Dynamical properties of greedy and lazy alternate base expansions

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Backgrounds

Let $\beta > 1$, a β-**representation** of a real number x is an infinite word $a \in (\mathbb{N})^{\mathbb{N}}$ such that

$$\begin{aligned} x &= \frac{a_0}{\beta} + \frac{a_1}{\beta^2} + \frac{a_2}{\beta^3} + \cdots \\ &= \sum_{i=0}^{\infty} \frac{a_i}{\beta^{i+1}} \end{aligned}$$

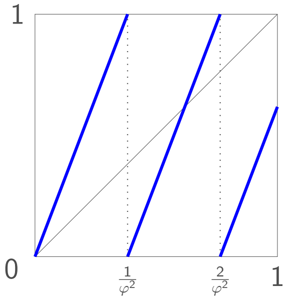
The **greedy** β-representation of $x \in [0, 1]$ is called greedy β-**expansion** of x and denoted $d_\beta(x) = a_0 a_1 \cdots \in [\![0, \lfloor \beta \rfloor]\!]^{\mathbb{N}}$

- $a_0 = \lfloor x\beta \rfloor$ and $r_0 = x\beta - a_0$
- $a_n = \lfloor r_{n-1}\beta \rfloor$ and $r_n = r_{n-1}\beta - a_n, \quad \forall n \geq 1$

On $[0,1)$, the greedy β-expansion is generated by iterating the transformation

$$T_\beta \colon\ [0,1) \to [0,1)\,,\ x \mapsto \beta x \bmod 1.$$

We have $a_n = \left\lfloor \beta T_\beta^n(x) \right\rfloor \in [\![0, \lceil \beta \rceil - 1]\!]$.



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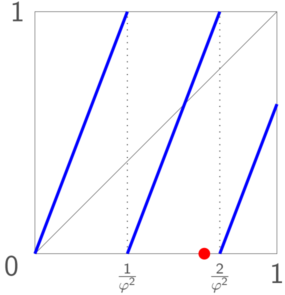

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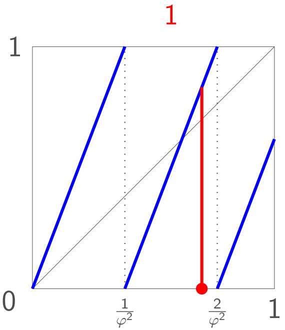

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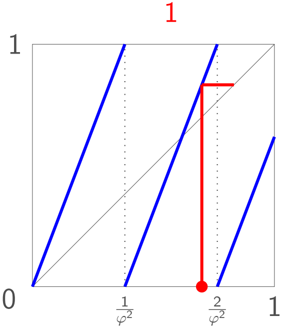

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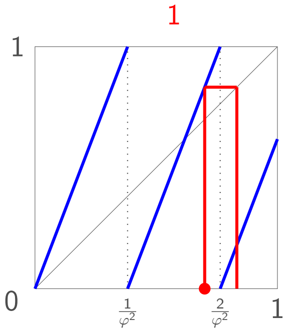

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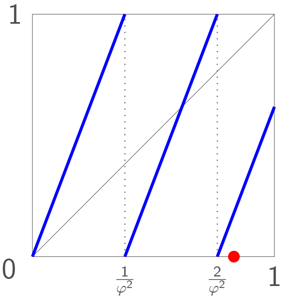

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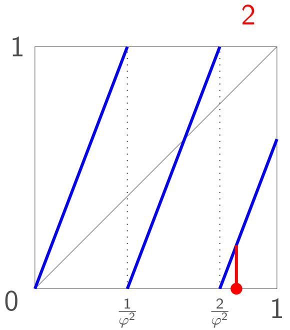

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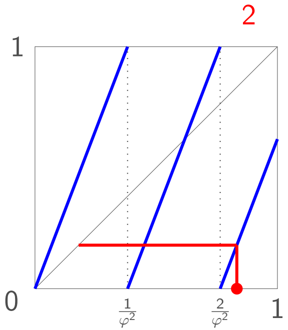

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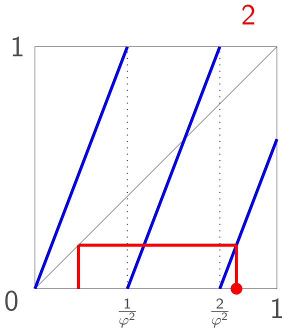

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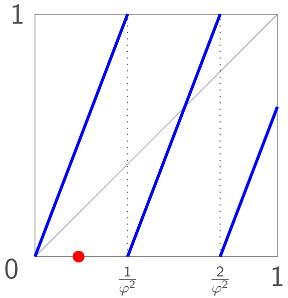

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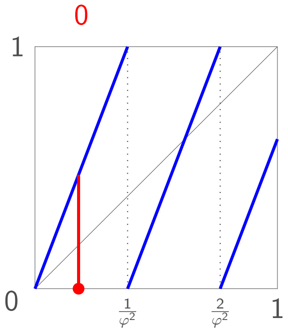

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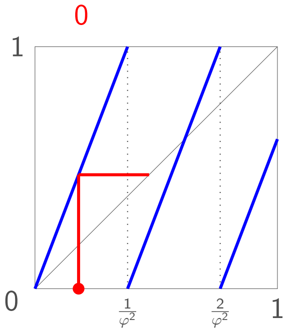

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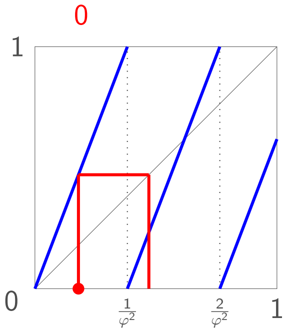

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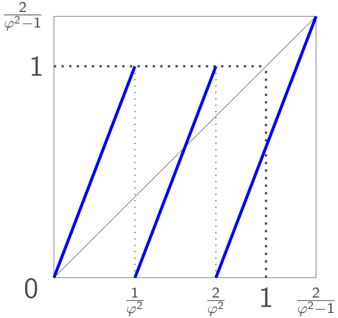
Let

$$x_\beta = \frac{\lceil \beta \rceil - 1}{\beta - 1}.$$

On $[0, x_\beta)$, the greedy β-expansion is generated by iterating the transformation

$$T_\beta \colon [0, x_\beta) \to [0, x_\beta),\ \ x \mapsto \begin{cases} \beta x - \lfloor \beta x \rfloor & \text{if } x \in [0,1) \\ \beta x - (\lceil \beta \rceil - 1) & \text{if } x \in [1, x_\beta). \end{cases}$$



The extended transformation T_{φ^2}.

Greedy algorithm:

If the first N digits of the greedy β-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest integer in $[\![0, \lceil\beta\rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} \leq x.$$

Lazy algorithm:

If the first N digits of the lazy β-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least integer in $[\![0, \lceil\beta\rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \sum_{n=N+1}^{\infty} \frac{\lceil\beta\rceil - 1}{\beta^{n+1}} \geq x.$$

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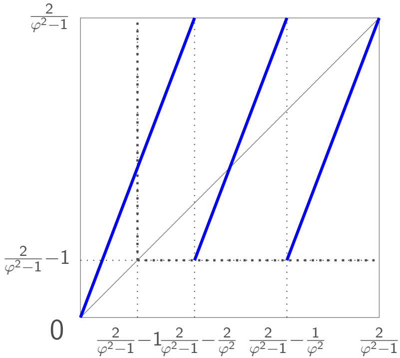
$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{\lceil\beta\rceil - 1}{\beta^{n+1}}}_{=\frac{x_\beta}{\beta^{N+1}}} \geq x.$$

On $(0, x_\beta]$, the lazy β-**expansion** is generated by iterating the transformation

$$L_\beta \colon (0, x_\beta] \to (0, x_\beta],\ x \mapsto \begin{cases} \beta x & \text{if } x \in (0, x_\beta - 1] \\ \beta x - \lceil \beta x - x_\beta \rceil & \text{if } x \in (x_\beta - 1, x_\beta]. \end{cases}$$



The transformation L_{φ^2}.

Alternate bases

Let $p \in \mathbb{N}_0$ and $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}) \in (\mathbb{R}_{>1})^p$.
A $\boldsymbol{\beta}$-representation of a real number x is a sequence $(a_i)_{i \in \mathbb{N}}$ such that

$$\begin{aligned} x = & \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \cdots + \frac{a_{p-1}}{\beta_0 \cdots \beta_{p-1}} \\ & + \frac{a_p}{(\beta_0 \cdots \beta_{p-1})\beta_0} + \frac{a_{p+1}}{(\beta_0 \cdots \beta_{p-1})\beta_0\beta_1} + \cdots + \frac{a_{2p-1}}{(\beta_0 \cdots \beta_{p-1})^2} \\ & + \cdots \end{aligned}$$

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Notation:
For all $i \in \mathbb{N}$, $\beta_i = \beta_{i \bmod p}$.

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The **greedy** $\boldsymbol{\beta}$-representation of $x \in [0,1]$, denoted $d_{\boldsymbol{\beta}}(x)$, is called the $\boldsymbol{\beta}$-**expansion** of x.

Let $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}) \in (\mathbb{R}_{>1})^p$, $x \in [0,1]$ and denote $d_{\boldsymbol{\beta}}(x) = (a_i)_{i \in \mathbb{N}}$.

- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
- $a_i = \lfloor r_{i-1}\beta_i \rfloor$ and $r_i = r_{i-1}\beta_i - a_i, \quad \forall i \geq 1$

In particular, for all $i \geq 0$, $a_i \in [\![0, \lfloor \beta_i \rfloor]\!]$.

Let $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_{\boldsymbol{\beta}}(1) = 2010^{\omega}.$$

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- $a_2 = \left\lfloor \left(\frac{-1+\sqrt{13}}{6}\right)\left(\frac{1+\sqrt{13}}{2}\right)\right\rfloor = 1, \quad r_2 = 0$

Combinatorial results (Charlier & C., 2020):

- The β-expansion of a real number $x \in [0,1]$ is the greatest of all the β-representations of x with respect to the lexicographic order.
- The β-expansion of 1 can't be purely periodic.
- The function d_β is increasing.
- Definition of a quasi-greedy representation $d^*_\beta(1)$.
- Generalization of Parry's theorem and its corollary.
- Characterization of the β-shift.
- Generalization of *Bertrand-Mathis'* theorem: The β-shift is sofic if and only if all quasi-greedy $\beta^{(i)}$-expansions of 1 are ultimately periodic, where $\beta^{(i)} = (\beta_i, \dots, \beta_{p-1}, \beta_0, \dots, \beta_{i-1})$.

$\vdots$

Let $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}) \in (\mathbb{R}_{\geq 1})^p$. We define the transformation

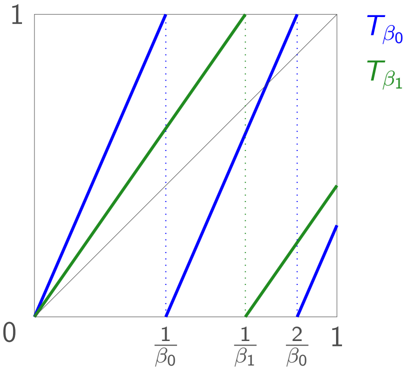
$$\begin{aligned} T_{\boldsymbol{\beta}} : [\![0, p-1]\!] \times [0,1) &\to [\![0, p-1]\!] \times [0,1)\,, \\ (i, x) &\mapsto (i+1 \bmod p, T_{\beta_i}(x))\,. \end{aligned}$$

The greedy $\boldsymbol{\beta}$-expansion of $x \in [0,1)$ is obtained by alternating the p maps

$$T_{\beta_0}, \ldots, T_{\beta_{p-1}}.$$

Consider $\beta = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_\beta(\tfrac{1+\sqrt{5}}{5}) =$$



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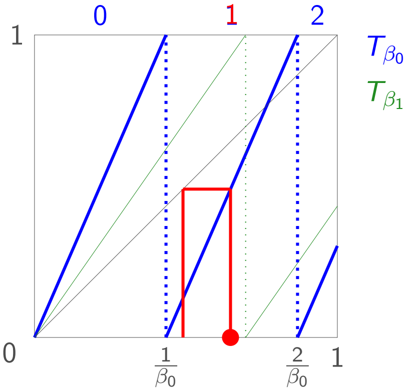
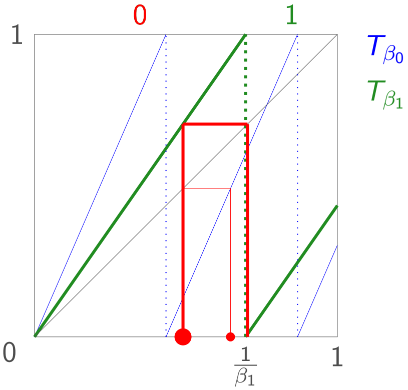
$$d_\beta(\tfrac{1+\sqrt{5}}{5}) = 1$$



Consider $\beta = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_\beta(\tfrac{1+\sqrt{5}}{5}) = 10$$



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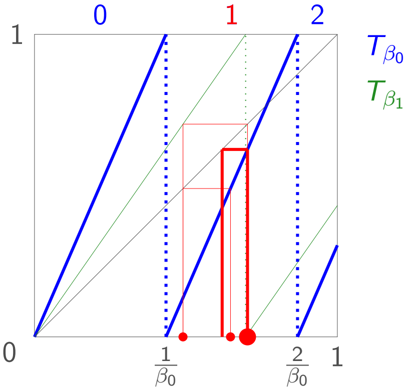
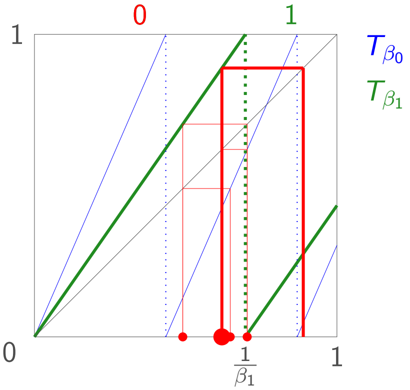


Consider $\beta = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_\beta(\frac{1+\sqrt{5}}{5}) = 1010$$



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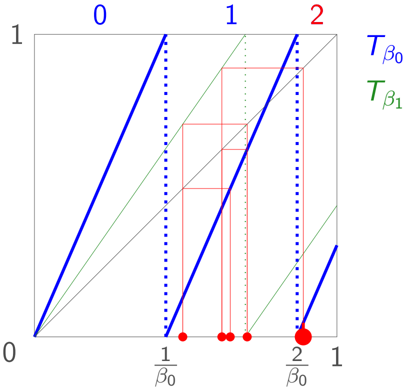
$$d_\beta(\tfrac{1+\sqrt{5}}{5}) = 10102\cdots$$



Let

$$x_{\boldsymbol{\beta}} = \sum_{n=0}^{\infty} \frac{\lceil \beta_n \rceil - 1}{\prod_{k=0}^{n} \beta_k}.$$

The extended greedy $\boldsymbol{\beta}$-expansion is generated by iterating the transformation

$$T_{\boldsymbol{\beta}} \colon \bigcup_{i=0}^{p-1} \left(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\right) \to \bigcup_{i=0}^{p-1} \left(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\right),$$
$$(i, x) \mapsto \begin{cases} \left((i+1) \bmod p, \beta_i x - \lfloor \beta_i x \rfloor\right) & \text{if } x \in [0,1) \\ \left((i+1) \bmod p, \beta_i x - (\lceil \beta_i \rceil - 1)\right) & \text{if } x \in [1, x_{\boldsymbol{\beta}^{(i)}}). \end{cases}$$

The greedy $\boldsymbol{\beta}$-expansion of $x \in [0, x_{\boldsymbol{\beta}})$ is obtained by alternating the p maps

$$\pi_2 \circ T_{\boldsymbol{\beta}} \circ \delta_i\big|_{[0,x_{\boldsymbol{\beta}^{(i)}})} \colon [0, x_{\boldsymbol{\beta}^{(i)}}) \to [0, x_{\boldsymbol{\beta}^{(i+1)}})$$

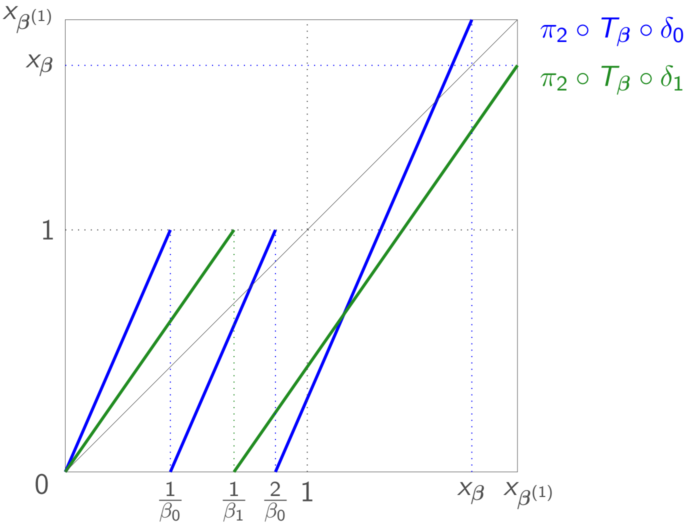
for $i \in [\![0, p-1]\!]$, where

$$\pi_2 \colon \mathbb{N} \times \mathbb{R} \to \mathbb{R},\ (i, x) \mapsto x$$

and

$$\delta_i \colon \mathbb{R} \to \{i\} \times \mathbb{R},\ x \mapsto (i, x).$$

The maps $\pi_2 \circ T_\beta \circ \delta_0\big|_{[0,x_\beta)}$ and $\pi_2 \circ T_\beta \circ \delta_1\big|_{[0,x_{\beta(1)})}$ with $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$



Greedy algorithm:

If the first N digits of the greedy $\boldsymbol{\beta}$-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest integer in $[\![0, \lceil \beta_N \rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} \leq x.$$

Lazy algorithm:

If the first N digits of the lazy $\boldsymbol{\beta}$-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least integer in $[\![0, \lceil \beta_N \rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} + \sum_{n=N+1}^{\infty} \frac{\lceil \beta_n \rceil - 1}{\prod_{k=0}^{n} \beta_k} \geq x.$$

Greedy algorithm:

If the first N digits of the greedy $\boldsymbol{\beta}$-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest integer in $[\![0, \lceil\beta_N\rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} \leq x.$$

Lazy algorithm:

If the first N digits of the lazy $\boldsymbol{\beta}$-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least integer in $[\![0, \lceil\beta_N\rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\prod_{k=0}^{n} \beta_k} + \underbrace{\sum_{n=N+1}^{\infty} \frac{\lceil\beta_n\rceil - 1}{\prod_{k=0}^{n} \beta_k}}_{=\frac{x_{\boldsymbol{\beta}^{(N+1)}}}{\prod_{k=0}^{N} \beta_k}} \geq x.$$

On $(0, x_{\boldsymbol{\beta}}]$, the lazy $\boldsymbol{\beta}$-**expansion** is generated by iterating the transformation

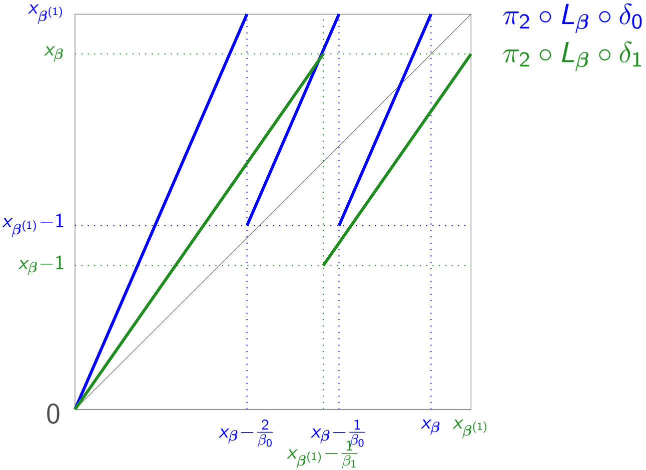
$$L_{\boldsymbol{\beta}} \colon \bigcup_{i=0}^{p-1} \left(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\right) \to \bigcup_{i=0}^{p-1} \left(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\right),$$
$$(i, x) \mapsto \begin{cases} \left((i+1) \bmod p, \beta_i x\right) & \text{if } x \in (0, x_{\boldsymbol{\beta}^{(i)}} - 1] \\ \left((i+1) \bmod p, \beta_i x - \lceil \beta_i x - x_{\boldsymbol{\beta}^{(i+1)}} \rceil\right) & \text{if } x \in \left(x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}\right]. \end{cases}$$

The lazy $\boldsymbol{\beta}$-expansion of $x \in (0, x_{\boldsymbol{\beta}}]$ is obtained by alternating the p maps

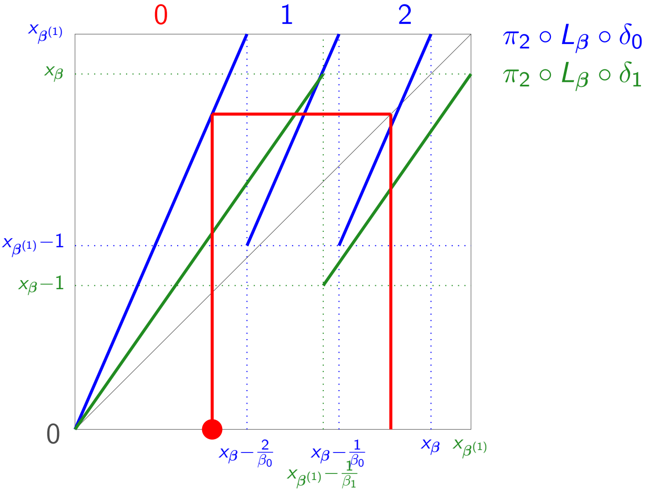
$$\pi_2 \circ L_{\boldsymbol{\beta}} \circ \delta_i \Big|_{(0, x_{\boldsymbol{\beta}^{(i)}}]} \colon (0, x_{\boldsymbol{\beta}^{(i)}}] \to (0, x_{\boldsymbol{\beta}^{(i+1)}}]$$

for $i \in [\![0, p-1]\!]$.

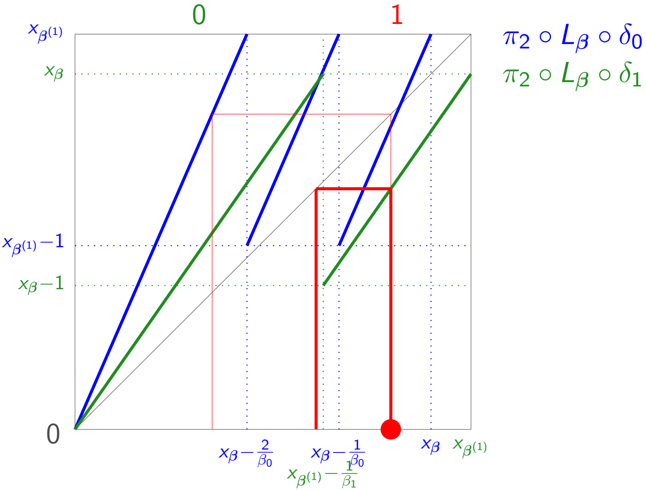
Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. The first five digits of the lazy β-expansion of $\frac{1+\sqrt{5}}{5}$ are



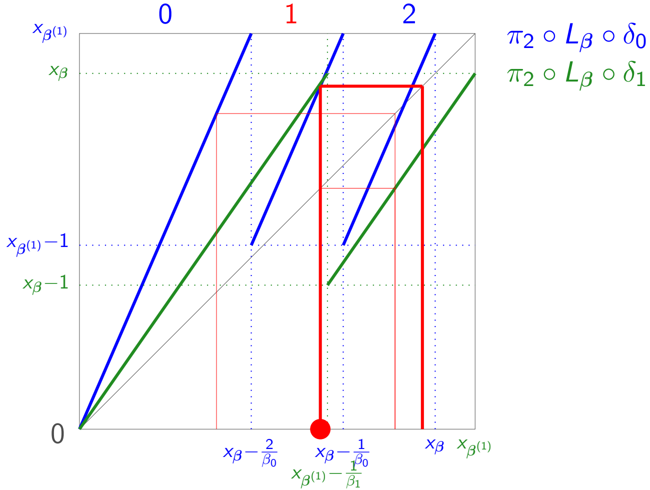
Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. The first five digits of the lazy β-expansion of $\frac{1+\sqrt{5}}{5}$ are 0



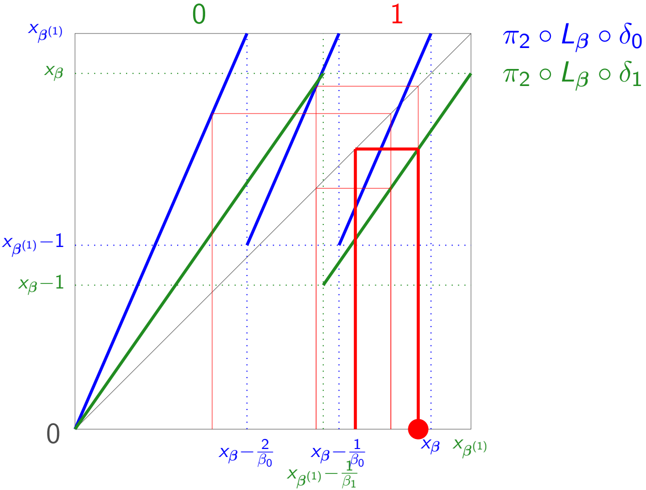
Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. The first five digits of the lazy β-expansion of $\frac{1+\sqrt{5}}{5}$ are 01



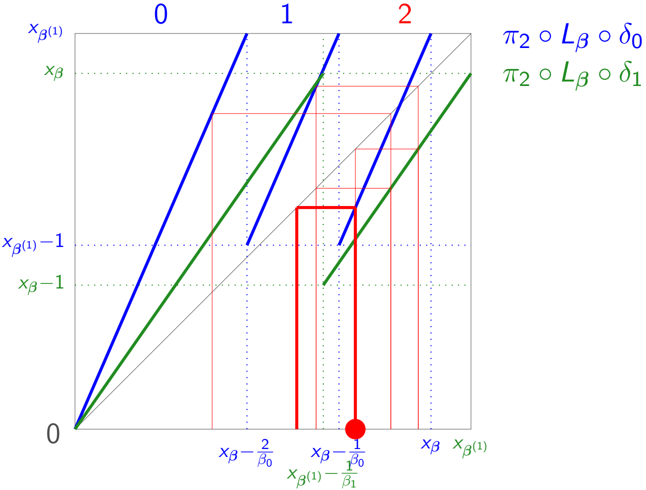
Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. The first five digits of the lazy β-expansion of $\frac{1+\sqrt{5}}{5}$ are 011



Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. The first five digits of the lazy β-expansion of $\frac{1+\sqrt{5}}{5}$ are 0111



Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. The first five digits of the lazy β-expansion of $\frac{1+\sqrt{5}}{5}$ are 01112.



Measure Theory

Consider a probability space $(X, \mathcal{F}, \mu)$ and a map $T\colon X \to X$.

- μ is T-invariant: $\mu(T^{-1}B) = \mu(B)$.
- μ absolutely continuous w.r.t ν: $\nu(B) = 0 \Rightarrow \mu(B) = 0$.
- μ is equivalent to ν: $\mu(B) = 0 \Leftrightarrow \nu(B) = 0$.
- T is ergodic: $T^{-1}B = B \Rightarrow \mu(B) = 0$ or 1.

Theorem (Rényi, 1957)

There exists a unique T_β-invariant absolutely continuous probability measure μ_β on $\mathcal{B}([0,1))$. Furthermore, the measure μ_β is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$ and the dynamical system $([0,1), \mathcal{B}([0,1)), \mu_\beta, T_\beta)$ is ergodic and has entropy $\log(\beta)$.

Question:

- Can we find a $T_{\boldsymbol{\beta}}$-invariant probability measure $\mu_{\boldsymbol{\beta}}$ equivalent to "Lebesgue" and such that the transformation $T_{\boldsymbol{\beta}}$ is ergodic?

Steps:

- For all $i \in [\![0, p-1]\!]$, we consider the alternate base

$$\boldsymbol{\beta}^{(i)} = (\beta_i, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{i-1})$$

  and we find a $\left(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_0} \circ T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_i}\right)$-invariant probability measure $\mu_{\boldsymbol{\beta},i}$, equivalent to Lebesgue and such that $T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_0} \circ T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_i}$ is ergodic.
- With the probability measures $\mu_{\boldsymbol{\beta},i}$ we construct a probability measure $\mu_{\boldsymbol{\beta}}$.
- We prove the properties on $\mu_{\boldsymbol{\beta}}$.

For all $n \in \mathbb{N}_{\geq 1}$ and all $\beta_0, \ldots, \beta_{n-1} > 1$, the map $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}$ is piecewise linear with a slope $\beta_0 \cdots \beta_{n-1} > 1$. From a result of Lasota and Yorke, we get the following.

Theorem (Charlier, C. & Dajani)

For all $n \in \mathbb{N}_{\geq 1}$ and all $\beta_0, \ldots, \beta_{n-1} > 1$, there exists a unique $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$-invariant absolutely continuous probability measure μ on $\mathcal{B}([0,1))$. Furthermore, the measure μ is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$, its density is bounded and decreasing, and the dynamical system $([0,1), \mathcal{B}([0,1)), \mu, T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$ is exact and has entropy $\log(\beta_0 \cdots \beta_{n-1})$.

Remark: Exactness implies ergodicity.

For all $i \in [\![0, p-1]\!]$, consider the alternate base

$$\boldsymbol{\beta}^{(i)} = (\beta_i, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{i-1})$$

and let $\mu_{\boldsymbol{\beta},i}$ be the measure given by the theorem for

$$T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_0} \circ T_{\beta_{p-1}} \circ \cdots \circ T_{\beta_i}.$$

We define the σ-algebra

$$\mathcal{T}_p = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) \colon \forall i \in [\![0, p-1]\!],\ B_i \in \mathcal{B}([0,1)) \right\}.$$

We define the probability measure $\mu_{\boldsymbol{\beta}}$ on $\mathcal{T}_p$ as follows:
For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$,

$$\mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1} (\{i\} \times B_i) \right) = \frac{1}{p} \sum_{i=0}^{p-1} \mu_{\boldsymbol{\beta},i}(B_i).$$

We define an *extended Lebesgue measure* λ_p over $\mathcal{T}_p$ as follows. For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$,

$$\lambda_p\Big(\bigcup_{i=0}^{p-1}(\{i\} \times B_i)\Big) = \frac{1}{p}\sum_{i=0}^{p-1}\lambda(B_i).$$

Theorem (Charlier, C. & Dajani)

The measure μ_{β} is the unique T_{β}-invariant probability measure on $\mathcal{T}_p$ that is absolutely continuous with respect to λ_p. Furthermore, μ_{β} is equivalent to λ_p on $\mathcal{T}_p$ and the dynamical system $([\![0, p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\beta}, T_{\beta})$ is ergodic and has entropy $\frac{1}{p}\log(\beta_0 \cdots \beta_{p-1})$.

The result can be extended on the dynamical system

$$\Big(\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol\beta^{(i)}})\big),\mathcal{T}_{\boldsymbol\beta},\mu_{\boldsymbol\beta},T_{\boldsymbol\beta}\Big)$$

where

$$\mathcal{T}_{\boldsymbol\beta}=\left\{\bigcup_{i=0}^{p-1}(\{i\}\times B_i)\colon \forall i\in[\![0,p-1]\!],\ B_i\in\mathcal{B}([0,x_{\boldsymbol\beta^{(i)}}))\right\}$$

by setting

$$\mu_{\boldsymbol\beta}(A)=\mu_{\boldsymbol\beta}\big(A\cap\big([\![0,p-1]\!]\times[0,1)\big)\big)$$

and

$$\lambda_p(A)=\lambda_p\big(A\cap\big([\![0,p-1]\!]\times[0,1)\big)\big).$$

Isomorphisms

Two dynamical systems

$$(X, \mathcal{F}_X, \mu_X, T_X)$$

and

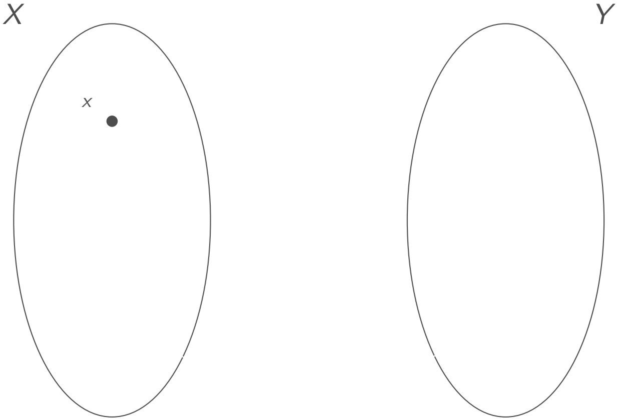
$$(Y, \mathcal{F}_Y, \mu_Y, T_Y)$$

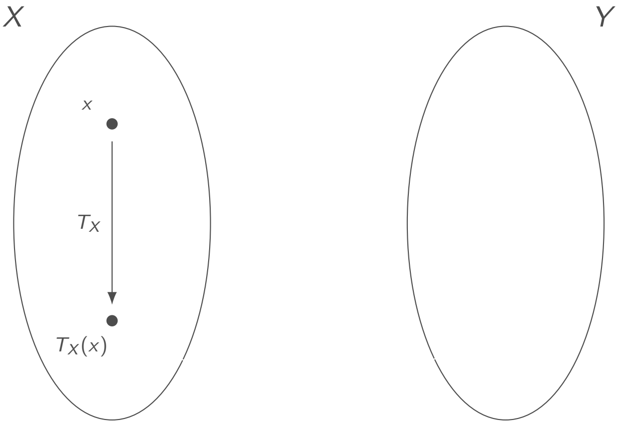
are (measurably) **isomorphic** if there exists an almost everywhere bijective measurable map $\psi \colon X \to Y$ such that

$$\psi \circ T_X = T_Y \circ \psi$$

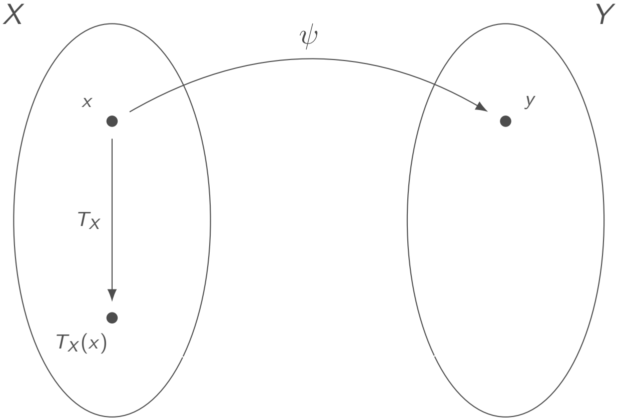
and

$$\mu_Y = \mu_X \circ \psi^{-1}.$$

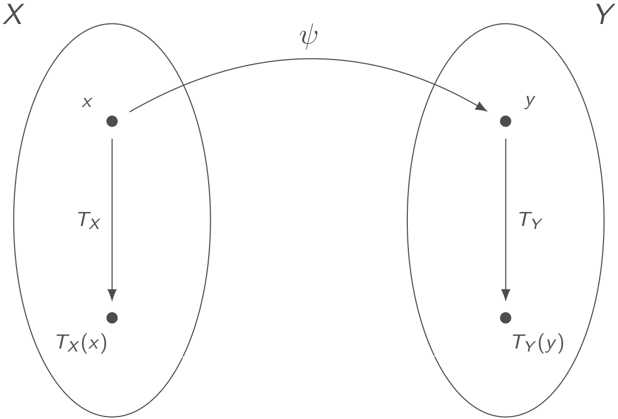

X
x
Y

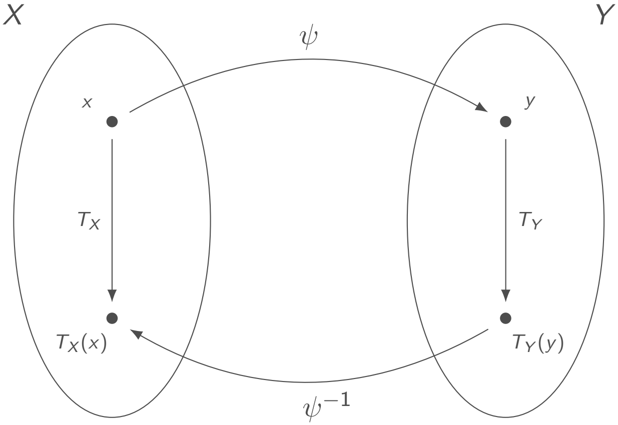


X
Y
x
T_X
T_X(x)

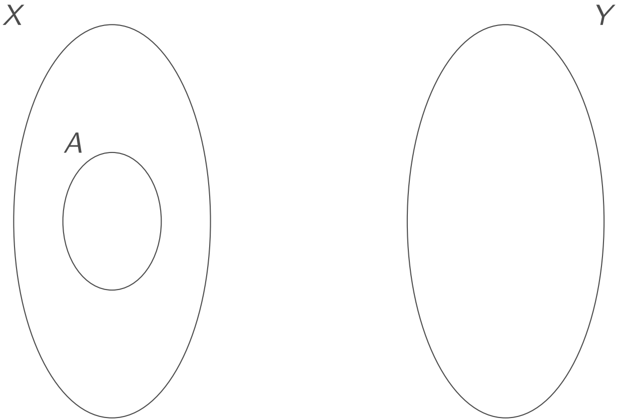


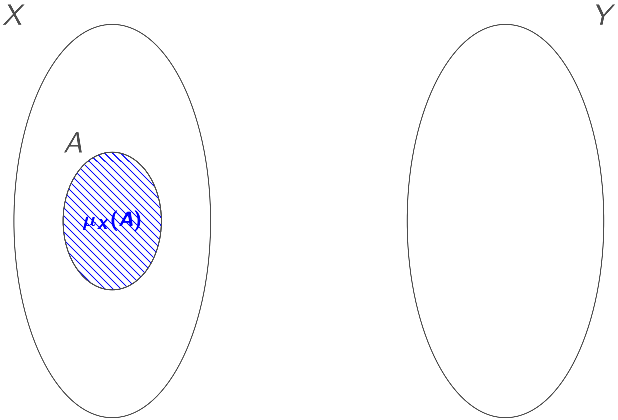
X
Y
ψ
x
y
T_X
$T_X(x)$

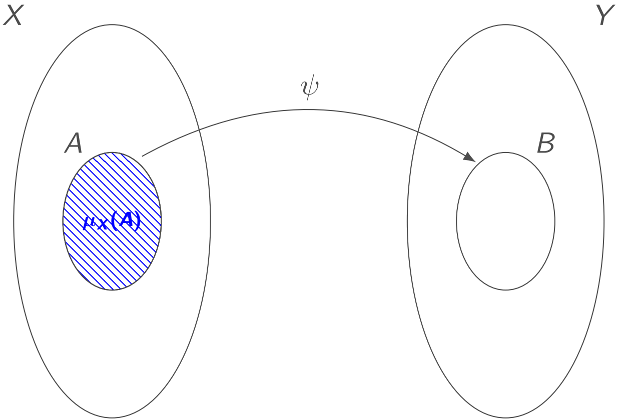

X
Y
ψ
x
y
T_X
T_Y
$T_X(x)$
$T_Y(y)$

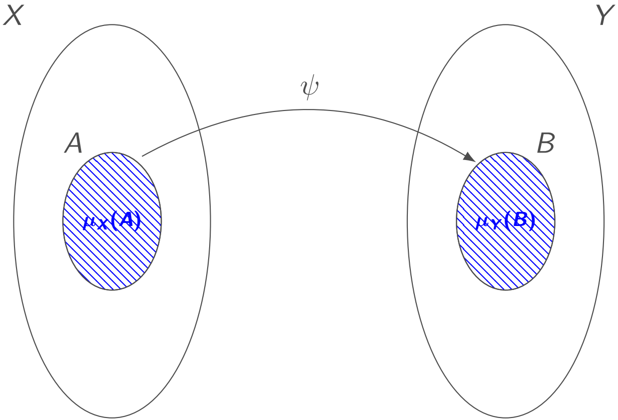


X
Y
ψ
x
y
T_X
T_Y
$T_X(x)$
$T_Y(y)$
ψ^{-1}


X
A
Y


X
A
$\mu_X(A)$
Y


X
Y
ψ
A
B
μX(A)


X
Y
ψ
A
B
μX(A)
μY(B)

Greedy-Lazy isomorphism

Consider the map

$$\phi_{\boldsymbol\beta}\colon \bigcup_{i=0}^{p-1} \left(\{i\}\times[0, x_{\boldsymbol\beta^{(i)}})\right) \to \bigcup_{i=0}^{p-1} \left(\{i\}\times(0, x_{\boldsymbol\beta^{(i)}}]\right),\ (i,x) \mapsto \left(i, x_{\boldsymbol\beta^{(i)}} - x\right).$$

We have

$$\phi_{\boldsymbol\beta} \circ T_{\boldsymbol\beta} = L_{\boldsymbol\beta} \circ \phi_{\boldsymbol\beta}.$$

Hence, if $x \in [0, x_{\boldsymbol\beta})$ has greedy $\boldsymbol\beta$-expansion

$$a_0 a_1 a_2 \cdots ,$$

then $\pi_2(\phi_{\boldsymbol\beta}(0,x)) = x_{\boldsymbol\beta} - x$ has lazy $\boldsymbol\beta$-expansion

$$(\lceil\beta_0\rceil - 1 - a_0)(\lceil\beta_1\rceil - 1 - a_1)(\lceil\beta_2\rceil - 1 - a_2)\cdots .$$

We define the σ-algebra

$$\mathcal{L}_{\beta} = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) \colon \forall i \in [\![0, p-1]\!],\ B_i \in \mathcal{B}\big((0, x_{\beta^{(i)}}]\big) \right\}$$

on $\bigcup_{i=0}^{p-1} \big(\{i\} \times \big(0, x_{\beta^{(i)}}\big]\big)$.

Theorem (Charlier, C. & Dajani)

The map ϕ_{β} is an isomorphism between the dynamical systems

$$\Big(\bigcup_{i=0}^{p-1} \big(\{i\} \times [0, x_{\beta^{(i)}})\big), \mathcal{T}_{\beta}, \mu_{\beta}, T_{\beta} \Big)$$

and

$$\Big(\bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\beta^{(i)}}]\big), \mathcal{L}_{\beta}, \mu_{\beta} \circ \phi_{\beta}^{-1}, L_{\beta} \Big).$$

Theorem (Charlier, C. & Dajani)

The measure $\mu_{\boldsymbol\beta} \circ \phi_{\boldsymbol\beta}^{-1}$ is the unique $L_{\boldsymbol\beta}$-invariant probability measure on $\mathcal{L}_{\boldsymbol\beta}$ that is absolutely continuous with respect to $\lambda_p \circ \phi_{\boldsymbol\beta}^{-1}$. Furthermore, $\mu_{\boldsymbol\beta} \circ \phi_{\boldsymbol\beta}^{-1}$ is equivalent to $\lambda_p \circ \phi_{\boldsymbol\beta}^{-1}$ on $\mathcal{L}_{\boldsymbol\beta}$ and the dynamical system $\left(\bigcup_{i=0}^{p-1} \left(\{i\} \times (0, x_{\boldsymbol\beta^{(i)}}]\right), \mathcal{L}_{\boldsymbol\beta}, \mu_{\boldsymbol\beta} \circ \phi_{\boldsymbol\beta}^{-1}, L_{\boldsymbol\beta}\right)$ is ergodic and has entropy $\frac{1}{p}\log(\beta_0 \cdots \beta_{p-1})$.

Remark: Analogue result on $\bigcup_{i=0}^{p-1} \left(\{i\} \times (x_{\boldsymbol\beta^{(i)}} - 1, x_{\boldsymbol\beta^{(i)}}]\right)$.

β-shift isomorphism

Let

$$D_\beta = \{d_\beta(x) \colon x \in [0,1)\} \quad \text{and} \quad S_\beta = \overline{D_\beta}.$$

Lemma (Charlier & C., 2020)

For all $n \in \mathbb{N}$, if $w \in D_{\beta^{(n)}}$ (resp. $S_{\beta^{(n)}}$) then $\sigma(w) \in D_{\beta^{(n+1)}}$ (resp. $S_{\beta^{(n+1)}}$).

Consider the σ-algebra

$$\mathcal{G}_{\boldsymbol{\beta}} = \left\{ \bigcup_{i=0}^{p-1} \left(\{i\} \times (C_i \cap S_{\boldsymbol{\beta}^{(i)}})\right) \colon C_i \in \mathcal{C}_{A_{\boldsymbol{\beta}}} \right\}$$

on $\bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}})$ where

$$A_{\boldsymbol{\beta}} = [\![0, \max_{i \in [\![0,p-1]\!]} \lceil \beta_i \rceil - 1]\!]$$

and, for all $\ell \in \mathbb{N}$ and $a_0, \ldots, a_{\ell-1} \in A_{\boldsymbol{\beta}}$

$$C_{A_{\boldsymbol{\beta}}}(a_0, \ldots, a_{\ell-1}) = \{w \in A_{\boldsymbol{\beta}}^{\mathbb{N}} \colon w[0] = a_0, \ldots, w[\ell-1] = a_{\ell-1}\}.$$

We define the maps

$$\begin{aligned}\sigma_p \colon \bigcup_{i=0}^{p-1}(\{i\} \times S_{\beta^{(i)}}) &\to \bigcup_{i=0}^{p-1}(\{i\} \times S_{\beta^{(i)}}),\\ (i,w) &\mapsto ((i+1) \bmod p, \sigma(w))\\ \psi_{\boldsymbol{\beta}} \colon [\![0, p-1]\!] \times [0,1) &\to \bigcup_{i=0}^{p-1}(\{i\} \times S_{\beta^{(i)}}),\\ (i,x) &\mapsto (i, d_{\beta^{(i)}}(x)).\end{aligned}$$

We have

$$\psi_{\boldsymbol{\beta}} \circ T_{\boldsymbol{\beta}} = \sigma_p \circ \psi_{\boldsymbol{\beta}}.$$

Theorem (Charlier, C. & Dajani)

The map ψ_β defines an isomorphism between the dynamical systems

$$\big([\![0, p-1]\!] \times [0,1), \mathcal{T}_p, \mu_\beta, T_\beta\big)$$

and

$$\Big(\bigcup_{i=0}^{p-1}(\{i\} \times S_{\beta^{(i)}}), \mathcal{G}_\beta, \mu_\beta \circ \psi_\beta^{-1}, \sigma_p\Big).$$

Remark: Analogue result on the measure $\mu_\beta \circ \psi_\beta^{-1}$.

$\boldsymbol{\beta}$-expansions and $(\beta_0\beta_1 \cdots \beta_{p-1})$-expansions

Let $\boldsymbol{\beta} = (\beta_0, \dots, \beta_{p-1}) \in (\mathbb{R}_{\geq 1})^p$. By definition of the $d_{\boldsymbol{\beta}}(x)$, we have

$$\begin{aligned} x = & \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \cdots + \frac{a_{p-1}}{\beta_0\cdots\beta_{p-1}} \\ & + \frac{a_p}{(\beta_0\cdots\beta_{p-1})\beta_0} + \frac{a_{p+1}}{(\beta_0\cdots\beta_{p-1})\beta_0\beta_1} + \cdots + \frac{a_{2p-1}}{(\beta_0\cdots\beta_{p-1})^2} \\ & + \cdots . \end{aligned}$$

That can be rewritten as follows

$$\begin{aligned} x = & \frac{a_0\beta_1\cdots\beta_{p-1} + a_1\beta_2\cdots\beta_{p-1} + \cdots + a_{p-1}}{\beta_0\beta_1\cdots\beta_{p-1}} \\ & + \frac{a_p\beta_1\cdots\beta_{p-1} + a_{p+1}\beta_2\cdots\beta_{p-1} + \cdots + a_{2p-1}}{(\beta_0\beta_1\cdots\beta_{p-1})^2} \\ & + \cdots . \end{aligned}$$

Define the digit set

$$\begin{aligned}\Delta_\beta =& \{a_0\beta_1 \cdots \beta_{p-1} + a_1\beta_2 \cdots \beta_{p-1} + \cdots + a_{p-2}\beta_{p-1} + a_{p-1} \colon \\ & a_j \in [\![0, \lceil \beta_j \rceil - 1]\!], j \in [\![0, p-1]\!]\} \\ =& \left\{ \sum_{j=0}^{p-1} a_j \beta_{j+1} \cdots \beta_{p-1} \colon a_j \in [\![0, \lceil \beta_j \rceil - 1]\!], j \in [\![0, p-1]\!] \right\}.\end{aligned}$$

Questions:

- Can we compare the maps $\pi_2 \circ T_{\boldsymbol{\beta}}^p \circ \delta_0$ and $T_{\beta_0 \cdots \beta_{p-1}, \Delta_{\boldsymbol{\beta}}}$ on $[0, x_{\boldsymbol{\beta}})$?
- Can we compare the maps $\pi_2 \circ L_{\boldsymbol{\beta}}^p \circ \delta_0$ and $L_{\beta_0 \cdots \beta_{p-1}, \Delta_{\boldsymbol{\beta}}}$ on $(0, x_{\boldsymbol{\beta}}]$?

Backgrounds on expansions over a general digit set

Consider $\beta > 1$ and a digit set

$$\Delta = \{0 = d_0 < d_1 < \cdots < d_m\} \subset \mathbb{R}\,.$$

The word $a = a_0 a_1 \cdots$ over Δ is a (β, Δ)-representation of $x \in \left[0, \frac{d_m}{\beta - 1}\right)$ if

$$x = \sum_{i=0}^{\infty} \frac{a_i}{\beta^{i+1}}.$$

Greedy algorithm:
If the first N digits of the greedy (β, Δ)-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest element in Δ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} \leq x.$$

Lazy algorithm:
If the first N digits of the lazy (β, Δ)-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least element in Δ such that

$$\sum_{i=0}^{N} \frac{a_i}{\beta^{i+1}} + \sum_{i=N+1}^{\infty} \frac{d_m}{\beta^{i+1}} \geq x.$$

Greedy algorithm:

If the first N digits of the greedy (β, Δ)-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest element in Δ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} \leq x.$$

Lazy algorithm:

If the first N digits of the lazy (β, Δ)-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least element in Δ such that

$$\sum_{i=0}^{N} \frac{a_i}{\beta^{i+1}} + \underbrace{\sum_{i=N+1}^{\infty} \frac{d_m}{\beta^{i+1}}}_{=\frac{\frac{d_m}{\beta-1}}{\beta^{N+1}}} \geq x.$$

The transformation that generates the greedy and lazy (β, Δ)-expansions are

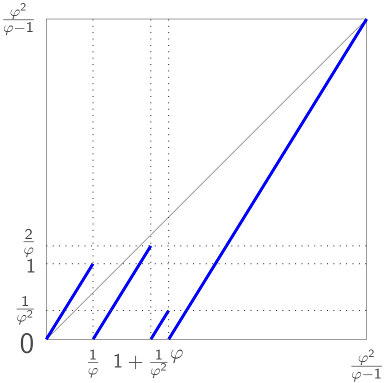
$$T_{\beta,\Delta} \colon [0, \tfrac{d_m}{\beta-1}) \to [0, \tfrac{d_m}{\beta-1}]),$$
$$x \mapsto \begin{cases} \beta x - d_i & \text{if } x \in \left[\frac{d_i}{\beta}, \frac{d_{i+1}}{\beta}\right), \text{ for } i \in [\![0, m-1]\!] \\ \beta x - d_m & \text{if } x \in \left[\frac{d_m}{\beta}, \frac{d_m}{\beta-1}\right) \end{cases}$$

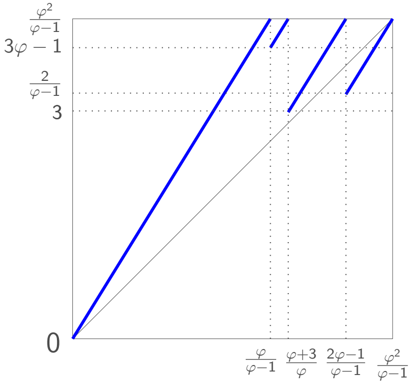
and

$$L_{\beta,\Delta} \colon (0, \tfrac{d_m}{\beta-1}] \to (0, \tfrac{d_m}{\beta-1}],$$
$$x \mapsto \begin{cases} \beta x & \text{if } x \in (0, \frac{d_m}{\beta-1} - \frac{d_m}{\beta}] \\ \beta x - d_k & \text{if } x \in (\frac{d_m}{\beta-1} - \frac{d_m - d_{k-1}}{\beta}, \frac{d_m}{\beta-1} - \frac{d_m - d_k}{\beta}], \ k \in [\![1, m]\!]. \end{cases}$$

The maps $T_{\beta,\Delta}$ and $L_{\beta,\widetilde{\Delta}}$ are isomorphic where

$$\widetilde{\Delta} := \{0, d_m - d_{m-1}, \ldots, d_m - d_1, d_m\}.$$





The transformations $T_{\varphi,\Delta}$ and $L_{\varphi,\widetilde{\Delta}}$ for $\Delta = \{0, 1, \frac{\varphi+1}{\varphi}, \varphi^2\}$.

β-Expansions and $(\beta_0 \cdots \beta_{p-1}, \Delta_\beta)$-expansions

Define

$$f_\beta \colon [\![0, \lceil \beta_0 \rceil - 1]\!] \times \cdots \times [\![0, \lceil \beta_{p-1} \rceil - 1]\!] \to \Delta_\beta,$$
$$(a_0, \dots, a_{p-1}) \mapsto \sum_{j=0}^{p-1} a_j \beta_{j+1} \cdots \beta_{p-1}.$$

We get $\Delta_\beta = \mathrm{im}(f_\beta)$ and

$$x_\beta = \frac{d_m}{\beta_0 \cdots \beta_{p-1} - 1}.$$

Lemma

For all $x \in [0, x_\beta)$, we have

$$\pi_2 \circ T_\beta^p \circ \delta_0(x) = \beta_{p-1} \cdots \beta_0 x - f_\beta(c)$$

where c is the lexicographically greatest p-tuple in $[\![0, \lceil \beta_0 \rceil - 1]\!] \times \cdots \times [\![0, \lceil \beta_{p-1} \rceil - 1]\!]$ such that $\frac{f_\beta(c)}{\beta_{p-1} \cdots \beta_0} \le x$.

Proposition (Charlier, C. & Dajani)

We have

$$T_{\beta_0 \cdots \beta_{p-1}, \Delta_\beta}(x) \le \pi_2 \circ T_\beta^p \circ \delta_0(x), \quad \forall x \in [0, x_\beta) \tag{1}$$

and

$$L_{\beta_0 \cdots \beta_{p-1}, \Delta_\beta}(x) \ge \pi_2 \circ L_\beta^p \circ \delta_0(x), \quad \forall x \in (0, x_\beta]. \tag{2}$$

Moreover, equality in (1) occurs if and only if equality in (2) do.

Lemma

The function f_β is non decreasing if and only if for all $j \in [\![1, p-2]\!]$,

$$\sum_{i=j}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} \leq \beta_j \cdots \beta_{p-1}.$$

Theorem (Charlier, C. & Dajani)

If the function f_β is non-decreasing then

$$T_{\beta_0 \cdots \beta_{p-1}, \Delta_\beta} = \pi_2 \circ T_\beta^p \circ \delta_0 \big|_{[0, x_\beta)}$$

and

$$L_{\beta_0 \cdots \beta_{p-1}, \Delta_\beta} = \pi_2 \circ L_\beta^p \circ \delta_0 \big|_{(0, x_\beta]}.$$

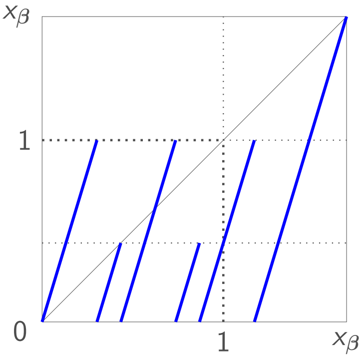
In particular, for all length-2 alternate bases equalities hold.

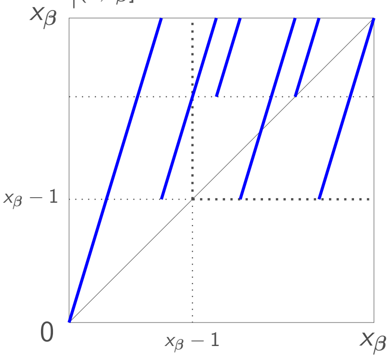
Let $\beta = (\beta_0, \beta_1) = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. Then

$$T_{\beta_0\beta_1,\Delta_\beta} = \pi_2 \circ T_\beta^2 \circ \delta_0\big|_{[0,x_\beta)}$$

and

$$L_{\beta_0\beta_1,\Delta_\beta} = \pi_2 \circ L_\beta^2 \circ \delta_0\big|_{(0,x_\beta]}.$$



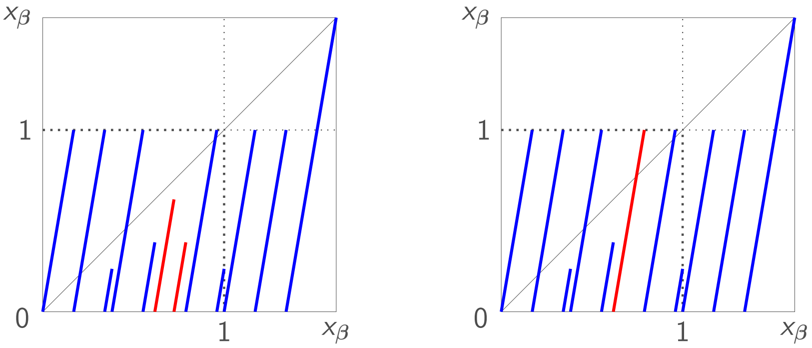


Let $\beta = (\varphi, \varphi, \sqrt{5})$. Then

$$T_{\sqrt{5}\varphi^2, \Delta_\beta} \neq \pi_2 \circ T_\beta^2 \circ \delta_0 \Big|_{[0, x_\beta)}$$

and

$$L_{\sqrt{5}\varphi^2, \Delta_\beta} \neq \pi_2 \circ L_\beta^2 \circ \delta_0 \Big|_{(0, x_\beta]}.$$

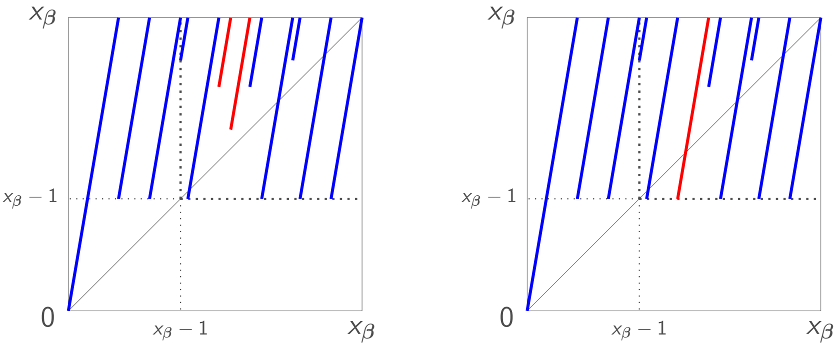


Let $\beta = (\varphi, \varphi, \sqrt{5})$. Then

$$T_{\sqrt{5}\varphi^2,\Delta_\beta} \neq \pi_2 \circ T_\beta^2 \circ \delta_0\Big|_{[0,x_\beta)}$$

and

$$L_{\sqrt{5}\varphi^2,\Delta_\beta} \neq \pi_2 \circ L_\beta^2 \circ \delta_0\Big|_{(0,x_\beta]}.$$

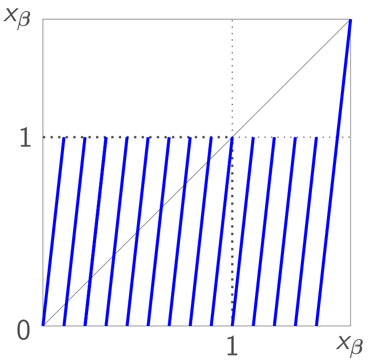


Let $\beta = (\frac{3}{2}, \frac{3}{2}, 4)$. We have $\Delta_\beta = [\![0, 13]\!]$. The map f_β is not non-decreasing:

$$f_\beta(0,1,3) = 7 > 6 = f_\beta(1,0,0).$$

However, we have

$$T_{9,\Delta_\beta} = \pi_2 \circ T_\beta^3 \circ \delta_0\big|_{[0,x_\beta)}.$$



Further work

- Numeration systems:
  1. Link between numeration systems without a dominant root and alternate bases ✓
  2. Convergence of the greatest words ✓
  3. Link between Ostrowski numeration systems and alternate bases ✓
  4. Hollander's conjecture : characterization of the linear numeration systems without a dominant root having a regular numeration language
  5. Correction of Bertrand's Theorem?
- Normalization:
  1. In alternate bases (Prague 🤞)
  2. In numeration systems without a dominant root
  3. In Ostrowski numeration systems

⋮