of the lazy $\boldsymbol{\beta}$-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least element in $[\![0, \lceil \beta_N \rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} + \sum_{n=N+1}^{\infty} \frac{\lceil \beta_n \rceil - 1}{\prod_{k=0}^{n} \beta_k} \geq x,$$

or equivalently,

$$\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} + \frac{x_{\boldsymbol{\beta}^{(N)}}}{\prod_{k=0}^{N} \beta_k} \geq x.$$

This algorithm is called the *lazy algorithm*. For all $N \in \mathbb{N}$, we have

$$\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} \leq x,$$

which implies that the lazy algorithm converges, that is,

$$x = \sum_{n=0}^{\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k}.$$

We now define the *lazy $\boldsymbol{\beta}$-transformation* by

$$L_{\boldsymbol{\beta}} : \bigcup_{i=0}^{p-1} (\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]) \to \bigcup_{i=0}^{p-1} (\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]),$$

$$(i,x)\mapsto\begin{cases}\big((i+1)\bmod p,\beta_i x\big) & \text{if } x\in(0,x_{\boldsymbol{\beta}^{(i)}}-1]\\ \big((i+1)\bmod p,\beta_i x-\lceil\beta_i x-x_{\boldsymbol{\beta}^{(i+1)}}\rceil\big) & \text{if } x\in(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}].\end{cases}$$

The lazy $\boldsymbol{\beta}$-expansion of $x\in(0,x_{\boldsymbol{\beta}}]$ is obtained by alternating the p maps

$$\pi_2\circ L_{\boldsymbol{\beta}}\circ\delta_i\big|_{(0,x_{\boldsymbol{\beta}^{(i)}}]}\colon(0,x_{\boldsymbol{\beta}^{(i)}}]\to(0,x_{\boldsymbol{\beta}^{(i+1)}}]$$

for $i\in[\![0,p-1]\!]$. The following proposition is the analogue of Proposition 3.2 for the lazy $\boldsymbol{\beta}$-transformation, which can be proved in a similar fashion.

Proposition 3.5. *For all $x\in(0,x_{\boldsymbol{\beta}}]$ and $n\in\mathbb{N}$, we have*

$$\pi_2\circ L^n_{\boldsymbol{\beta}}\circ\delta_0(x)=\beta_{n-1}\cdots\beta_0 x-\sum_{i=0}^{n-1}\beta_{n-1}\cdots\beta_{i+1}c_i$$

where $(c_0,\dots,c_{n-1})$ is the lexicographically least n-tuple in $\prod_{k=0}^{n-1}[\![0,\lceil\beta_k\rceil-1]\!]$ such that $\frac{\sum_{i=0}^{n-1}\beta_{n-1}\cdots\beta_{i+1}c_i}{\beta_{n-1}\cdots\beta_0}+\sum_{m=n}^{\infty}\frac{\lceil\beta_m\rceil-1}{\prod_{k=0}^{m}\beta_k}\geq x$.

Note that for each $i\in[\![0,p-1]\!]$,

$$L_{\boldsymbol{\beta}}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]\big)\subset\{(i+1)\bmod p\}\times(x_{\boldsymbol{\beta}^{(i+1)}}-1,x_{\boldsymbol{\beta}^{(i+1)}}].$$

Therefore, the lazy $\boldsymbol{\beta}$-transformation can be restricted to the domain $\bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]\big)$. The (restricted) lazy $\boldsymbol{\beta}$-transformation generates the digits of the lazy $\boldsymbol{\beta}$-expansions of real numbers in the interval $(x_{\boldsymbol{\beta}}-1,x_{\boldsymbol{\beta}}]$ as follows. For all $x\in(x_{\boldsymbol{\beta}}-1,x_{\boldsymbol{\beta}}]$ and $n\in\mathbb{N}$, the digit a_n in the lazy $\boldsymbol{\beta}$-expansion of x is equal to $\lceil\beta_n\pi_2\big(L^n_{\boldsymbol{\beta}}(0,x)\big)-x_{\boldsymbol{\beta}^{(n+1)}}\rceil$.

Similarly to the greedy case, we obtain that the subspace $\bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]\big)$ is an attractor of $L_{\boldsymbol{\beta}}$.

Proposition 3.6. *For each $(i,x)\in\bigcup_{i=0}^{p-1}\big(\{i\}\times(0,x_{\boldsymbol{\beta}^{(i)}}]\big)$, there exists $N\in\mathbb{N}$ such that for all $n\geq N$, $L^n_{\boldsymbol{\beta}}(i,x)\in\bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]\big)$.*

Proof. Let $(i,x)\in\bigcup_{i=0}^{p-1}\big(\{i\}\times(0,x_{\boldsymbol{\beta}^{(i)}}]\big)$. On the one hand, if $L^N_{\boldsymbol{\beta}}(i,x)\in\bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]\big)$ for some $N\in\mathbb{N}$, then clearly $L^n_{\boldsymbol{\beta}}(i,x)\in\bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]\big)$ for all $n\geq N$. On the other hand, if $L^n_{\boldsymbol{\beta}}(i,x)\notin\bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]\big)$ for all $n\in\mathbb{N}$, then we would get that $x=0$ since at each step, the lazy algorithm would pick the minimal digit, which is always 0. □

Example 3.7. Consider again the length-2 alternate base $\boldsymbol{\beta}=(\frac{1+\sqrt{13}}{2},\frac{5+\sqrt{13}}{6})$ from Examples 3.1 and 3.4. We have $x_{\boldsymbol{\beta}}=\frac{5+7\sqrt{13}}{18}\simeq1.67$ and $x_{\boldsymbol{\beta}^{(1)}}=\frac{2+\sqrt{13}}{3}\simeq1.86$. The maps $\pi_2\circ L_{\boldsymbol{\beta}}\circ\delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}\colon(0,x_{\boldsymbol{\beta}}]\to(0,x_{\boldsymbol{\beta}^{(1)}}]$ and $\pi_2\circ L_{\boldsymbol{\beta}}\circ\delta_1\big|_{(0,x_{\boldsymbol{\beta}^{(1)}}]}\colon(0,x_{\boldsymbol{\beta}^{(1)}}]\to(0,x_{\boldsymbol{\beta}}]$ are depicted in Figure 7. In Figure 8 we see the computation of the first five digits of the lazy $\boldsymbol{\beta}$-expansion of $\frac{1+\sqrt{5}}{5}$.

3.3. **A note on Cantor bases.** The greedy algorithm described in Sections 3.1 and 3.2 is well defined in the extended context of Cantor bases, i.e., sequences of real numbers $\boldsymbol{\beta}=(\beta_n)_{n\in\mathbb{N}}$ greater than 1 such that the product $\prod_{n=0}^{\infty}\beta_n$ is infinite [5]. In this case, the greedy algorithm converge on $[0,1)$: for all $x\in[0,1)$, the computed digits a_n are such that $\sum_{n=0}^{\infty}\frac{a_n}{\prod_{k=0}^{n}\beta_k}=x$. Therefore, the value $x_{\boldsymbol{\beta}}$ defined as in (3) is greater than or equal to 1. However, it might be that $x_{\boldsymbol{\beta}}=\infty$. For example, it is the case for the Cantor base given by $\beta_n=1+\frac{1}{n+1}$ for all $n\in\mathbb{N}$.

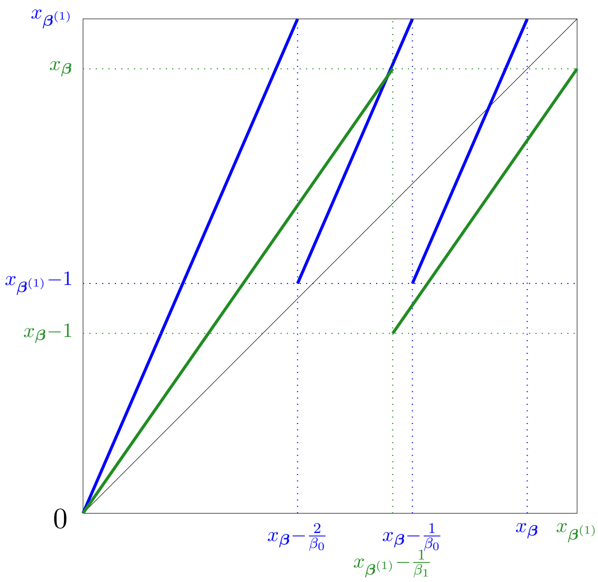

FIGURE 7. The maps $\pi_2 \circ L_{\boldsymbol{\beta}} \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$ (blue) and $\pi_2 \circ L_{\boldsymbol{\beta}} \circ \delta_1\big|_{(0,x_{\boldsymbol{\beta}^{(1)}}]}$ (green) with $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

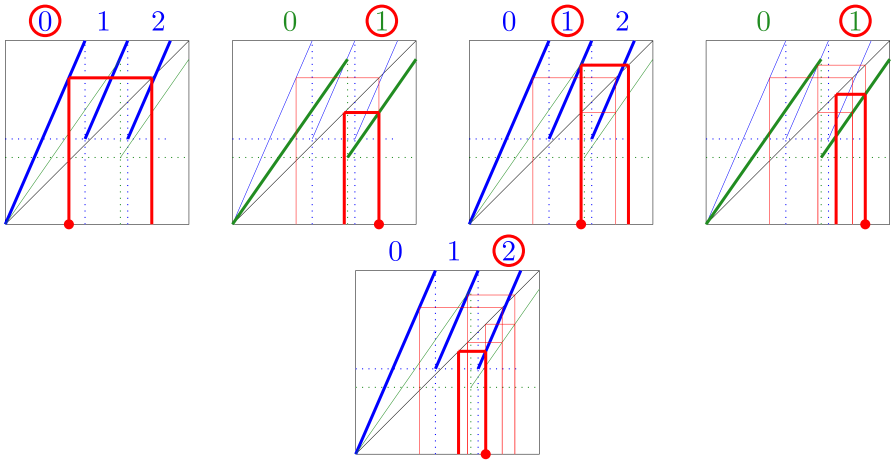

FIGURE 8. The first five digits of the lazy $\boldsymbol{\beta}$-expansion of $\frac{1+\sqrt{5}}{5}$ are 01112 for $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

Note that the restriction of the transformation $\pi_2 \circ (T^{\text{ext}}_{\boldsymbol{\beta}})^n \circ \delta_0$ to the unit interval $[0,1)$ coincide with the composition $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}$. Thus, when restricted to $[0,1)$, Proposition 3.2 can be reformulated as follows.

Proposition 3.8. *For all $x \in [0,1)$ and $n \in \mathbb{N}$, we have*

$$T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x) = \beta_{n-1} \cdots \beta_0 x - \sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k$$

where $(c_0, \ldots, c_{n-1})$ is the lexicographically greatest n-tuple in $\prod_{k=0}^{n-1} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k}{\beta_{n-1} \cdots \beta_0} \leq x$.

For all $k \in [\![0, n-1]\!]$, the transformation L_{β_k} is defined on $(0, x_{\beta_k}]$ and can be restricted to $(x_{\beta_k} - 1, x_{\beta_k}]$. So, the restricted transformations $L^{\text{restr}}_{\beta_0}, \ldots, L^{\text{restr}}_{\beta_{n-1}}$ cannot be composed to one another in general. Therefore, even if the lazy algorithm can be defined for Cantor bases, provided that $x_{\boldsymbol{\beta}} < \infty$, we cannot state an analogue of Proposition 3.8 in terms of the lazy transformations for Cantor bases.

Even though this paper is mostly concerned with alternate bases, let us emphasize that some results are indeed valid for any sequence $(\beta_n)_{n\in\mathbb{N}} \in (\mathbb{R}_{>1})^{\mathbb{N}}$, and hence for any Cantor base. This is the case of Proposition 3.8, Proposition 4.3, Corollary 4.4 and Proposition 4.14.

4. Dynamical properties of $T_{\boldsymbol{\beta}}$

In this section, we study the dynamics of the greedy $\boldsymbol{\beta}$-transformation. First, we generalize Theorem 2.2 to the transformation $T_{\boldsymbol{\beta}}$ on $[\![0, p-1]\!] \times [0, 1)$. Second, we extend the obtained result to the extended transformation $T_{\boldsymbol{\beta}}$. Third, we provide a formula for the density functions of the measures found in the first two parts. Finally, we compute the frequencies of the digits in the greedy $\boldsymbol{\beta}$-expansions.

4.1. **Unique absolutely continuous $T_{\boldsymbol{\beta}}$-invariant measure.** In order to generalize Theorem 2.2 to alternate bases, we start by recalling a result of Lasota and Yorke.

Theorem 4.1. [17, Theorem 4] *Let $T\colon [0, 1) \to [0, 1)$ be a transformation for which there exists a partition $[a_0, a_1), \ldots, [a_{K-1}, a_K)$ of the interval $[0, 1)$ with $a_0 < \cdots < a_K$ such that for each $k \in [\![0, K-1]\!]$, $T_{|[a_k,a_{k+1})}$ is convex, $T(a_k) = 0$, $T'(a_k) > 0$ and $T'(0) > 1$. Then there exists a unique T-invariant absolutely continuous probability measure. Furthermore, its density function is bounded and decreasing, and the corresponding dynamical system is exact.*

We then prove a stability lemma.

Lemma 4.2. *Let $\mathcal{I}$ be the family of transformations $T\colon [0, 1) \to [0, 1)$ for which there exist a partition $[a_0, a_1), \ldots, [a_{K-1}, a_K)$ of the interval $[0, 1)$ with $a_0 < \cdots < a_K$ and a slope $s > 1$ such that for all $k \in [\![0, K-1]\!]$, $a_{k+1} - a_k \le \frac{1}{s}$ and for all $x \in [a_k, a_{k+1})$, $T(x) = s(x - a_k)$. Then $\mathcal{I}$ is closed under composition.*

Proof. Let $S, T \in \mathcal{I}$. Let $[a_0, a_1), \ldots, [a_{K-1}, a_K)$ and $[b_0, b_1), \ldots, [b_{L-1}, b_L)$ be partitions of the interval $[0, 1)$ with $a_0 < \cdots < a_K$, $b_0 < \cdots < b_L$, and let $s, t > 1$ such that for all $k \in [\![0, K-1]\!]$, $a_{k+1} - a_k \le \frac{1}{s}$, for all $\ell \in [\![0, L-1]\!]$, $b_{\ell+1} - b_\ell \le \frac{1}{t}$ and for all $x \in [0, 1)$, $S(x) = s(x - a_k)$ if $x \in [a_k, a_{k+1})$ and $T(x) = t(x - b_\ell)$ if $x \in [b_\ell, b_{\ell+1})$. For each $k \in [\![0, K-1]\!]$, define L_k to be the greatest $\ell \in [\![0, L-1]\!]$ such that $a_k + \frac{b_\ell}{s} < a_{k+1}$. Consider the partition

$$\Big[a_0 + \frac{b_0}{s}, a_0 + \frac{b_1}{s}\Big), \ldots, \Big[a_0 + \frac{b_{L_0-1}}{s}, a_0 + \frac{b_{L_0}}{s}\Big), \Big[a_0 + \frac{b_{L_0}}{s}, a_1\Big)$$
$$\vdots$$
$$\Big[a_{K-1} + \frac{b_0}{s}, a_{K-1} + \frac{b_1}{s}\Big), \ldots, \Big[a_{K-1} + \frac{b_{L_{K-1}-1}}{s}, a_{K-1} + \frac{b_{L_{K-1}}}{s}\Big), \Big[a_{K-1} + \frac{b_{L_{K-1}}}{s}, a_K\Big)$$

of the interval $[0, 1)$. For each $k \in [\![0, K-1]\!]$ and $\ell \in [\![0, L_k-1]\!]$, $a_k + \frac{b_{\ell+1}}{s} - a_k - \frac{b_\ell}{s} \le \frac{1}{ts}$ and $a_{k+1} - a_k - \frac{b_{L_k}}{s} = (a_{k+1} - a_k - \frac{b_{L_k+1}}{s}) + \frac{b_{L_k+1} - b_{L_k}}{s} \le \frac{1}{ts}$. Now, let $x \in [0, 1)$ and $k \in [\![0, K-1]\!]$ be such that $x \in [a_k, a_{k+1})$. Then $S(x) = s(x - a_k) \in [0, 1)$. We distinguish two cases: either there exists $\ell \in [\![0, L_k - 1]\!]$ such that $x \in [a_k + \frac{b_\ell}{s}, a_k + \frac{b_{\ell+1}}{s})$, or $x \in [a_k + \frac{b_{L_k}}{s}, a_{k+1})$.

In the former case, $S(x) \in [b_\ell, b_{\ell+1})$ and $T \circ S(x) = t(S(x) - b_\ell) = ts(x - (a_k + \frac{b_\ell}{s}))$. In the latter case, since $a_{k+1} - a_k \le \frac{b_{L_k+1}}{s}$, we get that $S(x) \in [b_{L_k}, b_{L_k+1})$ and hence that $T \circ S(x) = t(S(x) - b_{L_k}) = ts(x - (a_k + \frac{b_{L_k}}{s}))$. This shows that the composition $T \circ S$ belongs to $\mathcal{I}$. □

The following proposition provides us with the main tool for the construction of a $T_{\boldsymbol{\beta}}$-invariant measure.

Proposition 4.3. *For all $n \in \mathbb{N}_{\ge 1}$ and all $\beta_0, \ldots, \beta_{n-1} > 1$, there exists a unique $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$-invariant absolutely continuous probability measure μ on $\mathcal{B}([0,1))$. Furthermore, the measure μ is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$, its density function is bounded and decreasing, and the dynamical system $([0,1), \mathcal{B}([0,1)), \mu, T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$ is exact and has entropy $\log(\beta_{n-1} \cdots \beta_0)$.*

Proof. The existence of a unique $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$-invariant absolutely continuous probability measure μ on $\mathcal{B}([0,1))$, the fact that its density function is bounded and decreasing, and the exactness of the corresponding dynamical system follow from Theorem 4.1 and Lemma 4.2. With a similar argument as in [8, Lemma 2.6], we can conclude that $\frac{d\mu}{d\lambda} > 0$ λ-almost everywhere on $[0,1)$. It follows that μ is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$. Moreover, the entropy equals $\log(\beta_{n-1} \cdots \beta_0)$ since $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}$ is a piecewise linear transformation of constant slope $\beta_{n-1} \cdots \beta_0$ [9, 24]. □

The following consequence of Proposition 4.3 will be useful for proving our generalization of Theorem 2.2.

Corollary 4.4. *Let $n \in \mathbb{N}_{\ge 1}$ and $\beta_0, \ldots, \beta_{n-1} > 1$. Then for all $B \in \mathcal{B}([0,1))$ such that $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})^{-1}(B) = B$, we have $\lambda(B) \in \{0,1\}$.*

For each $i \in [\![0, p-1]\!]$, we let $\mu_{\boldsymbol{\beta},i}$ denote the unique $(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}})$-invariant absolutely continuous probability measure given by Proposition 4.3. We use the convention that for all $n \in \mathbb{Z}$, $\mu_{\boldsymbol{\beta},n} = \mu_{\boldsymbol{\beta},n \bmod p}$. Let us define a probability measure $\mu_{\boldsymbol{\beta}}$ on the σ-algebra

$$\mathcal{T}_p = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) : \forall i \in [\![0, p-1]\!],\ B_i \in \mathcal{B}([0,1)) \right\} \tag{6}$$

over $[\![0, p-1]\!] \times [0,1)$ as follows. For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$, we set

$$\mu_{\boldsymbol{\beta}} \left(\bigcup_{i=0}^{p-1} (\{i\} \times B_i) \right) = \frac{1}{p} \sum_{i=0}^{p-1} \mu_{\boldsymbol{\beta},i}(B_i). \tag{7}$$

We now study the properties of the probability measure $\mu_{\boldsymbol{\beta}}$.

Lemma 4.5. *For $i \in [\![0, p-1]\!]$, we have $\mu_{\boldsymbol{\beta},i} = \mu_{\boldsymbol{\beta},i-1} \circ T_{\beta_{i-1}}^{-1}$.*

Proof. Let $i \in [\![0, p-1]\!]$. By the definition of $\mu_{\boldsymbol{\beta},i}$ and by Proposition 4.3, it suffices to show that $\mu_{\boldsymbol{\beta},i-1} \circ T_{\beta_{i-1}}^{-1}$ is a $(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}})$-invariant absolutely continuous probability measure on $\mathcal{B}([0,1))$. First, we have $\mu_{\boldsymbol{\beta},i-1}\big(T_{\beta_{i-1}}^{-1}([0,1))\big) = \mu_{\boldsymbol{\beta},i-1}([0,1)) = 1$. Second, for all $B \in \mathcal{B}([0,1))$, we have

$$\begin{aligned} \mu_{\boldsymbol{\beta},i-1} \circ T_{\beta_{i-1}}^{-1}\big((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}})^{-1}(B)\big) &= \mu_{\boldsymbol{\beta},i-1}\big((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}} \circ T_{\beta_{i-p-1}})^{-1}(B)\big) \\ &= \mu_{\boldsymbol{\beta},i-1}\big((T_{\beta_{i-2}} \circ \cdots \circ T_{\beta_{i-p-1}})^{-1}(T_{\beta_{i-1}}^{-1}(B))\big) \\ &= \mu_{\boldsymbol{\beta},i-1}\big(T_{\beta_{i-1}}^{-1}(B)\big). \end{aligned}$$

Third, for all $B \in \mathcal{B}([0,1))$ such that $\lambda(B)=0$, we get that $\lambda(T_{\beta_{i-1}}^{-1}(B))=0$ by Remark 2.3, and hence that $\mu_{\boldsymbol{\beta},i-1}(T_{\beta_{i-1}}^{-1}(B))=0$ since $\mu_{\boldsymbol{\beta},i-1}$ is absolutely continuous. □

Proposition 4.6. *The measure $\mu_{\boldsymbol{\beta}}$ is $T_{\boldsymbol{\beta}}$-invariant.*

Proof. For all $B_0,\dots,B_{p-1} \in \mathcal{B}([0,1))$,

$$\begin{aligned}
\mu_{\boldsymbol{\beta}}\left(T_{\boldsymbol{\beta}}^{-1}\left(\bigcup_{i=0}^{p-1}(\{i\}\times B_i)\right)\right) &= \mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1}T_{\boldsymbol{\beta}}^{-1}(\{i\}\times B_i)\right)\\
&= \mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1}\left(\{(i-1) \bmod p\}\times T_{\beta_{i-1}}^{-1}(B_i)\right)\right)\\
&= \frac{1}{p}\sum_{i=0}^{p-1}\mu_{\boldsymbol{\beta},i-1}(T_{\beta_{i-1}}^{-1}(B_i))\\
&= \frac{1}{p}\sum_{i=0}^{p-1}\mu_{\boldsymbol{\beta},i}(B_i)\\
&= \mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1}(\{i\}\times B_i)\right)
\end{aligned}$$

where we applied Lemma 4.5 for the fourth equality. □

Corollary 4.7. *The quadruple $\big([\![0,p-1]\!]\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol{\beta}},T_{\boldsymbol{\beta}}\big)$ is a dynamical system.*

Let us define a new measure λ_p over the σ-algebra $\mathcal{T}_p$. For all $B_0,\dots,B_{p-1}\in\mathcal{B}([0,1))$, we set

$$\lambda_p\left(\bigcup_{i=0}^{p-1}(\{i\}\times B_i)\right)=\frac{1}{p}\sum_{i=0}^{p-1}\lambda(B_i). \tag{8}$$

We call this measure the *p-Lebesgue measure* on $\mathcal{T}_p$.

Proposition 4.8. *The measure $\mu_{\boldsymbol{\beta}}$ is equivalent to the p-Lebesgue measure on $\mathcal{T}_p$.*

Proof. This follows from the fact that the p measures $\mu_{\boldsymbol{\beta},0},\dots,\mu_{\boldsymbol{\beta},p-1}$ are equivalent to the Lebesgue measure λ on $\mathcal{B}([0,1))$. □

Next, we compute the entropy of the dynamical system $\big([\![0,p-1]\!]\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol{\beta}},T_{\boldsymbol{\beta}}\big)$. To do so, we consider the p induced transformations

$$T_{\boldsymbol{\beta},i}\colon \{i\}\times[0,1)\to\{i\}\times[0,1),\ (i,x)\mapsto T_{\boldsymbol{\beta}}^p(i,x)$$

for $i\in[\![0,p-1]\!]$. Note that indeed, for all $(i,x)\in[\![0,p-1]\!]\times[0,1)$, the first return of (i,x) to $\{i\}\times[0,1)$ is equal to p. Thus $T_{\boldsymbol{\beta},i}=T_{\boldsymbol{\beta}}^p\big|_{\{i\}\times[0,1)}$. For each $i\in[\![0,p-1]\!]$, the induced transformation $T_{\boldsymbol{\beta},i}$ is measure preserving with respect to the measure $\nu_{\boldsymbol{\beta},i}$ on the σ-algebra $\{\{i\}\times B : B\in\mathcal{B}([0,1))\}$ defined as follows: for all $B\in\mathcal{B}([0,1))$,

$$\nu_{\boldsymbol{\beta},i}(\{i\}\times B)=p\mu_{\boldsymbol{\beta}}(\{i\}\times B).$$

Lemma 4.9. *For every $i\in[\![0,p-1]\!]$, the map $\delta_i\big|_{[0,1)}\colon[0,1)\to\{i\}\times[0,1),\ x\mapsto(i,x)$ defines an isomorphism between the dynamical systems*

$$\big([0,1),\mathcal{B}([0,1)),\mu_{\boldsymbol{\beta},i},T_{\beta_{i-1}}\circ\cdots\circ T_{\beta_{i-p}}\big)$$

and

$$\big(\{i\}\times[0,1),\{\{i\}\times B : B\in\mathcal{B}([0,1))\},\nu_{\boldsymbol{\beta},i},T_{\boldsymbol{\beta},i}\big)$$

Proof. This is a straightforward verification. □

Proposition 4.10. *The entropy of the dynamical system* $\big([\![0,p-1]\!]\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol\beta},T_{\boldsymbol\beta}\big)$ *is* $\frac{1}{p}\log(\beta_{p-1}\cdots\beta_0)$.

Proof. Let $i\in[\![0,p-1]\!]$. By Abramov's formula (see Section 2.1), we have

$$h_{\mu_{\boldsymbol\beta}}(T_{\boldsymbol\beta})=\mu_{\boldsymbol\beta}(\{i\}\times[0,1))\,h_{\nu_{\boldsymbol\beta,i}}(T_{\boldsymbol\beta,i})=\frac{1}{p}\,h_{\nu_{\boldsymbol\beta,i}}(T_{\boldsymbol\beta,i}).$$

Since the entropy is invariant under isomorphism, it follows from Proposition 4.3 and Lemma 4.9 that $h_{\nu_{\boldsymbol\beta,i}}(T_{\boldsymbol\beta,i})=\log(\beta_{p-1}\cdots\beta_0)$. Hence the conclusion. □

Finally, we prove that any $T_{\boldsymbol\beta}$-invariant set has p-Lebesgue measure 0 or 1.

Proposition 4.11. *For all* $A\in\mathcal{T}_p$ *such that* $T_{\boldsymbol\beta}^{-1}(A)=A$*, we have* $\lambda_p(A)\in\{0,1\}$.

Proof. Let $B_0,\dots,B_{p-1}$ be sets in $\mathcal{B}([0,1))$ such that

$$T_{\boldsymbol\beta}^{-1}\left(\bigcup_{i=0}^{p-1}(\{i\}\times B_i)\right)=\bigcup_{i=0}^{p-1}(\{i\}\times B_i).$$

This implies that

$$T_{\beta_{i-1}}^{-1}(B_i)=B_{(i-1)\bmod p}\quad\text{for all } i\in[\![0,p-1]\!]. \tag{9}$$

We use the convention that $B_n=B_{n\bmod p}$ for all $n\in\mathbb{Z}$. An easy induction yields that for all $i\in[\![0,p-1]\!]$ and $n\in\mathbb{N}$, $(T_{\beta_{i-1}}\circ\cdots\circ T_{\beta_{i-n}})^{-1}(B_i)=B_{i-n}$. In particular, for $n=p$, we get that for each $i\in[\![0,p-1]\!]$, $(T_{\beta_{i-1}}\circ\cdots\circ T_{\beta_{i-p}})^{-1}(B_i)=B_i$. By Corollary 4.4, for each $i\in[\![0,p-1]\!]$, $\lambda(B_i)\in\{0,1\}$. By definition of λ_p, in order to conclude, it suffices to show that either $\lambda(B_i)=0$ for all $i\in[\![0,p-1]\!]$, or $\lambda(B_i)=1$ for all $i\in[\![0,p-1]\!]$. From (9) and Remark 2.3, we get that for each $i\in[\![0,p-1]\!]$, $\lambda(B_i)=0$ if and only if $\lambda(B_{i+1})=0$. The conclusion follows. □

We are now able to state the announced generalization of Theorem 2.2 to alternate bases.

Theorem 4.12. *The measure* $\mu_{\boldsymbol\beta}$ *is the unique* $T_{\boldsymbol\beta}$*-invariant probability measure on* $\mathcal{T}_p$ *that is absolutely continuous with respect to* λ_p*. Furthermore,* $\mu_{\boldsymbol\beta}$ *is equivalent to* λ_p *on* $\mathcal{T}_p$ *and the dynamical system* $([\![0,p-1]\!]\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol\beta},T_{\boldsymbol\beta})$ *is ergodic and has entropy* $\frac{1}{p}\log(\beta_{p-1}\cdots\beta_0)$.

Proof. By Propositions 4.6 and 4.8, $\mu_{\boldsymbol\beta}$ is a $T_{\boldsymbol\beta}$-invariant probability measure that is absolutely continuous with respect to λ_p on $\mathcal{B}([0,1))$. Then we get from Proposition 4.11 that for all $A\in\mathcal{T}_p$ such that $T_{\boldsymbol\beta}^{-1}(A)=A$, we have $\mu_{\boldsymbol\beta}(A)\in\{0,1\}$. Therefore, the dynamical system $([\![0,p-1]\!]\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol\beta},T_{\boldsymbol\beta})$ is ergodic. Now, we obtain that the measure $\mu_{\boldsymbol\beta}$ is unique as a well-known consequence of the Ergodic Theorem, see [9, Theorem 3.1.2]. The equivalence between $\mu_{\boldsymbol\beta}$ and λ_p and the entropy of the system were already obtained in Propositions 4.8 and 4.10. □

For p greater than 1, the dynamical system $([\![0,p-1]\!]\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol\beta},T_{\boldsymbol\beta})$ is not exact even though for all $i\in[\![0,p-1]\!]$, the dynamical systems $([0,1),\mathcal{B}([0,1)),\mu_{\boldsymbol\beta,i},T_{\beta_{i-1}}\circ\cdots\circ T_{\beta_{i-p}})$ are exact. It suffices to note that the dynamical system $([\![0,p-1]\!]\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol\beta},T_{\boldsymbol\beta}^p)$ is not ergodic for $p>1$. Indeed, $T_{\boldsymbol\beta}^{-p}(\{0\}\times[0,1))=\{0\}\times[0,1)$ whereas $\mu_{\boldsymbol\beta}(\{0\}\times[0,1))=\frac{1}{p}$.

4.2. **Extended measure.** In order to study the dynamics of the extended greedy $\boldsymbol{\beta}$-transformation defined in (5), we define *extended measures* $\mu_{\boldsymbol{\beta}}^{\text{ext}}$ and $\lambda_{\boldsymbol{\beta}}^{\text{ext}}$ by extending the domain of the measures $\mu_{\boldsymbol{\beta}}$ and λ_p defined in (7) and (8) respectively. First, we define a new σ-algebra $\mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}$ on $\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big)$ as follows:

$$\mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}} = \left\{ \bigcup_{i=0}^{p-1}(\{i\}\times B_i) : \forall i \in [\![0,p-1]\!],\ B_i \in \mathcal{B}([0,x_{\boldsymbol{\beta}^{(i)}})) \right\}.$$

Second, for $A \in \mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}$, we set $\mu_{\boldsymbol{\beta}}^{\text{ext}}(A) = \mu_{\boldsymbol{\beta}}\big(A\cap\big([\![0,p-1]\!]\times[0,1)\big)\big)$ and $\lambda_{\boldsymbol{\beta}}^{\text{ext}}(A) = \lambda_p\big(A\cap\big([\![0,p-1]\!]\times[0,1)\big)\big)$.

Note that, in the previous section, we could have denoted the σ-algebra $\mathcal{T}_p$ by $\mathcal{T}_{\boldsymbol{\beta}}$ and similarly, the measure λ_p by $\lambda_{\boldsymbol{\beta}}$. We chose to only emphasize the dependence in p since the definitions of $\mathcal{T}_p$ and λ_p indeed only depend on the length p of the corresponding alternate base $\boldsymbol{\beta}$.

Theorem 4.13. *The measure $\mu_{\boldsymbol{\beta}}^{\text{ext}}$ is the unique $T_{\boldsymbol{\beta}}^{\text{ext}}$-invariant probability measure on $\mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}$ that is absolutely continuous with respect to $\lambda_{\boldsymbol{\beta}}^{\text{ext}}$. Furthermore, $\mu_{\boldsymbol{\beta}}^{\text{ext}}$ is equivalent to $\lambda_{\boldsymbol{\beta}}^{\text{ext}}$ on $\mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}$ and the dynamical system $(\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big), \mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}, \mu_{\boldsymbol{\beta}}^{\text{ext}}, T_{\boldsymbol{\beta}}^{\text{ext}})$ is ergodic and has entropy $\frac{1}{p}\log(\beta_{p-1}\cdots\beta_0)$.*

Proof. Clearly, $\mu_{\boldsymbol{\beta}}^{\text{ext}}$ is a probability measure on $\mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}$. For all $A \in \mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}$, we have

$$\begin{aligned}
\mu_{\boldsymbol{\beta}}^{\text{ext}}((T_{\boldsymbol{\beta}}^{\text{ext}})^{-1}(A)) &= \mu_{\boldsymbol{\beta}}\big((T_{\boldsymbol{\beta}}^{\text{ext}})^{-1}(A)\cap([\![0,p-1]\!]\times[0,1))\big)\\
&= \mu_{\boldsymbol{\beta}}\big((T_{\boldsymbol{\beta}}^{\text{ext}})^{-1}\big(A\cap([\![0,p-1]\!]\times[0,1))\big)\cap([\![0,p-1]\!]\times[0,1))\big)\\
&= \mu_{\boldsymbol{\beta}}\big(T_{\boldsymbol{\beta}}^{-1}\big(A\cap([\![0,p-1]\!]\times[0,1))\big)\big)\\
&= \mu_{\boldsymbol{\beta}}\big(A\cap([\![0,p-1]\!]\times[0,1))\big)\\
&= \mu_{\boldsymbol{\beta}}^{\text{ext}}(A)
\end{aligned}$$

where we used Proposition 4.6 for the fourth equality. This shows that $\mu_{\boldsymbol{\beta}}^{\text{ext}}$ is $T_{\boldsymbol{\beta}}^{\text{ext}}$-invariant on $\mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}$. The conclusion then follows from the fact that the identity map from $[\![0,p-1]\!]\times[0,1)$ to $\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big)$ defines an isomorphism between the dynamical systems $([\![0,p-1]\!]\times[0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}})$ and $(\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big), \mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}, \mu_{\boldsymbol{\beta}}^{\text{ext}}, T_{\boldsymbol{\beta}}^{\text{ext}})$. □

4.3. **Density functions.** In the next proposition, we express the density function of the unique measure given in Proposition 4.3.

Proposition 4.14. *Consider $n \in \mathbb{N}_{\geq 1}$ and $\beta_0,\dots,\beta_{n-1} > 1$. Suppose that*

- *K is the number of not onto branches of $T_{\beta_{n-1}}\circ\cdots\circ T_{\beta_0}$*
- *for $j \in [\![1,K]\!]$, c_j is the right-hand side endpoint of the domain of the j-th not onto branche of $T_{\beta_{n-1}}\circ\cdots\circ T_{\beta_0}$*
- *$T\colon[0,1)\to[0,1)$ is the transformation defined by $T(x) = T_{\beta_{n-1}}\circ\cdots\circ T_{\beta_0}(x)$ for $x \notin \{c_1,\dots,c_K\}$ and $T(c_j) = \lim_{x\to c_j^-} T_{\beta_{n-1}}\circ\cdots\circ T_{\beta_0}(x)$ for $j \in [\![1,K]\!]$*
- *S is the matrix defined by $S = (S_{i,j})_{1\leq i,j,\leq K}$ where*
$$S_{i,j} = \sum_{m=1}^{\infty}\frac{\delta(T^m(c_i) > c_j)}{(\beta_{n-1}\cdots\beta_0)^m},$$
where $\delta(P)$ equals 1 when the property P is satisfied and 0 otherwise
- *1 is not an eigenvalue of S*
- *$d_0 = 1$ and $\big(d_1\cdots d_K\big) = \big(1\cdots 1\big)\,(-S + Id_K)^{-1}$*

- $C = \int_0^1 \left(d_0 + \sum_{j=1}^K d_j \sum_{m=1}^\infty \frac{\chi_{[0,T^m(c_j)]}}{(\beta_{n-1}\cdots\beta_0)^m}\right) d\lambda$ *is the normalization constant.*

Then the density function of the $(T_{\beta_{n-1}}\circ\cdots\circ T_{\beta_0})$*-invariant measure given by Proposition 4.3 with respect to the Lebesgue measure is*

$$\frac{1}{C}\left(d_0 + \sum_{j=1}^K d_j \sum_{m=1}^\infty \frac{\chi_{[0,T^m(c_j)]}}{(\beta_{n-1}\cdots\beta_0)^m}\right). \tag{10}$$

Proof. This is an application of the formula given in [14, Theorem 2]. □

In [14] Gora conjectured that 1 is not an eigenvalue of the matrix S if and only if the dynamical system is exact. Thus, if Gora's conjecture were true, thanks to Proposition 4.3, the hypothesis that 1 is not an eigenvalue of the matrix S could be removed from the statement of Proposition 4.14. In particular, Proposition 4.14 would then provide the density function of the $(T_{\beta_{n-1}}\circ\cdots\circ T_{\beta_0})$-invariant measure given by Proposition 4.3 without any further conditions.

Example 4.15. Consider once again the alternate base $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. The composition $T_{\beta_1}\circ T_{\beta_0}$ is depicted in Figure 9. Since $\frac{1}{\beta_0} = \beta_1 - 1$, keeping the notation of

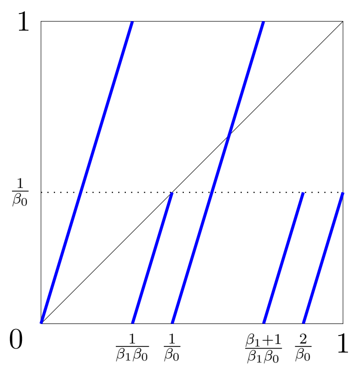

FIGURE 9. The composition $T_{\beta_1}\circ T_{\beta_0}$ with $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

Proposition 4.14, we have $K = 3$, $c_1 = \frac{1}{\beta_0}$, $c_2 = \frac{2}{\beta_0}$ and $c_3 = 1$. We have $T(c_1) = T(c_2) = T(c_3) = c_1$. Therefore, all elements in S equal 0, $d_0 = d_1 = d_2 = d_3 = 1$ and $C = 1 + \frac{3}{\beta_0(\beta_1\beta_0 - 1)} = 1 + \frac{3}{\beta_0^2}$. The density of the unique absolutely continuous $(T_{\beta_1}\circ T_{\beta_0})$-invariant probability measure is

$$\frac{1}{C}\left(1 + \frac{3}{\beta_0}\chi_{[0,\frac{1}{\beta_0}]}\right).$$

For example, $\mu\big([0,\frac{1}{\beta_0})\big) = \frac{13+\sqrt{13}}{26}$. Moreover, it can be checked that $\mu\big((T_{\beta_1}\circ T_{\beta_0})^{-1}[0,\frac{1}{\beta_0})\big) = \mu\big([0,\frac{1}{\beta_0})\big)$.

We obtain a formula for the density function $\frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p}$ by using the density functions $\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda}$ for $i \in [\![0, p-1]\!]$. We first need a lemma.

Lemma 4.16. *For all* $i \in [\![0,p-1]\!]$*, all sets* $B \in \mathcal{B}([0,1))$ *and all* $\mathcal{B}([0,1))$*-measurable functions* $f\colon [0,1) \to [0,\infty)$*, the map* $f\circ\pi_2\colon [\![0,p-1]\!]\times[0,1) \to [0,\infty)$ *is* $\mathcal{T}_p$*-measurable and*

$$\int_{\{i\}\times B} f\circ\pi_2 \, d\lambda_p = \frac{1}{p}\int_B f d\lambda.$$

Proof. This follows from standard arguments by using the definition of the Lebesgue integral via simple functions. □

Proposition 4.17. *The density function* $\frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p}$ *of* $\mu_{\boldsymbol{\beta}}$ *with respect to the* p*-Lebesgue measure on* $\mathcal{T}_p$ *is*

$$\sum_{i=0}^{p-1}\left(\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda}\circ\pi_2\right)\cdot\chi_{\{i\}\times[0,1)}. \tag{11}$$

Proof. Let $A\in\mathcal{T}_p$ and let $B_0,\ldots,B_{p-1}\in\mathcal{B}([0,1))$ such that $A=\bigcup_{i=0}^{p-1}(\{i\}\times B_i)$. It follows from Lemma 4.16 that

$$\begin{aligned}\int_A\sum_{i=0}^{p-1}\left(\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda}\circ\pi_2\right)\cdot\chi_{\{i\}\times[0,1)}\,d\lambda_p &=\sum_{i=0}^{p-1}\int_{\{i\}\times B_i}\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda}\circ\pi_2\,d\lambda_p\\ &=\frac{1}{p}\sum_{i=0}^{p-1}\int_{B_i}\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda}\,d\lambda\\ &=\frac{1}{p}\sum_{i=0}^{p-1}\mu_{\boldsymbol{\beta},i}(B_i)\\ &=\mu_{\boldsymbol{\beta}}(A).\end{aligned}$$

□

Note that the formula (11) also holds for the extended measures $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}$ and $\lambda_{\boldsymbol{\beta}}^{\mathrm{ext}}$ on $\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$.

4.4. **Frequencies.** We now turn to the frequencies of the digits in the greedy $\boldsymbol{\beta}$-expansions of real numbers in the interval $[0,1)$. Recall that the frequency of a digit d occurring in the greedy $\boldsymbol{\beta}$-expansion $a_0a_1a_2\cdots$ of a real number x in $[0,1)$ is equal to

$$\lim_{n\to\infty}\frac{1}{n}\#\{0\le k<n: a_k=d\},$$

provided that this limit exists.

Proposition 4.18. *For* λ*-almost all* $x\in[0,1)$*, the frequency of any digit* d *occurring in the greedy* $\boldsymbol{\beta}$*-expansion of* x *exists and is equal to*

$$\frac{1}{p}\sum_{i=0}^{p-1}\mu_{\boldsymbol{\beta},i}\Big(\big[\tfrac{d}{\beta_i},\tfrac{d+1}{\beta_i}\big)\cap[0,1)\Big).$$

Proof. Let $x\in[0,1)$ and let d be a digit occurring in $d_{\boldsymbol{\beta}}(x)=a_0a_1a_2\cdots$. Then for all $k\in\mathbb{N}$, $a_k=d$ if and only if $\pi_2(T_{\boldsymbol{\beta}}^k(0,x))\in[\frac{d}{\beta_k},\frac{d+1}{\beta_k})\cap[0,1)$. Moreover, since for all $k\in\mathbb{N}$, $T_{\boldsymbol{\beta}}^k(0,x)\in\{k\bmod p\}\times[0,1)$, we have

$$\begin{aligned}\chi_{[\frac{d}{\beta_k},\frac{d+1}{\beta_k})\cap[0,1)}\big(\pi_2\big(T_{\boldsymbol{\beta}}^k(0,x)\big)\big)&=\chi_{\{k\bmod p\}\times\left([\frac{d}{\beta_k},\frac{d+1}{\beta_k})\cap[0,1)\right)}\big(T_{\boldsymbol{\beta}}^k(0,x)\big)\\ &=\sum_{i=0}^{p-1}\chi_{\{i\}\times\left([\frac{d}{\beta_i},\frac{d+1}{\beta_i})\cap[0,1)\right)}\big(T_{\boldsymbol{\beta}}^k(0,x)\big).\end{aligned}$$

Therefore, if it exists, the frequency of d in $d_{\boldsymbol{\beta}}(x)$ is equal to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sum_{i=0}^{p-1}\chi_{\{i\}\times\left([\frac{d}{\beta_i},\frac{d+1}{\beta_i})\cap[0,1)\right)}\big(T_{\boldsymbol{\beta}}^k(0,x)\big).$$

Yet, for each $i \in [\![0,p-1]\!]$ and for $\mu_{\boldsymbol{\beta}}$-almost all $y \in [\![0,p-1]\!] \times [0,1)$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{\{i\}\times\left([\frac{d}{\beta_i},\frac{d+1}{\beta_i})\cap[0,1)\right)}\big(T_{\boldsymbol{\beta}}^k(y)\big) &= \int_{[\![0,p-1]\!]\times[0,1)} \chi_{\{i\}\times\left([\frac{d}{\beta_i},\frac{d+1}{\beta_i})\cap[0,1)\right)}\, d\mu_{\boldsymbol{\beta}} \\ &= \mu_{\boldsymbol{\beta}}\Big(\{i\}\times\big(\big[\tfrac{d}{\beta_i},\tfrac{d+1}{\beta_i}\big)\cap[0,1)\big)\Big) \\ &= \frac{1}{p}\mu_{\boldsymbol{\beta},i}\Big(\big[\tfrac{d}{\beta_i},\tfrac{d+1}{\beta_i}\big)\cap[0,1)\Big)\end{aligned}$$

where we used Theorem 4.12 and the Ergodic Theorem for the first equality. The conclusion now follows from Proposition 4.8. □

Note that, when $p=1$, Proposition 4.18 gives back the classical formula $\mu_\beta\big([\frac{d}{\beta},\frac{d+1}{\beta})\cap[0,1)\big)$ for the frequency of the digit d, where μ_β is the measure given in Theorem 2.2.

5. Isomorphism between greedy and lazy $\boldsymbol{\beta}$-transformations

In this section, we show that

$$\phi_{\boldsymbol{\beta}}\colon \bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big) \to \bigcup_{i=0}^{p-1}\big(\{i\}\times(0,x_{\boldsymbol{\beta}^{(i)}}]\big),\ (i,x)\mapsto \big(i,x_{\boldsymbol{\beta}^{(i)}}-x\big) \tag{12}$$

defines an isomorphism between the greedy $\boldsymbol{\beta}$-transformation and the lazy $\boldsymbol{\beta}$-transformation.

We consider the σ-algebra

$$\mathcal{L}_{\boldsymbol{\beta}} = \left\{\bigcup_{i=0}^{p-1}(\{i\}\times B_i) : \forall i\in[\![0,p-1]\!],\ B_i\in\mathcal{B}\big((0,x_{\boldsymbol{\beta}^{(i)}}]\big)\right\}$$

on $\bigcup_{i=0}^{p-1}\big(\{i\}\times\big(0,x_{\boldsymbol{\beta}^{(i)}}\big]\big)$.

Proposition 5.1. *The map $\phi_{\boldsymbol{\beta}}$ is an isomorphism between the dynamical systems $\big(\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big),\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}},\mu_{\boldsymbol{\beta}}^{\mathrm{ext}},T_{\boldsymbol{\beta}}^{\mathrm{ext}}\big)$ and $\big(\bigcup_{i=0}^{p-1}\big(\{i\}\times(0,x_{\boldsymbol{\beta}^{(i)}}]\big),\mathcal{L}_{\boldsymbol{\beta}},\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}\circ\phi_{\boldsymbol{\beta}}^{-1},L_{\boldsymbol{\beta}}\big)$.*

Proof. Clearly, $\phi_{\boldsymbol{\beta}}$ is a bimeasurable bijective map. Hence, we only have to show that $\phi_{\boldsymbol{\beta}}\circ T_{\boldsymbol{\beta}}^{\mathrm{ext}} = L_{\boldsymbol{\beta}}\circ\phi_{\boldsymbol{\beta}}$. Let $(i,x)\in\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big)$. First, suppose that $x\in[0,1)$. Then

$$\phi_{\boldsymbol{\beta}}\circ T_{\boldsymbol{\beta}}^{\mathrm{ext}}(i,x) = \big((i+1)\bmod p,\, x_{\boldsymbol{\beta}^{(i+1)}}-\beta_i x+\lfloor\beta_i x\rfloor\big)$$

and

$$L_{\boldsymbol{\beta}}\circ\phi_{\boldsymbol{\beta}}(i,x) = \big((i+1)\bmod p,\, \beta_i(x_{\boldsymbol{\beta}^{(i)}}-x)-\lceil\beta_i(x_{\boldsymbol{\beta}^{(i)}}-x)-x_{\boldsymbol{\beta}^{(i+1)}}\rceil\big).$$

Second, suppose that $x\in[1,x_{\boldsymbol{\beta}^{(i)}})$. Then

$$\phi_{\boldsymbol{\beta}}\circ T_{\boldsymbol{\beta}}^{\mathrm{ext}}(i,x) = \big((i+1)\bmod p,\, x_{\boldsymbol{\beta}^{(i+1)}}-\beta_i x+\lfloor\beta_i\rfloor-1\big)$$

and

$$L_{\boldsymbol{\beta}}\circ\phi_{\boldsymbol{\beta}}(i,x) = \big((i+1)\bmod p,\, \beta_i(x_{\boldsymbol{\beta}^{(i)}}-x)\big).$$

In both cases, we easily get that $\phi_{\boldsymbol{\beta}}\circ T_{\boldsymbol{\beta}}^{\mathrm{ext}}(i,x) = L_{\boldsymbol{\beta}}\circ\phi_{\boldsymbol{\beta}}(i,x)$ by using (4). □

Thanks to Proposition 5.1, we obtain an analogue of Theorem 4.13 for the lazy $\boldsymbol{\beta}$-transformation.

Theorem 5.2. *The measure* $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \phi_{\boldsymbol{\beta}}^{-1}$ *is the unique* $L_{\boldsymbol{\beta}}$*-invariant probability measure on* $\mathcal{L}_{\boldsymbol{\beta}}$ *that is absolutely continuous with respect to* $\lambda_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \phi_{\boldsymbol{\beta}}^{-1}$*. Furthermore,* $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \phi_{\boldsymbol{\beta}}^{-1}$ *is equivalent to* $\lambda_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \phi_{\boldsymbol{\beta}}^{-1}$ *on* $\mathcal{L}_{\boldsymbol{\beta}}$ *and the dynamical system* $\big(\bigcup_{i=0}^{p-1}\big(\{i\}\times(0,x_{\boldsymbol{\beta}^{(i)}}]\big), \mathcal{L}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \phi_{\boldsymbol{\beta}}^{-1}, L_{\boldsymbol{\beta}}\big)$ *is ergodic and has entropy* $\frac{1}{p}\log(\beta_{p-1}\cdots\beta_0)$*.*

Similarly, we have an analogue of Theorem 4.12 for the lazy $\boldsymbol{\beta}$-transformation, by considering the σ-algebra

$$\mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{restr}} = \left\{ \bigcup_{i=0}^{p-1}(\{i\}\times B_i) : \forall i \in [\![0,p-1]\!],\ B_i \in \mathcal{B}\big((x_{\boldsymbol{\beta}^{(i)}}-1, x_{\boldsymbol{\beta}^{(i)}}]\big) \right\}.$$

Remark that in the lazy case, we denote the restricted σ-algebra by $\mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{restr}}$ since there is a dependence on the alternate base $\boldsymbol{\beta}$ and not only on its length p as in the greedy case. We also set

$$\phi_{\boldsymbol{\beta}}^{\mathrm{restr}} \colon [\![0,p-1]\!]\times[0,1) \to \bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1, x_{\boldsymbol{\beta}^{(i)}}]\big),\ (i,x)\mapsto \big(i, x_{\boldsymbol{\beta}^{(i)}}-x\big)$$

and

$$L_{\boldsymbol{\beta}}^{\mathrm{restr}} \colon \bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1, x_{\boldsymbol{\beta}^{(i)}}]\big) \to \bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1, x_{\boldsymbol{\beta}^{(i)}}]\big),$$
$$(i,x)\mapsto \big((i+1) \bmod p, \beta_i x - \lceil \beta_i x - x_{\boldsymbol{\beta}^{(i+1)}}\rceil\big).$$

Theorem 5.3. *The measure* $\mu_{\boldsymbol{\beta}} \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{restr}})^{-1}$ *is the unique* $L_{\boldsymbol{\beta}}^{\mathrm{restr}}$*-invariant probability measure on* $\mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{restr}}$ *that is absolutely continuous with respect to* $\lambda_p \circ \phi_{\boldsymbol{\beta}}^{-1}$*. Furthermore,* $\mu_{\boldsymbol{\beta}} \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{restr}})^{-1}$ *is equivalent to* $\lambda_p \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{restr}})^{-1}$ *on* $\mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{restr}}$ *and the dynamical system* $\big(\bigcup_{i=0}^{p-1}\big(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1, x_{\boldsymbol{\beta}^{(i)}}]\big), \mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{restr}}, \mu_{\boldsymbol{\beta}} \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{restr}})^{-1}, L_{\boldsymbol{\beta}}^{\mathrm{restr}}\big)$ *is ergodic and has entropy* $\frac{1}{p}\log(\beta_{p-1}\cdots\beta_0)$*.*

Remark 5.4. We deduce from Proposition 5.1 that if the greedy $\boldsymbol{\beta}$-expansion of a real number $x \in [0, x_{\boldsymbol{\beta}})$ is $a_0a_1a_2\cdots$, then the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - x$ is $(\lceil\beta_0\rceil - 1 - a_0)(\lceil\beta_1\rceil - 1 - a_1)(\lceil\beta_2\rceil - 1 - a_2)\cdots$.

6. Isomorphism with the $\boldsymbol{\beta}$-shift

The aim of this section is to generalize the isomorphism between the greedy β-transformation and the β-shift to the framework of alternate bases. We start by providing some background of the real base case.

Let D_β denote the set of all greedy β-expansions of real numbers in the interval $[0,1)$. The *β-shift* is the set S_β defined as the topological closure of D_β with respect to the prefix distance of infinite words. For an alphabet A, we let $\mathcal{C}_A$ denote the σ-algebra generated by the *cylinders*

$$C_A(a_0,\ldots,a_{\ell-1}) = \{w \in A^{\mathbb{N}} : w[0]=a_0,\ldots,w[\ell-1]=a_{\ell-1}\}$$

for all $\ell \in \mathbb{N}$ and $a_0,\ldots,a_{\ell-1} \in A$, where the notation $w[k]$ designates the letter at position k in the infinite word w, and we call

$$\sigma_A \colon A^{\mathbb{N}} \to A^{\mathbb{N}},\ a_0a_1a_2\cdots \mapsto a_1a_2a_3\cdots$$

the *shift operator* over A. If no confusion is possible, we simply write σ instead of σ_A. Then, it is a folklore fact (similar to [10, Example 1.2.19]) that the map $\psi_\beta \colon [0,1) \to S_\beta,\ x \mapsto d_\beta(x)$ defines an isomorphism between the dynamical systems $([0,1), \mathcal{B}([0,1)), \mu_\beta, T_\beta)$ and $(S_\beta, \mathcal{C}_{A_\beta} \cap S_\beta, \mu_\beta \circ \psi_\beta^{-1}, \sigma_{|S_\beta})$ where A_β denote the alphabet of digits $[\![0, \lceil\beta\rceil - 1]\!]$.

Now, let us extend the previous notation to the framework of alternate bases. Let $A_{\boldsymbol{\beta}}$ denote the alphabet $[\![0, \max_{i\in[\![0,p-1]\!]} \lceil \beta_i \rceil - 1]\!]$, let $D_{\boldsymbol{\beta}}$ denote the subset of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ made of all greedy $\boldsymbol{\beta}$-expansions of real numbers in $[0,1)$ and let $S_{\boldsymbol{\beta}}$ denote the topological closure of $D_{\boldsymbol{\beta}}$ with respect to the prefix distance of infinite words:

$$D_{\boldsymbol{\beta}} = \{d_{\boldsymbol{\beta}}(x) : x \in [0,1)\} \quad \text{and} \quad S_{\boldsymbol{\beta}} = \overline{D_{\boldsymbol{\beta}}}.$$

The following lemma was proved in [5, Proposition 32].

Lemma 6.1. *For all $n \in \mathbb{N}$, if $w \in S_{\boldsymbol{\beta}^{(n)}}$ then $\sigma(w) \in S_{\boldsymbol{\beta}^{(n+1)}}$.*

Consider the σ-algebra

$$\mathcal{G}_{\boldsymbol{\beta}} = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times C_i) : \forall i \in [\![0, p-1]\!],\ C_i \in \mathcal{C}_{A_{\boldsymbol{\beta}}} \cap S_{\boldsymbol{\beta}^{(i)}} \right\}$$

on $\bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}})$. We define

$$\sigma_p \colon \bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}}) \to \bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}}),\ (i,w) \mapsto ((i+1) \bmod p, \sigma(w))$$

$$\psi_{\boldsymbol{\beta}} \colon [\![0, p-1]\!] \times [0,1) \to \bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}}),\ (i,x) \mapsto (i, d_{\boldsymbol{\beta}^{(i)}}(x)).$$

Note that the transformation σ_p is well defined by Lemma 6.1.

Proposition 6.2. *The map $\psi_{\boldsymbol{\beta}}$ defines an isomorphism between the dynamical systems*

$$\big([\![0, p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}}\big) \quad \text{and} \quad \left(\bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}, \sigma_p \right).$$

Proof. It is easily seen that $\psi_{\boldsymbol{\beta}} \circ T_{\boldsymbol{\beta}} = \sigma_p \circ \psi_{\boldsymbol{\beta}}$ and that $\psi_{\boldsymbol{\beta}}$ is injective. Moreover, $\psi_{\boldsymbol{\beta}}\big([\![0,p-1]\!] \times [0,1)\big) = \cup_{i=0}^{p-1} (\{i\} \times D_{\boldsymbol{\beta}^{(i)}})$ and $\mu_{\boldsymbol{\beta}}(\psi_{\boldsymbol{\beta}}^{-1}(\cup_{i=0}^{p-1}(\{i\} \times D_{\boldsymbol{\beta}^{(i)}})) = 1$. □

However, although $\psi_{\boldsymbol{\beta}}$ is continuous, it does not define a topological isomorphism since it is not surjective.

Remark 6.3. In view of Proposition 6.2, the set $\bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}})$ can be seen as the $\boldsymbol{\beta}$-*shift*, that is, the generalization of the β-shift to alternate bases. However, in the previous work [5], what we called the $\boldsymbol{\beta}$-shift is the union $\bigcup_{i=0}^{p-1} S_{\boldsymbol{\beta}^{(i)}}$. This definition was motivated by the following combinatorial result [5, Theorem 48] : the set $\bigcup_{i=0}^{p-1} S_{\boldsymbol{\beta}^{(i)}}$ is sofic if and only if for every $i \in [\![0, p-1]\!]$, the quasi-greedy $\boldsymbol{\beta}^{(i)}$-representation of 1 is ultimately periodic. In summary, we can say that there are two ways to extend the notion of β-shift to alternate bases $\boldsymbol{\beta}$, depending on the way we look at it: either as a dynamical object or as a combinatorial object.

Thanks to Proposition 6.2, we obtain an analogue of Theorem 4.12 for the transformation σ_p.

Theorem 6.4. *The measure $\mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}$ is the unique σ_p-invariant probability measure on $\mathcal{G}_{\boldsymbol{\beta}}$ that is absolutely continuous with respect to $\lambda_p \circ \psi_{\boldsymbol{\beta}}^{-1}$. Furthermore, $\mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}$ is equivalent to $\lambda_p \circ \psi_{\boldsymbol{\beta}}^{-1}$ on $\mathcal{G}_{\boldsymbol{\beta}}$ and the dynamical system $\big(\bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}, \sigma_p\big)$ is ergodic and has entropy $\frac{1}{p} \log(\beta_{p-1} \cdots \beta_0)$.*

Remark 6.5. Let $D'_{\boldsymbol{\beta}}$ denote the subset of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ made of all lazy $\boldsymbol{\beta}$-expansions of real numbers in $(x_{\boldsymbol{\beta}}-1, x_{\boldsymbol{\beta}}]$ and let $S'_{\boldsymbol{\beta}}$ denote the topological closure of $D'_{\boldsymbol{\beta}}$ with respect to the prefix distance of infinite words. From Remark 5.4, it is easily seen that

$$\theta_{\boldsymbol{\beta}}\colon \bigcup_{i=0}^{p-1}(\{i\}\times S_{\boldsymbol{\beta}^{(i)}}) \to \bigcup_{i=0}^{p-1}(\{i\}\times S'_{\boldsymbol{\beta}^{(i)}}),\ (i,a_0a_1\cdots)\mapsto (i,(\lceil\beta_i\rceil-1-a_0)(\lceil\beta_{i+1}\rceil-1-a_2)\cdots)$$

defines an isomorphism from $\big(\bigcup_{i=0}^{p-1}(\{i\}\times S_{\boldsymbol{\beta}^{(i)}}),\mathcal{G}_{\boldsymbol{\beta}},\mu_{\boldsymbol{\beta}}\circ\psi_{\boldsymbol{\beta}}^{-1},\sigma_p\big)$ to $\big(\bigcup_{i=0}^{p-1}(\{i\}\times S'_{\boldsymbol{\beta}^{(i)}}),\mathcal{G}'_{\boldsymbol{\beta}},$ $\mu_{\boldsymbol{\beta}}\circ\psi_{\boldsymbol{\beta}}^{-1}\circ\theta_{\boldsymbol{\beta}}^{-1},\sigma'_p\big)$ where

$$\mathcal{G}'_{\boldsymbol{\beta}} = \left\{\bigcup_{i=0}^{p-1}\left(\{i\}\times(C_i\cap S'_{\boldsymbol{\beta}^{(i)}})\right) : C_i\in\mathcal{C}_{A_{\boldsymbol{\beta}}}\right\}$$

$$\sigma'_p\colon \bigcup_{i=0}^{p-1}(\{i\}\times S'_{\boldsymbol{\beta}^{(i)}}) \to \bigcup_{i=0}^{p-1}(\{i\}\times S'_{\boldsymbol{\beta}^{(i)}}),\ (i,w)\mapsto((i+1)\bmod p,\sigma(w)).$$

We then deduce from Propositions 5.1 and 6.2 that $\theta_{\boldsymbol{\beta}}\circ\psi_{\boldsymbol{\beta}}\circ(\phi_{\boldsymbol{\beta}}^{\mathrm{restr}})^{-1}$ is an isomorphism from $\big(\bigcup_{i=0}^{p-1}\left(\{i\}\times(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]\right),\mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{restr}},\mu_{\boldsymbol{\beta}}\circ(\phi_{\boldsymbol{\beta}}^{\mathrm{restr}})^{-1},L_{\boldsymbol{\beta}}^{\mathrm{restr}}\big)$ to $\big(\bigcup_{i=0}^{p-1}(\{i\}\times S'_{\boldsymbol{\beta}^{(i)}}),\mathcal{G}'_{\boldsymbol{\beta}},$ $\mu_{\boldsymbol{\beta}}\circ\psi_{\boldsymbol{\beta}}^{-1}\circ\theta_{\boldsymbol{\beta}}^{-1},\sigma'_p\big)$. It is easy to check that, as expected, that for all $(i,x)\in\bigcup_{i=0}^{p-1}\big(\{i\}\times$ $(x_{\boldsymbol{\beta}^{(i)}}-1,x_{\boldsymbol{\beta}^{(i)}}]$, we have $\theta_{\boldsymbol{\beta}}\circ\psi_{\boldsymbol{\beta}}\circ(\phi_{\boldsymbol{\beta}}^{\mathrm{restr}})^{-1}(i,x)=(i,\ell_{\boldsymbol{\beta}^{(i)}}(x))$ where $\ell_{\boldsymbol{\beta}}(x)$ denoted the lazy $\boldsymbol{\beta}$-expansion of x.

7. $\boldsymbol{\beta}$-expansions and $(\beta_{p-1}\cdots\beta_0,\Delta_{\boldsymbol{\beta}})$-expansions

By rewriting Equality (1) from Section 3 as

$$x = \frac{\beta_{p-1}\cdots\beta_1 a_0+\beta_{p-1}\cdots\beta_2 a_1+\cdots+a_{p-1}}{\beta_{p-1}\cdots\beta_0} + \frac{\beta_{p-1}\cdots\beta_1 a_p+\beta_{p-1}\cdots\beta_1 a_{p+1}+\cdots+a_{2p-1}}{(\beta_{p-1}\cdots\beta_0)^2} + \cdots \tag{13}$$

we can see the greedy and lazy $\boldsymbol{\beta}$-expansions of real numbers as $(\beta_{p-1}\cdots\beta_0)$-representations over the digit set

$$\Delta_{\boldsymbol{\beta}} = \left\{\sum_{i=0}^{p-1}\beta_{p-1}\cdots\beta_{i+1}c_i : \forall i\in[\![0,p-1]\!],\ c_i\in[\![0,\lceil\beta_i\rceil-1]\!]\right\}.$$

In this section, we examine some cases where by considering the greedy (resp. lazy) $\boldsymbol{\beta}$-expansion and rewriting it as (13), the obtained representation is the greedy (resp. lazy) $(\beta_{p-1}\cdots\beta_0,\Delta_{\boldsymbol{\beta}})$-expansion. We first recall the formalism of β-expansions of real numbers over a general digit set [22].

7.1. **Real base expansions over general digit sets.** Consider an arbitrary finite set $\Delta=\{d_0,d_1,\ldots,d_m\}\subset\mathbb{R}$ where $0=d_0<d_1<\cdots<d_m$. Then a (β,Δ)-*representation* of a real number x in the interval $[0,\frac{d_m}{\beta-1})$ is an infinite sequence $a_0a_1a_2\cdots$ over Δ such that $x=\sum_{n=0}^{\infty}\frac{a_n}{\beta^{n+1}}$. Such a set Δ is called an *allowable digit set for* β if

$$\max_{k\in[\![0,m-1]\!]}(d_{k+1}-d_k)\le\frac{d_m}{\beta-1}. \tag{14}$$

In this case, the *greedy* (β, Δ)*-expansion* of a real number $x \in [0, \frac{d_m}{\beta-1})$ is defined recursively as follows: if the first N digits of the greedy (β, Δ)-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest element in Δ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} \leq x.$$

The greedy (β, Δ)-expansion can also be obtained by iterating the *greedy* (β, Δ)*-transformation*

$$T_{\beta,\Delta} \colon [0, \tfrac{d_m}{\beta-1}) \to [0, \tfrac{d_m}{\beta-1}),\ x \mapsto \begin{cases} \beta x - d_k & \text{if } x \in [\frac{d_k}{\beta}, \frac{d_{k+1}}{\beta}),\ k \in [\![0, m-1]\!] \\ \beta x - d_m & \text{if } x \in [\frac{d_m}{\beta}, \frac{d_m}{\beta-1}) \end{cases}$$

as follows: for all $n \in \mathbb{N}$, a_n is the greatest digit d in Δ such that $\frac{d}{\beta} \leq T^n_{\beta,\Delta}(x)$ [7].

Example 7.1. Consider the digit set $\Delta = \{0, 1, \varphi + \frac{1}{\varphi}, \varphi^2\}$. It is easily checked that Δ is an allowable digit set for φ. The greedy (φ, Δ)-transformation

$$T_{\varphi,\Delta} \colon [0, \tfrac{\varphi^2}{\varphi-1}) \to [0, \tfrac{\varphi^2}{\varphi-1}),\ x \mapsto \begin{cases} \varphi x & \text{if } x \in [0, \frac{1}{\varphi}) \\ \varphi x - 1 & \text{if } x \in [\frac{1}{\varphi}, 1 + \frac{1}{\varphi^2}) \\ \varphi x - (\varphi + \frac{1}{\varphi}) & \text{if } x \in [1 + \frac{1}{\varphi^2}, \varphi) \\ \varphi x - \varphi^2 & \text{if } x \in [\varphi, \frac{\varphi^2}{\varphi-1}) \end{cases}$$

is depicted in Figure 10.

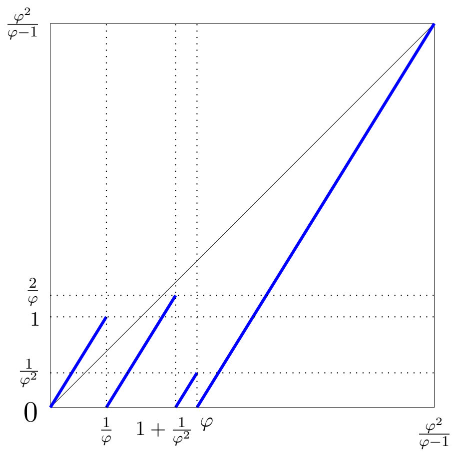

FIGURE 10. The transformation $T_{\varphi,\Delta}$ for $\Delta = \{0, 1, \frac{\varphi+1}{\varphi}, \varphi^2\}$.

Similarly, if Δ is an allowable digit set for β, then the *lazy* (β, Δ)*-expansion* of a real number $x \in (0, \frac{d_m}{\beta-1}]$ is defined recursively as follows: if the first N digits of the lazy (β, Δ)-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least element in Δ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \sum_{n=N+1}^{\infty} \frac{d_m}{\beta^{n+1}} \geq x.$$

The *lazy* (β, Δ)*-transformation*

$$L_{\beta,\Delta} \colon (0, \tfrac{d_m}{\beta-1}] \to (0, \tfrac{d_m}{\beta-1}],\ x \mapsto \begin{cases} \beta x & \text{if } x \in (0, \frac{d_m}{\beta-1} - \frac{d_m}{\beta}] \\ \beta x - d_k & \text{if } x \in (\frac{d_m}{\beta-1} - \frac{d_m - d_{k-1}}{\beta}, \frac{d_m}{\beta-1} - \frac{d_m - d_k}{\beta}],\ k \in [\![1, m]\!] \end{cases}$$

can be used to obtain the digits of the lazy (β, Δ)-expansions: for all $n \in \mathbb{N}$, a_n is the least digit d in Δ such that $\frac{d}{\beta} + \sum_{k=1}^{\infty} \frac{d_m}{\beta^{k+1}} \geq L_{\beta,\Delta}^n(x)$ [7].

In [7, Proposition 2.2], it is shown that if Δ is an allowable digit set for β then so is the set $\widetilde{\Delta} := \{0, d_m - d_{m-1}, \dots, d_m - d_1, d_m\}$ and

$$\phi_{\beta,\Delta} \colon [0, \tfrac{d_m}{\beta-1}) \to (0, \tfrac{d_m}{\beta-1}],\ x \mapsto \tfrac{d_m}{\beta-1} - x$$

is a bicontinuous bijection satisfying $L_{\beta,\widetilde{\Delta}} \circ \phi_{\beta,\Delta} = \phi_{\beta,\Delta} \circ T_{\beta,\Delta}$.

Example 7.2. Consider the digit set $\widetilde{\Delta}$ where Δ is the digit set from Example 7.1. We get $\widetilde{\Delta} = \{0, 1 - \frac{1}{\varphi}, \varphi, \varphi^2\}$. The lazy $(\varphi, \widetilde{\Delta})$-transformation

$$L_{\varphi,\widetilde{\Delta}} \colon (0, \tfrac{\varphi^2}{\varphi-1}] \to (0, \tfrac{\varphi^2}{\varphi-1}],\ x \mapsto \begin{cases} \varphi x & \text{if } x \in (0, \frac{\varphi}{\varphi-1}] \\ \varphi x - (1 - \frac{1}{\varphi}) & \text{if } x \in (\frac{\varphi}{\varphi-1}, \frac{\varphi+3}{\varphi}] \\ \varphi x - \varphi & \text{if } x \in (\frac{\varphi+3}{\varphi}, \frac{2\varphi-1}{\varphi-1}] \\ \varphi x - \varphi^2 & \text{if } x \in (\frac{2\varphi-1}{\varphi-1}, \frac{\varphi^2}{\varphi-1}] \end{cases}$$

is depicted in Figure 11. It is conjugate to the greedy (φ, Δ)-transformation $T_{\varphi,\Delta}$ by $\phi_{\varphi,\Delta} \colon [0, \frac{\varphi^2}{\varphi-1}) \to (0, \frac{\varphi^2}{\varphi-1}],\ x \mapsto \frac{\varphi^2}{\varphi-1} - x$.

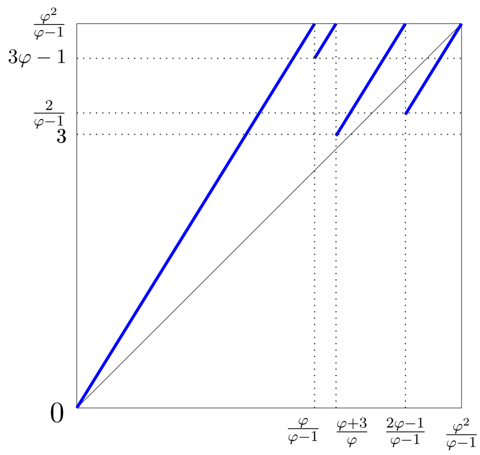

FIGURE 11. The transformation $L_{\varphi,\widetilde{\Delta}}$ for $\Delta = \{0, 1, \varphi + \frac{1}{\varphi}, \varphi^2\}$.

7.2. **Comparison between $\boldsymbol{\beta}$-expansions and $(\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}})$-expansions.** The digit set $\Delta_{\boldsymbol{\beta}}$ has cardinality at most $\prod_{i=0}^{p-1} \lceil \beta_i \rceil$ and can be rewritten $\Delta_{\boldsymbol{\beta}} = \mathrm{im}(f_{\boldsymbol{\beta}})$ where

$$f_{\boldsymbol{\beta}} \colon \prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!] \to \mathbb{R},\ (c_0, \dots, c_{p-1}) \mapsto \sum_{i=0}^{p-1} \beta_{p-1} \cdots \beta_{i+1} c_i.$$

Note that $f_{\boldsymbol{\beta}}$ is not injective in general. Let us write $\Delta_{\boldsymbol{\beta}} = \{d_0, d_1, \dots, d_m\}$ with $d_0 < d_1 < \cdots < d_m$. We have $d_0 = f_{\boldsymbol{\beta}}(0, \dots, 0) = 0$, $d_1 = f_{\boldsymbol{\beta}}(0, \dots, 0, 1) = 1$ and $d_m = f_{\boldsymbol{\beta}}(\lceil \beta_0 \rceil - 1, \dots, \lceil \beta_{p-1} \rceil - 1)$. In what follows, we suppose that $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ is equipped with the lexicographic order: $(c_0, \dots, c_{p-1}) <_{\text{lex}} (c'_0, \dots, c'_{p-1})$ if there exists $i \in [\![0, p-1]\!]$ such that $c_0 = c'_0, \dots, c_{i-1} = c'_{i-1}$ and $c_i < c'_i$.

Lemma 7.3. *The set $\Delta_{\boldsymbol{\beta}}$ is an allowable digit set for $\beta_{p-1} \cdots \beta_0$.*

Proof. We need to check Condition (14). We have $d_0 = 0$ and

$$d_m = f_{\boldsymbol{\beta}}(\lceil \beta_0 \rceil - 1, \ldots, \lceil \beta_{p-1} \rceil - 1) \geq \sum_{i=0}^{p-1} \beta_{p-1} \cdots \beta_{i+1}(\beta_i - 1) = \beta_{p-1} \cdots \beta_0 - 1,$$

Therefore, it suffices to show that for all $k \in [\![0, m-1]\!]$, $d_{k+1} - d_k \leq 1$. Thus, we only have to show that $f(c'_0, \ldots, c'_{p-1}) - f(c_0, \ldots, c_{p-1}) \leq 1$ where $(c_0, \ldots, c_{p-1})$ and $(c'_0, \ldots, c'_{p-1})$ are lexicographically consecutive elements of $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$. For such p-tuples, there exists $j \in [\![0, p-1]\!]$ such that $c_0 = c'_0, \ldots, c_{j-1} = c'_{j-1}$, $c_j = c'_j - 1$, $c_{j+1} = \lceil \beta_{j+1} \rceil - 1, \ldots, c_{p-1} = \lceil \beta_{p-1} \rceil - 1$ and $c'_{j+1} = \cdots = c'_{p-1} = 0$. Then

$$\begin{aligned} f(c'_0, \ldots, c'_{p-1}) - f(c_0, \ldots, c_{p-1}) &= \beta_{p-1} \cdots \beta_{j+1} - \sum_{i=j+1}^{p-1} \beta_{p-1} \cdots \beta_{i+1}(\lceil \beta_i \rceil - 1) \\ &\leq \beta_{p-1} \cdots \beta_{j+2} - \sum_{i=j+2}^{p-1} \beta_{p-1} \cdots \beta_{i+1}(\lceil \beta_i \rceil - 1) \\ &\ \ \vdots \\ &\leq \beta_{p-1} - (\lceil \beta_{p-1} \rceil - 1) \\ &\leq 1. \end{aligned}$$

□

Since $x_{\boldsymbol{\beta}} = \frac{d_m}{\beta_{p-1} \cdots \beta_0 - 1}$, it follows from Lemma 7.3 that every point in $[0, x_{\boldsymbol{\beta}})$ admits a greedy $(\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}})$-expansion.

Proposition 7.4. *For all $x \in [0, x_{\boldsymbol{\beta}})$, we have $T_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}(x) \leq \pi_2 \circ (T^{\text{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0(x)$ and $L_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}(x) \geq \pi_2 \circ L^p_{\boldsymbol{\beta}} \circ \delta_0(x)$.*

Proof. Let $x \in [0, x_{\boldsymbol{\beta}})$. On the one hand, $T_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}(x) = \beta_{p-1} \cdots \beta_0 x - d$ where d is the greatest digit in $\Delta_{\boldsymbol{\beta}}$ such that $\frac{d}{\beta_{p-1} \cdots \beta_0} \leq x$. On the other hand, by rephrasing Proposition 3.2 in terms of the map $f_{\boldsymbol{\beta}}$ when n equals p, we get $\pi_2 \circ (T^{\text{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0(x) = \beta_{p-1} \cdots \beta_0 x - f_{\boldsymbol{\beta}}(c)$ where c is the lexicographically greatest p-tuple in $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ such that $\frac{f_{\boldsymbol{\beta}}(c)}{\beta_{p-1} \cdots \beta_0} \leq x$. By definition of d, we get $d \geq f_{\boldsymbol{\beta}}(c)$. Therefore, we obtain that $T_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}(x) \leq \pi_2 \circ (T^{\text{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0(x)$. The inequality $L_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}(x) \geq \pi_2 \circ L^p_{\boldsymbol{\beta}} \circ \delta_0(x)$ then follows from Proposition 5.1. □

In what follows, we provide some conditions under which the inequalities of Proposition 7.4 happen to be equalities.

Proposition 7.5. *The transformations $T_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ (T^{\text{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0\big|_{[0, x_{\boldsymbol{\beta}})}$ coincide if and only if the transformations $L_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ L^p_{\boldsymbol{\beta}} \circ \delta_0\big|_{(0, x_{\boldsymbol{\beta}}]}$ do.*

Proof. We only show the forward direction, the backward direction being similar. Suppose that $T_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}} = \pi_2 \circ (T^{\text{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0\big|_{[0, x_{\boldsymbol{\beta}})}$ and let $x \in (0, x_{\boldsymbol{\beta}}]$. Since $x_{\boldsymbol{\beta}} = \frac{d_m}{\beta_{p-1} \cdots \beta_0 - 1}$ and $\Delta_{\boldsymbol{\beta}} = \widetilde{\Delta_{\boldsymbol{\beta}}}$, we successively obtain that

$$\begin{aligned} L_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}(x) &= L_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}} \circ \phi_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}(x_{\boldsymbol{\beta}} - x) \\ &= \phi_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}} \circ T_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}}(x_{\boldsymbol{\beta}} - x) \\ &= \phi_{\beta_{p-1} \cdots \beta_0, \Delta_{\boldsymbol{\beta}}} \circ \pi_2 \circ (T^{\text{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0(x_{\boldsymbol{\beta}} - x) \end{aligned}$$

$$= \pi_2 \circ \phi_{\boldsymbol{\beta}} \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^p \circ \delta_0(x_{\boldsymbol{\beta}} - x)$$
$$= \pi_2 \circ L_{\boldsymbol{\beta}}^p \circ \phi_{\boldsymbol{\beta}} \circ \delta_0(x_{\boldsymbol{\beta}} - x)$$
$$= \pi_2 \circ L_{\boldsymbol{\beta}}^p \circ \delta_0(x).$$

□

The next result provides us with a sufficient condition under which the transformations $T_{\beta_{p-1}\cdots\beta_0,\Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ coincide. Here, the non-decreasingness of the map $f_{\boldsymbol{\beta}}$ refers to the lexicographic order: for all $c, c' \in \prod_{i=0}^{p-1} [\![0, \lceil\beta_i\rceil - 1]\!]$, $c <_{\text{lex}} c' \implies f_{\boldsymbol{\beta}}(c) \le f_{\boldsymbol{\beta}}(c')$.

Theorem 7.6. *If the map $f_{\boldsymbol{\beta}}$ is non-decreasing then $T_{\beta_{p-1}\cdots\beta_0,\Delta_{\boldsymbol{\beta}}} = \pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$.*

Proof. We keep the same notation as in the proof of Proposition 7.4. Let $c' \in \prod_{i=0}^{p-1} [\![0, \lceil\beta_i\rceil - 1]\!]$ such that $d = f_{\boldsymbol{\beta}}(c')$. By definition of c, we get $c \ge_{\text{lex}} c'$. Now, if $f_{\boldsymbol{\beta}}$ is non-decreasing then $f_{\boldsymbol{\beta}}(c) \ge f_{\boldsymbol{\beta}}(c') = d$. Hence the conclusion. □

The following example shows that considering the length-p alternate base $\boldsymbol{\beta} = (\beta, \ldots, \beta)$ with $p \in \mathbb{N}_{\ge 3}$, it may happen that $T_{\beta^p,\Delta_{\boldsymbol{\beta}}}$ differs from $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$. This result was already proved in [6].

Example 7.7. Consider the alternate base $\boldsymbol{\beta} = (\varphi^2, \varphi^2, \varphi^2)$. Then $\Delta_{\boldsymbol{\beta}} = \{\varphi^4 c_0 + \varphi^2 c_1 + c_2 : c_0, c_1, c_2 \in \{0, 1, 2\}\}$. In [6, Proposition 2.1], it is proved that $T_{\beta^n,\Delta_{\boldsymbol{\beta}}} = T_\beta^n$ for all $n \in \mathbb{N}$ if and only if $f_{\boldsymbol{\beta}}$ is non-decreasing. Since $f_{\boldsymbol{\beta}}(0, 2, 2) = 2\varphi^2 + 2 > \varphi^4 = f_{\boldsymbol{\beta}}(1, 0, 0)$, the tranformations $T_{\varphi^6,\Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ differ by [6, Proposition 2.1].

Whenever $f_{\boldsymbol{\beta}}$ is not non-decreasing, the transformations $T_{\beta_{p-1}\cdots\beta_0,\Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ can either coincide or not. The following two examples illustrate both cases. In particular, Example 7.9 shows that the sufficient condition given in Theorem 7.6 is not necessary.

Example 7.8. Consider the alternate base $\boldsymbol{\beta} = (\varphi, \varphi, \sqrt{5})$. Then $\Delta_{\boldsymbol{\beta}} = \{\sqrt{5}\varphi c_0 + \sqrt{5} c_1 + c_2 : c_0, c_1 \in \{0, 1\},\ c_2 \in \{0, 1, 2\}\}$. However, $f_{\boldsymbol{\beta}}(0, 1, 2) = \sqrt{5} + 2 \simeq 4.23$ and $f_{\boldsymbol{\beta}}(1, 0, 0) = \sqrt{5}\varphi \simeq 3.61$. It can be easily check that there exists $x \in [0, x_{\boldsymbol{\beta}})$ such that $T_{\sqrt{5}\varphi^2,\Delta_{\boldsymbol{\beta}}}(x) \ne \pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^3 \circ \delta_0(x)$. For example, we can compute $T_{\sqrt{5}\varphi^2,\Delta_{\boldsymbol{\beta}}}(0.75) \simeq 0.15$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^3 \circ \delta_0(0.75) \simeq 0.77$. The transformations $T_{\sqrt{5}\varphi^2,\Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ are depicted in Figure 12, where the red lines show the images of the interval $\big[\frac{\sqrt{5}+2}{\sqrt{5}\varphi^2}, \frac{\sqrt{5}\varphi+1}{\sqrt{5}\varphi^2}\big) \simeq [0.72, 0.78)$, that is where the two transformations differ. Similarly, the transformations $L_{\sqrt{5}\varphi^2,\Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ L_{\boldsymbol{\beta}}^3 \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$ are depicted in Figure 13. As illustrated in red, the two transformations differ on the interval $\phi_{\sqrt{5}\varphi^2,\Delta_{\boldsymbol{\beta}}}\Big(\big[\frac{\sqrt{5}+2}{\sqrt{5}\varphi^2}, \frac{\sqrt{5}\varphi+1}{\sqrt{5}\varphi^2}\big)\Big) \simeq (0.82, 0.89]$.

Example 7.9. Consider the alternate base $\boldsymbol{\beta} = (\frac{3}{2}, \frac{3}{2}, 4)$. We have $\Delta_{\boldsymbol{\beta}} = [\![0, 13]\!]$. The map $f_{\boldsymbol{\beta}}$ is not non-decreasing since we have $f_{\boldsymbol{\beta}}(0, 1, 3) = 7$ and $f_{\boldsymbol{\beta}}(1, 0, 0) = 6$. However, $T_{9,\Delta_{\boldsymbol{\beta}}} = \pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ and $L_{9,\Delta_{\boldsymbol{\beta}}} = \pi_2 \circ L_{\boldsymbol{\beta}}^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$. The transformation $T_{9,\Delta_{\boldsymbol{\beta}}}$ is depicted in Figure 14.

The next example illustrates that it may happen that the transformations $T_{\beta_{p-1}\cdots\beta_0,\Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ indeed coincide on $[0, 1)$ but not on $[0, x_{\boldsymbol{\beta}})$.

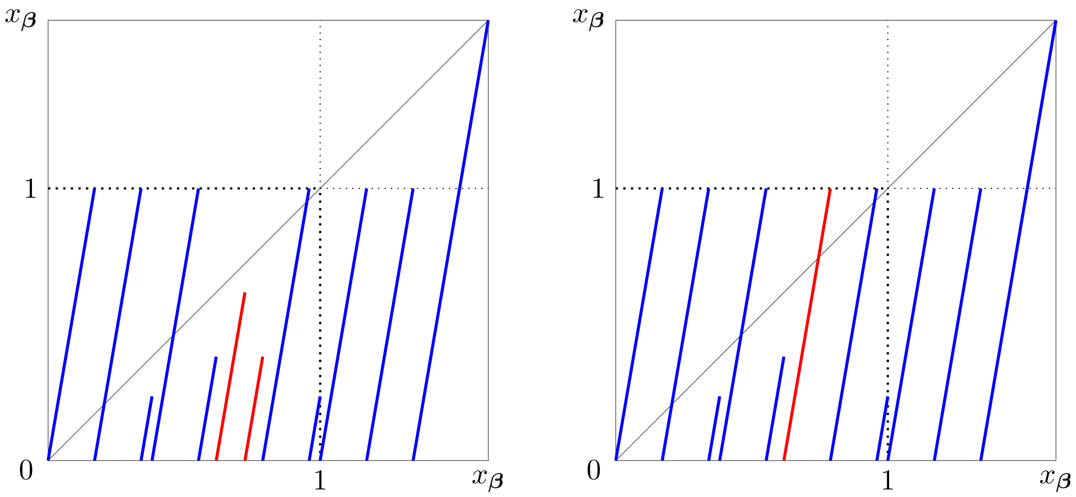

FIGURE 12. The transformations $T_{\sqrt{5}\varphi^2,\Delta_{\boldsymbol{\beta}}}$ (left) and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ (right) with $\boldsymbol{\beta} = (\varphi, \varphi, \sqrt{5})$.

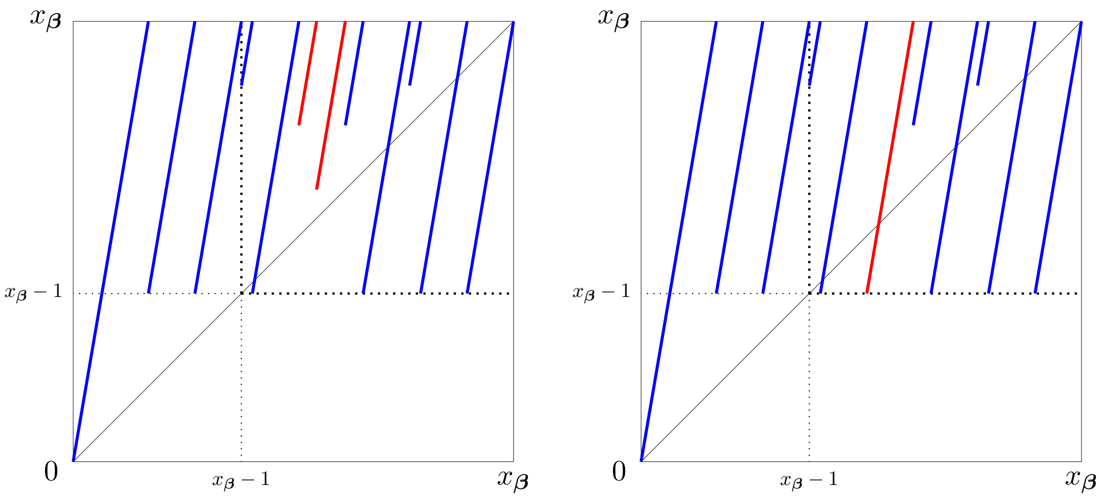

FIGURE 13. The transformations $L_{\sqrt{5}\varphi^2,\Delta_{\boldsymbol{\beta}}}$ (left) and $\pi_2 \circ L_{\boldsymbol{\beta}}^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ (right) with $\boldsymbol{\beta} = (\varphi, \varphi, \sqrt{5})$.

Example 7.10. Consider the alternate base $\boldsymbol{\beta} = (\frac{\sqrt{5}}{2}, \frac{\sqrt{6}}{2}, \frac{\sqrt{7}}{2})$. Then $f_{\boldsymbol{\beta}}(0,1,1) > f_{\boldsymbol{\beta}}(1,0,0)$ and it can be checked that the maps $T_{\frac{\sqrt{210}}{8},\Delta_{\boldsymbol{\beta}}}$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ differ on the interval $\big[\frac{f_{\boldsymbol{\beta}}(0,1,1)}{\beta_2\beta_1\beta_0}, \frac{f_{\boldsymbol{\beta}}(1,0,1)}{\beta_2\beta_1\beta_0}\big) \simeq [1.28, 1.44)$. However, the two maps coincide on $[0,1)$.

Finally, we provide a necessary and sufficient condition for the map $f_{\boldsymbol{\beta}}$ to be non-decreasing.

Proposition 7.11. *The map $f_{\boldsymbol{\beta}}$ is non-decreasing if and only if for all $j \in [\![1, p-2]\!]$,*

$$\sum_{i=j}^{p-1} \beta_{p-1} \cdots \beta_{i+1}(\lceil \beta_i \rceil - 1) \leq \beta_{p-1} \cdots \beta_j. \tag{15}$$

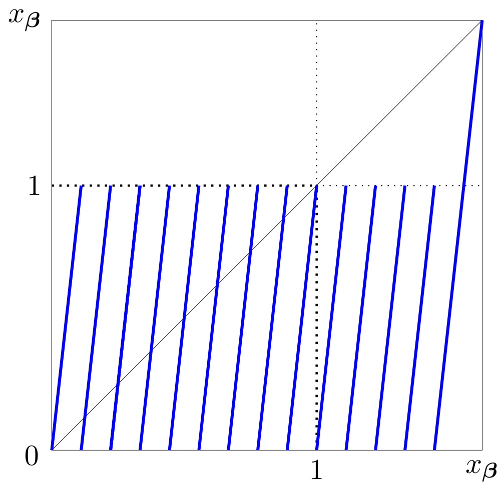

FIGURE 14. The transformations $T_{9,\Delta_{\boldsymbol{\beta}}}$ where $\boldsymbol{\beta}=(\frac{3}{2},\frac{3}{2},4)$.

Proof. If the map $f_{\boldsymbol{\beta}}$ is non-decreasing then for all $j\in[\![1,p-2]\!]$,

$$\begin{aligned}\sum_{i=j}^{p-1}\beta_{p-1}\cdots\beta_{i+1}(\lceil\beta_i\rceil-1)&=f_{\boldsymbol{\beta}}(0,\dots,0,0,\lceil\beta_j\rceil-1,\dots,\lceil\beta_{p-1}\rceil-1)\\&\le f_{\boldsymbol{\beta}}(0,\dots,0,1,0,\dots,0)\\&=\beta_{p-1}\cdots\beta_j.\end{aligned}$$

Conversely, suppose that (15) holds for all $j\in[\![1,p-2]\!]$ and that $(c_0,\dots,c_{p-1})$ and $(c'_0,\dots,c'_{p-1})$ are p-tuples in $\prod_{i=0}^{p-1}[\![0,\lceil\beta_i\rceil-1]\!]$ such that $(c_0,\dots,c_{p-1})<_{\text{lex}}(c'_0,\dots,c'_{p-1})$. Then there exists $j\in[\![0,p-1]\!]$ such that $c_0=c'_0,\dots,c_{j-1}=c'_{j-1}$ and $c_j\le c'_j-1$. We get

$$\begin{aligned}f_{\boldsymbol{\beta}}(c_0,\dots,c_{p-1})&\le\sum_{i=0}^{j}\beta_{p-1}\cdots\beta_{i+1}c'_i-\beta_{p-1}\cdots\beta_{j+1}+\sum_{i=j+1}^{p-1}\beta_{p-1}\cdots\beta_{i+1}(\lceil\beta_i\rceil-1)\\&\le\sum_{i=0}^{j}\beta_{p-1}\cdots\beta_{i+1}c'_i\\&\le f_{\boldsymbol{\beta}}(c'_0,\dots,c'_{p-1}).\end{aligned}$$

□

Corollary 7.12. *If $p=2$ then $T_{\beta_1\beta_0,\Delta_{\boldsymbol{\beta}}}=\pi_2\circ(T^{\text{ext}}_{\boldsymbol{\beta}})^2\circ\delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$. In particular, $T_{\beta_1\beta_0,\Delta_{\boldsymbol{\beta}}}\big|_{[0,1)}=T_{\beta_1}\circ T_{\beta_0}$.*

Proof. This follows from Theorem 7.6 and Proposition 7.11. □

Example 7.13. Consider once more the alternate base $\boldsymbol{\beta}=(\frac{1+\sqrt{13}}{2},\frac{5+\sqrt{13}}{6})$ from Example 3.1. Then $\Delta_{\boldsymbol{\beta}}=\{0,1,\beta_1,\beta_1+1,2\beta_1,2\beta_1+1\}$ and $x_{\boldsymbol{\beta}}=\frac{2\beta_1+1}{\beta_1\beta_0-1}=\frac{5+7\sqrt{13}}{18}$. The transformations $\pi_2\circ(T^{\text{ext}}_{\boldsymbol{\beta}})^2\circ\delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ and $\pi_2\circ L^2_{\boldsymbol{\beta}}\circ\delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$ are depicted in Figure 15. By Corollary 7.12, they coincides with $T_{\beta_1\beta_0,\Delta_{\boldsymbol{\beta}}}$ and $L_{\beta_1\beta_0,\Delta_{\boldsymbol{\beta}}}$ respectively.

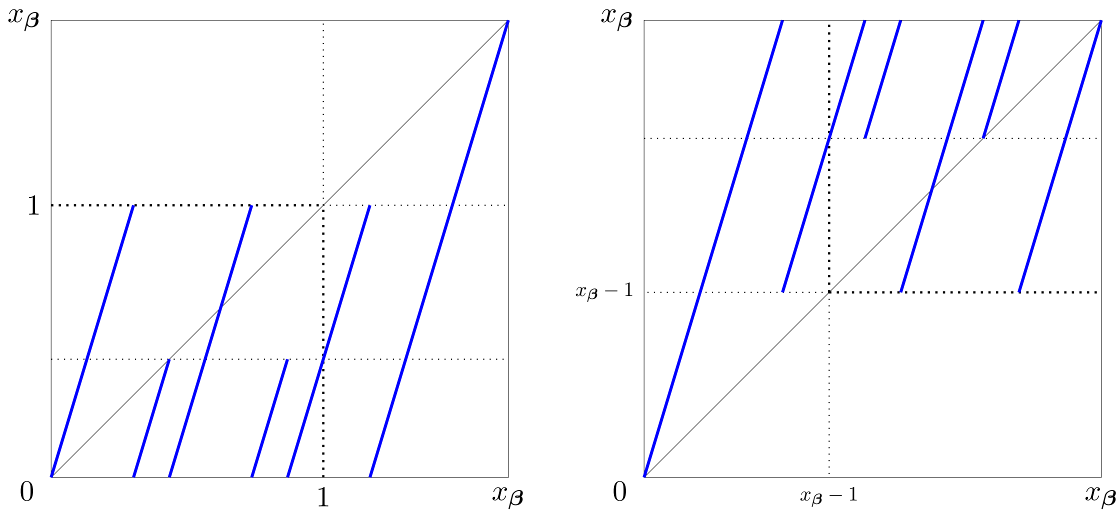

FIGURE 15. The transformations $\pi_2 \circ (T^{\text{ext}}_{\boldsymbol{\beta}})^2 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ (left) and $\pi_2 \circ L^2_{\boldsymbol{\beta}} \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$ (right) for $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$.

8. Further work

In this work, we concentrated on measure theoretical aspects of alternate base expansions. A natural question would be to consider the topological point of view. For example, it would be of interest to prove that the topological entropies of the topological dynamical systems under consideration coincide with the measure theoretical entropy $\frac{1}{p}\log(\beta_{p-1}\cdots\beta_0)$ found in this paper. In particular, this would prove that the measure theoretical dynamical systems studied in this paper are all of maximal entropy.

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