About some notions of regularity for functions

Dissertation presented by
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for the degree of Doctor in Sciences

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LIÈGE université
Mathematics

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Pointwise Hölder spaces

Let $x_0 \in \mathbb{R}^d$; a function $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ belongs to the Hölder space $\Lambda^\alpha(x_0)$ ($\alpha > 0$) if there exist $C > 0$ and a polynomial P_{x_0} of degree less than α s.t., for j large enough,

$$\|f - P_{x_0}\|_{L^\infty(B(x_0,2^{-j}))} \leq C2^{-j\alpha}.$$

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$$\sup_{|h|\leq 2^{-j}} \|\Delta^{[\alpha]+1} f\|_{L^\infty(B_h^{[\alpha]}(x_0,2^{-j}))} \leq C 2^{-j\alpha}.$$

with

$$\Delta_h^1 f(x) = f(x+h) - f(x) \qquad \text{and} \qquad \Delta_h^{n+1} = \Delta_h^1 \Delta_h^n f(x),$$

and $B_h^M(x_0, 2^{-j}) = \{x : [x_0, x_0 + (M+1)h] \subset B(x_0, 2^{-j})\}$.

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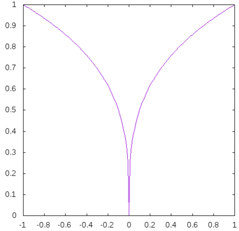

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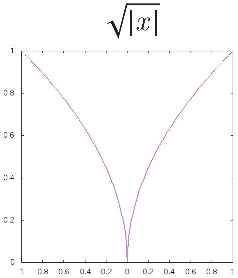

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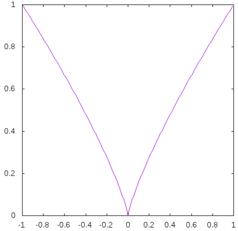

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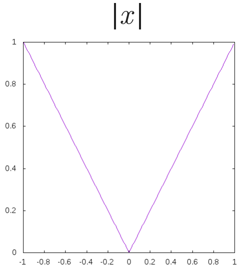

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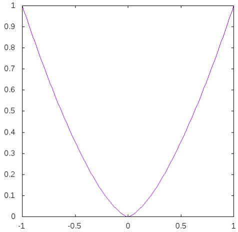

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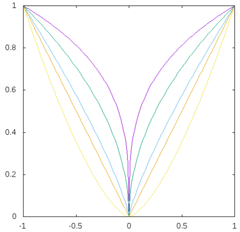
$|x|^{1/3}$, $\sqrt{|x|}$, $|x|^{0.8}$, $|x|$, $|x|^{3/2}$.



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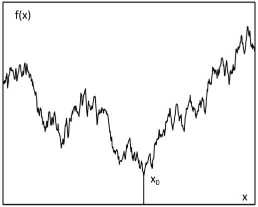

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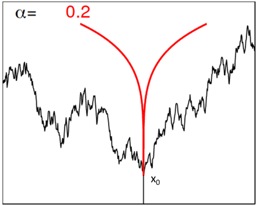

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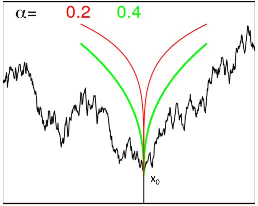

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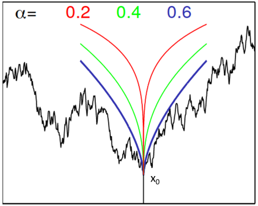

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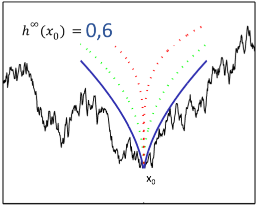

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$$D^{(\infty)}(h) = \dim_{\mathcal{H}}\{x : h^{(\infty)}(x) = h\}.$$

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Kreit and Nicolay
replace the dyadic sequence appearing in the definition by a more general sequence, called admissible.

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- Hölder spaces are a pointwise version of some Besov spaces.
- The multifractal formalism of Jaffard and Frayse is based on the belonging to some Besov spaces.
- Besov spaces were generalized using admissible sequences.

Some equivalent definitions of Besov spaces of generalized smoothness

Besov spaces

Historically Besov spaces were first defined using interpolation spaces

$$B^s_{p,q} = [H^t_p, H^u_p]_{\alpha,q},$$

with $s = (1-\alpha)t + \alpha u$, where H^t_p and H^u_p are Sobolev spaces

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or equivalently using Litllewood-Paley Theory

$$B^s_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \ : \ \left(2^{js} \| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)) \|_{L^p(\mathbb{R}^d))} \right)_j \in \ell^q \right\}$$

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$$\operatorname{supp} \varphi_j \subseteq \{\xi \in \mathbb{R}^d \ : \ 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$$

Admissible sequences

A sequence $\boldsymbol{\sigma} = (\sigma_j)_{j \in \mathbb{N}_0}$ of real positive numbers is called admissible if there exists a positive constant C such that

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$$\underline{\sigma}_j := \inf_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k} \qquad \text{and} \qquad \overline{\sigma}_j := \sup_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k}$$

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so that for any $\varepsilon > 0$, there exists $C > 0$ s.t. for all j, k

$$C^{-1}2^{j(\underline{s}(\sigma)-\varepsilon)} \leq \frac{\sigma_{j+k}}{\sigma_k} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

Admissible sequences

Example

If $s \in \mathbb{R}$, $\boldsymbol{s} = (2^{sj})_j$ is admissible with $\underline{s}(\boldsymbol{s}) = \overline{s}(\boldsymbol{s}) = s$

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$$\lim_{t \to 0} \frac{\psi(rt)}{\psi(t)} = 1,$$

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If ψ is a slowly varying function and $u \in \mathbb{R}$, the sequence $\boldsymbol{\sigma} = (2^{ju}\psi(2^j))_j$ is admissible with $\underline{s}(\boldsymbol{\sigma}) = \overline{s}(\boldsymbol{\sigma}) = u$.

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$$B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \;:\; \left(\sigma_j \| \mathcal{F}^{-1}(\varphi_j^{\boldsymbol{\gamma},J} \mathcal{F} f) \|_{L^p(\mathbb{R}^d)} \right)_j \in \ell^q \right\}$$

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- $\operatorname{supp}(\varphi^{\gamma,J}_j) \subseteq \{\xi \in \mathbb{R}^d \; : \; |\xi| \leq \gamma_{j+Jk_0}\} \; \forall j \in \{0, \ldots, Jk_0 - 1\}$,
- $\operatorname{supp}(\varphi^{\gamma,J}_j) \subseteq \{\xi \in \mathbb{R}^d \; : \; \gamma_{j-Jk_0} \leq |\xi| \leq \gamma_{j+Jk_0}\} \; \forall j \geq Jk_0$,

Generalized Besov spaces

Moura, 2007

Let $p, q \in [1,\infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and $0 < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1}$. For any $n \in \mathbb{N}$ such that $\overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n$, we have

$$B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q} = \{f \in L^p : (\sigma_j \sup_{|h|\leq \gamma_j^{-1}} \|\Delta^n_h f\|_{L^p})_j \in \ell^q\}.$$

Generalized Besov spaces and convolution

Characterization of Generalized Besov spaces in terms of convolution
L.L. & S. Nicolay (2019)

Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and $\underline{s}(\boldsymbol{\sigma}) > 0$; we have

$$B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q} = \{f \in L^p : \exists \phi \in \mathcal{D} \text{ such that } (\sigma_j \| f * \phi_{\gamma_j^{-1}} - f \|_{L^p})_j \in \ell^q\}.$$

Generalized Besov spaces and derivatives

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Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. Let the numbers $k, n \in \mathbb{N}_0$ be such that

$$k < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1} \leq \overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n.$$

We have

$$B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q} = \{f \in W^k_p : (\gamma_j^{-|\alpha|}\sigma_j \sup_{|h| \leq \gamma_j^{-1}} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p})_j \in \ell^q \quad \forall |\alpha| = k\}.$$

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Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. Let $n \in \mathbb{N}$ be such that $n < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1} \leq \overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n + 1$; the following assertions are equivalent:

1. The function f belongs to $B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q}$;

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2. The function f belongs to W_p^n and, for all $h \in \mathbb{R}^d$ and almost every $x \in \mathbb{R}^d$, we have

$$f(x+h) = \sum_{|\alpha| \leq n} D^\alpha f(x) \frac{h^\alpha}{|\alpha|!} + R_n(x,h)\frac{|h|^n}{n!},$$

where

$$(\sigma_j \gamma_j^{-n} \sup_{|h| \leq \gamma_j^{-1}} \|R_n(\cdot, h)\|_{L^p})_j \in \ell^q;$$

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3. If, given $j \in \mathbb{N}_0$, π_j is a net of $\mathbb{R}^d$ made of cubes of diagonal γ_j^{-1}, then for all $j \in \mathbb{N}_0$, there exists g_{π_j} such that

- the trace of g_{π_j} in each cube of π_j is a polynomial of degree at most n,
- one has $(\sigma_j \|f - g_{\pi_j}\|_{L^p})_j \in \ell^q$.

Interpolation methods

Let A_0, A_1 be two normed vector spaces which are continuously embedded in a Hausdorff topological vector space V. As a consequence, the spaces $A_0 \cap A_1$ and $A_0 + A_1$ are also normed vector spaces.

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- $\exists (u_j)_{j\in\mathbb{Z}} \subset A_0 \cap A_1$ such that

$$a = \sum_{j\in\mathbb{Z}} u_j \text{ with convergence in } A_0 + A_1$$

and

$$(2^{-\alpha j} \max\{\|u_j\|_{A_0}, 2^j \|u_j\|_{A_1}\})_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

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OR

- $\forall j \in \mathbb{Z}$, there exist $a_{0,j} \in A_0$ and $a_{1,j} \in A_1$ such that $a = a_{0,j} + a_{1,j}$ and

  $$(2^{-\alpha j}(\|a_{0,j}\|_{A_0} + 2^j \|a_{1,j}\|_{A_1}))_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

Interpolation methods

Let A_0, A_1 be two normed vector spaces which are continuously embedded in a Hausdorff topological vector space V. As a consequence, the spaces $A_0 \cap A_1$ and $A_0 + A_1$ are also normed vector spaces.We say that $a \in [A_0, A_1]_{\alpha,q}$ $(0 < \alpha < 1$ and $1 \leq q \leq \infty)$ if

- $\exists (u_j)_{j\in\mathbb{Z}} \subset A_0 \cap A_1$ such that

$$a = \sum_{j\in\mathbb{Z}} u_j \text{ with convergence in } A_0 + A_1$$

  and

$$(2^{-\alpha j} \max\{\|u_j\|_{A_0}, 2^j \|u_j\|_{A_1}\})_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

OR

- $\forall j \in \mathbb{Z}$, there exist $a_{0,j} \in A_0$ and $a_{1,j} \in A_1$ such that $a = a_{0,j} + a_{1,j}$ and

$$(2^{-\alpha j}(\|a_{0,j}\|_{A_0} + 2^j \|a_{1,j}\|_{A_1}))_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

Generalized interpolation methods

Let $r, s \in \mathbb{R}$ and σ, γ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and

$$r < \min\{\underline{s}(\sigma)\underline{s}(\gamma)^{-1}, \underline{s}(\sigma)\overline{s}(\gamma)^{-1}\} \leq \max\{\overline{s}(\sigma)\underline{s}(\gamma)^{-1}, \overline{s}(\sigma)\overline{s}(\gamma)^{-1}\} < s.$$

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$$\theta_j := \begin{cases} \gamma_{-j}^{-r}\sigma_{-j} & \text{if } -j \in \mathbb{N}_0 \\ \gamma_j^r \sigma_j^{-1} & \text{if } j \in \mathbb{N} \end{cases} \qquad \psi_j := \begin{cases} \gamma_{-j}^{-(s-r)} & \text{if } -j \in \mathbb{N}_0 \\ \gamma_j^{(s-r)} & \text{if } j \in \mathbb{N} \end{cases}$$

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- $a \in [A_0, A_1]^{\theta,\psi}_{J,q}$ if there exists $(u_j)_{j\in\mathbb{Z}} \subset A_0 \cap A_1$ such that $a = \sum_{j\in\mathbb{Z}} u_j$, with convergence in $A_0 + A_1$ and
$$(\theta_j \max\{\|u_j\|_{A_0}, \psi_j \|u_j\|_{A_1}\})_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$
- $a \in [A_0, A_1]^{\theta,\psi}_{K,q}$ if $\forall j \in \mathbb{Z}$, there exist $a_{0,j} \in A_0$ and $a_{1,j} \in A_1$ such that $a = a_{0,j} + a_{1,j}$ and
$$(\theta_j (\|a_{0,j}\|_{A_0} + \psi_j \|a_{1,j}\|_{A_1}))_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

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$$[A_0, A_1]^{\theta,\psi}_{J,q} = [A_0, A_1]^{\theta,\psi}_{K,q} =: [A_0, A_1]^{\sigma,\gamma}_q$$

Generalized Besov spaces and interpolation

Characterization of Generalized Besov spaces in terms of generalized interpolation L.L. & S. Nicolay (2019)

Let $p, q \in [1, \infty]$, $r, s \in \mathbb{R}$, and $\boldsymbol{\sigma}$, $\boldsymbol{\gamma}$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and

$$r < \min\{\underline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1}, \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1}\} \leq \max\{\overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1}, \overline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1}\} < s;$$

we have

$$B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q} = [H^r_p, H^s_p]^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_q.$$

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Characterization of Generalized Besov spaces in terms of generalized interpolation L.L. & S. Nicolay (2019)

Let $p, q \in [1, \infty]$, $r, s \in \mathbb{R}$, and $\boldsymbol{\sigma}$, $\boldsymbol{\gamma}$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and

$$k < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1} \leq \overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n,$$

we have

$$B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q} = [\, W^k_p, W^n_p \,]^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_q .$$

Pointwise spaces of generalized smoothness

The space $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$

Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -\frac{d}{p}$, $f \in L^p_{\text{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$ whenever

$$(\sigma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))})_j \in \ell^q,$$

where, given $r > 0$, if $\overline{s}(\boldsymbol{\sigma}) > 0$, we have

$$B_h(x_0, r) = \{x : [x, x + (\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1)h] \subset B(x_0, r)\},$$

and $B_h(x_0, r) = B(x_0, r)$ otherwise.

The space $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$

Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > 0$, $f \in L^p_{\text{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$ whenever
there exists a sequence of polynomials $(P_{j,x_0})_j$ of degree less than or equal to $\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor$ such that

$$(\sigma_j 2^{jd/p} \| f - P_{j,x_0} \|_{L^p(B(x_0, 2^{-j}))})_j \in \ell^q.$$

The space $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$

Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > 0$, $f \in L^p_{\text{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$ whenever
there exists a sequence of polynomials $(P_{j,x_0})_j$ of degree less than or equal to $\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor$ such that

$$(\sigma_j 2^{jd/p} \|f - P_{j,x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^q.$$

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Moreover, if $0 \leq n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor < \underline{s}(\boldsymbol{\sigma})$, there exists a unique polynomial P_{x_0} of degree less than or equal to n such that

$$(\sigma_j 2^{jd/p} \|f - P_{x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^q.$$

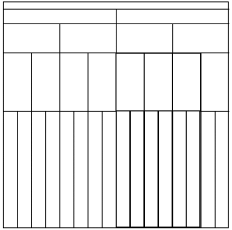
Wavelet leaders

Given a dyadic cube $\lambda \in \Lambda_j$ at scale j, the p-wavelet leader of λ ($p \in [1, \infty]$) is defined by

$$d_\lambda^p = \sup_{j' \geq j} \Big(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} (2^{(j-j')d/p} |c_{\lambda'}|)^p \Big)^{1/p}.$$

Given $x_0 \in \mathbb{R}^d$, we set

$$d_j^p(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda^p.$$



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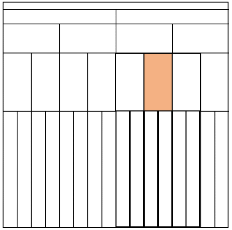

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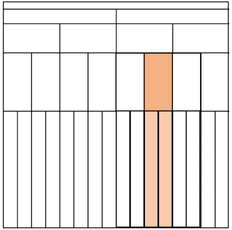

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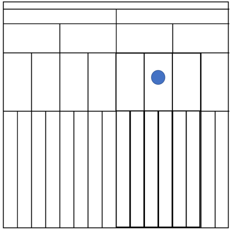

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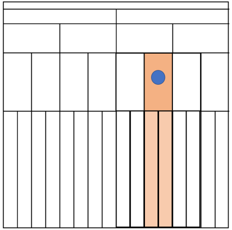
Given $x_0 \in \mathbb{R}^d$, we set

$$d^p_j(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d^p_\lambda.$$



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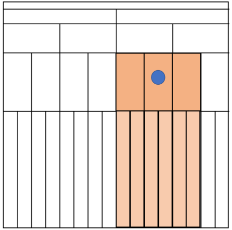

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$$d_j^p(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda^p.$$

L.L. & S. Nicolay (2020)

If f belongs to the space $T_{p,q}^{\boldsymbol{\sigma}}(x_0)$, then

$$(\sigma_j d_j^p(x_0))_j \in \ell^q.$$

Conversely, if $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to ∞ and $\underline{\sigma}_1 > 2^{-d/p}$, if f belongs to $B_{p,q}^s(\mathbb{R}^d)$ for some $s > 0$, then $(\sigma_j d_j^p(x_0))_j \in \ell^q$ implies $f \in T_{p,q,\log}^{\boldsymbol{\sigma}}(x_0)$.

Wavelet leaders

Let $p, q \in [1, \infty]$, $x_0 \in \mathbb{R}^d$ and f be a function from L^p_{loc}; if σ is an admissible sequence such that $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to ∞, we say that f belongs to $T^{\boldsymbol{\sigma}}_{p,q,\log}(x_0)$ if there exists $J \in \mathbb{N}$ for which

$$\Big(\frac{2^{jd/p}\sigma_j}{|\log_2(2^{-jd/p}\sigma_j^{-1})|} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))}\Big)_{j\geq J} \in \ell^q.$$

Wavelet leaders

f belongs to $T^{\sigma}_{p,q,\log}(x_0)$ if there exists $J \in \mathbb{N}$ for which

$$\Big(\frac{2^{jd/p}\sigma_j}{|\log_2(2^{-jd/p}\sigma_j^{-1})|}\sup_{|h|\le 2^{-j}}\|\Delta_h^{\lfloor \overline{s}(\sigma)\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))}\Big)_{j\ge J} \in \ell^q.$$

$$E^{\varepsilon}_{\infty}(x_0) = \{f \in B^{\varepsilon}_{\infty,\infty}(\mathbb{R}^d) \ : \ (\sigma_j d_j^{\infty}(x_0))_j \in \ell^{\infty}\},$$

equipped with the norm

$$\|\cdot\|_{E^{\varepsilon}_{\infty}(x_0)} \ : \ E^{\varepsilon}_{\infty}(x_0) \to [0,+\infty) \ : \ f \mapsto \|f\|_{B^{\varepsilon}_{\infty,\infty}} + \|(\sigma_j d_j^{\infty}(x_0))_j\|_{\ell^{\infty}}.$$

Wavelet leaders

f belongs to $T^{\boldsymbol{\sigma}}_{p,q,\log}(x_0)$ if there exists $J \in \mathbb{N}$ for which

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$$E^{\varepsilon}_{\infty}(x_0) = \{f \in B^{\varepsilon}_{\infty,\infty}(\mathbb{R}^d) \; : \; (\sigma_j\, d_j^{\infty}(x_0))_j \in \ell^{\infty}\},$$

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$$\|\cdot\|_{E^{\varepsilon}_{\infty}(x_0)} \; : \; E^{\varepsilon}_{\infty}(x_0) \to [0,+\infty) \; : \; f \mapsto \|f\|_{B^{\varepsilon}_{\infty,\infty}} + \|(\sigma_j\, d_j^{\infty}(x_0))_j\|_{\ell^{\infty}}.$$

L.L. & S. Nicolay (2020)

If $x_0 \in \mathbb{R}^d$, for all $0 < \varepsilon < \frac{\underline{s}(\boldsymbol{\sigma})}{4}$, from the the prevalence point of view, almost every function of $E^{\varepsilon}_{\infty}(x_0)$ belongs to $T^{\boldsymbol{\sigma}}_{\infty,\log}(x_0) \setminus T^{\boldsymbol{\sigma},\infty}_{/_s\log}(x_0)$.

Wavelet leaders

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$$E_1^{\varepsilon}(x_0) = \{f \in B^{\varepsilon}_{1,\infty}(\mathbb{R}^d) \ : \ (\sigma_j\, d_j^1(x_0))_j \in \ell^\infty\},$$

equipped with the norm

$$\|\cdot\|_{E_1^{\varepsilon}(x_0)} \ : \ E_1^{\varepsilon}(x_0) \to [0,+\infty) \ : \ f \mapsto \|f\|_{B^{\varepsilon}_{1,\infty}} + \|(\sigma_j\, d_j^1(x_0))_j\|_{\ell^\infty}.$$

L.L. & S. Nicolay (2020)

If $x_0 \in \mathbb{R}^d$, for all $0 < \varepsilon < \frac{\underline{s}(\boldsymbol{\sigma})+d}{4}$, from the the prevalence point of view, almost every function of $E_1^{\varepsilon}(x_0)$ belongs to $T^{\boldsymbol{\sigma}}_{1,\log}(x_0) \setminus T^{\boldsymbol{\sigma},1}_{/_s\log}(x_0)$.

Generalized Hölder exponent

Decreasing family of admissible sequences

Let $p, q \in [1, \infty]$; if, given $h > -d/p$, $\boldsymbol{\gamma}^{(h)}$ is an admissible sequence, the family of admissible sequences $h \mapsto \boldsymbol{\gamma}^{(h)}$ is (p, q)-decreasing if it satisfies $\underline{s}(\boldsymbol{\gamma}^{(h)}) > -d/p$, $\underline{\gamma}_1^{(h)} > 2^{-d/p}$ for any $h > -d/p$ and if $-d/p < h < h'$ implies

$$T_{p,q}^{\boldsymbol{\gamma}^{(h')}}(x_0) \subset T_{p,q}^{\boldsymbol{\gamma}^{(h)}}(x_0).$$

Generalized Hölder exponent

Decreasing family of admissible sequences and associated exponent

Let $p, q \in [1, \infty]$; if, given $h > -d/p$, $\boldsymbol{\gamma}^{(h)}$ is an admissible sequence, the family of admissible sequences $h \mapsto \boldsymbol{\gamma}^{(h)}$ is (p, q)-decreasing if it satisfies $\underline{s}(\boldsymbol{\gamma}^{(h)}) > -d/p$, $\underline{\gamma}_1^{(h)} > 2^{-d/p}$ for any $h > -d/p$ and if $-d/p < h < h'$ implies

$$T_{p,q}^{\boldsymbol{\gamma}^{(h')}}(x_0) \subset T_{p,q}^{\boldsymbol{\gamma}^{(h)}}(x_0).$$

$$h_{p,q}(x_0) := \sup\{h > -d/p : f \in T_{p,q}^{\boldsymbol{\gamma}^{(h)}}(x_0)\}.$$

Generalized Hölder exponent

Decreasing family of admissible sequences and associated exponent and spectrum

Let $p, q \in [1, \infty]$; if, given $h > -d/p$, $\boldsymbol{\gamma}^{(h)}$ is an admissible sequence, the family of admissible sequences $h \mapsto \boldsymbol{\gamma}^{(h)}$ is (p, q)-decreasing if it satisfies $\underline{s}(\boldsymbol{\gamma}^{(h)}) > -d/p$, $\underline{\gamma}_1^{(h)} > 2^{-d/p}$ for any $h > -d/p$ and if $-d/p < h < h'$ implies

$$T_{p,q}^{\boldsymbol{\gamma}^{(h')}}(x_0) \subset T_{p,q}^{\boldsymbol{\gamma}^{(h)}}(x_0).$$

$$h_{p,q}(x_0) := \sup\{h > -d/p : f \in T_{p,q}^{\boldsymbol{\gamma}^{(h)}}(x_0)\}.$$

$$D_{p,q}(h) := \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) = h\}).$$

Generalized Hölder exponent

Decreasing family of admissible sequences and associated spectrum

If $(\boldsymbol{\gamma}^{(h)})_h$ is a (p,q)-decreasing family of admissible sequences

$$D_{p,q}(h) := \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) = h\}).$$

L.L. & S. Nicolay (2020)

It $\boldsymbol{\sigma}$ is an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) - \frac{d}{r} > -\frac{d}{p}$ and if $s \leq q$ then, for all $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have

$$\dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\}) \leq d + r\overline{s}(\frac{\boldsymbol{\gamma}^{(h)}}{\boldsymbol{\sigma}}).$$

Multifractal formalism

Compatibility conditions

An admissible sequence $\boldsymbol{\sigma}$ and a family of admissible sequences $\boldsymbol{\gamma}^{(\cdot)}$ are compatible for $p, q, r, s \in [1, \infty]$ with $s \leq q$ if

- $\underline{s}(\boldsymbol{\sigma}) > 0$,
- $\underline{s}(\boldsymbol{\sigma}) - d/r > -d/p$,
- the function ζ defined on $(-d/p, \infty)$ by

$$\zeta(h) := \underline{s}\Big(\frac{\boldsymbol{\gamma}^{(h)}}{\boldsymbol{\sigma}}\Big) = \overline{s}\Big(\frac{\boldsymbol{\gamma}^{(h)}}{\boldsymbol{\sigma}}\Big)$$

  is non decreasing, continuous and such that

$$\{h > -d/p : \zeta(h) < -d/r\} \neq \emptyset.$$

We call ζ the ratio function and set $h_{\min}(r) := \sup\{h > -d/p : \zeta(h) < -d/r\}$.

Multifractal formalism

$$\zeta(h) := \underline{s}(\frac{\boldsymbol{\gamma}^{(h)}}{\boldsymbol{\sigma}}) = \overline{s}(\frac{\boldsymbol{\gamma}^{(h)}}{\boldsymbol{\sigma}})$$

Multifractal formalism

$$\zeta(h) := \underline{s}(\frac{\gamma^{(h)}}{\sigma}) = \overline{s}(\frac{\gamma^{(h)}}{\sigma})$$

L.L. & S. Nicolay (2020)

Let $p, q, r, s \in [1, \infty]$ with $s \leq q$, σ be an admissible sequence and $\gamma^{(\cdot)}$ be a family of admissible sequences compatible with σ. From the prevalence point of view, for almost every $f \in B^{\sigma}_{r,s}$, $D_{p,q}$ is defined on $I = [\zeta^{-1}(-d/r), \zeta^{-1}(0)]$ and

$$D_{p,q}(h) = d + r\zeta(h),$$

for any $h \in I$.

Moreover, for almost every $x_0 \in \mathbb{R}^d$, we have $h_{p,q}(x_0) = \zeta^{-1}(0)$.

Multifractal formalism

L.L. & S. Nicolay (2020)

Let $p, q, r, s \in [1, \infty]$ with $s \leq q$, $\boldsymbol{\sigma}$ be an admissible sequence and $\boldsymbol{\gamma}^{(\cdot)}$ be a family of admissible sequences compatible with $\boldsymbol{\sigma}$. From the prevalence point of view, for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, $D_{p,q}$ is defined on $I = [\zeta^{-1}(-d/r), \zeta^{-1}(0)]$ and

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Moreover, for almost every $x_0 \in \mathbb{R}^d$, we have $h_{p,q}(x_0) = \zeta^{-1}(0)$.

If $p = q = \infty$, $(\boldsymbol{\gamma}^{(h)})_{h>0}$ is the usual family $(2^{jh})_{h>0}$ and $\sigma = (2^{sj})_j$

$$\zeta(h) = \underline{s}((2^{(h-s)j})_j) = h - s$$

Multifractal formalism

L.L. & S. Nicolay (2020)

Let $p, q, r, s \in [1, \infty]$ with $s \leq q$, $\boldsymbol{\sigma}$ be an admissible sequence and $\boldsymbol{\gamma}^{(\cdot)}$ be a family of admissible sequences compatible with $\boldsymbol{\sigma}$. From the prevalence point of view, for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, $D_{p,q}$ is defined on $I = [\zeta^{-1}(-d/r), \zeta^{-1}(0)]$ and

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Moreover, for almost every $x_0 \in \mathbb{R}^d$, we have $h_{p,q}(x_0) = \zeta^{-1}(0)$.

If $p = q = \infty$, $(\boldsymbol{\gamma}^{(h)})_{h>0}$ is the usual family $(2^{jh})_{h>0}$ and $\sigma = (2^{sj})_j$

$$\forall h \in [s - \frac{d}{r}, s] \; : \; D(h) = d + r(h - s)$$

About the results of Calderón and Zygmund

Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty]$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$. A function $f \in L^p(\mathbb{R}^d)$ belongs to the space $T^p_\phi(x_0)$ if there exist a polynomial P of degree strictly less than $\underline{b}(\phi)$ and a constant $C > 0$ such that

$$r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \leq C\phi(r) \qquad \forall r > 0.$$

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A function $\phi : (0,+\infty) \to (0,+\infty)$ is a *Boyd function* if $\phi(1) = 1$, ϕ is continuous and, for all $x \in (0,+\infty)$,

$$\overline{\phi}(x) := \sup_{y>0} \frac{\phi(xy)}{\phi(y)} < \infty. \tag{1}$$

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A sequence $\sigma = (\sigma_j)_j$ of real positive numbers is admissible if and only if there exists a Boyd function ϕ such that, for any j, $\phi(2^j) = \sigma_j$.

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Equipped with this norm, $T^p_\phi(x_0)$ is a Banach space

$$\|f\|_{T^p_\phi(x_0)} := \|f\|_{L^p(\mathbb{R}^d)} + \sum_{|\alpha|<\underline{b}(\phi)} \frac{|D^\alpha P(x_0)|}{\alpha!} + \sup_{r>0} \phi(r)^{-1} r^{-d/p}\|f - P\|_{L^p(B(x_0,r))}.$$

Elliptic partial differential equations

An *elliptic partial differential equation at* $x_0 \in \mathbb{R}^d$ *of order* $m \in \mathbb{N}$ is a partial differentiable equation of the form

$$\mathcal{E}f = \sum_{|\alpha| \leq m} a_\alpha D^\alpha f = g$$

where for all $|\alpha| \leq m$, a_α is an $s \times r$ matrix of functions, f and g are vector valued functions with $f_j \in W_m^p(\mathbb{R}^d)$ for all $j \in \{1, \ldots, r\}$ and

$$\mu(x_0) := \inf_{|\xi|=1} \det[(\sum_{|\alpha|=m} a_\alpha^*(x_0)\xi^\alpha)(\sum_{|\alpha|=m} a_\alpha(x_0)\xi^\alpha)] > 0$$

is the ellipticity constant of $\mathcal{E}$ at x_0.

Operators

$$\mathcal{J}^s f := \mathcal{F}^{-1}\big((1+|\cdot|^2)^{-s/2}\mathcal{F}f\big) \qquad (s \in \mathbb{R},\ f \in \mathcal{S}')$$

and

$$\mathcal{K}f = p.v. \int k(\cdot - y) f(y)\, dy.$$

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If $\overline{b}(\phi) + s < 0$ or $\exists n \in \mathbb{N}$ s.t. $n < \underline{b}(\phi) + s \leq \overline{b}(\phi) + s < n+1$ and $p \in (1, \infty]$.

$$\begin{tikzcd} T^p_\phi(x_0) \arrow[r, "\mathcal{J}^s"] \arrow[loop below, "\mathcal{K}"] & T^q_{\phi_s}(x_0) \end{tikzcd}$$

where $\phi_s : (0, +\infty) \to (0, +\infty) \quad x \mapsto \phi(x) x^s$ and

- $1/p \geq 1/q \geq \frac{1}{p} - \frac{s}{d}$ if $p < d/s$,
- $p \leq q \leq \infty$ if $d/s < p \leq \infty$,
- $p \leq q < \infty$ if $d/s = p$

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$$\mathcal{K}f = p.v.\int k(\cdot - y)f(y)\,dy.$$

If $k\in C^\infty(\mathbb{R}^d\setminus\{0\})$ is homogeneous of degree $-d$ with mean value zero on the sphere Σ, $\overline{b}(\phi)<0$ or $\exists n\in\mathbb{N}$ s.t. $n<\underline{b}(\phi)\le\overline{b}(\phi)<n+1$ and $p\in(1,\infty)$

$$\begin{array}{ccc} T^p_\phi(x_0) & \xrightarrow{\mathcal{J}^s} & T^q_{\phi_s}(x_0) \\ \circlearrowleft & & \\ \mathcal{K} & & \end{array}$$

with

$$\|\mathcal{K}f\|_{T^p_\phi(x_0)} \le C_{\phi,p}\Big(\sup_{\substack{|x|=1\\ 0\le|\alpha|\le\lceil\overline{b}(\phi)\rceil_{\mathbb{N}}}}|D^\alpha k(x)|\Big)\|f\|_{T^p_\phi(x_0)},$$

Generalization of the main result of Calderón and Zygmund

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Our variant

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Let, $p \in (1, \infty)$, $\phi, \varphi \in \mathcal{B}$ be such that $0 < \underline{b}(\phi)$, $-d/p < \underline{b}(\varphi)$ and such that there exists $n \in \mathbb{Z}$ such that $n < \underline{b}(\varphi) \leq \overline{b}(\varphi) < n + 1$; let us define k_p as follows:

- if $\underline{b}(\varphi) = \overline{b}(\varphi)$, $\quad k_p(\phi, \varphi) := \min\{k \in \mathbb{N} : \frac{1}{k}(\underline{b}(\varphi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\}\}$,
- if $n < \underline{b}(\varphi) < \overline{b}(\varphi) < n + 1$,

$$k_p(\phi, \varphi) := k_p(\phi, \ \cdot \ ^{\underline{b}(\varphi)}) + \min\{k \in \mathbb{N} : \frac{\overline{b}(\varphi) - \underline{b}(\varphi)}{k} < \min\{1, \underline{b}(\phi)\}\}.$$

Calderón & Zygmund (1960)

Let $p \in]1, \infty[$, $x_0 \in \mathbb{R}^d$, $u > 0$ and v be a non integer such that $-\frac{d}{p} \leq v \leq u$. Let $\mathcal{E}f = g$ be an elliptic differentiable equation of order m at x_0 such that $f \in W^p_m(\mathbb{R}^d)$, the coefficients of $\mathcal{E}$ are functions in $T^\infty_u(x_0)$ and $g \in T^p_v(x_0)$. Then, for all $|\alpha| \leq m$, $D^\alpha f \in T^q_{v+m-|\alpha|}(x_0)$ with

$$\|D^\alpha f\|_{T^q_{v+m-|\alpha|}(x_0)} \leq C \left(||g||_{T^p_v(x_0)} + ||f||_{W^p_m(\mathbb{R}^d)} \right)$$

where q is determined by p, m and α

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L.L. & S. Nicolay (2020)

Let $p \in]1,\infty[$, $q \in]1,\infty]$, $x_0 \in \mathbb{R}^d$ and $\phi, \varphi \in \mathcal{B}$ be such that $-\frac{d}{p} < \underline{b}(\varphi)$, $0 < \underline{b}(\phi)$ and there exists $n \in \mathbb{Z}$ such that $n < \underline{b}(\varphi) < \overline{b}(\varphi) < n+1$. Let $\mathcal{E}f = g$ be an elliptic differentiable equation of order m at x_0 such that the coefficients of $\mathcal{E}$ are functions in $T^q_\phi(x_0)$ whose x_0 is a Lebesgue point. Let us suppose that:

- $g \in T^{p_1}_\varphi(x_0)$ with $\frac{1}{p_1} := \frac{1}{p} + \frac{1}{q}$
- $\phi \preccurlyeq \varphi$ and $\overline{b}(\varphi) \leq \underline{b}(\phi)$ or $\overline{b}(\varphi) - \underline{b}(\varphi) \leq \min\{1, \underline{b}(\phi)\}$,
- $f \in W^s_m(\mathbb{R}^d)$ for all $s \in [p', p]$ with $0 < \frac{1}{p'} := \frac{k_p(\phi,\varphi)}{q} + \frac{1}{p} < 1$

L.L. & S. Nicolay (2020)

There exists $C_{p',\phi,\varphi,m}$ such that for all $|\alpha| \leq m$, $D^\alpha f \in T^{q'}_{\varphi_{m-|\alpha|}}(x_0)$ for all $q' \geq 1$ such that

- $\frac{1}{p'} \geq \frac{1}{q'} \geq \frac{1}{p'} - \frac{m-|\alpha|}{d}$ if $\frac{1}{p'} > \frac{m-|\alpha|}{d}$,
- $p' \leq q' \leq \infty$ if $\frac{1}{p'} < \frac{m-|\alpha|}{d}$,
- $p' \leq q' < \infty$ if $\frac{1}{p'} = \frac{m-|\alpha|}{d}$.

Moreover, we have

$$||D^\alpha f||_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} \leq C_{p',\phi,\varphi}(M(1+MN)^{k_p(\phi,\varphi)-1}||g||_{T^{p_1}_{\varphi}(x_0)}$$

$$+ k_p(\phi,\varphi)(1+MN)^{k_p(\phi,\varphi)}(||f||_{W^p_m(\mathbb{R}^d)} + ||f||_{W^{p'}_m(\mathbb{R}^d)})$$

Continuously differentiable functions on compact sets

Defining continuously differentiable functions on compact sets

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Whitney (1934)

A function f is the restriction to K of a continuously differentiable function on $\mathbb{R}^d$, with continuous derivative df, if and only if

$$\lim_{\substack{y \to x \\ y \in K}} \frac{f(y) - f(x) - \langle df(x), y - x \rangle}{|y - x|} = 0,$$

uniformly on $x \in K$.

Defining continuously differentiable functions on compact sets

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A function $f \in C^1(\mathbb{R}^d|K)$, with continuous derivative df, if and only if

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- If the compact set K is topologically regular (the closure of its interior)

$C^1_{\mathrm{int}}(K) = \{f \in C(K) : f|_{\mathring{K}} \in C^1(\mathring{K})$ and df extends continuously to $K\}$.

Our proposition

$C^1(K)$

A function f, continuous on K, belongs to $C^1(K)$ if there exits a continuous function df on K with values in the linear maps from $\mathbb{R}^d$ to $\mathbb{R}$ such that, for all $x \in K$,

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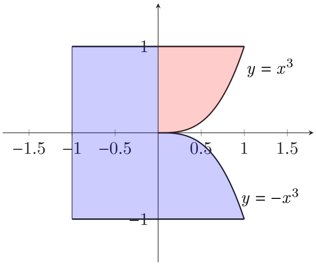

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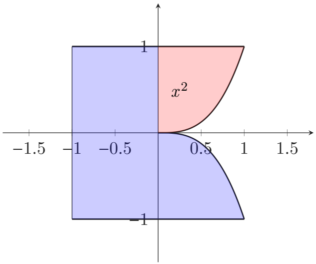

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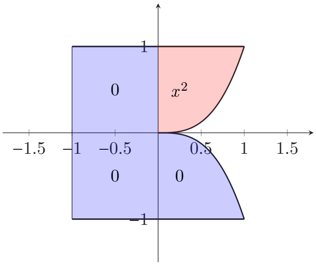

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In general, a derivative need not be unique. For this reason, a good tool to study $C^1(K)$ is the jet space

$$\mathcal{J}^1(K) = \{(f, df) : df \text{ is a continuous derivative of } f \text{ on } K\}$$

endowed with the norm

$$\|(f, df)\|_{\mathcal{J}^1(K)} = \|f\|_K + \|df\|_K,$$

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In general, a derivative need not be unique. For this reason, a good tool to study $C^1(K)$ is the jet space $\mathcal{J}^1(K)$ for which

$$C^1(K) = \pi(\mathcal{J}^1(K)) \text{ for the projection } \pi(f, df) = f$$

and we equip $C^1(K)$ with the norm

$$\|f\|_{C^1(K)} = \|f\|_K + \inf\{\|df\|_K : df \text{ is a continuous derivative of } f \text{ on } K\}.$$

Banach space?

Banach space?

L. Frerick, L. L., J. Wengenroth (2020)
If K is a compact set with infinitely many connected components, then $(C^1(K), \|\cdot\|_{C^1(K)})$ is incomplete.

Banach space?

Pointwise (Whitney) regularity

We say that A is *pointwise (Whitney) regular* if for any $x \in A$ there exist a neighbourhood V_x of x in A and $C_x > 0$ such that any $y \in V_x$ is joined to x by a rectifiable path in A of length bounded by $C_x|x - y|$.

If, for all $x \in A$, $V_x = A$ and C_x is uniform, A is (Whitney) regular.

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K is pointwise regular $\Longrightarrow$ $(\mathcal{J}^1(K), \|\cdot\|_{\mathcal{J}^1(K)})$ B.S.

$\Uparrow$ $\quad$ $\Nearrow$ $\quad$ $\Downarrow$

For all $x \in K$, there exists $C_x > 0$ such that $\sup_{\substack{y \in K \\ y \neq x}} \frac{|f(y) - f(x)|}{|y - x|} \leq C_x \|f\|_{C^1(K)}$ $\Longleftarrow$ $(C^1(K), \|\cdot\|_{C^1(K)})$ B.S.

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L. Frerick, L. L., J. Wengenroth (2020)
$(C^1(K), \|\cdot\|_{C^1(K)})$ is complete if and only if K has finitely many components which are pointwise Whitney regular.

$C^1_{\mathrm{int}}(K)$ and $C^1(K)$

If K is topologically regular, $C^1(K) \subsetneq C^1_{\mathrm{int}}(K)$.

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If K is topologically regular, $C^1(K) \subsetneq C^1_{\text{int}}(K)$.

Whitney (1934)

Let K be a topologically regular compact set. If $\mathring{K}$ is Whitney regular, then $C^1_{\text{int}}(K) = C^1(\mathbb{R}^d|K)$.

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Whitney conjecture: What can be said about the reverse implication?

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Whitney conjecture: What can be said about the reverse implication?

L. Frerick, L. L., J. Wengenroth (2020)
Let K be a topologically regular compact set and assume that, for all $x \in \partial K$, there exist $C_x > 0$ and a neighbourhood V_x of x in K such that each $y \in V_x$ can be joined from x by a rectifiable path in $\mathring{K} \cup \{x, y\}$ of length bounded by $C_x|x - y|$. Then $C^1_{\text{int}}(K) = C^1(K)$.

$C^1_{\mathrm{int}}(K)$ and $C^1(K)$

If K is topologically regular, $C^1(K) \subsetneq C^1_{\mathrm{int}}(K)$.

Whitney (1934)

Let K be a topologically regular compact set. If $\mathring{K}$ is Whitney regular, then $C^1_{\mathrm{int}}(K) = C^1(\mathbb{R}^d|K)$.

Whitney conjecture: What can be said about the reverse implication?

L. Frerick, L. L., J. Wengenroth (2020)

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L. Frerick, L. L., J. Wengenroth (2020)

Construction of a compact set K for which $C^1_{\mathrm{int}}(K) = C^1(K) = C^1(\mathbb{R}^d|K)$ but $\mathring{K}$ is not Whitney regular.

$C^1(\mathbb{R}^d|K)$ and $C^1(K)$

$C^1(\mathbb{R}^d|K)$ and $C^1(K)$

L. Frerick, L. L., J. Wengenroth (2020)
For each compact set K, the space $C^1(\mathbb{R}^d|K)$ is dense in $C^1(K)$.

$C^1(\mathbb{R}^d|K)$ and $C^1(K)$

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L. Frerick, L. L., J. Wengenroth (2020)
$C^1(K) = C^1(\mathbb{R}^d|K)$ with equivalent norms if and only if K has only finitely many components which are all Whitney regular.

$C^1(\mathbb{R}|K)$ and $C^1(K)$

$C^1(\mathbb{R}|K)$ and $C^1(K)$

Whitney (1934)

A function $f \in C^1(\mathbb{R}|K)$ if and only if, for all non-isolated $\xi \in K$,

$$\lim_{x,y\to\xi} \frac{f(x)-f(y)}{x-y} = f'(\xi),$$

$C^1(\mathbb{R}|K)$ and $C^1(K)$

Whitney (1934)
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The simple idea is that small gaps are dangerous for the Lipschitz continuity on K which is a necessary condition for C^1-extendability.

$C^1(\mathbb{R}|K)$ and $C^1(K)$

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The gap structure function

$$\sigma(\xi) = \lim_{\varepsilon\to 0} \sup \left\{ \frac{\sup\{|y-\xi| : y \in G\}}{\ell(G)} : G \subseteq (\xi-\varepsilon, \xi+\varepsilon) \text{ is a gap of } K \right\}.$$

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L. Frerick, L. L., J. Wengenroth (2020)

For a compact set $K \subseteq \mathbb{R}$ we have $C^1(K) = C^1(\mathbb{R}|K)$ if and only if $\sigma(\xi) < \infty$ for all $\xi \in K$.

About some notions of regularity for functions

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Mathematics

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