DYNAMICAL BEHAVIOR OF ALTERNATE BASE EXPANSIONS

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Abstract. We generalize the greedy and lazy \( \beta \)-transformations for a real base \( \beta \) to the setting of alternate bases \( \beta = (\beta_0, \ldots, \beta_{p-1}) \), which were recently introduced by the first and second authors as a particular case of Cantor bases. As in the real base case, these new transformations, denoted \( T_\beta \) and \( L_\beta \) respectively, can be iterated in order to generate the digits of the greedy and lazy \( \beta \)-expansions of real numbers. The aim of this paper is to describe the dynamical behaviors of \( T_\beta \) and \( L_\beta \). We first prove the existence of a unique absolutely continuous (with respect to an extended Lebesgue measure, called the \( p \)-Lebesgue measure) \( T_\beta \)-invariant measure. We then show that this unique measure is in fact equivalent to the \( p \)-Lebesgue measure and that the corresponding dynamical system is ergodic and has entropy \( \frac{1}{p} \log(\beta_{p-1} \cdots \beta_0) \). We then express the density of this measure and compute the frequencies of letters in the greedy \( \beta \)-expansions. We obtain the dynamical properties of \( L_\beta \) by showing that the lazy dynamical system is isomorphic to the greedy one. We also provide an isomorphism with a suitable extension of the \( \beta \)-shift. Finally, we show that the \( \beta \)-expansions can be seen as \( (\beta_{p-1} \cdots \beta_0) \)-representations over general digit sets and we compare both frameworks.

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1. Introduction

A representation of a nonnegative real number \( x \) in a real base \( \beta > 1 \) is an infinite sequence \( a_0a_1a_2\cdots \) of nonnegative integers such that \( x = \sum_{i=0}^{\infty} \frac{a_i}{\beta^i} \). Among all \( \beta \)-representations, the greedy and lazy ones play a special role. They can be generated by iterating the so-called greedy \( \beta \)-transformations \( T_\beta \) and lazy \( \beta \)-transformations \( L_\beta \) respectively. The dynamical properties of \( T_\beta \) and \( L_\beta \) are now well understood since the seminal works of Rényi [17] and Parry [15]; for example, see [9].

In a recent work, the first two authors introduced the notion of expansions of real numbers in a real Cantor base, that is, an infinite sequence of real bases \( \beta = (\beta_n)_{n \geq 0} \) satisfying \( \prod_{n=0}^{\infty} \beta_n = \infty \) [3]. In this initial work, the focus was on the combinatorial properties of these expansions. In particular, generalizations of several combinatorial results of real base

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expansions were obtained, such as Parry’s criterion for greedy $\beta$-expansions or Bertrand-Mathis characterization of sofic $\beta$-shifts. The latter result was obtained for the subclass of periodic Cantor bases, namely the alternate bases.

The aim of this paper is to study the dynamical behaviors of the greedy and lazy expansions in an alternate base $\beta = (\beta_0, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots)$. It is organized as follows. In Section 2, we provide the necessary background on measure theory and on expansions of real numbers in a real base. In Section 3, we introduce the greedy and lazy alternate base expansions and define the associated transformations $T_\beta$ and $L_\beta$. Section 4 is concerned with the dynamical properties of the greedy transformation. We first prove the existence of a unique absolutely continuous (with respect to an extended Lebesgue measure, called the $p$-Lebesgue measure) $T_\beta$-invariant measure and then prove that this measure is equivalent to the $p$-Lebesgue measure and that the corresponding dynamical system is ergodic. We then express the density of this measure and compute the frequencies of letters in the greedy $\beta$-expansions. In Section 5 and 6, we prove that the greedy dynamical system is isomorphic to the lazy one, as well as to a suitable extension of the $\beta$-shift. In Section 7, we show that the $\beta$-expansions can be seen as $(\beta_{p-1} \cdots \beta_0)$-representations over general digit sets and we compare both frameworks.

2. Preliminaries

2.1. Measure preserving dynamical systems. A probability space is a triplet $(X, \mathcal{F}, \mu)$ where $X$ is a set, $\mathcal{F}$ is a $\sigma$-algebra over $X$ and $\mu$ is a measure on $\mathcal{F}$ such that $\mu(X) = 1$. For a measurable transformation $T: X \to X$ and a measure $\mu$ on $\mathcal{F}$, the measure $\mu$ is $T$-invariant, or equivalently, the transformation $T: X \to X$ is measure preserving with respect to $\mu$, if for all $B \in \mathcal{F}$, $\mu(T^{-1}(B)) = \mu(B)$. A dynamical system is a quadruple $(X, \mathcal{F}, (X, \mathcal{F}, \mu, T))$ where $(X, \mathcal{F}, \mu)$ is a probability space and $T: X \to X$ is a measure preserving transformation with respect to $\mu$. A dynamical system $(X, \mathcal{F}, (X, \mathcal{F}, \mu, T))$ is ergodic if for all $B \in \mathcal{F}$, $T^{-1}(B) = B$ implies $\mu(B) \in \{0, 1\}$, and is exact if $\bigcap_{n=0}^{\infty} T^{-n}(B) \in \mathcal{F}$ only contains sets of measure 0 or 1. Clearly, any exact dynamical system is ergodic. Two dynamical systems $(X, \mathcal{F}_X, \mu_X, T_X)$ and $(Y, \mathcal{F}_Y, \mu_Y, T_Y)$ are (measure preservingly) isomorphic if there exists a $\mu_X$-a.e. injective measurable map $\psi: X \to Y$ such that $\mu_Y = \mu_X \circ \psi^{-1}$ and $\psi \circ T_X = T_Y \circ \psi \circ \mu_X$-a.e. For two measures $\mu$ and $\nu$ on the same $\sigma$-algebra $\mathcal{F}$, $\mu$ is absolutely continuous with respect to $\nu$ if for all $B \in \mathcal{F}$, $\nu(B) = 0$ implies $\mu(B) = 0$, and $\mu$ and $\nu$ are equivalent if they are absolutely continuous with respect to each other. In what follows, we will be concerned by the Borel $\sigma$-algebras $\mathcal{B}(A)$, where $A \subset \mathbb{R}$. In particular, a measure on $\mathcal{B}(A)$ is absolutely continuous if it is absolutely continuous with respect to the Lebesgue measure $\lambda$ restricted to $\mathcal{B}(A)$. The Radon-Nikodym theorem states that $\mu$ and $\nu$ are two probability measures such that $\mu$ is absolutely continuous with respect to $\nu$, then there exists a $\nu$-integrable map $f:\ X \to [0, +\infty)$ such that for all $B \in \mathcal{F}$, $\mu(B) = \int_B f \, d\nu$. Moreover, the map $f$ is $\nu$-a.e. unique. It is called the density of the measure $\mu$ with respect to $\nu$ and is usually denoted $\frac{d\mu}{d\nu}$.

For more details on measure theory and ergodic theory, we refer the reader to [2, 8, 11].

2.2. Real base expansions. Let $\beta$ be a real number greater than 1. A $\beta$-representation of a nonnegative real number $x$ is an infinite sequence $a_0 a_1 a_2 \cdots$ over $\mathbb{N}$ such that $x = \sum_{i=0}^{\infty} \frac{a_i}{\beta^i}$. For $x \in [0, 1)$, a particular $\beta$-representation of $x$, called the greedy $\beta$-expansion of $x$, is obtained by using the greedy algorithm. If the first $N$ digits of the $\beta$-expansion of $x$ are given by $a_0, \ldots, a_{N-1}$, then the next digit $a_N$ is the greatest integer in $[0, [\beta]^{-1}]$.
such that
\[ \sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} \leq x. \]
The greedy \( \beta \)-expansion can also be obtained by iterating the greedy \( \beta \)-transformation
\[ T_\beta : [0,1) \to [0,1) \text{, } x \mapsto \beta x - \lfloor \beta x \rfloor, \]
by setting \( a_n = \lfloor \beta T_\beta^n(x) \rfloor \) for all \( n \in \mathbb{N} \).

**Example 1.** In this example and throughout the paper, \( \varphi \) designates the golden ratio, i.e., \( \varphi = \frac{1+\sqrt{5}}{2} \). The transformation \( T_{\varphi^2} \) is depicted in Figure 1.

![Figure 1](image-url)

**Figure 1.** The transformation \( T_{\varphi^2} \).

Real base expansions have been studied through various points of view. We refer the reader to [14, Chapter 7] for a survey on their combinatorial properties and [8] for a survey on their dynamical properties. A fundamental dynamical result is the following. This summarizes results from [15, 17, 18].

**Theorem 2.** There exists a unique \( T_\beta \)-invariant absolutely continuous probability measure \( \mu_\beta \) on \( B([0,1)) \). Furthermore, the measure \( \mu_\beta \) is equivalent to the Lebesgue measure on \( B([0,1)) \) and the dynamical system \( ([0,1), B([0,1)), \mu_\beta, T_\beta) \) is ergodic and has entropy \( \log(\beta) \).

**Remark 3.** It follows from Theorem 2 that \( T_\beta \) is non-singular with respect to the Lebesgue measure, i.e., for all \( B \in B([0,1)) \), \( \lambda(B) = 0 \) if and only if \( \lambda(T_\beta^{-1}(B)) = 0 \).

In what follows, we let
\[ x_\beta = \frac{[\beta] - 1}{\beta - 1}. \]
This value corresponds to the greatest real number that has a \( \beta \)-representation over the alphabet \( [0, [\beta] - 1] \). Clearly, we have \( x_\beta \geq 1 \). The extended greedy \( \beta \)-transformation, still denoted \( T_\beta \), is defined in [9] as
\[ T_\beta : [0, x_\beta) \to [0, x_\beta) \text{, } x \mapsto \begin{cases} \beta x - [\beta x] & \text{if } x \in [0,1) \\ \beta x - ([\beta] - 1) & \text{if } x \in [1, x_\beta). \end{cases} \]
Note that for all \( x \in \left[ \frac{[\beta] - 1}{\beta}, \frac{[\beta]}{\beta} \right) \), the two cases of the definition coincide since \( [\beta x] = [\beta] - 1 \). The extended \( \beta \)-transformation restricted to the interval \([0,1)\) gives back the classical greedy \( \beta \)-transformation defined above. Moreover, for all \( x \in [0, x_\beta) \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( T_\beta^n(x) \in [0,1) \).
Figure 2. The extended transformation $T_{\varphi^2}$.

**Example 4.** We continue Example 1. The extended greedy transformation $T_{\varphi^2}$ is depicted in Figure 2.

In the greedy algorithm, each digit is chosen as the largest possible among $0, 1, \ldots, \lceil \beta \rceil - 1$ at the considered position. At the other extreme, the lazy algorithm picks the least possible digit at each step [10]: if the first $N$ digits of the expansion of a real number $x \in (0, x_\beta]$ are given by $a_0, \ldots, a_{N-1}$, then the next digit $a_N$ is the least element in $[0, \lceil \beta \rceil - 1]$ such that

$$
\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \sum_{n=N+1}^{\infty} \frac{\lceil \beta \rceil - 1}{\beta^{n+1}} \geq x,
$$
or equivalently,

$$
\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \frac{x_\beta}{\beta^{N+1}} \geq x.
$$

The so-obtained $\beta$-representation is called the lazy $\beta$-expansion of $x$. The lazy $\beta$-transformation dynamically generating the lazy $\beta$-expansion is the transformation $L_\beta$ defined as follows [8]:

$$
L_\beta: (0, x_\beta] \to (0, x_\beta], \quad x \mapsto \begin{cases} 
\beta x & \text{if } x \in (0, x_\beta - 1) \\
\beta x - \lceil \beta x - x_\beta \rceil & \text{if } x \in (x_\beta - 1, x_\beta]. 
\end{cases}
$$

Observe that for all $x \in \left(\frac{x_\beta-1}{\beta}, \frac{x_\beta}{\beta}\right]$, the two cases of the definition coincide since $\lceil \beta x - x_\beta \rceil = 0$. Moreover, since $L_\beta((x_\beta - 1, x_\beta]) = (x_\beta - 1, x_\beta]$, the lazy transformation $L_\beta$ can be restricted to the length-one interval $(x_\beta - 1, x_\beta]$. Also note that for all $x \in (0, x_\beta]$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $L^n_\beta(x) \in (x_\beta - 1, x_\beta]$. Furthermore, for all $x \in (x_\beta - 1, x_\beta]$ and $n \in \mathbb{N}$, we have $a_n = \lfloor \beta L^n_\beta(x) - x_\beta \rfloor$.

**Example 5.** The lazy transformation $L_{\varphi^2}$ is depicted in Figure 3.

It is proven in [9] that there is an isomorphism between the greedy and the lazy $\beta$-transformations. As a direct consequence of this property, an analogue of Theorem 2 is obtained for the lazy transformation restricted to the interval $(x_\beta - 1, x_\beta]$.
3. ALTERNATE BASE EXPANSIONS

Let \( p \) be a positive integer and \( \beta = (\beta_0, \ldots, \beta_{p-1}) \) be a \( p \)-tuple of real numbers greater than 1. Such a \( p \)-tuple \( \beta \) is called an alternate base and \( p \) is called its length. A \( \beta \)-representation of a nonnegative real number \( x \) is an infinite sequence \( a_0a_1a_2\cdots \) over \( \mathbb{N} \) such that

\[
x = \frac{a_0}{\beta_0} + \frac{a_1}{\beta_1\beta_0} + \cdots + \frac{a_{p-1}}{\beta_{p-1}\cdots\beta_0} + \frac{a_p}{\beta_0(\beta_{p-1}\cdots\beta_0)} + \cdots + \frac{a_{p+1}}{\beta_1\beta_0(\beta_{p-1}\cdots\beta_0)} + \cdots + \frac{a_{2p-1}}{(\beta_{p-1}\cdots\beta_0)^2} + \cdots
\]

We use the convention that for all \( n \in \mathbb{Z} \), \( \beta_n = \beta_{n \mod p} \) and \( \beta^{(n)} = (\beta_n, \ldots, \beta_{n+p-1}) \). Therefore, the equality (1) can be rewritten

\[
x = \sum_{n=0}^{+\infty} \frac{a_n}{\prod_{k=0}^{n-1} \beta_k}.
\]

The alternate bases are particular cases of Cantor real bases, which were introduced and studied in [3].

In this paper, our aim is to study the dynamics behind some distinguished representation in alternate bases, namely the greedy and lazy \( \beta \)-expansions. First, we recall the notion of greedy \( \beta \)-expansions defined in [3] and we introduce the greedy \( \beta \)-transformation dynamically generating the digits of the greedy \( \beta \)-expansions. Second, we introduce the notion of lazy \( \beta \)-expansions and the corresponding lazy \( \beta \)-transformation.

3.1. The greedy \( \beta \)-expansion. For \( x \in [0,1) \), a distinguished \( \beta \)-representation, called the greedy \( \beta \)-expansion of \( x \), is obtained from the greedy algorithm. If the first \( N \) digits of the greedy \( \beta \)-expansion of \( x \) are given by \( a_0, \ldots, a_{N-1} \), then the next digit \( a_N \) is the greatest integer in \([0, \lfloor \beta_N \rfloor - 1]\) such that

\[
\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n-1} \beta_k} \leq x.
\]

The greedy \( \beta \)-expansion can also be obtained by alternating the \( \beta_i \)-transformations: for all \( x \in [0,1) \) and \( n \in \mathbb{N} \), \( a_n = \lfloor \beta_n(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)) \rfloor \). The greedy \( \beta \)-expansion of \( x \) is
denoted $d_\beta(x)$. In particular, if $p = 1$ then it corresponds to the usual greedy $\beta$-expansion as defined in Section 2.2.

**Example 6.** Consider the alternate base $\beta = (\frac{1+\sqrt{13}}{2}, 5+\sqrt{13})$ already studied in [3]. The greedy $\beta$-expansions are obtained by alternating the transformations $T_{\frac{1+\sqrt{13}}{2}}$ and $T_{5+\sqrt{13}}$, which are both depicted in Figure 4. Moreover, in Figure 5 we see the computation of the first five digits of the greedy $\beta$-expansion of $\frac{1+\sqrt{5}}{8}$.

![Figure 4. The transformations $T_{\frac{1+\sqrt{13}}{2}}$ (blue) and $T_{5+\sqrt{13}}$ (green).](image)

**Figure 5.** The first five digits of the greedy $\beta$-expansion of $\frac{1+\sqrt{5}}{8}$ are 10102 for $\beta = (\frac{1+\sqrt{13}}{2}, 5+\sqrt{13})$.

We now define the **greedy $\beta$-transformation** by

$$T_\beta : [0, p-1] \times [0, 1) \to [0, p-1] \times [0, 1), \ (i, x) \mapsto ((i + 1) \mod p, T_\beta(i, x)).$$

The greedy $\beta$-transformation generates the digits of the greedy $\beta$-expansions as follows. For all $x \in [0, 1)$ and $n \in \mathbb{N}$, the digit $a_n$ of $d_\beta(x)$ is equal to $\lfloor \beta_n \pi_2(T_\beta(0, x)) \rfloor$ where

$$\pi_2 : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \ (n, x) \mapsto x.$$

As in Section 2.2, the greedy $\beta$-transformation can be extended to an interval of real numbers bigger than $[0, 1)$. To do so, we define

$$x_\beta = \sum_{n=0}^{\infty} \frac{\lfloor \beta_n \rfloor - 1}{\prod_{k=0}^{n} \beta_k}.$$  

It can be easily seen that $1 \leq x_\beta < \infty$. This value corresponds to the greatest real number that has a $\beta$-representation $a_0a_1a_2 \cdots$ such that each letter $a_n$ belongs to the alphabet.
Proposition 7. For all \(i\) for \(\beta_1\) \(\in\) \([0, [\beta_n] - 1]\). Moreover, for all \(n \in \mathbb{Z}\),

\[
x_{\beta^{(n)}} = \frac{x_{\beta^{(n+1)}} + [\beta_n] - 1}{\beta_n}.
\]

We define the extended greedy \(\beta\)-transformation, still denoted \(T_\beta\), by

\[
T_\beta: \bigcup_{i=0}^{p-1} \{i\} \times [0, x_{\beta^{(i)}}) \rightarrow \bigcup_{i=0}^{p-1} \{i\} \times [0, x_{\beta^{(i)}}),
\]

\[
(i, x) \mapsto \begin{cases} (i+1) \rem p, \beta_i x - \lceil \beta_i x \rceil & \text{if } x \in [0, 1) \\ (i+1) \rem p, \beta_i x - ((\beta_i) - 1) & \text{if } x \in [1, x_{\beta^{(i)}}). \end{cases}
\]

The greedy \(\beta\)-expansion of \(x \in [0, x_\beta)\) is obtained by alternating the \(p\) maps

\[
\pi_2 \circ T_\beta \circ \delta_i: [0, x_{\beta^{(i)}}) \rightarrow [0, x_{\beta^{(i+1)}})
\]

for \(i \in [0, p-1]\), where

\[
\delta_i: \mathbb{R} \rightarrow \{i\} \times \mathbb{R}, \; x \mapsto (i, x).
\]

Proposition 7. For all \(x \in [0, x_\beta)\) and \(n \in \mathbb{N}\), we have

\[
\pi_2 \circ T_\beta^n \circ \delta_0(x) = \beta_{n-1} \cdots \beta_0 x - \sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k
\]

where \((c_0, \ldots, c_{n-1})\) is the lexicographically greatest \(n\)-tuple in \(\prod_{k=0}^{n-1} [0, [\beta_k] - 1]\) such that

\[
\sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k \leq x.
\]

Proof. We proceed by induction on \(n\). The base case \(n = 0\) is immediate. Now, suppose that the result is satisfied for some \(n \in \mathbb{N}\). Let \(x \in [0, x_\beta)\). Let \((c_0, \ldots, c_{n-1})\) is the lexicographically greatest \(n\)-tuple in \(\prod_{k=0}^{n-1} [0, [\beta_k] - 1]\) such that

\[
\sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k \leq x.
\]

Then it is easily seen that for all \(m < n\), \((c_0, \ldots, c_{m})\) is the lexicographically greatest \((m+1)\)-tuple in \(\prod_{k=0}^{m} [0, [\beta_k] - 1]\) such that \(\sum_{k=0}^{m} \beta_{m-1} \cdots \beta_{k+1} c_k \leq x\). Now, set \(y = \pi_2 \circ T_\beta^n \circ \delta_0(x)\). Then \(y \in [0, x_{\beta^{(n)}})\) and by induction hypothesis, we obtain that \(y = \beta_{n-1} \cdots \beta_0 x - \sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k\). Then, by setting

\[
c_n = \begin{cases} \lfloor \beta_n y \rfloor & \text{if } y \in [0, 1) \\ \lfloor \beta_n \rfloor - 1 & \text{if } y \in [1, x_{\beta^{(n)}}) \end{cases}
\]

we obtain that \(\pi_2 \circ T_\beta^{n+1} \circ \delta_0(x) = \beta_n \cdots \beta_0 x - \sum_{k=0}^{n} \beta_{n-1} \cdots \beta_{k+1} c_k\). In order to conclude, we have to show that

a) \(\sum_{k=0}^{n} \beta_{n-1} \cdots \beta_{k+1} c_k \leq x\)

b) \((c_0, \ldots, c_n)\) is the lexicographically greatest \((n+1)\)-tuple in \(\prod_{k=0}^{n} [0, [\beta_k] - 1]\) such that a) holds.

By definition of \(c_n\), we have \(c_n \leq \beta_n y\). Therefore,

\[
\sum_{k=0}^{n} \beta_{n-1} \cdots \beta_{k+1} c_k = \beta_n \sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k + c_n = \beta_n (\beta_{n-1} \cdots \beta_0 x - y) + c_n \leq \beta_n \cdots \beta_0 x.
\]

This shows that a) holds.

Let us show b) by contradiction. Suppose that there exists \((c_0', \ldots, c_n') \in \prod_{k=0}^{n} [0, [\beta_k] - 1]\) such that \((c_0', \ldots, c_n') > \text{lex} (c_0, \ldots, c_n)\) and \(\sum_{k=0}^{n} \frac{\beta_{n-1} \cdots \beta_{k+1} c_k}{\beta_{n-1} \cdots \beta_0} \leq x\). Then there exists \(m \leq n\) such that \(c_0' = c_0, \ldots, c_m-1 = c_{m-1}\) and \(c_m' \geq c_m + 1\). We again consider two cases. First,
suppose that \( m < n \). Since \((c'_0, \ldots, c'_m) \succ_{\text{lex}} (c_0, \ldots, c_m)\), we get \( \sum_{k=0}^{m} \frac{\beta_{m+1} c'_k}{\beta_{n-1}} > x \). But then
\[
\sum_{k=0}^{n} \beta_n \cdot \beta_{k+1} c'_k \geq \beta_n \cdot \beta_{m+1} \sum_{k=0}^{m} \frac{\beta_{m+1} c'_k}{\beta_{n-1}} > \beta_n \cdot \beta_0 x,
\]
a contradiction. Second, suppose that \( m = n \). Then
\[
\beta_n \cdots \beta_0 x \geq \sum_{k=0}^{n} \beta_n \cdot \beta_{k+1} c'_k \geq \sum_{k=0}^{n-1} \beta_n \cdot \beta_{k+1} c_k + c_n + 1,
\]
hence \( \beta_n y \geq c_n + 1 \). If \( y \in [0, 1) \) then \( c_n + 1 = \lfloor \beta_n y \rfloor + 1 > \beta_n y \), a contradiction. Otherwise, \( y \in [1, x_{\beta(\pi)}) \) and \( c_n + 1 = \lfloor \beta_n \rfloor \). But then \( c'_n \geq \lfloor \beta_n \rfloor \), which is impossible since \( c'_n \in [0, \lfloor \beta_n \rfloor - 1] \). This shows b) and ends the proof.

The restriction of the extended greedy \( \beta \)-transformation to the domain \([0, p-1] \times [0, 1)\) gives back the greedy \( \beta \)-transformation initially defined in 2. Moreover, for all \((i, x) \in \bigcup_{i=0}^{p-1} ([i] \times [0, x_{\beta(i)})\), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( T_{\beta}^n(i, x) \in [0, p-1] \times [0, 1) \).

**Example 8.** Let \( \beta = \left( \frac{1 + \sqrt{13}}{2}, \frac{5 + \sqrt{13}}{6} \right) \) be the alternate base of Example 6. The maps \( \pi_2 \circ T_{\beta} \circ \delta_0 \mid_{[0, x_{\beta})} : [0, x_{\beta}) \to [0, x_{\beta(\pi)}) \) and \( \pi_2 \circ T_{\beta} \circ \delta_1 \mid_{[0, x_{\beta(\pi)})} : [0, x_{\beta(\pi)}) \to [0, x_{\beta}) \) are depicted in Figure 6.

**Figure 6.** The maps \( \pi_2 \circ T_{\beta} \circ \delta_0 \mid_{[0, x_{\beta})} \) (blue) and \( \pi_2 \circ T_{\beta} \circ \delta_1 \mid_{[0, x_{\beta(\pi)})} \) (green) with \( \beta = \left( \frac{1 + \sqrt{13}}{2}, \frac{5 + \sqrt{13}}{6} \right) \).

### 3.2. The lazy \( \beta \)-expansion

As in the real base case, in the greedy \( \beta \)-expansion, each digit is chosen as the largest possible at the considered position. Here, we define and study the other extreme \( \beta \)-representation, called the lazy \( \beta \)-expansion, taking the least possible digit at each step. For \( x \in [0, x_{\beta}) \), if the first \( N \) digits of the lazy \( \beta \)-expansion of \( x \) are given by \( a_0, \ldots, a_{N-1} \), then the next digit \( a_N \) is the least element in \([0, \lfloor \beta_N \rfloor - 1]\) such that
\[
\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} + \sum_{n=N+1}^{\infty} \frac{[\beta_n] - 1}{\prod_{k=0}^{n} \beta_k} \geq x,
\]
or equivalently,
\[ \sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} + \frac{x^{\beta(n)}}{\prod_{k=0}^{n} \beta_k} \geq x. \]

This algorithm is called the lazy algorithm. For all \( N \in \mathbb{N} \), we have
\[ \sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} \leq x, \]
which implies that the lazy algorithm converges, that is,
\[ x = \sum_{n=0}^{\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k}. \]

We now define the lazy \( \beta \)-transformation by
\[ L_{\beta} : \bigcup_{i=0}^{p-1} \{(i) \times (0, x_{\beta(i)})\} \rightarrow \bigcup_{i=0}^{p-1} \{(i) \times (0, x_{\beta(i)})\}, \]
\[ (i, x) \mapsto \begin{cases} ((i+1) \mod p, \beta_i x) & \text{if } x \in (0, x_{\beta(i)} - 1] \\ ((i+1) \mod p, \beta_i x - [\beta_i x - x_{\beta(i+1)}]) & \text{if } x \in (x_{\beta(i)} - 1, x_{\beta(i)}]. \end{cases} \]

The lazy \( \beta \)-expansion of \( x \in (0, x_{\beta}) \) is obtained by alternating the \( p \) maps
\[ \pi_2 \circ L_{\beta} \circ \delta_1 |_{(0, x_{\beta(i)})} : (0, x_{\beta(i)}) \rightarrow (0, x_{\beta(i+1)}) \]
for \( i \in [0, p-1] \). The following proposition is the analogue of Proposition 7 for the lazy \( \beta \)-transformation, which can be proved in a similar fashion.

**Proposition 9.** For all \( x \in (0, x_{\beta}) \) and \( n \in \mathbb{N} \), we have
\[ \pi_2 \circ L_{\beta}^n \circ \delta_0 (x) = \beta_{n-1} \cdots \beta_0 x - \sum_{i=0}^{n-1} \beta_{n-1} \cdots \beta_{i+1} c_i \]
where \((c_0, \ldots, c_{n-1})\) is the lexicographically least \( n \)-tuple in \( \prod_{k=0}^{n-1} [0, [\beta_k] - 1] \) such that
\[ \frac{\sum_{i=0}^{n-1} \beta_{n-1} \cdots \beta_{i+1} c_i}{\beta_{n-1} \cdots \beta_0} + \sum_{m=n}^{\infty} \frac{[\beta_m]^{-1}}{\prod_{k=m}^{\infty} \beta_k} \geq x. \]

Note that for each \( i \in [0, p-1] \),
\[ L_{\beta} (\{i\} \times (x_{\beta(i)} - 1, x_{\beta(i)}]) \subseteq \{(i+1) \mod p\} \times (x_{\beta(i+1)} - 1, x_{\beta(i+1)}]. \]

Therefore, the lazy \( \beta \)-transformation can be restricted to the domain \( \bigcup_{i=0}^{p-1} \{(i) \times (x_{\beta(i)} - 1, x_{\beta(i)}]\). The (restricted) lazy \( \beta \)-transformation generates the digits of the lazy \( \beta \)-expansions of real numbers in the interval \((x_{\beta} - 1, x_{\beta})\) as follows. For all \( x \in (x_{\beta} - 1, x_{\beta}) \) and \( n \in \mathbb{N} \), the digit \( a_n \) in the lazy \( \beta \)-expansion of \( x \) is equal to \([\beta_n \pi_2(L_{\beta}^n(0, x)) - x_{\beta(n+1)}] \).

Finally, observe that for all \((i, x)\in \bigcup_{i=0}^{p-1} \{(i) \times (x_{\beta(i)}), \) there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( L_{\beta}^n (i, x) \in \bigcup_{i=0}^{p-1} \{(i) \times (x_{\beta(i)} - 1, x_{\beta(i)}]. \)

**Example 10.** Consider again the length-2 alternate base \( \beta = (1+\sqrt{13}, \frac{5+\sqrt{13}}{18}) \) from Examples 6 and 8. We have \( x_{\beta} = \frac{5+7\sqrt{13}}{18} \approx 1.67 \) and \( x_{\beta(i)} = \frac{2+4\sqrt{13}}{3} \approx 1.86 \). The maps \( \pi_2 \circ L_{\beta} \circ \delta_0 |_{(0, x_{\beta})} : (0, x_{\beta}) \rightarrow (0, x_{\beta(i)}) \) and \( \pi_2 \circ L_{\beta} \circ \delta_1 |_{(0, x_{\beta(i)})} : (0, x_{\beta(i)}) \rightarrow (0, x_{\beta}) \) are depicted in Figure 7. In Figure 8 we see the computation of the first five digits of the lazy \( \beta \)-expansion of \( \frac{1+\sqrt{13}}{8} \).
Figure 7. The maps \( \pi_2 \circ L_\beta \circ \delta_0 \big|_{[0,x_\beta]} \) (blue) and \( \pi_2 \circ L_\beta \circ \delta_1 \big|_{[0,x_\beta(1)]} \) (green) with \( \beta = \left( \frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6} \right) \).

Figure 8. The first five digits of the lazy \( \beta \)-expansion of \( \frac{1+\sqrt{5}}{2} \) are 01112 for \( \beta = \left( \frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6} \right) \).

3.3. A note on Cantor bases. The greedy algorithm described in Sections 3.1 and 3.2 is well defined in the extended context of Cantor bases, i.e., sequences of real numbers \( \beta = (\beta_n)_{n \in \mathbb{N}} \) greater than 1 such that the product \( \prod_{n=0}^{\infty} \beta_n \) is infinite [3]. In this case, the greedy algorithm converge on \([0,1)\): for all \( x \in [0,1) \), the computed digits \( a_n \) are such that \( \sum_{n=0}^{\infty} a_n \beta_n = x \). Therefore, the value \( x_\beta \) defined as in (3) is greater than or equal to 1. However, it might be that \( x_\beta = \infty \). For example, it is the case for the Cantor base given by \( \beta_n = 1 + \frac{1}{n+1} \) for all \( n \in \mathbb{N} \).

Note that the restriction of the transformation \( \pi_2 \circ T_\beta^n \circ \delta_0 \) to the unit interval \([0,1)\) coincide with the composition \( T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0} \). Thus, when restricted to \([0,1)\), Proposition 7 can be reformulated as follows.

Proposition 11. For all \( x \in [0,1) \) and \( n \in \mathbb{N} \), we have

\[
T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x) = \beta_{n-1} \cdots \beta_0 x - \sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_{k+1} c_k
\]
where \((c_0, \ldots, c_{n-1})\) is the lexicographically greatest n-tuple in \(\prod_{k=0}^{n-1} [0, [\beta_k] - 1]\) such that
\[
\sum_{k=0}^{n-1} \beta_{n-1} \cdots \beta_0 c_k \leq x.
\]

For all \(k \in [0, n-1]\), the transformation \(L_{\beta_k}\) is defined on \((0, x_{\beta_k})\) and can be restricted to \((x_{\beta_k} - 1, x_{\beta_k})\). So, the restricted transformations \(L_{\beta_0}, \ldots, L_{\beta_{n-1}}\) cannot be composed to one another in general. Therefore, even if the lazy algorithm can be defined for Cantor bases, provided that \(x_\beta < \infty\), we cannot state an analogue of Proposition 11 in terms of the lazy transformations for Cantor bases.

Even though this paper is mostly concerned with alternate bases, let us emphasize that some results are indeed valid for any sequence \((\beta_n)_{n \in \mathbb{N}} \in (\mathbb{R}_{>1})^\mathbb{N}\), and hence for any Cantor base. This is the case of Proposition 11, Theorem 14, Corollary 15 and Proposition 25.

4. Dynamical properties of \(T_\beta\)

In this section, we study the dynamics of the greedy \(\beta\)-transformations. First, we generalize Theorem 2 to the transformation \(T_\beta\) on \([0, p - 1] \times [0, 1)\). Second, we extend the obtained result to the extended transformation \(T_\beta\). Third, we provide a formula for the densities of the measures found in the first two parts. Finally, we compute the frequencies of the digits in the greedy \(\beta\)-expansions.

4.1. Unique absolutely continuous \(T_\beta\)-invariant measure.

In order to generalize Theorem 2 to alternate bases, we start by recalling a result of Lasota and Yorke.

**Theorem 12.** [13, Theorem 4] Let \(T : [0, 1) \to [0, 1)\) be a transformation for which there exists a partition \([a_0, a_1], \ldots, [a_{K-1}, a_K]\) of the interval \([0, 1)\) with \(a_0 < \cdots < a_K\) such that for each \(k \in [0, K - 1]\), \(T([a_k, a_{k+1})] \) is convex, \(T(a_k) = 0\), \(T'(a_k) > 0\) and \(T'(0) > 1\). Then there exists a unique \(T\)-invariant absolutely continuous probability measure. Furthermore, its density is bounded and decreasing, and the corresponding dynamical system is exact.

We then prove a stability lemma.

**Lemma 13.** Let \(I\) be the family of transformations \(T : [0, 1) \to [0, 1)\) for which there exist a partition \([a_0, a_1], \ldots, [a_{K-1}, a_K]\) of the interval \([0, 1)\) with \(a_0 < \cdots < a_K\) and a slope \(s > 1\) such that for all \(k \in [0, K - 1]\), \(a_{k+1} - a_k \leq \frac{1}{s}\) and for all \(x \in (a_k, a_{k+1})\), \(T(x) = s(x - a_k)\). Then \(I\) is closed under composition.

**Proof.** Let \(S, T \in I\). Let \([a_0, a_1], \ldots, [a_{K-1}, a_K]\) and \([b_0, b_1], \ldots, [b_{L-1}, b_L]\) be partitions of the interval \([0, 1)\) with \(a_0 < \cdots < a_K\), \(b_0 < \cdots < b_L\), and let \(s, t > 1\) such that for all \(k \in [0, K - 1]\), \(a_{k+1} - a_k \leq \frac{1}{s}\), for all \(\ell \in [0, L - 1]\), \(b_{\ell+1} - b_\ell \leq \frac{1}{t}\) and for all \(x \in [0, 1), S(x) = s(x - a_k)\) if \(x \in (a_k, a_{k+1})\) and \(T(x) = t(x - b_\ell)\) if \(x \in (b_\ell, b_{\ell+1})\). For each \(k \in [0, K - 1]\), define \(L_k\) to be the greatest \(\ell \in [0, L - 1]\) such that \(a_k + \frac{b_\ell}{s} < a_{k+1}\). Consider the partition

\[
[a_0 + \frac{b_0}{s}, a_0 + \frac{b_1}{s}), \ldots, [a_0 + \frac{b_{L_0-1}}{s}, a_0 + \frac{b_{L_0}}{s}), [a_0 + \frac{b_{L_0}}{s}, a_1) \]
\]

\[
[a_{K-1} + \frac{b_0}{s}, a_{K-1} + \frac{b_1}{s}), \ldots, [a_{K-1} + \frac{b_{L_{K-1}-1}}{s}, a_{K-1} + \frac{b_{L_{K-1}}}{s}), [a_{K-1} + \frac{b_{L_{K-1}}}{s}, a_K) \]
\]

of the interval \([0, 1)\). For each \(k \in [0, K - 1]\) and \(\ell \in [0, L_k - 1]\), \(a_k + \frac{b_{\ell+1}}{s} - a_k - \frac{b_\ell}{s} \leq \frac{1}{ts}\) and \(a_{k+1} - a_k - \frac{b_{L_k}}{s} = (a_{k+1} - a_k) - \frac{b_{L_k}}{s} \leq \frac{1}{ts}\). Now, let \(x \in [0, 1)\) and \(k \in [0, K - 1]\) be such that \(x \in (a_k, a_{k+1})\). Then \(S(x) = s(x - a_k) \in [0, 1)\). We distinguish two cases:
either there exists $\ell \in [0, L_k - 1]$ such that $x \in [a_k + \frac{b_{k+1}}{s}, a_k + \frac{b_{k+1}}{s}]$, or $x \in [a_k + \frac{b_{k+1}}{s}, a_{k+1}]$.

In the former case, $S(x) \in [b_k, b_{k+1}]$ and $T \circ S(x) = \ell \circ (S(x) - b_k) = \ell s(x - (a_k + \frac{b_{k+1}}{s}))$. In the latter case, since $a_{k+1} - a_k \leq \frac{b_{k+1}}{s}$, we get that $S(x) \in [b_{L_k}, b_{L_k+1})$ and hence that $T \circ S(x) = \ell \circ (S(x) - b_{L_k}) = \ell s(x - (a_k + \frac{b_{L_k}}{s}))$. This shows that the composition $T \circ S$ belongs to $I$. \hfill $\square$

The following theorem provides us with the main tool for the construction of a $T_{\beta}$-invariant measure.

**Theorem 14.** For all $n \in \mathbb{N}_{\geq 1}$ and all $\beta_0, \ldots, \beta_{n-1} > 1$, there exists a unique $(T_{\beta_0} \circ \cdots \circ T_{\beta_n})$-invariant absolutely continuous probability measure $\mu$ on $\mathcal{B}([0,1])$. Furthermore, the measure $\mu$ is equivalent to the Lebesgue measure on $\mathcal{B}([0,1])$, its density is bounded and decreasing, and the dynamical system $(\{0,1\}, \mathcal{B}([0,1]), \mu, T_{\beta_0} \circ \cdots \circ T_{\beta_n})$ is exact and has entropy $\log(\beta_{n-1} \cdots \beta_0)$.

**Proof.** The existence of a unique $(T_{\beta_0} \circ \cdots \circ T_{\beta_n})$-invariant absolutely continuous probability measure $\mu$ on $\mathcal{B}([0,1])$, the fact that its density is bounded and decreasing, and the exactness of the corresponding dynamical system follow from Theorem 12 and Lemma 13. With a similar argument as in [6], we can conclude that $\frac{d\mu}{dx} > 0 \ \lambda$-a.e. on $[0,1]$. It follows that $\mu$ is equivalent to the Lebesgue measure on $\mathcal{B}([0,1])$. Moreover, the entropy equals $\log(\beta_{n-1} \cdots \beta_0)$ since $T_{\beta_{n-1}} \cdots \circ T_{\beta_0}$ is a piecewise linear transformation of constant slope $\beta_{n-1} \cdots \beta_0$ \hfill $\square$

The following consequence of Theorem 14 will be useful for proving our generalization of Theorem 2.

**Corollary 15.** Let $n \in \mathbb{N}_{\geq 1}$ and $\beta_0, \ldots, \beta_{n-1} > 1$. Then for all $B \in \mathcal{B}([0,1])$ such that $(T_{\beta_0} \circ \cdots \circ T_{\beta_n})^{-1}(B) = B$, we have $\lambda(B) \in \{0,1\}$.

For each $i \in [0, p - 1]$, we let $\mu_{\beta,i}$ denote the unique $(T_{\beta_1} \circ \cdots \circ T_{\beta_p})$-invariant absolutely continuous probability measure given by Theorem 14. We use the convention that for all $n \in \mathbb{Z}$, $\mu_{\beta,n} = \mu_{\beta,n \mod p}$. Let us define a probability measure $\mu_{\beta}$ on the $\sigma$-algebra

$$T_p = \left\{ \bigcup_{i=0}^{p-1} (i \times B_i) : \forall i \in [0, p - 1], \ B_i \in \mathcal{B}([0,1]) \right\}$$

over $[0, p - 1] \times [0,1)$ as follows. For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1])$, we set

$$\mu_{\beta} \left( \bigcup_{i=0}^{p-1} (i \times B_i) \right) = \frac{1}{p} \sum_{i=0}^{p-1} \mu_{\beta,i}(B_i).$$

We now study the properties of the probability measure $\mu_{\beta}$.

**Lemma 16.** For $i \in [0, p - 1]$, we have $\mu_{\beta,i} = \mu_{\beta,i-1} \circ T_{\beta_i}^{-1}$.

**Proof.** Let $i \in [0, p - 1]$. By definition of $\mu_{\beta,i}$, it suffices to show that $\mu_{\beta,i-1} \circ T_{\beta_i}^{-1}$ is a $(T_{\beta_1} \circ \cdots \circ T_{\beta_p})$-invariant absolutely continuous probability measure on $\mathcal{B}([0,1])$. First, we have $\mu_{\beta,i-1}(T_{\beta_i}^{-1}([0,1])) = \mu_{\beta,i-1}(0,1)) = 1$. Second, for all $B \in \mathcal{B}([0,1])$, we have

$$\mu_{\beta,i-1} \circ T_{\beta_i}^{-1}(T_{\beta_1} \circ \cdots \circ T_{\beta_p})^{-1}(B)) = \mu_{\beta,i-1}((T_{\beta_1} \circ \cdots \circ T_{\beta_p} \circ T_{\beta_{i-1}})^{-1}(B)) = \mu_{\beta,i-1}(T_{\beta_1} \circ \cdots \circ T_{\beta_{i-1}})^{-1}(T_{\beta_i}^{-1}(B)) = \mu_{\beta,i-1}(T_{\beta_i}^{-1}(B)).$$
Third, for all $B \in \mathcal{B}([0, 1))$ such that $\lambda(B) = 0$, we get that $\lambda(T_{\beta_{i-1}}^{-1}(B)) = 0$ by Remark 3, and hence that $\mu_{\beta,i-1}(T_{\beta_{i-1}}^{-1}(B)) = 0$ since $\mu_{\beta,i-1}$ is absolutely continuous. \hfill \Box

**Proposition 17.** The measure $\mu_\beta$ is $T_\beta$-invariant.

**Proof.** For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0, 1))$,

$$\mu_\beta \left( T_\beta^{-1} \left( \bigcup_{i=0}^{p-1} \{i\} \times B_i \right) \right) = \mu_\beta \left( \bigcup_{i=0}^{p-1} T_\beta^{-1} \{i\} \times B_i \right)$$

$$= \mu_\beta \left( \bigcup_{i=0}^{p-1} \{(i-1) \mod p\} \times T_{\beta_{i-1}}^{-1}(B_i) \right)$$

$$= \frac{1}{p} \sum_{i=0}^{p-1} \mu_{\beta,i}(B_i)$$

$$= \frac{1}{p} \sum_{i=0}^{p-1} \mu_{\beta,i-1}(T_{\beta_{i-1}}^{-1}(B_i))$$

$$= \mu_\beta \left( \bigcup_{i=0}^{p-1} \{i\} \times B_i \right)$$

where we applied Lemma 16 for the fourth equality. \hfill \Box

**Corollary 18.** The quadruple $([0, p - 1] \times [0, 1), T_p, \mu_\beta, T_\beta)$ is a dynamical system.

Let us define a new measure $\lambda_p$ over the $\sigma$-algebra $T_p$. For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0, 1))$, we set

$$\lambda_p \left( \bigcup_{i=0}^{p-1} \{i\} \times B_i \right) = \frac{1}{p} \sum_{i=0}^{p-1} \lambda(B_i).$$

We call this measure the $p$-Lebesgue measure on $T_p$.

**Proposition 19.** The measure $\mu_\beta$ is equivalent to the $p$-Lebesgue measure on $T_p$.

**Proof.** This follows from the fact that the $p$ measures $\mu_{\beta,0}, \ldots, \mu_{\beta,p-1}$ are equivalent to the Lebesgue measure $\lambda$ on $\mathcal{B}([0, 1))$. \hfill \Box

Next, we compute the entropy of the dynamical system $([0, p - 1] \times [0, 1), T_p, \mu_\beta, T_\beta)$. To do so, we consider the $p$ induced transformations

$$T_{\beta,i}: \{i\} \times [0, 1) \to \{i\} \times [0, 1), \quad (i, x) \mapsto T_p^n(i, x)$$

for $i \in [0, p - 1]$. Note that indeed, for all $(i, x) \in [0, p - 1] \times [0, 1)$, the first return of $(i, x)$ to $\{i\} \times [0, 1)$ is equal to $p$. Thus $T_{\beta,i} = T_{\beta,i}^\frac{1}{p}$ for all $i \in [0, p - 1]$. As is well known [7], for each $i \in [0, p - 1]$, the induced transformation $T_{\beta,i}$ is measure preserving with respect to the measure $\nu_{\beta,i}$ on the $\sigma$-algebra $\{\{i\} \times B : B \in \mathcal{B}([0, 1))\}$ defined as follows: for all $B \in \mathcal{B}([0, 1))$,

$$\nu_{\beta,i}(\{i\} \times B) = p \mu_\beta(\{i\} \times B).$$

**Lemma 20.** For every $i \in [0, p - 1]$, the map $\delta_i|_{[0, 1)}: [0, 1) \to \{i\} \times [0, 1)$, $x \mapsto (i, x)$ defines an isomorphism between the dynamical systems

$$([0, 1), \mathcal{B}([0, 1)), \mu_{\beta,i}, T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{1-p}})$$

and

$$([0, 1), \mathcal{B}([0, 1)), \mu_{\beta,i}, T_{\beta,i})$$
Theorem 23. The measure \( \mu_\beta \) is the unique \( T_\beta \)-invariant probability measure on \( T_p \) that is absolutely continuous with respect to \( \lambda_p \). Furthermore, \( \mu_\beta \) is equivalent to \( \lambda_p \) on \( T_p \) and the dynamical system \([0, p-1] \times [0, 1), T_p, \mu_\beta, T_\beta\) is ergodic and has entropy \( \frac{1}{p} \log(\beta_{p-1} \cdots \beta_0) \).

Proof. By Propositions 17 and 19, \( \mu_\beta \) is a \( T_\beta \)-invariant probability measure that is absolutely continuous with respect to \( \lambda_p \) on \( B([0, 1)) \). Then we get from Proposition 22 that for all \( A \in T_p \) such that \( T_\beta^{-1}(A) = A \), we have \( \mu_\beta(A) \in \{0, 1\} \). Therefore, the dynamical system \([0, p-1] \times [0, 1), T_p, \mu_\beta, T_\beta\) is ergodic. Now, we obtain that the measure \( \mu_\beta \) is unique as a well-known consequence of the Ergodic Theorem, see [7, Theorem 3.1.2]. The equivalence between \( \mu_\beta \) and \( \lambda_p \) and the entropy of the system were already obtained in Propositions 19 and 21.

For \( p \) greater than 1, the dynamical system \([0, p-1] \times [0, 1), T_p, \mu_\beta, T_\beta\) is not exact even though for all \( i \in [0, p-1] \), the dynamical systems \((0, 1], B([0, 1)), \mu_\beta,i,T_\beta^{-i} \cdots T_\beta^{-p}\) are exact. It suffices to note that the dynamical system \([0, p-1] \times [0, 1), T_p, \mu_\beta, T_\beta^p\) is not ergodic for \( p > 1 \). Indeed, \( T_\beta^{-p}(\{0\} \times [0, 1)) = \{0\} \times [0, 1) \) whereas \( \mu_\beta(\{0\} \times [0, 1)) = \frac{1}{p} \).
4.2. Extended measure. In order to study the dynamics of the extended greedy $\beta$-transformation, we extend the definitions of the measures $\mu_\beta$ and $\lambda_\beta$. First, we define a new $\sigma$-algebra $\mathcal{T}_\beta$ on $\bigcup_{i=0}^{p-1}(\{i\} \times [0,x_{\beta(i)})]$ as follows:

$$
\mathcal{T}_\beta = \left\{ \bigcup_{i=0}^{p-1}(\{i\} \times B_i) : \forall i \in [0,p-1], B_i \in \mathcal{B}([0,x_{\beta(i)})] \right\}.
$$

Second, we extend the domain of the measures $\mu_\beta$ and $\lambda_\beta$ to $\mathcal{T}_\beta$ (while keeping the same notation) as follows. For $A \in \mathcal{T}_\beta$, we set $\mu_\beta(A) = \mu_\beta(A \cap ([0,p-1] \times [0,1]))$ and $\lambda_\beta(A) = \lambda_\beta(A \cap ([0,p-1] \times [0,1]))$.

**Theorem 24.** The measure $\mu_\beta$ is the unique $\mathcal{T}_\beta$-invariant probability measure on $\mathcal{T}_\beta$ that is absolutely continuous with respect to $\lambda_\beta$. Furthermore, $\mu_\beta$ is equivalent to $\lambda_\beta$ on $\mathcal{T}_\beta$ and the dynamical system $(\bigcup_{i=0}^{p-1}(\{i\} \times [0,x_{\beta(i)}]), \mathcal{T}_\beta, \mu_\beta, T_\beta)$ is ergodic and has entropy $\frac{1}{p} \log(\beta_{p-1} \cdots \beta_0)$.

**Proof.** Clearly, $\mu_\beta$ is a probability measure on $\mathcal{T}_\beta$. For all $A \in \mathcal{T}_\beta$, we have

$$
\mu_\beta(T_\beta^{-1}(A)) = \mu_\beta(T_\beta^{-1}(A) \cap ([0,p-1] \times [0,1]))
$$

$$
= \mu_\beta(T_\beta^{-1}(A \cap ([0,p-1] \times [0,1])) \cap ([0,p-1] \times [0,1]))
$$

$$
= \mu_\beta(T_\beta^{-1}(A \cap ([0,p-1] \times [0,1])))
$$

$$
= \mu_\beta(A \cap ([0,p-1] \times [0,1]))
$$

$$
= \mu_\beta(A)
$$

where we used Proposition 17 for the fourth equality. This shows that $\mu_\beta$ is $T_\beta$-invariant on $\mathcal{T}_\beta$. The conclusion then follows from the fact that the identity map from $[0,p-1] \times [0,1)$ to $\bigcup_{i=0}^{p-1}(\{i\} \times [0,x_{\beta(i)}])$ defines an isomorphism between the dynamical systems $(\bigcup_{i=0}^{p-1}(\{i\} \times [0,x_{\beta(i)}]), \mathcal{T}_\beta, \mu_\beta, T_\beta)$ and $(\bigcup_{i=0}^{p-1}(\{i\} \times [0,x_{\beta(i)}]), \mathcal{T}_\beta, \mu_\beta, T_\beta)$. \qed

4.3. Densities. In the next proposition, we express the density of the unique measure given in Theorem 14.

**Proposition 25.** Consider $n \in \mathbb{N}_{\geq 1}$ and $\beta_0, \ldots, \beta_{n-1} > 1$. Suppose that

- $K$ is the number of not onto branches of $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}$
- for $j \in [1,K]$, $c_j$ is the right-hand side endpoint of the domain of the $j$-th not onto branch of $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}$
- $T : [0,1) \to [0,1)$ is the transformation defined by $T(x) = T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)$ for $x \notin \{c_1, \ldots, c_K\}$ and $T(c_j) = \lim_{x \to c_j^-} T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)$ for $j \in [1,K]$
- $S$ is the matrix defined by $S = (S_{i,j})_{1 \leq i,j \leq K}$ where

$$
S_{i,j} = \sum_{m=1}^{\infty} \frac{\delta(T^m(c_i) > c_j)}{(\beta_{n-1} \cdots \beta_0)^m},
$$

where $\delta(P)$ equals 1 when the property $P$ is satisfied and 0 otherwise

- $1$ is not an eigenvalue of $S$
- $d_0 = 1$ and $(d_1 \cdots d_K) = (1 \cdots 1) (-S + Id_K)^{-1}$
- $C = \int_0^1 \left( d_0 + \sum_{j=1}^K d_j \sum_{m=1}^{\infty} \frac{\chi_{0,T^m(c_j)}}{(\beta_{n-1} \cdots \beta_0)^m} \right) \, d\lambda$ is the normalization constant.
Then the density of the \( (T_{β_{n-1}} \circ \cdots \circ T_{β_0}) \)-invariant measure given by Theorem 14 with respect to the Lebesgue measure is

\[
\frac{1}{C} \left( d_0 + \sum_{j=1}^{K} d_j \sum_{m=1}^{∞} \chi_{[0,T^m(c_j)]}(β_{n-1} \cdots β_0)^m \right).
\]

**Proof.** This is an application of the formula given in [12]. □

Note that the only hypothesis in the statement of Proposition 25 is that 1 is not an eigenvalue of the matrix \( S \). In [12] Gora conjectured that this condition is equivalent to the exactness of the dynamical system, which is a property we know to be satisfied by Theorem 14.

**Example 26.** Consider once again the alternate base \( β = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right) \). The composition \( T_{β_1} \circ T_{β_0} \) is depicted in Figure 9. Since \( \frac{1}{β_0} = β_1 - 1 \), keeping the notation of Proposition 25, we have \( K = 3 \), \( c_1 = \frac{1}{β_0} \), \( c_2 = \frac{2}{β_0} \) and \( c_3 = 1 \). We have \( T(c_1) = T(c_2) = T(c_3) = c_1 \).

Therefore, all elements in \( S \) equal 0, \( d_0 = d_1 = d_2 = d_3 = 1 \) and \( C = 1 + \frac{3}{β_0(β_1β_0-1)} = 1 + \frac{3}{β_0} \).

The density of the unique absolutely continuous \( (T_{β_1} \circ T_{β_0}) \)-invariant probability measure is

\[
\frac{1}{C} \left( 1 + \frac{3}{β_0} \chi_{[0,\frac{1}{β_0}]} \right).
\]

For example, \( µ([0, \frac{1}{β_0}]) = \frac{13+\sqrt{13}}{26} \). Moreover, it can be checked that \( µ((T_{β_1} \circ T_{β_0})^{-1}[0, \frac{1}{β_0}]) = µ([0, \frac{1}{β_0}]) \).

We obtain a formula for the density \( \frac{dµ}{dλ} \) by using the densities \( \frac{dµ_i}{dλ_p} \) for \( i ∈ [0, p - 1] \) given in Proposition 25. We first need a lemma.

**Lemma 27.** For all \( i ∈ [0, p - 1] \), all sets \( B ∈ B([0,1]) \) and all \( B([0,1]) \)-measurable functions \( f: [0,1) → [0,∞) \), the map \( f \circ π_2: [0,p-1] × [0,1) → [0,∞) \) is \( T_p \)-measurable and

\[
\int_{\{i\}×B} f \circ π_2 \ dλ_p = \frac{1}{p} \int_B f \ dλ.
\]

**Proof.** This follows from the definition of the Lebesgue integral via simple functions. □
Proposition 28. The density $\frac{d\mu_\beta}{dx_p}$ of $\mu_\beta$ with respect to the $p$-Lebesgue measure on $T_p$ is

$$
\sum_{i=0}^{p-1} \left( \frac{d\mu_{\beta,i}}{d\lambda} \circ \pi_2 \right) \cdot \chi_{\{i\} \times [0,1)}.
$$

Proof. Let $A \in T_p$ and let $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$ such that $A = \bigcup_{i=0}^{p-1} \{i\} \times B_i$. It follows from Lemma 27 that

$$
\int_A \sum_{i=0}^{p-1} \left( \frac{d\mu_{\beta,i}}{d\lambda} \circ \pi_2 \right) \cdot \chi_{\{i\} \times [0,1)} \ d\lambda_p = \sum_{i=0}^{p-1} \int_{B_i} \frac{d\mu_{\beta,i}}{d\lambda} \circ \pi_2 \ d\lambda_p = \frac{1}{p} \sum_{i=0}^{p-1} \mu_{\beta,i}(B_i) = \mu_\beta(A).
$$

Note that the formula (7) also holds for the extended measures $\mu_\beta$ and $\lambda_p$ on $T_\beta$.

4.4. Frequencies. We now turn to the frequencies of the digits in the greedy $\beta$-expansions of real numbers in the interval $[0,1)$. Recall that the frequency of a digit $d$ occurring in the greedy $\beta$-expansion $a_0a_1a_2\cdots$ of a real number $x$ in $[0,1)$ is equal to

$$
\lim_{n \to \infty} \frac{1}{n} \#\{0 \leq k < n : a_k = d\},
$$

provided that this limit converges.

Proposition 29. For $\lambda$-almost all $x \in [0,1)$, the frequency of any digit $d$ occurring in the greedy $\beta$-expansion of $x$ exists and is equal to

$$
\frac{1}{p} \sum_{i=0}^{p-1} \mu_{\beta,i} \left( \left\lfloor \frac{d}{\beta^k} \right\rfloor \cap [0,1) \right).
$$

Proof. Let $x \in [0,1)$ and let $d$ be a digit occurring in $d_\beta(x) = a_0a_1a_2\cdots$. Then for all $k \in \mathbb{N}$, $a_k = d$ if and only if $\pi_2(T^k_\beta(0,x)) \in \left(\left\lfloor \frac{d}{\beta^k} \right\rfloor, \left\lceil \frac{d}{\beta^k} \right\rceil \right) \cap [0,1)$. Moreover, since for all $k \in \mathbb{N}$, $T^k_\beta(0,x) \in \{k \mod p\} \times [0,1)$, we have

$$
\chi_{\left(\left\lfloor \frac{d}{\beta^k} \right\rfloor, \left\lceil \frac{d}{\beta^k} \right\rceil \right) \cap [0,1)}(\pi_2(T^k_\beta(0,x))) = \chi_{\{k \mod p\} \times \left(\left\lfloor \frac{d}{\beta^k} \right\rfloor, \left\lceil \frac{d}{\beta^k} \right\rceil \cap [0,1)\right)}(T^k_\beta(0,x)) = \sum_{i=0}^{p-1} \chi_{\{i\} \times \left(\left\lfloor \frac{d}{\beta^k} \right\rfloor, \left\lceil \frac{d}{\beta^k} \right\rceil \cap [0,1)\right)}(T^k_\beta(0,x)).
$$

Therefore, if it exists, the frequency of $d$ in $d_\beta(x)$ is equal to

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{i=0}^{p-1} \chi_{\{i\} \times \left(\left\lfloor \frac{d}{\beta^k} \right\rfloor, \left\lceil \frac{d}{\beta^k} \right\rceil \cap [0,1)\right)}(T^k_\beta(0,x)).
$$
Yet, for each \( i \in [0, p - 1] \) and for \( \mu_\beta \)-almost all \( y \in [0, p - 1] \times [0, 1) \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{\{i\}} \times \left( \left( \frac{y}{\beta_i}, \frac{y+1}{\beta_i} \right) \cap [0, 1) \right) \left( T_{\beta_i}^k(y) \right) = \int_{[0, p-1] \times [0, 1)} \chi_{\{i\}} \times \left( \left( \frac{y}{\beta_i}, \frac{y+1}{\beta_i} \right) \cap [0, 1) \right) d\mu_\beta
\]

\[
= \mu_\beta \left( \{i\} \times \left( \left( \frac{d}{\beta_i}, \frac{d+1}{\beta_i} \right) \cap [0, 1) \right) \right)
\]

\[
= \frac{1}{p} \mu_{\beta^i} \left( \left( \frac{d}{\beta_i}, \frac{d+1}{\beta_i} \right) \cap [0, 1) \right)
\]

where we used Theorem 23 and the Ergodic Theorem for the first equality. The conclusion now follows from Proposition 19.

\[\square\]

5. ISOMORPHISM BETWEEN GREEDY AND LAZY \( \beta \)-TRANSFORMATIONS

In this section, we show that

\[
\phi_\beta : \bigcup_{i=0}^{p-1} \left( \{i\} \times [0, x_{\beta(i)}) \right) \to \bigcup_{i=0}^{p-1} \left( \{i\} \times (0, x_{\beta(i)}) \right), \quad (i, x) \mapsto (i, x_{\beta(i)} - x)
\]

defines an isomorphism between the greedy \( \beta \)-transformation and the lazy \( \beta \)-transformation.

We consider the \( \sigma \)-algebra

\[
\mathcal{L}_\beta = \left\{ \bigcup_{i=0}^{p-1} \left( \{i\} \times B_i \right) : \forall i \in \{0, p-1\}, B_i \in \mathcal{B}((0, x_{\beta(i)})) \right\}
\]

on \( \bigcup_{i=0}^{p-1} \left( \{i\} \times (0, x_{\beta(i)}) \right) \).

**Theorem 30.** The map \( \phi_\beta \) is an isomorphism between the dynamical systems \( \left( \bigcup_{i=0}^{p-1} \left( \{i\} \times [0, x_{\beta(i)}) \right), T_\beta, \mu_\beta, T_\beta \right) \) and \( \left( \bigcup_{i=0}^{p-1} \left( \{i\} \times (0, x_{\beta(i)}) \right), \mathcal{L}_\beta, \mu_\beta \circ \phi_\beta^{-1}, L_\beta \right) \).

**Proof.** Clearly, \( \phi_\beta \) is a bijective map. Hence, we only have to show that \( \phi_\beta \circ T_\beta = L_\beta \circ \phi_\beta \).

Let \( (i, x) \in \bigcup_{i=0}^{p-1} \left( \{i\} \times [0, x_{\beta(i)}) \right) \). First, suppose that \( x \in [0, 1) \). Then

\[
\phi_\beta \circ T_\beta(i, x) = ((i+1) \mod p, x_{\beta(i+1)} - \beta_i x + [\beta_i x])
\]

and

\[
L_\beta \circ \phi_\beta(i, x) = ((i+1) \mod p, \beta_i (x_{\beta(i)} - x) - [\beta_i (x_{\beta(i)} - x) - x_{\beta(i+1)}]).
\]

Second, suppose that \( x \in [1, x_{\beta(i)}) \). Then

\[
\phi_\beta \circ T_\beta(i, x) = ((i+1) \mod p, x_{\beta(i+1)} - \beta_i x + [\beta_i] - 1)
\]

and

\[
L_\beta \circ \phi_\beta(i, x) = ((i+1) \mod p, \beta_i (x_{\beta(i)} - x)).
\]

In both cases, we easily get that \( \phi_\beta \circ T_\beta(i, x) = L_\beta \circ \phi_\beta(i, x) \) by using (4).

\[\square\]

Thanks to Theorem 30, we obtain an analogue of Theorem 24 for the lazy \( \beta \)-transformation.

**Theorem 31.** The measure \( \mu_\beta \circ \phi_\beta^{-1} \) is the unique \( L_\beta \)-invariant probability measure on \( \mathcal{L}_\beta \) that is absolutely continuous with respect to \( \lambda_p \circ \phi_\beta^{-1} \). Furthermore, \( \mu_\beta \circ \phi_\beta^{-1} \) is equivalent to \( \lambda_p \circ \phi_\beta^{-1} \) on \( \mathcal{L}_\beta \) and the dynamical system \( \left( \bigcup_{i=0}^{p-1} \left( \{i\} \times (0, x_{\beta(i)}) \right), \mathcal{L}_\beta, \mu_\beta \circ \phi_\beta^{-1}, L_\beta \right) \) is ergodic and has entropy \( \frac{1}{p} \log(\beta_{p-1} \cdots \beta_0) \).
Similarly, we have an analogue of Theorem 23 for the lazy $\beta$-transformation, by considering the $\sigma$-algebra

$$\mathcal{L}'_\beta = \left\{ \bigcup_{i=0}^{p-1} \{i\} \times B_i : \forall i \in [0, p-1], \quad B_i \in \mathcal{B}(x_{\beta(i)} - 1, x_{\beta(i)}) \right\}.$$ 

**Theorem 32.** The measure $\mu_\beta \circ \phi_\beta^{-1}$ is the unique $L_\beta$-invariant probability measure on $\mathcal{L}'_\beta$ that is absolutely continuous with respect to $\lambda_p \circ \phi_\beta^{-1}$. Furthermore, $\mu_\beta \circ \phi_\beta^{-1}$ is equivalent to $\lambda_p \circ \phi_\beta^{-1}$ on $\mathcal{L}'_\beta$ and the dynamical system $\left( \bigcup_{i=0}^{p-1} \{i\} \times (x_{\beta(i)} - 1, x_{\beta(i)}), \mathcal{L}'_\beta, \mu_\beta \circ \phi_\beta^{-1}, L_\beta \right)$ is ergodic and has entropy $\frac{1}{p} \log(\beta_{p-1} \cdots \beta_0)$.

**Remark 33.** We deduce from Theorem 30 that if the greedy $\beta$-expansion of a real number $x \in [0, x_{\beta})$ is $a_0 a_1 a_2 \cdots$, then the lazy $\beta$-expansion of $x_{\beta} - x$ is $([\beta_0] - 1 - a_0)([\beta_1] - 1 - a_1)([\beta_2] - 1 - a_2) \cdots$.

### 6. Isomorphism with the $\beta$-shift

The aim of this section is to generalize the isomorphism between the greedy $\beta$-transformation and the $\beta$-shift to the framework of alternate bases. We start by providing some background of the real base case.

Let $D_\beta$ denote the set of all greedy $\beta$-expansions of real numbers in the interval $[0, 1)$. The $\beta$-shift is the set $S_\beta$ defined as the topological closure of $D_\beta$ with respect to the prefix distance of infinite words. For an alphabet $A$, we let $C_A$ denote the $\sigma$-algebra generated by the cylinders

$$C_A(a_0, \ldots, a_{\ell-1}) = \{ w \in A^\mathbb{N} : w[0] = a_0, \ldots, w[\ell - 1] = a_{\ell - 1} \}$$

for all $\ell \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{\ell - 1} \in A$, where the notation $w[k]$ designates the letter at position $k$ in the infinite word $w$, and we call

$$\sigma_A : A^\mathbb{N} \to A^\mathbb{N}, \quad a_0 a_1 a_2 \cdots \mapsto a_1 a_2 a_3 \cdots$$

the shift operator over $A$. If no confusion is possible, we simply write $\sigma$ instead of $\sigma_A$.

Then the map $\psi_\beta : [0, 1) \to S_\beta, x \mapsto d_\beta(x)$ defines an isomorphism between the dynamical systems $([0, 1), B([0, 1)), \mu_\beta, T_\beta)$ and $(S_\beta, \{ C \cap S_\beta : C \in C_A \beta \}, \mu_\beta \circ \psi_\beta^{-1}, \sigma_{S_\beta})$ where $A_\beta$ denote the alphabet of digits $[0, [\beta] - 1]$.

Now, let us extend the previous notation to the framework of alternate bases. Let $A_\beta$ denote the alphabet $[0, \max_{i \in [0, p-1]} [\beta_i] - 1]$, let $D_\beta$ denote the subset of $A_\beta^\mathbb{N}$ made of all greedy $\beta$-expansions of real numbers in $[0, 1)$ and let $S_\beta$ denote the topological closure of $D_\beta$ with respect to the prefix distance of infinite words:

$$D_\beta = \{ d_\beta(x) : x \in [0, 1) \} \quad \text{and} \quad S_\beta = \overline{D_\beta}.$$ 

The following lemma was proved in [3].

**Lemma 34.** For all $n \in \mathbb{N}$, if $w \in S_\beta^{(n)}$ then $\sigma(w) \in S_\beta^{(n+1)}$.

Consider the $\sigma$-algebra

$$\mathcal{G}_\beta = \left\{ \bigcup_{i=0}^{p-1} \{i\} \times (C_i \cap S_{\beta(i)}) : C_i \in C_{A_\beta} \right\}$$

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on $\bigcup_{i=0}^{p-1}\{i\} \times S_{\beta(i)}$. We define
\[
\sigma_p: \bigcup_{i=0}^{p-1}\{i\} \times S_{\beta(i)} \to \bigcup_{i=0}^{p-1}\{i\} \times S_{\beta(i)}, \ (i, w) \mapsto ((i + 1) \mod p, \sigma(w))
\]
\[
\psi_{\beta}: [0, p - 1] \times [0, 1) \to \bigcup_{i=0}^{p-1}\{i\} \times S_{\beta(i)}, \ (i, x) \mapsto (i, d_{\beta(i)}(x)).
\]
Note that the transformation $\sigma_p$ is well defined by Lemma 34.

**Theorem 35.** The map $\psi_{\beta}$ defines an isomorphism between the dynamical systems
\[
([0, p - 1] \times [0, 1), T_p, \mu_\beta, T_\beta) \text{ and } \left( \bigcup_{i=0}^{p-1}\{i\} \times S_{\beta(i)}, G_\beta, \mu_\beta \circ \psi_{\beta}^{-1}, \sigma_p \right).
\]

**Proof.** It is easily seen that $\psi_{\beta} \circ T_\beta = \sigma_p \circ \psi_{\beta}$ and that $\psi_{\beta}$ is injective. \qed

However, since $\psi_{\beta}$ is not surjective, it does not define a topological isomorphism.

**Remark 36.** In view of Theorem 35, the set $\bigcup_{i=0}^{p-1}\{i\} \times S_{\beta(i)}$ can be seen as the $\beta$-shift, that is, the generalization of the $\beta$-shift to alternate bases. However, in the previous work [3], what we called the $\beta$-shift is the union $\bigcup_{i=0}^{p-1}S_{\beta(i)}$. This definition was motivated by the following combinatorial result: the set $\bigcup_{i=0}^{p-1}S_{\beta(i)}$ is sofic if and only if for every $i \in [0, p - 1]$, the quasi-greedy $\beta^{(i)}$-representation of 1 is ultimately periodic. In summary, we can say that there are two ways to extend the notion of $\beta$-shift to alternate bases $\beta$, depending on the way we look at it: either as a dynamical object or as a combinatorial object.

Thanks to Theorem 35, we obtain an analogue of Theorem 23 for the transformation $\sigma_p$.

**Theorem 37.** The measure $\rho_\beta$ is the unique $\sigma_p$-invariant probability measure on $G_\beta$ that is absolutely continuous with respect to $\lambda_p \circ \psi_{\beta}^{-1}$. Furthermore, $\rho_\beta$ is equivalent to $\lambda_p \circ \psi_{\beta}^{-1}$ on $G_\beta$ and the dynamical system $\left( \bigcup_{i=0}^{p-1}\{i\} \times S_{\beta(i)}, G_\beta, \rho_\beta, \sigma_p \right)$ is ergodic and has entropy $\frac{1}{p} \log(\beta_{p-1} \cdots \beta_0)$.

**Remark 38.** Let $D'_{\beta}$ denote the subset of $A_{\beta}^\infty$ made of all lazy $\beta$-expansions of real numbers in $(x_\beta - 1, x_\beta]$ and let $S'_{\beta}$ denote the topological closure of $D'_{\beta}$ with respect to the prefix distance of infinite words. From Remark 33, it is easily seen that
\[
\theta_{\beta}: \bigcup_{i=0}^{p-1}\{i\} \times S'_{\beta(i)} \to \bigcup_{i=0}^{p-1}\{i\} \times S'_{\beta(i)}, \ (i, a_0a_1 \cdots) \mapsto (i, ([\beta_i] - 1 - a_0)([\beta_{i+1}] - 1 - a_2) \cdots)
\]
defines an isomorphism from $\left( \bigcup_{i=0}^{p-1}\{i\} \times S_{\beta(i)}, G_\beta, \rho_\beta, \sigma_p \right)$ to $\left( \bigcup_{i=0}^{p-1}\{i\} \times S'_{\beta(i)}, G'_\beta, \rho_\beta \circ \theta_{\beta}^{-1}, \sigma'_p \right)$ where
\[
G'_\beta = \left\{ \bigcup_{i=0}^{p-1}\{i\} \times (C_i \cap S'_{\beta(i)})) : C_i \in C_{A_\beta} \right\}
\]
\[
\sigma'_p: \bigcup_{i=0}^{p-1}\{i\} \times S'_{\beta(i)} \to \bigcup_{i=0}^{p-1}\{i\} \times S'_{\beta(i)}, \ (i, w) \mapsto ((i + 1) \mod p, \sigma(w)).
\]
We then deduce from Theorem 30 and 35 that \( \theta_\beta \circ \psi_\beta \circ \phi_\beta^{-1} \) is an isomorphism from 
\( (\bigcup_{i=0}^{p-1} \{i\} \times (x_{\beta^{(i)}} - 1, x_{\beta^{(i)}}]) \), \( L_\beta, \mu_\beta \circ \phi_\beta^{-1}, L_\beta \) to \( (\bigcup_{i=0}^{p-1} \{i\} \times S'_{\beta^{(i)}}, G'_\beta, \rho_\beta \circ \theta_\beta^{-1}, \sigma'_{\beta}) \) 
where here \( \phi_\beta \) denoted the restricted map
\[
\phi_\beta : \bigcup_{i=0}^{p-1} \{i\} \times [0, 1) \rightarrow \bigcup_{i=0}^{p-1} \{i\} \times (x_{\beta^{(i)}} - 1, x_{\beta^{(i)}}), \ (i, x) \mapsto (i, x_{\beta^{(i)}} - x).
\]

It is easy to check that, as expected, that for all \( (i, x) \in \bigcup_{i=0}^{p-1} \{i\} \times (x_{\beta^{(i)}} - 1, x_{\beta^{(i)}}) \), we 
have \( \theta_\beta \circ \psi_\beta \circ \phi_\beta^{-1}(i, x) = (i, \ell_\beta(x)) \) where \( \ell_\beta(x) \) denoted the lazy \( \beta \)-expansion of \( x \).

7. \( \beta \)-EXPANSIONS AND \((\beta_{p-1} \cdots \beta_0, \Delta_\beta)\)-EXPANSIONS

By rewriting Equality (1) from Section 3 as
\[
(9) \quad x = \frac{\beta_{p-1} \cdots \beta_1 a_0 + \beta_{p-1} \cdots \beta_2 a_1 + \cdots + a_{p-1}}{\beta_{p-1} \cdots \beta_0}
+ \frac{\beta_{p-1} \cdots \beta_1 a_0 + \beta_{p-1} \cdots \beta_1 a_{p+1} + \cdots + a_{2p-1}}{(\beta_{p-1} \cdots \beta_0)^2}
+ \cdots
\]
we can see the greedy and lazy \( \beta \)-expansions of real numbers as \((\beta_{p-1} \cdots \beta_0)\)-representations 
over the digit set
\[
\Delta_\beta = \left\{ \sum_{i=0}^{p-1} \beta_{p-1} \cdots \beta_{i+1} c_i : \forall i \in [0, p-1], \ c_i \in [0, \lfloor \beta_i \rfloor - 1] \right\}.
\]

In this section, we examine some cases where by considering the greedy (resp. lazy) \( \beta \)-expansion 
and rewriting it as (9), the obtained representation is the greedy (resp. lazy) 
\((\beta_{p-1} \cdots \beta_0, \Delta_\beta)\)-expansion. We first recall the formalism of \( \beta \)-expansions of real numbers 
over a general digit set [16].

7.1. Real base expansions over general digit sets. Consider an arbitrary finite set 
\( \Delta = \{d_0, d_1, \ldots, d_m\} \subset \mathbb{R} \) where \( 0 = d_0 < d_1 < \cdots < d_m \). Then a \((\beta, \Delta)\)-representation 
of a real number \( x \) in the interval \([0, \frac{d_m}{\beta - 1}]\) is an infinite sequence \( a_0 a_1 a_2 \cdots \) over \( \Delta \) such that 
\( x = \sum_{n=0}^{\infty} \frac{a_n}{\beta^n} \). Such a set \( \Delta \) is called an allowable digit set for \( \beta \) if
\[
(10) \quad \max_{k \in [0, m-1]} (d_{k+1} - d_k) \leq \frac{d_m}{\beta - 1}.
\]
In this case, the greedy \((\beta, \Delta)\)-expansion of a real number \( x \in [0, \frac{d_m}{\beta - 1}] \) is defined recursively as follows: if the first \( N \) digits of the greedy \((\beta, \Delta)\)-expansion of \( x \) are given by \( a_0, \ldots, a_{N-1} \), 
then the next digit \( a_N \) is the greatest element in \( \Delta \) such that 
\[
\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} \leq x.
\]
The greedy \((\beta, \Delta)\)-expansion can also be obtained by iterating the greedy \((\beta, \Delta)\)-transformation 
\( T_{\beta, \Delta} : [0, \frac{d_m}{\beta - 1}] \rightarrow [0, \frac{d_m}{\beta - 1}], \ x \mapsto \begin{cases} 
\beta x - d_k & \text{if } x \in [\frac{d_k}{\beta}, \frac{d_{k+1}}{\beta}], \ k \in [0, m-1] \\
\beta x - d_m & \text{if } x \in [\frac{d_m}{\beta}, \frac{d_{m+1}}{\beta}] \end{cases} \)
as follows: for all \( n \in \mathbb{N} \), \( a_n \) is the greatest digit \( d \) in \( \Delta \) such that \( \frac{d}{\beta} \leq T_{\beta, \Delta}^n(x) \) [5].
**Example 39.** Consider the digit set $\Delta = \{0, 1, \varphi + \frac{1}{\varphi}, \varphi^2\}$. It is easily checked that $\Delta$ is an allowable digit set for $\varphi$. The greedy $(\varphi, \Delta)$-transformation

$$T_{\varphi, \Delta} : [0, \frac{\varphi^2}{\varphi-1}) \to [0, \frac{\varphi^2}{\varphi-1}), x \mapsto \begin{cases} \varphi x & \text{if } x \in [0, \frac{1}{\varphi}) \\ \varphi x - 1 & \text{if } x \in [\frac{1}{\varphi}, 1 + \frac{1}{\varphi}) \\ \varphi x - (\varphi + \frac{1}{\varphi}) & \text{if } x \in [1 + \frac{1}{\varphi}, \varphi) \\ \varphi x - \varphi^2 & \text{if } x \in [\varphi, \frac{\varphi^2}{\varphi-1}) \end{cases}$$

is depicted in Figure 10.

![Figure 10. The transformation $T_{\varphi, \Delta}$ for $\Delta = \{0, 1, \varphi + \frac{1}{\varphi}, \varphi^2\}$.](image)

Similarly, if $\Delta$ is an allowable digit set for $\beta$, then the lazy $(\beta, \Delta)$-expansion of a real number $x \in (0, \frac{d_m}{\beta-1}]$ is defined recursively as follows: if the first $N$ digits of the lazy $(\beta, \Delta)$-expansion of $x$ are given by $a_0, \ldots, a_{N-1}$, then the next digit $a_N$ is the least element in $\Delta$ such that

$$\frac{N}{\beta^{n+1}} + \sum_{n=N+1}^{\infty} \frac{d_m}{\beta^{n+1}} \geq x.$$ 

The lazy $(\beta, \Delta)$-transformation

$$L_{\beta, \Delta} : (0, \frac{d_m}{\beta-1}] \to (0, \frac{d_m}{\beta-1}), x \mapsto \begin{cases} \beta x & \text{if } x \in (0, \frac{d_m}{\beta-1} - \frac{d_m}{\beta-1}) \\ \beta x - d_k & \text{if } x \in \left(\frac{d_m}{\beta-1} - \frac{d_m - d_k}{\beta}, \frac{d_m - d_k}{\beta} - \frac{d_m - d_k}{\beta} \right], k \in [1, m] \end{cases}$$

can be used to obtain the digits of the lazy $(\beta, \Delta)$-expansions: for all $n \in \mathbb{N}$, $a_n$ is the least digit $d$ in $\Delta$ such that $\frac{d}{\beta} + \sum_{k=1}^{\infty} \frac{d_m}{\beta^{n+1}} \geq L_{\beta, \Delta}(x)$ [5].

In [5], it is shown that if $\Delta$ is an allowable digit set for $\beta$ then so is the set $\bar{\Delta} := \{0, d_m - d_{m-1}, \ldots, d_m - d_1, d_m\}$ and

$$\phi_{\beta, \Delta} : [0, \frac{d_m}{\beta-1}) \to (0, \frac{d_m}{\beta-1}), x \mapsto \frac{d_m}{\beta-1} - x$$

is a bicontinuous bijection satisfying $L_{\beta, \bar{\Delta}} \circ \phi_{\beta, \Delta} = \phi_{\beta, \Delta} \circ T_{\beta, \Delta}$. 
Example 40. Consider the digit set $\tilde{\Delta}$ where $\Delta$ is the digit set from Example 39. We get $\tilde{\Delta} = \{0, 1 - \frac{1}{\varphi}, \varphi, \varphi^2\}$. The lazy $(\varphi, \tilde{\Delta})$-transformation

$$L_{\varphi, \tilde{\Delta}} : (0, \frac{\varphi^2}{\varphi - 1}) \rightarrow (0, \frac{\varphi^2}{\varphi - 1}], \ x \mapsto \begin{cases} \varphi x & \text{if } x \in (0, \frac{\varphi}{\varphi - 1}] \\ \varphi x - (1 - \frac{1}{\varphi}) & \text{if } x \in (\frac{\varphi}{\varphi - 1}, \frac{\varphi + 3}{\varphi - 1}] \\ \varphi x - \varphi & \text{if } x \in (\frac{\varphi + 3}{\varphi - 1}, \frac{2\varphi - 1}{\varphi - 1}] \\ \varphi x - \varphi^2 & \text{if } x \in (\frac{2\varphi - 1}{\varphi - 1}, \frac{\varphi^2}{\varphi - 1}] \end{cases}$$

is depicted in Figure 11. It is conjugate to the greedy $(\varphi, \Delta)$-transformation $T_{\varphi, \Delta}$ by $\phi_{\varphi, \Delta} : [0, \frac{\varphi^2}{\varphi - 1}) \rightarrow (0, \frac{\varphi^2}{\varphi - 1}], \ x \mapsto \frac{\varphi^2}{\varphi - 1} - x$. 

![Figure 11](image_url)

**Figure 11.** The transformation $L_{\varphi, \tilde{\Delta}}$ for $\Delta = \{0, 1, \varphi + \frac{1}{\varphi}, \varphi^2\}$.

7.2. Comparison between $\beta$-expansions and $(\beta_{p-1} \cdots \beta_1, \Delta_\beta)$-expansions. The digit set $\Delta_\beta$ has cardinality at most $\prod_{i=0}^{p-1} |\beta_i|$ and can be rewritten $\Delta_\beta = \text{im}(f_\beta)$ where

$$f_\beta : \prod_{i=0}^{p-1} [0, [\beta_i] - 1] \rightarrow \mathbb{R}, \ (c_0, \ldots, c_{p-1}) \mapsto \sum_{i=0}^{p-1} \beta_{p-1} \cdots \beta_i c_i.$$ 

Note that $f_\beta$ is not injective in general. Let us write $\Delta_\beta = \{d_0, d_1, \ldots, d_m\}$ with $d_0 < d_1 < \cdots < d_m$. We have $d_0 = f_\beta(0, \ldots, 0) = 0$, $d_1 = f_\beta(0, \ldots, 0, 1) = 1$ and $d_m = f_\beta([\beta_0] - 1, \ldots, [\beta_{p-1}] - 1)$. In what follows, we suppose that $\prod_{i=0}^{p-1} [0, [\beta_i] - 1]$ is equipped with the lexicographic order: $(c_0, \ldots, c_{p-1}) <_\text{lex} (c'_0, \ldots, c'_{p-1})$ if there exists $i \in [0, p-1]$ such that $c_0 = c'_0, \ldots, c_{i-1} = c'_{i-1}$ and $c_i < c'_i$.

**Lemma 41.** The set $\Delta_\beta$ is an allowable digit set for $\beta_{p-1} \cdots \beta_0$.

**Proof.** We need to check Condition (10). We have $d_0 = 0$ and

$$d_m = f_\beta([\beta_0] - 1, \ldots, [\beta_{p-1}] - 1) \geq \sum_{i=0}^{p-1} \beta_{p-1} \cdots \beta_{i+1} [\beta_i - 1] = \beta_{p-1} \cdots \beta_0 - 1,$$

Therefore, it suffices to show that for all $k \in [0, m - 1]$, $d_{k+1} - d_k \leq 1$. Thus, we only have to show that $f(c_0, \ldots, c'_{p-1}) - f(c_0, \ldots, c_{p-1}) \leq 1$ where $(c_0, \ldots, c_{p-1})$ and $(c'_0, \ldots, c'_{p-1})$ are lexicographically consecutive elements of $\prod_{i=0}^{p-1} [0, [\beta_i] - 1]$. For such $p$-tuples, there exists
Proof. Let $x_β = \frac{d_m}{β_{p-1}...β_0-1}$, it follows from Lemma 41 that every point in $[0, x_β)$ admits a greedy $(β_{p-1}...β_0, Δ_β)$-expansion.

Let us restate Proposition 7 when $n$ equals $p$ in terms of the map $f_β$.

**Lemma 42.** For all $x \in [0, x_β)$, we have

$$π_2 \circ T_β^p \circ δ_0(x) = β_{p-1}...β_0x - f_β(c)$$

where $c$ is the lexicographically greatest $p$-tuple in $\prod_{i=0}^{p-1} [0, [β_i] - 1]$ such that $\frac{f_β(c)}{β_{p-1}...β_0} \leq x$.

**Proposition 43.** For all $x \in [0, x_β)$, we have $T_{β_{p-1}...β_0, Δ_β}(x) \leq π_2 \circ T_β^p \circ δ_0(x)$ and $L_{β_{p-1}...β_0, Δ_β}(x) \geq π_2 \circ L_β^p \circ δ_0(x)$.

**Proof.** Let $x \in [0, x_β)$. On the one hand, $T_{β_{p-1}...β_0, Δ_β}(x) = β_{p-1}...β_0x - d$ where $d$ is the greatest digit in $Δ_β$ such that $\frac{d}{β_{p-1}...β_0} \leq x$. On the other hand, by Lemma 42, $π_2 \circ T_β^p \circ δ_0(x) = β_{p-1}...β_0x - f_β(c)$ where $c$ is the greatest $p$-tuple in $\prod_{i=0}^{p-1} [0, [β_i] - 1]$ such that $\frac{f_β(c)}{β_{p-1}...β_0} \leq x$. By definition of $d$, we get $d \geq f_β(c)$. Therefore, we obtain that $T_{β_{p-1}...β_0, Δ_β}(x) \leq π_2 \circ T_β^p \circ δ_0(x)$. The inequality $L_{β_{p-1}...β_0, Δ_β}(x) \geq π_2 \circ L_β^p \circ δ_0(x)$ then follows from Theorem 30. □

In what follows, we provide some conditions under which the inequalities of Proposition 43 happen to be equalities.

**Proposition 44.** The transformations $T_{β_{p-1}...β_0, Δ_β}$ and $π_2 \circ T_β^p \circ δ_0|_{[0, x_β)}$ coincide if and only if the transformations $L_{β_{p-1}...β_0, Δ_β}$ and $π_2 \circ L_β^p \circ δ_0|_{[0, x_β)}$ coincide.

**Proof.** We only show the forward direction, the backward direction being similar. Suppose that $T_{β_{p-1}...β_0, Δ_β} = π_2 \circ T_β^p \circ δ_0|_{[0, x_β)}$ and let $x \in (0, x_β]$. Since $x_β = \frac{d_m}{β_{p-1}...β_0 - 1}$ and $Δ_β = \tilde{Δ}_β$, we successively obtain that

$$L_{β_{p-1}...β_0, Δ_β}(x) = L_{β_{p-1}...β_0, Δ_β} \circ T_{β_{p-1}...β_0, Δ_β}(x_β - x)$$

$$= φ_{β_{p-1}...β_0, Δ_β} \circ T_{β_{p-1}...β_0, Δ_β}(x_β - x)$$

$$= φ_{β_{p-1}...β_0, Δ_β} \circ T_β^p \circ δ_0(x_β - x)$$

$$= π_2 \circ φ_β \circ T_β^p \circ δ_0(x_β - x)$$

$$= π_2 \circ L_β^p \circ φ_β \circ δ_0(x_β - x)$$

$$= π_2 \circ T_β^p \circ δ_0(x_β - x)$$

$$= π_2 \circ L_β^p \circ δ_0(x_β - x)$$

$$= \tilde{π}_2 \circ L_β^p \circ δ_0(x_β - x)$$
The next result provides us with a sufficient condition under which the transformations $T_{\beta_{p-1} \ldots \beta_0, \Delta_\beta}$ and $\pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0 |_{[0,x_\beta]}$ coincide. Here, the non-decreasingness of the map $f_\beta$ refers to the lexicographic order: for all $c, c' \in \prod_{i=0}^{P-1} [0, [\beta_i] - 1]$, $c <_{\text{lex}} c' \implies f_\beta(c) \leq f_\beta(c')$.

**Theorem 45.** If the map $f_\beta$ is non-decreasing then $T_{\beta_{p-1} \ldots \beta_0, \Delta_\beta} = \pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0 |_{[0,x_\beta]}$.

**Proof.** We keep the same notation as in the proof of Proposition 43. Let $d = f_\beta(c')$. By definition of $c$, we get $c \geq_{\text{lex}} c'$. Now, if $f_\beta$ is non-decreasing then $f_\beta(c) \leq f_\beta(c') = d$. Hence the conclusion. □

The following example shows that considering the length-$p$ alternate base $\beta = (\beta, \ldots, \beta)$ with $p \in \mathbb{N}_{\geq 3}$, it may happen that $T_{\beta_{p-1} \ldots \beta_0, \Delta_\beta}$ differs from $\pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0 |_{[0,x_\beta]}$. This result was already proved in [4].

**Example 46.** Consider the alternate base $\beta = (\varphi^2, \varphi^2, \varphi^2)$. Then $\Delta_\beta = \{\varphi^4 c_0 + \varphi^2 c_1 + c_2 : c_0, c_1, c_2 \in \{0, 1\}\}$. In [4, Proposition 2.1], it is proved that $T_{\beta_{p-1} \ldots \beta_0, \Delta_\beta}$ coincides with $\pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0 |_{[0,x_\beta]}$ if and only if $f_\beta$ is non-decreasing. Since $f_\beta(0, 2, 2) = 2\varphi^2 + 2 > \varphi^2 = f_\beta(1, 0, 0)$, the transformations $T_{\varphi^2, \Delta_\beta}$ and $\pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0 |_{[0,x_\beta]}$ differ by [4, Proposition 2.1].

Whenever $f_\beta$ is not non-decreasing, the transformations $T_{\beta_{p-1} \ldots \beta_0, \Delta_\beta}$ and $\pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0 |_{[0,x_\beta]}$ can either coincide or not. The following two examples illustrate both cases. In particular, Example 48 shows that the sufficient condition given in Theorem 45 is not necessary.

**Example 47.** Consider the alternate base $\beta = (\varphi, \varphi, \sqrt{5})$. Then $\Delta_\beta = \{\sqrt{5}c_0 + \sqrt{5}c_1 + c_2 : c_0, c_1 \in \{0, 1\}, c_2 \in \{0, 1, 2\}\}$. However, $f_\beta(0, 1, 2) = \sqrt{5} + 2 \approx 4.23$ and $f_\beta(1, 0, 0) = \sqrt{5}\varphi \approx 3.61$. It can be easily check that there exists $x \in [0, x_\beta]$ such that $T_{\sqrt{5}\varphi^2, \Delta_\beta}(x) \neq \pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0(x)$. For example, we can compute $T_{\sqrt{5}\varphi^2, \Delta_\beta}(0.75) \approx 0.15$ and $\pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0(0.75) \approx 0.77$. The transformations $T_{\sqrt{5}\varphi^2, \Delta_\beta}$ and $\pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0 |_{[0,x_\beta]}$ are depicted in Figure 12, where the red lines show the images of the interval $[\sqrt{5\varphi^2}, \frac{\sqrt{5\varphi^2} + 1}{\sqrt{5\varphi^2}}] \approx [0.72, 0.78]$, that is where the two transformations differ. Similarly, the transformations $L_{\sqrt{5}\varphi^2, \Delta_\beta}$ and $\pi_2 \circ L_{\Delta_\beta}^3 \circ \delta_0 |_{[0,x_\beta]}$ are depicted in Figure 13. As illustrated in red, the two transformations differ on the interval $\left(\frac{\sqrt{5\varphi^2} + 2}{\sqrt{5\varphi^2}}, \frac{\sqrt{5\varphi^2} + 1}{\sqrt{5\varphi^2}}\right) \approx (0.82, 0.89).

**Example 48.** Consider the alternate base $\beta = (\frac{1}{2}, \frac{1}{2}, 4)$. We have $\Delta_\beta = [0, 13]$. The map $f_\beta$ is not non-decreasing since we have $f_\beta(0, 1, 3) = 7$ and $f_\beta(1, 0, 0) = 6$. However, $T_{9, \Delta_\beta} = \pi_2 \circ T_{\Delta_\beta}^3 \circ \delta_0 |_{[0,x_\beta]}$ and $L_{9, \Delta_\beta} = \pi_2 \circ L_{\Delta_\beta}^3 \circ \delta_0 |_{[0,x_\beta]}$. The transformation $T_{9, \Delta_\beta}$ is depicted in Figure 14.

The next example illustrates that it may happen that the transformations $T_{\beta_{p-1} \ldots \beta_0, \Delta_\beta}$ and $\pi_2 \circ T_{\Delta_\beta}^p \circ \delta_0 |_{[0,x_\beta]}$ indeed coincide on $[0, 1)$ but not on $[0, x_\beta]$.

**Example 49.** Consider the alternate base $\beta = (\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2})$. Then $f_\beta(0, 1, 1) > f_\beta(1, 0, 0)$ and it can be checked that the maps $T_{\frac{\sqrt{5}}{2}, \Delta_\beta}$ and $\pi_2 \circ T_{\Delta_\beta}^3 \circ \delta_0 |_{[0,x_\beta]}$ differ on the interval $[f_\beta(0, 1, 1), f_\beta(1, 0, 1)] \approx [1.28, 1.44]$. However, the two maps coincide on $[0, 1)$. 

\[ \pi_2 \circ L_\beta^p \circ \delta_0(x) = \pi_2 \circ L_\beta^p \circ \delta_0(x). \]
Finally, we provide a necessary and sufficient condition for the map $f_\beta$ to be non-decreasing.

**Proposition 50.** The map $f_\beta$ is non-decreasing if and only if for all $j \in [1, p - 2]$,

\[
\sum_{i=j}^{p-1} \beta_{p-1} \cdots \beta_i ([\beta_i] - 1) \leq \beta_{p-1} \cdots \beta_j.
\]

(11)

**Proof.** If the map $f_\beta$ is non-decreasing then for all $j \in [1, p - 2]$,

\[
\sum_{i=j}^{p-1} \beta_{p-1} \cdots \beta_i ([\beta_i] - 1) = f_\beta(0, \ldots, 0, [\beta_j] - 1, \ldots, [\beta_{p-1}] - 1).
\]
Figure 14. The transformations $T_{9, \Delta \beta}$ where $\beta = (\frac{3}{2}, \frac{3}{2}, 4)$.

\[
\leq f_\beta(0, \ldots, 0, 1, 0, \ldots, 0) = \beta_{p-1} \cdots \beta_j.
\]

Conversely, suppose that (11) holds for all $j \in [1, p-2]$ and that $(c_0, \ldots, c_{p-1})$ and $(c'_0, \ldots, c'_{p-1})$ are $p$-tuples in $\prod_{i=0}^{p-1} \lfloor 0, [\beta_i] - 1 \rfloor$ such that $(c_0, \ldots, c_{p-1}) \prec_{\text{lex}} (c'_0, \ldots, c'_{p-1})$. Then there exists $j \in [0, p-1]$ such that $c_0 = c'_0, \ldots, c_{j-1} = c'_{j-1}$ and $c_j \leq c'_j - 1$. We get

\[
f_\beta(c_0, \ldots, c_{p-1}) \leq \sum_{i=0}^j \beta_{p-1} \cdots \beta_i c'_i - \beta_{p-1} \cdots \beta_{j+1} + \sum_{i=j+1}^{p-1} \beta_{p-1} \cdots \beta_{i+1}([\beta_i] - 1)
\]

\[
\leq \sum_{i=0}^j \beta_{p-1} \cdots \beta_i c'_i
\]

\[
\leq f_\beta(c'_0, \ldots, c'_{p-1}).
\]

\[\square\]

**Corollary 51.** If $p = 2$ then $T_{\beta_1 \beta_0, \Delta \beta} = \pi_2 \circ T_{\beta}^2 \circ \delta_0|_{[0, x_\beta]}$. In particular, $T_{\beta_1 \beta_0, \Delta \beta}|_{[0, 1)} = T_{\beta_1} \circ T_{\beta_0}.$

**Proof.** This follows from Theorem 45 and Proposition 50. \[\square\]

**Example 52.** Consider once more the alternate base $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$ from Example 6. Then $\Delta \beta = \{0, 1, \beta_1, \beta_1 + 1, 2\beta_1, 2\beta_1 + 1\}$ and $x_\beta = \frac{2\beta_1 + 1}{n_{\beta_1 \beta_0} - 1} = \frac{5+7\sqrt{13}}{18}$. The transformations $\pi_2 \circ T_{\beta}^2 \circ \delta_0|_{[0, x_\beta]}$ and $\pi_2 \circ L_{\beta}^2 \circ \delta_0|_{[0, x_\beta]}$ are depicted in Figure 15. By Corollary 51, they coincide with $T_{\beta_1 \beta_0, \Delta \beta}$ and $L_{\beta_1 \beta_0, \Delta \beta}$ respectively.

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Figure 15. The transformations $\pi_2 \circ T_2^\beta \circ \delta_0 \mid_{(0,x_\beta)}$ (left) and $\pi_2 \circ L_2^\beta \circ \delta_0 \mid_{(0,x_\beta)}$ (right) for $\beta = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$.

References