Université de Liège

Faculté des Sciences
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About some notions of regularity for functions

Dissertation présentée le 10 mars 2021 par

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en vue de l'obtention du grade de Docteur en Sciences

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“La science peut expliquer l’univers sans avoir besoin d’un créateur”
Stephen Hawking.

Abstract

Given a function, a first natural desire is to know its "behaviour". To achieve this goal, different notions, such as differentiability, Lipschitz or Hölderian conditions, have been introduced through the time, with more and more preciseness. In this thesis, we aim at characterizing the regularity of functions from different points of view that generalize the precited ones, and using different associated functional spaces.

First, we focus on uniform regularity, investigated through Besov spaces of generalized smoothness. These spaces were originally defined in terms of Littlewood-Paley decompositions and, quickly afterwards, a characterization using finite differences was given. Using this last one, we present some alternative definitions for Besov spaces of generalized smoothness, involving elementary objects: (weak) derivatives, polynomials and convolution. This is made in order to understand as precisely as possible what means the belonging to a given Besov space. Initially, these spaces are known to be interpolation spaces between Sobolev spaces. A first generalization was obtained by introducing a function parameter in the interpolation formula. The spaces we consider here are even more general and, as an intent to "close the circle", we define a new method of interpolation for which Besov spaces of generalized smoothness are still linked to Sobolev spaces.

Then, we study pointwise regularity by defining functional spaces that generalize both the ones of Hölder and Calderón and Zygmund. After nearly characterizing them by the mean of wavelet coefficients, we establish a multifractal formalism particularly well adapted to explore the pointwise regularity through our new spaces. In fact, as their definition is a kind of localization around the point of interest of generalized Besov conditions, it is not a surprise that Besov spaces of generalized smoothness play a major role in this formalism. After investigating the multifractal nature of pointwise spaces of generalized smoothness, we focus, in a more functional analysis point of view, on their interaction with partial differential equations. This follows the trail of Calderón and Zygmund as we link generalized pointwise smoothness with some families of operators. This leads to a theorem that allows to give the regularity of the solution of an elliptic partial differential equation by formulating it from the regularity of the coefficients and the right-hand side of the equation.

Finally, at a midpoint between uniform and pointwise regularites, we study functions that are continuously differentiable on a compact set. Even if the question seems naive and harmless at first look, all good habits from open sets are missing and a whole new theory needs to be established. Based on deep results of functional analysis, we characterize the completeness of the defined functional space and show that, for any compact set, the restrictions on it of the continuously differentiable functions on $\mathbb{R}^d$ are dense in our space. Finally, the latter is compared with other spaces, more frequently met in the literature.

Résumé

Étant donnée une fonction, un premier désir naturel est de connaitre son “comportement”. Pour atteindre cet objectif, différentes notions telles que la différentiabilité, les conditions de Lipschitz ou de Hölder, ont été introduites à travers le temps, avec de plus en plus de précision. Dans cette thèse, nous souhaitons caractériser la régularité de fonctions depuis différents points de vue, qui généralisent les précédents, et en utilisant divers espaces fonctionnels.

Premièrement, nous nous intéressons à la régularité uniforme, étudiée à travers les espaces de Besov de régularité généralisée. Ces espaces ont originalement été définis en termes de décomposition de Littlewood-Paley et, peu de temps après, une caractérisation utilisant les différences finies était obtenu. En exploitant cette dernière, nous présentons des définitions alternatives pour les espaces de Besov généralisés, au moyen d’objets élémentaires : les dérivées (faibles), les polynômes et la convolution. Cela est fait en vue de comprendre, aussi précisément que possible, ce que signifie l’appartenance à un espace de Besov donné. Initialement, ces espaces sont connus pour être des espaces d’interpolation entre les espaces de Sobolev. Une première généralisation a été obtenue en introduisant une fonction en paramètre de la formule d’interpolation. Les espaces que nous considérons ici sont encore plus généraux et, dans une tentative de “boucler la boucle”, nous définissons une nouvelle méthode d’interpolation réelle pour laquelle les espaces de Besov de régularité généralisée sont toujours liés aux espaces de Sobolev.

Ensuite, nous étudions la régularité ponctuelle en définissant des espaces fonctionnels qui généralisent à la fois les espaces de Hölder et de Calderón et Zygmund. Après avoir (presque) caractérisé ceux-ci au moyen de coefficients en ondelettes, nous établissons un formalisme multifractal particulièrement bien adapté pour explorer la régularité ponctuelle au travers de nos espaces. En fait, vu que leur définition est une sorte de localisation autour du point d’intérêt de la condition d’appartenance aux espaces de Besov généralisés, c’est sans surprise que ces derniers jouent un rôle majeur dans ce formalisme. Après avoir étudié la nature multifractale des espaces ponctuels de régularité généralisée, nous nous focalisons, d’un point de vue plus tourné vers l’analyse fonctionnelle, sur leurs interactions avec les équations aux dérivées partielles. Cela suit le chemin tracé par Calderón et Zygmund puisque nous lions la régularité

généralisée avec des familles d'opérateurs. Cela conduit à un théorème qui permet de donner la régularité de la solution d'une équation différentielle elliptique en la formulant à partir de la régularité des coefficients et du membre de droite de l'équation.

Finalement, en guise d'intermédiaire entre les régularités uniformes et ponctuelles, nous étudions les fonctions continûment dérivables sur un ensemble compact. Même si cette question semble naïve et inoffensive au premier coup d'oeil, toutes les bonnes habitudes acquises sur les ensembles ouverts manquent à l'appel et il est nécessaire d'établir entièrement une nouvelle théorie. En s'appuyant sur des résultats profonds d'analyse fonctionnelle, nous caractérisons la complétude de l'espace fonctionnel que nous définissons et montrons que, pour tout ensemble compact, les restrictions à ce dernier des fonctions continûment dérivables sur $\mathbb{R}^d$ sont denses dans notre espace. Finalement, ce dernier est comparé avec d'autres espaces, plus fréquemment rencontrés dans la littérature.

Remerciements

Je tiens naturellement à remercier en premier mon promoteur, Samuel Nicolay, qui a été un formidable encadrant lors de mes quatre années de thèse. Tout d'abord car il a su me transmettre son intérêt pour l'analyse multifractale et ce manuscrit est le fruit de nos discussions passionnées. Ensuite car il a toujours veillé à me conseiller efficacement, me forçant parfois à sortir de mes zones de confort, mais en ayant toujours en vue mon intérêt personnel et la construction de ma carrière scientifique. Finalement car, bien plus qu'un simple directeur de thèse, il est devenu un ami fidèle qui compte désormais beaucoup pour moi. Son humour et ses passages dans mon bureau pour discuter de tout et de rien auront égayé mon travail.

Outre Samuel Nicolay, Jean-Pierre Schneiders et Françoise Bastin (alias *Mamy* pour les intimes) m'ont enseigné l'analyse durant mes études. Leur rigueur et leurs connaissances m'ont donné l'envie d'entreprendre une thèse dans ce domaine. Je les remercie d'avoir accepté d'être membres de mon jury et en particulier Jean-Pierre Schneiders de le présider. J'espère pouvoir leur rendre l'intérêt qu'ils ont éveillé en moi !

Tout au long de ma thèse, Céline Esser a été une compagne d'aventures particulièrement agréable, de par sa bonne humeur constante, ses nombreux encouragements mais aussi ses moqueries toujours bien placées. Je suis heureux qu'elle soit secrétaire de mon jury et je la remercie d'avoir accepté cette tâche.

Leonhard Frerick et Jochen Wengenroth m'ont invité à effectuer un séjour de recherche à leurs côtés à Trèves d'octobre 2019 à janvier 2020. Pouvoir bénéficier de leurs énormes connaissances de l'analyse fonctionnelle dans ses résultats, ses techniques et sa littérature a été une expérience extrêmement enrichissante. Je suis fier du travail que nous avons fait ensemble et je les remercie du temps qu'ils m'ont consacré et d'avoir accepté d'être membres de mon jury de thèse.

Stéphane Jaffard et Jasson Vindas sont des spécialistes renommés dans l'univers de l'analyse multifractale et des ondelettes. Je les remercie de me faire l'honneur d'avoir accepté d'être rapporteurs de cette thèse et par conséquent d'être membres de mon jury.

J'adresse également mes remerciements les plus affectueux à ma famille et en particulier mes parents. Tout au long de ma vie et donc de mes études, ils ont axé tous les moyens en leur possession vers ma réussite, personnelle et professionnelle. Sans eux, cette thèse n'aurait jamais vu le jour. Je pense également à ma grand-mère qui n'a jamais eu de cesse de me soutenir dans ma démarche scientifique mais qui s'en est allée en cours de route. Elle aurait été si fière de tenir ce manuscrit entre ses mains. Elle m'aurait avoué, avec un sourire, ne pas pouvoir le comprendre mais aurait reconnu qu'il représente la conclusion d'une belle aventure. Cette thèse lui est dédiée.

Je remercie chaleureusement ma fiancée, Marie Lejeune, qui, en plus de devoir me supporter tous les jours, n'a jamais cessé de m'encourager. Elle sait ô combien le bon déroulement d'un projet de thèse repose avant tout sur le bien-être personnel. Elle en a été la garante, par sa présence, sa complicité et sa quiétude. Je me dois également de la remercier pour son précieux travail de relecture, cette thèse en a grandement bénéficié.

Je tiens également à remercier mes collègues du B37. L'ambiance de travail ces dernières années m'a semblé excellente et je ne garderai que des bons souvenirs de nos relations. Cela a toujours été un vrai moment de plaisir de passer du temps tous ensemble à la "salle détente", même s'il faut bien avouer que ça nous a aussi vraisemblablement permis de nous partager l'épidémie du moment... Parmi le monde du B37, certaines personnes se sont distinguées et ces remerciements sont l'occasion parfaite de les citer, pour immortaliser nos amitiés dans le marbre de ma thèse. Laurent De Rudder et Sophie Klenkenberg ayant appris à vivre avec mes jeux de mots et blagues nulles, ils méritent bien d'être mis à l'honneur. Avec Marie, nous formons désormais un chouette quadruo ! Arman Molla a effectué ses études puis sa thèse à mes côtés, cela a été un véritable plaisir de partager des moments aussi bien scientifiques que festifs ensemble. Danielle Bartholomeus, en plus d'être la secrétaire la plus efficace de l'univers, est une incroyable conteuse d'aventures, merci pour ces moments de rigolade. Je remercie Vinciane Godfrind pour sa constante gentillesse à mon égard. J'ai également particulièrement apprécié prolonger mes temps de midi avec Patricia Tossings, englués dans des discussions toujours plus intéressantes. Je n'oublie finalement pas mes moniteurs de Whist Loïc Demeulenaere et Julien Raskin, je suis heureux d'avoir obtenu mon permis sous leur supervision. De manière générale, j'ai particulièrement apprécié être "de l'autre côté du rideau" et ainsi de pouvoir devenir le collègue de mes anciens professeurs. Je me dois de les remercier, ainsi que toutes les personnes qui m'ont enseigné les mathématiques, pour la passion qu'ils ont éveillée en moi.

Je remercie également le FNRS d'avoir financé ma thèse ainsi que mon séjour à Trèves. Le mandat d'"aspirant" est une aubaine pour un jeune chercheur désireux de s'émanciper dans le milieu scientifique, notamment par les possibilités accrues de voyages qu'il offre. Ces derniers ont été l'occasion de rencontrer des scientifiques renommés mais surtout des collègues chaleureux. Parmi ceux-ci, je tiens à remercier Guillaume Havard pour sa bonne humeur, son attention et son précieux travail de guide parisien.

En dehors du monde des mathématiques, j'ai pu compter sur des amis fidèles pour m'entourer. Je remercie en particulier Guillaume, Simon et Adrien pour leur bienveillance. Loin des yeux mais proche du cœur, je remercie finalement Maud, pour nos contacts constants, notamment à refaire le monde.

De manière cette fois tout à fait symbolique, je remercie Henri Vernes, formidable conteur d'aventures, pour m'avoir transporté au travers de ses romans, ainsi que tous les artistes de ma playlist, qui m'ont constamment accompagné dans mon travail.

Contents

Introduction

Given $x_0 \in \mathbb{R}^d$ and $\alpha \geq 0$, we say that a function $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ belongs to the pointwise Hölder space of order α at x_0, which is noted $f \in \Lambda^\alpha(x_0)$ following [86], if there exist a polynomial P_{x_0} of degree stricly less than α and a constant $C > 0$ such that, for all $j \in \mathbb{N}$,

$$\|f - P_{x_0}\|_{L^\infty(B(x_0,2^{-j}))} \leq C 2^{-\alpha j}, \tag{1}$$

where, as usual, $B(x_0, r)$ is the open ball centered at x_0 with radius r. When, for all $x_0 \in \mathbb{R}^d$, f belongs to $\Lambda^\alpha(x_0)$, with an uniform constant C, we say that f is uniformly Hölder of order α and we note $f \in \Lambda^\alpha(\mathbb{R}^d)$.

The aim of those spaces is to define intermediate regularities between the more standard spaces $C^p(\Omega)$ of p-times continuously differentiable functions on the open set Ω. In equation (1), P_{x_0} is the Taylor polynomial of f at x_0, so that one removes the smoothness part from f around x_0 to measure the regularity of what remains. If $0 < \alpha < \beta$, $\Lambda^\beta(x_0) \subseteq \Lambda^\alpha(x_0)$ and one can characterize the regularity at x_0 of a given function f by its Hölder exponent

$$h_f(x_0) = \sup\{\alpha \geq 0\,;\, f \in \Lambda^\alpha(x_0)\}.$$

Hölderian regularity is in particular well-adapted to study the so-called monsters of analysis: everywhere continuous but nowhere differentiable functions [110]. For instance, for all $a \in (0,1)$ and $b > 1$ such that $ab > 1$, the Weierstraß function [127]

$$\mathcal{W}_{a,b} \;:\; \mathbb{R} \to \mathbb{R} \;:\; x \mapsto \sum_{j=0}^{+\infty} a^j \cos(b^j \pi x)$$

satisfies, for all $x_0 \in \mathbb{R}^d$,

$$h_{\mathcal{W}_{a,b}}(x_0) = -\frac{\log(a)}{\log(b)},$$

see [57, 3].

Unfortunately, in general, the function $x_0 \mapsto h_f(x_0)$ can be itself very irregular. For this reason, one prefers to compute the Hausdorff dimension $\dim_{\mathcal{H}}$ of the isohölder sets, i.e. the sets of points sharing the same Hölder exponent. The spectrum of the function f is then defined by

$$D \;:\; [0,+\infty] \to [0,d] \cup \{-\infty\} \;:\; h \mapsto \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d \;:\; h_f(x_0) = h\}),$$

with the convention that $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

If the spectrum D of a function admits a unique finite value, this function is called monofractal (of exponent h). For example, $\mathcal{W}_{a,b}$ is monofractal of exponent $-\frac{\log(a)}{\log(b)}$ and $D(-\frac{\log(a)}{\log(b)}) = 1$. At the opposite, the spectrum of a multifractal function admits different values h for which $D(h) \neq -\infty$. For instance, the Riemann function

$$\mathcal{R}(x) = \sum_{j=1}^{+\infty} \frac{\sin(j^2 \pi x)}{j^2}$$

is multifractal because its spectrum is

$$D(h) = \begin{cases} 4h-2 & \text{if } h \in [\frac{1}{2}, \frac{3}{4}] \\ 0 & \text{if } h = \frac{3}{2} \\ -\infty & \text{otherwise,} \end{cases}$$

as established in [68].

One then has to find conditions to determine spectra of functions. A formula, aimed at obtaining such a spectrum, is called a multifractal formalism and we wish this formula to be valid for a large class of functions. Therefore, we look for determining regularity spaces in which "most" of the functions which belong to them satisfy the corresponding formalism. To formalize this "most", one can speaks in terms of prevalence, a probabilistic notion that generalizes to infinite dimension spaces the notion of "almost everywhere" provided by the Lebesgue measure.

A good tool to establish a multifractal formalism is to use the (discrete) wavelet transform: the associated wavelet coefficients of a function can be used to study the Holdërian regularity, by defining the so-called wavelet leaders. Wavelets can also be used to characterize the belonging to some functional spaces such as, for instance, the Besov spaces ([102]) . Thanks to these two facts, Jaffard and Fraysse proved [72, 70] that, from the prevalence point of view, almost every function $f \in B^s_{p,\infty}(\mathbb{R}^d)$, with $p \in [1,\infty]$ and $s > \frac{d}{p}$, verifies the following multifractal formalism:

$$\forall h \in [s - d/p, s], \quad D(h) = hp - sp + d. \tag{2}$$

Unfortunately, Hölder spaces can only be used for functions that are locally bounded almost everywhere, , as inequality (1) relies on a L^∞ norm. This assumption is, of course, not always satisfied. For this reason, Jaffard and Mellot suggested in [74, 75] to use in the multifractal analysis theory some functional spaces originally defined by Calderón and Zygmund [26]. If $p \in [1,\infty]$, $x_0 \in \mathbb{R}^d$ and $\alpha \geq -d/p$, a function $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ belongs to the space $T^p_\alpha(x_0)$ if there exist a polynomial P_{x_0} of degree stricly less than α and a constant $C > 0$ such that, for all $j \in \mathbb{N}$,

$$2^{jd/p} \|f - P_{x_0}\|_{L^p(B(x_0, 2^{-j}))} \leq C 2^{-\alpha j}. \tag{3}$$

Another drawback from Hölder spaces is their inability to precisely characterize the pointwise behaviour of some remarkable functions and therefore distinguish them. For example, it is well-known that, if $(\Omega, \mathcal{B}, \mathbb{P})$ is a probability space and if $B_\cdot(\cdot)$ is the Brownian motion on it, almost surely for all $\omega \in \Omega$ and for almost every $t_0 \in \mathbb{R}$, there exists $C > 0$ such that

$$|B_t(\omega) - B_{t_0}(\omega)| \leq C|t_0 - t|^{\frac{1}{2}}\sqrt{\log\log|t - t_0|^{-1}}$$

while, for all $t_0 \in \mathbb{R}$, $h_{B_\cdot(\omega)} = \frac{1}{2}$, see [82, 60]. To overcome this problem, Kreit and Nicolay generalized the Hölder spaces in [87, 88, 90] by replacing the dyadic sequence which appears in the right-hand side of (1) by an admissible sequence $\boldsymbol{\sigma} = (\sigma_j)_j$, i.e. a sequence of strictly positive real numbers such that the sequence $(\sigma_{j+1}/\sigma_j)_j$ is bounded. Such sequences are quite easy to handle with as, for instance, they have the advantage that their asymptotic behaviour can be characterized directly, using their so-called Boyd indices. Admissible sequences have already been used to define generalized Besov spaces ([44]).

The initial objective of this thesis was to combine these two methods in order to define functional spaces better suited to characterize more precisely the pointwise behaviour of a given function, even if it is not locally bounded. Thanks to the generalized Besov spaces, we are able to provide a new multifractal formalism, which generalizes (2), and a general framework for the wavelet leaders method. This establishes the theoretical background needed to implement some methods that could be used, for example, to detect if a process is a Brownian motion, or not. Combined with the Black-Scholes model ([13, 58]), this could help to predict the dynamics of a financial market.

Chapter 1 of this thesis is devoted to the presentation of the main tools we will use. Some of them have been briefly quoted in this introduction, more details and references can be found there.

Besov spaces of generalized smoothness play a central role in the multifractal formalism we present. It is thus natural to start by a in-depth study of those spaces, in order to exactly understand the properties of the functions which belong to them. Chapter 2 presents some alternative definitions of the Besov spaces of generalized smoothness, mostly by connecting them to the well-known Sobolev spaces. The uniform regularity of functions is discussed in terms of derivatives, polynomials and convolution. Moreover, we present a generalized method of interpolation, based on admissible sequences, which is particularly wellsuited to our context. Using it, we show that Besov spaces of generalized smoothness are interpolation spaces which "lie" in between two Sobolev spaces.

Chapter 3 to 6 focus on pointwise smoothness.

In Chapter 3, we introduce the generalized pointwise Hölder spaces we are working with. After discussing their definitions, we nearly characterize them with some wavelet coefficients, the p-wavelet leaders, which generalize the wavelet leaders. The properties obtained are used to link pointwise and uniform regularities, thanks to the wavelet characterization of the generalized Besov spaces, proved by Almeida in [2]. Our multifractal formalism is then presented and we show its validity, from the prevalence point of view.

The definition of the p-wavelet leaders given in this thesis is slighty different from the one proposed in [75] and used by him and his co-authors, see [94, 76, 93]. Our choice seems, in our eyes, more relevant and easier to handle with. To convince the reader, in addition to the nearly characterization of the generalized regularity established in Chapter 3, we discuss in Chapter 4 some other nice facts that "our" leaders can provide, concerning the so-called irregularity spaces and a result of prevalence in multifractal analysis.

As already stated, the pointwise spaces of p-regularity were originally introduced by Calderón and Zygmund. They used them to characterize the regularity of the solutions of some partial differential equations. Thus, a natural question is to know whether their results extend to our generalized spaces or not. In Chapter 5, we give an alternative definition of them, using functions instead of sequences to measure the regularity, which is more suited to this context of differentiation. Afterwards, we state the elementary properties of those spaces, needed in the sequel. By the way, we proved a generalization of Whitney extension theorem, which, originally, gives a characterization of the functions which are p-times continuously differentiable on a closed set which are in fact the restriction on it of a p-times continuously differentiable function on the whole Euclidean space ([19]).

Once done, in Chapter 6, connections between generalized pointwise smoothness and elliptic partial differential equations are explored. This follows the trail of Calderón and Zygmund: they showed that such equations can be reduced in terms of fundamental operators that we first need to handle.

A midpoint between uniform and pointwise regularites is to consider functions defined on a compact set. Then, using the structure and the geometry of the compact, we are able to define and use richer operators acting on the functions, such as the Fréchet derivative. Nevertheless, all the good habits acquired while considering functions in $C^p(\Omega)$ have to be dropped, just because we don't work on an open set anymore. In Chapter 7, we propose to start by considering functions which are continuously differentiable on a compact set. The completeness of the obtained space, equipped with a natural norm, is discussed while the density of the restriction to K of the continuously differentiable functions on $\mathbb{R}^d$ is established. This last point gives another connection between this thesis and Whitney extension theorem. We finish by comparing our notion of differentiability on compact sets with others, previously considered in the literature.

1 The tools

In this thesis, one of our main goals is to study some functional spaces in order to use them to capture information about the regularity of a given function better. Their definition will rely on standard functional spaces and especially their norms.

The main idea idea will be to compare relevant quantities to admissible sequences or Boyd functions, depending on the context. They generalize dyadic sequences and power functions respectively without being too far away from them, which makes them a good tool to use in our analysis.

From the wavelet transform, we will use particularly well-chosen coefficients to extract information about the pointwise regularity of a function. Together with the Hausdorff dimension, this will allow us to estimate the "size" of the set of points sharing the same given regularity. This estimation will rely on a so-called *multifractal formalism* whose validity will be ensured by prevalence.

Of course, to prove our results, we will need some fundamental theorems of functional analysis.

All those tools are gathered in this first chapter which may be seen as the foundation of our work.

Contents

1.1 Some standard functional spaces and notations

For a given non-empty open set $\Omega \subset \mathbb{R}^d$ and $p \in [1,\infty]$, $L^p(\Omega)$ is the Lebesgue space of the measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega)} := (\int_\Omega |f|^p \, dx)^{1/p} < \infty$$

if $p < \infty$ and

$$\|f\|_{L^\infty(\Omega)} := \mathrm{supess}_{x\in\Omega} |f(x)| < \infty,$$

otherwise. One sets $L^p := L^p(\mathbb{R}^d)$. As usual, $\ell^p(\mathbb{K})$ (where $\mathbb{K}$ is either $\mathbb{N}$, $\mathbb{N}_0$ or $\mathbb{Z}$) is the Banach subspace of $\mathbb{R}^{\mathbb{K}}$ consisting of all sequences $(x_j)_j$ satisfying

$$\|(x_j)_j\|_{\ell^p(\mathbb{K})} := (\sum_{j\in\mathbb{K}} |x_j|^p)^{1/p} < \infty$$

if p is finite or $\|(x_j)_j\|_{\ell^\infty(\mathbb{K})} = \sup_{j\in\mathbb{K}} |x_j| < \infty$. One sets $\ell^p := \ell^p(\mathbb{N}_0)$.

Let $k \in \mathbb{N}_0$ and $p \in [1,\infty]$; the (historical) Sobolev space $W_p^k(\Omega)$ is defined as

$$W_p^k(\Omega) := \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \, \forall |\alpha| \leq k\},$$

where $D^\alpha f$ is the weak derivative of order α of f ($D^\alpha f$ will denote either the usual derivative or the weak derivative, depending on the context). Equipped with the norm

$$\|f\|_{W_p^k(\Omega)} := \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(\Omega)},$$

$W_p^k(\Omega)$ is a Banach space (see e.g. [1, 121]). We set $W_p^k := W_p^k(\mathbb{R}^d)$.

As usual, $\mathcal{D}$ (resp. $\mathcal{S}$) is the space of infinitely differentiable functions with compact support (resp. the Schwartz space of rapidly decreasing infinitely differentiable functions) on $\mathbb{R}^d$ equipped with the usual topology and $\mathcal{D}'$ (resp. $\mathcal{S}'$) denotes its topological dual, i.e. the space of distributions (resp. tempered distributions) on $\mathbb{R}^d$. If $f \in \mathcal{S}'$, then $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ denote its Fourier transform and its inverse Fourier transform, respectively.

Given $s \in \mathbb{R}$, let u_s be the tempered distribution defined by

$$\mathcal{F}u_s = (1 + |\cdot|^2)^{s/2}.$$

Of course, one has $u_{-s} * u_s = \delta$, where δ is the Dirac delta "function"(see e.g. [116]). Given $s \in \mathbb{R}$ and $p \in [1,\infty]$, the (fractional) Sobolev space H_p^s is defined as

$$H_p^s := \{f \in \mathcal{S}' : \|f\|_{H_p^s} = \|u_s * f\|_{L^p} < \infty\}.$$

Given $s \in \mathbb{N}_0$ and $1 < p < \infty$, one has $H^s_p = W^s_p$. Among the most common properties of these Sobolev spaces, the one that will be used mostly is maybe the continuous embedding $H^s_p \hookrightarrow H^r_p$, valid whenever $r \le s$ [1, 7, 124, 91, 121]. Using Calderón-Zygmund theory, one can show that fractional Sobolev spaces correspond to Bessel potential spaces (see e.g. [1, 7]): if $\mathcal{J}$ is the Bessel operator of order s:

$$\mathcal{J}^s f = \mathcal{F}^{-1}\Big((1+|\cdot|^2)^{-s/2}\mathcal{F}f\Big) \qquad (s \in \mathbb{R},\ f \in \mathcal{S}'),$$

one has

$$H^s_p = \{f \in \mathcal{S}' : \|\mathcal{J}^{-s} f\|_{L^p} < \infty\} \qquad (s \in \mathbb{R},\ 1 \le p \le \infty). \tag{1.1}$$

1.2 Admissible sequences and Boyd functions

Definition 1.2.1. A sequence $\boldsymbol{\sigma} = (\sigma_j)_j$ of real positive numbers is called *admissible* if there exists a positive constant C such that

$$C^{-1}\sigma_j \le \sigma_{j+1} \le C\sigma_j,$$

for any $j \in \mathbb{N}$.

If $\boldsymbol{\sigma}$ is such a sequence, we set

$$\underline{\sigma}_j = \inf_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j = \sup_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k}.$$

Since $(\log \underline{\sigma}_j)_j$ is a subadditive (resp. $(\log \overline{\sigma}_j)_j$ is a superadditive) sequence, Fekete's lemma [48] states that the limits

$$\underline{s}(\boldsymbol{\sigma}) = \lim_j \frac{\log_2 \underline{\sigma}_j}{j} \quad \text{and} \quad \overline{s}(\boldsymbol{\sigma}) = \lim_j \frac{\log_2 \overline{\sigma}_j}{j}$$

exist and are finite. They are defined as the *lower and upper Boyd indices* of $\boldsymbol{\sigma}$. It is well known (see e.g. [87]) that, if $\boldsymbol{\sigma}$ is an admissible sequence and $\varepsilon > 0$, there exists a positive constant C such that

$$C^{-1}2^{j(\underline{s}(\boldsymbol{\sigma})-\varepsilon)} \le \underline{\sigma}_j \le \frac{\sigma_{j+k}}{\sigma_k} \le \overline{\sigma}_j \le C2^{j(\overline{s}(\boldsymbol{\sigma})+\varepsilon)}, \tag{1.2}$$

for any $j, k \in \mathbb{N}$.

In the following, $\boldsymbol{\sigma}$ will always stand for an admissible sequence and, given $s \in \mathbb{R}$, we set $\boldsymbol{s} = (2^{sj})_j$. Of course, we have $\underline{s}(\boldsymbol{s}) = \overline{s}(\boldsymbol{s}) = s$.

It is straightforward to note that, if $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ are two admissible sequences, then:

- the sequence $\boldsymbol{\sigma} + \boldsymbol{\gamma} = (\sigma_j + \gamma_j)_j$ is admissible,

- the sequence $\boldsymbol{\sigma\gamma} = (\sigma_j\gamma_j)_j$ is admissible with $\underline{s}(\boldsymbol{\sigma\gamma}) \geq \underline{s}(\boldsymbol{\sigma}) + \underline{s}(\boldsymbol{\gamma})$ and $\overline{s}(\boldsymbol{\sigma\gamma}) \leq \overline{s}(\boldsymbol{\sigma}) + \overline{s}(\boldsymbol{\gamma})$,
- if $u > 0$, the sequence $\boldsymbol{\sigma^u} = (\sigma_j^u)_j$ is admissible with $\underline{s}(\boldsymbol{\sigma^u}) = u\underline{s}(\boldsymbol{\sigma})$ and $\overline{s}(\boldsymbol{\sigma^u}) = u\overline{s}(\boldsymbol{\sigma})$,
- if $u < 0$, the sequence $\boldsymbol{\sigma^u} = (\sigma_j^u)_j$ is admissible with $\underline{s}(\boldsymbol{\sigma^u}) = u\overline{s}(\boldsymbol{\sigma})$ and $\overline{s}(\boldsymbol{\sigma^u}) = u\underline{s}(\boldsymbol{\sigma})$.

In the sequel, we will very often work with admissible sequences $\boldsymbol{\gamma} = (\gamma_j)_j$ such that $\underline{\gamma}_1 > 1$. Such a sequence is strongly increasing (following [44]), i.e. there exists a number $k_0 \in \mathbb{N}$ such that

$$2\gamma_j \leq \gamma_k \ \forall j,k \in \mathbb{N}_0 \text{ s.t. } j + k_0 \leq k.$$

As equation (1.2) suggests, Boyd indices are good indicators to measure the growth of an admissible sequence. For instance, they give some conditions to bound sums in which dyadic, admissible and ℓ^q sequences appear.

Lemma 1.2.2. *Let $m \in \mathbb{N}$, $\boldsymbol{\sigma}$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}^{-1}) > m$ and $\boldsymbol{\varepsilon} \in \ell^q$ with $q \in [1,\infty]$; there exists a sequence $\boldsymbol{\xi} \in \ell^q$ such that*

$$\sum_{j=J}^{\infty} \varepsilon_j 2^{jm} \sigma_j \leq \xi_J 2^{Jm} \sigma_J,$$

for all $J \in \mathbb{N}$.

Proof. Let $\delta, \delta' > 0$ be such that $-2\delta' > m + \overline{s}(\sigma) + \delta$; given $J \in \mathbb{N}$, we have, using Hölder's inequality,

$$\begin{aligned}\sum_{j=J}^{\infty} \varepsilon_j 2^{jm} \sigma_j &\leq C \sum_{j=J}^{\infty} \varepsilon_j 2^{(j-J)(m+\overline{s}(\sigma)+\delta)} 2^{Jm} \sigma_J \\ &\leq C (\sum_{j=J}^{\infty} (\varepsilon_j 2^{-\delta'(j-J)})^q)^{1/q} (\sum_{j=J}^{\infty} 2^{-p\delta'(j-J)})^{1/p} 2^{Jm} \sigma_J,\end{aligned}$$

where p is the conjugate exponent of q (with the usual modification if one of the indices is ∞). It remains to check that the sequence $\boldsymbol{\xi}$ defined by

$$\xi_j = C (\sum_{k=j}^{\infty} (\varepsilon_j 2^{-\delta'(j-J)})^q)^{1/q}$$

belongs to ℓ^q, which is easy. □

In the same way, we can get the following result.

Lemma 1.2.3. *Let $m \in \mathbb{N}$, $\boldsymbol{\sigma}$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}^{-1}) < m$ and $\boldsymbol{\varepsilon} \in \ell^q$ with $q \in [1,\infty]$; there exists a sequence $\boldsymbol{\xi} \in \ell^q$ such that*

$$\sum_{j=0}^{J} \varepsilon_j 2^{jm} \sigma_j \leq \xi_J 2^{Jm} \sigma_J,$$

for all $J \in \mathbb{N}$.

Admissible sequences are strongly related to so-called Boyd functions.

Definition 1.2.4. A function $\phi : (0,+\infty) \to (0,+\infty)$ is a *Boyd function* if $\phi(1) = 1$, ϕ is continuous and, for all $x \in (0,+\infty)$,

$$\overline{\phi}(x) := \sup_{y>0} \frac{\phi(xy)}{\phi(y)} < \infty. \tag{1.3}$$

We denote by $\mathcal{B}$ the set of Boyd functions.

Let us highlight the fact that Boyd functions are part of the general theory of regular variation for real functions, founded by Jovan Karamata [81] with the introduction of slowly varying functions, see Definition 1.2.7 below. A comprehensive study of these notions can be read in [12].

If $\phi \in \mathcal{B}$, then

- $\overline{\phi}$ is submultiplicative; this follows from the fact that

$$\frac{\phi(xyz)}{\phi(z)} = \frac{\phi(xz)}{\phi(z)} \frac{\phi(xzy)}{\phi(xz)} \leq \overline{\phi}(x)\overline{\phi}(y),$$

  for any $x, y, z > 0$,

- $\overline{\phi}$ is Lebesgue-measurable, since ϕ is continuous,

- one has $\overline{\phi}(x) \geq \phi(x)$ and $\overline{\phi}(1/x) \geq 1/\phi(x)$, for any $x > 0$.

The fact that $\overline{\phi}$ is submultiplicative allows us to introduce the following notion (see e.g. [31] or [12] where the terminology Matuszewska indices is used):

Definition 1.2.5. The *lower and upper Boyd indices of the function $\phi \in \mathcal{B}$* are respectively defined by

$$\underline{b}(\phi) := \sup_{x\in(0,1)} \frac{\log \overline{\phi}(x)}{\log x} = \lim_{x\to 0} \frac{\log \overline{\phi}(x)}{\log x}$$

and

$$\overline{b}(\phi) := \inf_{x\in(1,+\infty)} \frac{\log \overline{\phi}(x)}{\log x} = \lim_{x\to +\infty} \frac{\log \overline{\phi}(x)}{\log x}.$$

The change of supremum and infimum into limits in the previous equalities again comes from a classical result (see e.g. Theorem 7.6.2 in [61]). Let us point out that we have $-\infty < \underline{b}(\phi) \leq \overline{b}(\phi) < +\infty$, since if b is defined as

$$b(x) := \frac{\log \overline{\phi}(x)}{\log x},$$

we have $b(x) \geq b(1/x)$ for $x > 1$.

Similarly to admissible sequences, Boyd indices also allow us to estimate the asymptotic behaviour of functions in $\mathcal{B}$, near the origin and at infinity. The following proposition also appears in [12].

Proposition 1.2.6. *Let $\phi \in \mathcal{B}$, $\varepsilon > 0$ and $R > 0$; there exist $C_1, C_2, C_3, C_4 > 0$ such that*

1. *for all $r \in (0, R]$,*
$$C_1 r^{\overline{b}(\phi)+\varepsilon} \leq \phi(r) \leq C_2 r^{\underline{b}(\phi)-\varepsilon}, \tag{1.4}$$

2. *for all $r \in [R, +\infty)$,*
$$C_3 r^{\underline{b}(\phi)-\varepsilon} \leq \phi(r) \leq C_4 r^{\overline{b}(\phi)+\varepsilon}. \tag{1.5}$$

Proof. Let us prove the first assertion. There exists $R_0 \in (0,1)$ such that, for all $r \in (0, R_0)$,

$$\underline{b}(\phi) - \frac{\log \overline{\phi}(r)}{\log r} \leq \varepsilon,$$

which implies that, for such r,

$$\overline{\phi}(r) \leq r^{\underline{b}(\phi)-\varepsilon}. \tag{1.6}$$

Similarly, there exists $R_1 > 1$ such that, for all $r \in (R_1, \infty)$,

$$\overline{\phi}(r) \leq r^{\overline{b}(\phi)+\varepsilon}. \tag{1.7}$$

Now, using (1.3), we have

$$\overline{\phi}(1/r)^{-1} \leq \phi(r) \leq \overline{\phi}(r), \tag{1.8}$$

for all $r > 0$ and from inequalities (1.6), (1.7) and (1.8), we get,

$$r^{\overline{b}(\phi)+\varepsilon} \leq \phi(r) \leq r^{\underline{b}(\phi)-\varepsilon},$$

for $0 < r \leq \min\{R_0, 1/R_1\}$. If $R \leq \min\{R_0, 1/R_1\}$, one can take $C_1 = C_2 = 1$; otherwise we can use the continuity of the functions

$$r \mapsto \frac{\phi(r)}{r^{\overline{b}(\phi)+\varepsilon}} \qquad \text{and} \qquad r \mapsto \frac{\phi(r)}{r^{\underline{b}(\phi)-\varepsilon}}$$

on the compact set $[\min\{R_0, 1/R_1\}, R]$ to find two constants $C_1, C_2 > 0$ such that (1.4) holds. Inequality (1.5) can be obtained by an analogous reasoning. $\square$

Definition 1.2.7. A strictly positive function ψ is a *slowly varying function* if

$$\lim_{t\to 0}\frac{\psi(rt)}{\psi(t)} = 1,$$

for any $r > 0$.

Slowly varying functions give rise to fundamental examples of Boyd functions.

Example 1.2.8. If ψ is any slowly varying function, then, for any $u \in \mathbb{R}$, the function $\phi : (0,+\infty) \to (0,+\infty) : r \mapsto r^u\psi(r)$ is a Boyd function with $\underline{b}(\phi) = \overline{b}(\phi) = u$ (see [87] for example). Such functions are known as Karamata regularly varying functions (with index u), see [12]. A standard possibility is to take $\psi = |\ln|^s$, for some $s > 0$.

Remark 1.2.9. Inequality (1.6) can be extended in the following way: for all $\varepsilon > 0$ and $R > 0$, there exists $C > 0$ such that for all $r \in (0,R]$,

$$\overline{\phi}(r) \leq Cr^{\underline{b}(\phi)-\varepsilon}.$$

If $R > R_0$, we can use the submultiplicativity of $\overline{\phi}$ to see that for all $r \in (0,R]$,

$$\overline{\phi}(r) \leq \overline{\phi}(\frac{R}{R_0})\overline{\phi}(\frac{R_0}{R}r) \leq \overline{\phi}(\frac{R}{R_0})(\frac{R_0}{R})^{\underline{b}(\phi)-\varepsilon}r^{\underline{b}(\phi)-\varepsilon}.$$

Similarly, we can extend inequality (1.7) using the same approach: for all $\varepsilon > 0$ and $R > 0$, there exists $C > 0$ such that, for all $r \in [R,\infty)$,

$$\overline{\phi}(r) \leq Cr^{(\overline{b}(\phi)+\varepsilon)}.$$

As a corollary to this remark, we have the following result (see e.g. [31], [101]), showing that the Boyd indices give an integrability criterion for Boyd functions.

Proposition 1.2.10. *Let $\phi \in \mathcal{B}$; if $\overline{b}(\phi) < 0$, then $\int_1^{+\infty}\overline{\phi}(x)/x\,dx < \infty$ and if $\underline{b}(\phi) > 0$, then $\int_0^1\overline{\phi}(x)/x\,dx < \infty$.*

Again, one can note that, if ϕ_1 and ϕ_2 are Boyd functions, $\phi_1\phi_2$, ϕ_1/ϕ_2, ϕ_1^u ($u \in \mathbb{R}$) and $\phi_1(1/\cdot)$ are Boyd functions [101].

Boyd functions and admissible sequences are connected by dyadic sequences.

Proposition 1.2.11. *A sequence $\boldsymbol{\sigma} = (\sigma_j)_j$ of real positive numbers is admissible if and only if there exists a Boyd function ϕ such that, for any j, $\phi(2^j) = \sigma_j$. Moreover, in this case, we have $\underline{b}(\phi) = \underline{s}(\boldsymbol{\sigma})$ and $\overline{b}(\phi) = \overline{s}(\boldsymbol{\sigma})$.*

Proof. The sufficiency of the condition is straightforward. For the necessity, if $\boldsymbol{\sigma} = (\sigma_j)_j$ is an admissible sequence, one can check that the function ϕ defined on $(0,+\infty)$ by

$$\phi(x) = \begin{cases} \frac{\sigma_{j+1}-\sigma_j}{2^j}(x-2^j)+\sigma_j & \text{if } x \in [2^j, 2^{j+1}) \text{ (for } j \in \mathbb{N}_0) \\ 1 & \text{if } x \in (0,1) \end{cases}$$

is a Boyd function satisfying, for any j, the equality $\phi(2^j) = \sigma_j$.

For the "moreover part", of course, for any j, $\overline{\sigma}_j \leq \overline{\phi}(2^j)$ and, for any $r \geq 1$, if $2^j \leq r < 2^{j+1}$, $\overline{\phi}(r) \leq \overline{\phi}(2^j)\overline{\phi}(2^{-j}r) \leq C\overline{\phi}(2^j)$, by the submultiplicativity of ϕ and Remark 1.2.9. This is enough to show that $\overline{b}(\phi) = \overline{s}(\sigma)$. The equality $\underline{b}(\phi) = \underline{s}(\sigma)$ is obtained in the same way, after noting that $\underline{\phi} = \overline{\phi^{-1}}(1/\cdot)$. □

Using Example 1.2.8, the next corollary is obvious.

Corollary 1.2.12. *If ψ is a slowly varying function and $u \in \mathbb{R}$, the sequence $\boldsymbol{\sigma} = (2^{ju}\psi(2^j))_j$ is admissible with $\underline{s}(\boldsymbol{\sigma}) = \overline{s}(\boldsymbol{\sigma}) = u$.*

1.3 Finite differences

We will heavily use the finite differences in the sequel (see e.g. [16, 79, 107]). Given a function f defined on $\mathbb{R}^d$ and $x, h \in \mathbb{R}^d$, the finite difference $\Delta_h^n f$ of f is defined as follows

$$\Delta_h^1 f(x) = f(x+h) - f(x) \qquad \text{and} \qquad \Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x),$$

for any $n \in \mathbb{N}$. It is easy to check that the following formula holds:

$$\Delta_h^n f(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x + (n-j)h). \tag{1.9}$$

In this thesis, by convention, if $n \leq 0$, $\Delta_h^n f = f$.

The centered finite difference $\delta_h^n f$ is obtained in the same way:

$$\delta_h^1 f(x) = f(x + h/2) - f(x - h/2) \qquad \text{and} \qquad \delta_h^{n+1} f(x) = \delta_h^1 \delta_h^n f(x).$$

Since we have $\delta_h^n f(x) = \Delta_h^n f(x - nh/2)$, these two notions will lead to the same definitions; for example, we obviously have

$$\|\Delta_h^n f\|_{L^p} = \|\delta_h^n f\|_{L^p},$$

for any $h \in \mathbb{R}^d$, any $n \in \mathbb{N}$, and any $p \in [1, \infty]$. If $f \in W_p^k$ ($k \in \mathbb{N}$, $p \in [1, \infty]$), for all $1 \leq n \leq k$, there exists a constant $C > 0$, not depending on the function f, such that

$$\|\Delta_h^n f\|_{L^p} \leq C|h|^n \sup_{|\alpha|=n} \|D^\alpha f\|_{L^p},$$

for all $h \in \mathbb{R}^d$.

1.4 Wavelets

Let us briefly recall some definitions and notations about wavelets (for more precisions, see e.g. [36, 102, 100]). Under some general assumptions, there exist a function φ and $2^d - 1$ functions $(\psi^{(i)})_{1\leq i<2^d}$, called wavelets, such that

$$\{\varphi(x-k) : k \in \mathbb{Z}^d\} \cup \{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^d, k \in \mathbb{Z}^d, j \in \mathbb{N}\} \tag{1.10}$$

form an orthogonal basis of L^2. Any function $f \in L^2$ can be decomposed as follows,

$$f(x) = \sum_{k\in\mathbb{Z}^d} C_k \varphi(x-k) + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}^d} \sum_{1\leq i<2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int_{\mathbb{R}^d} f(x) \psi^{(i)}(2^j x - k)\, dx$$

and

$$C_k = \int_{\mathbb{R}^d} f(x) \varphi(x-k)\, dx. \tag{1.11}$$

Let us remark that we do not choose the L^2 normalization for the wavelets, but rather an L^∞ normalization, which is better fitted to the study of the Hölderian regularity.

In this thesis, we will consider two families of wavelets:

- compactly supported wavelets, built in [35],
- wavelets in the Schwartz space of rapidly decreasing infinitely differentiable functions, built in [92].

If ψ is such a wavelet, $\psi^{(i)}(2^j \cdot -k)$ is localized around the dyadic cube

$$\lambda_{j,k}^{(i)} := \frac{i}{2^{j+1}} + \frac{k}{2^j} + [0, \frac{1}{2^{j+1}})^d.$$

In the sequel, we will often omit any reference to the indices i, j and k for such cubes by writing $\lambda = \lambda_{j,k}^{(i)}$. We will also index the wavelet coefficients of a function f with the dyadic cubes λ so that c_λ will refer to the quantity $c_{j,k}^{(i)}$. The notation Λ_j will stand for the set of dyadic cubes λ of $\mathbb{R}^d$ with side length 2^{-j} and the unique dyadic cube from Λ_j containing the point $x_0 \in \mathbb{R}^d$ will be denoted $\lambda_j(x_0)$. The set of dyadic cubes is $\Lambda := \cup_{j\in\mathbb{N}} \Lambda_j$. Two dyadic cubes λ and λ' are adjacent if there exists $j \in \mathbb{N}$ such that $\lambda, \lambda' \in \Lambda_j$ and $\operatorname{dist}(\lambda, \lambda') = 0$. The set of the 3^d dyadic cubes adjacent to λ will be denoted by 3λ.

In practice, a real-life signal is often modelled as a finite sequence $x_1, \ldots, x_J$ of real numbers that can be interpreted as the realisation of a function f defined on the interval $[0,1]$ for which, for all $1 \leq j \leq J$, $f(\frac{j}{J}) = x_j$. Then, one can assume that f belongs to $L^2(\mathbb{T}^d)$, the set of 1-periodic functions which are in $L^2_{\text{loc}}(\mathbb{R}^d)$.

To deal with $L^2(\mathbb{T}^d)$ functions, one can use the periodization operator

$$[\cdot] : f \mapsto \sum_{l \in \mathbb{Z}^d} f(\cdot - l)$$

to obtain an orthonormal basis of $L^2(\mathbb{T}^d)$ from the one defined in (1.10). Indeed, as shown in [36, 33], the set

$$\{[\psi_\lambda] : 1 \leq i < 2^d, j \in \mathbb{N} \text{ and } k \in \{0, \ldots, 2^{j-1}\}^d\},$$

together with the function 1, form an orthonormal basis of $L^2(\mathbb{T}^d)$. The wavelet coefficients are then defined in the same way:

$$c_\lambda^{\text{per}} = 2^{dj} \int_{[0,1]^d} f(x)[\psi_\lambda](x)\, dx. \tag{1.12}$$

One often omits the mention "per" and any references to the periodization operator in (1.12) and c_λ can denote both the wavelet coefficients or the periodized wavelet coefficients, depending on the context. In the sequel, the established results concern both families of coefficients.

1.5 Hausdorff measure and dimension

The "size" of the sets of points sharing the same regularity will be estimated with the help of the Hausdorff dimension which is defined from the homonym measure. Details about what it summarized here can be found in [42, 115, 59]. Let us fix X a separable metric space; if $\varepsilon, h > 0$, we first define

$$\mathcal{H}_\varepsilon^h : \wp(X) \to [0, +\infty] : A \mapsto \inf\{\sum_j \operatorname{diam}^h(A_j) : A \subseteq \bigcup_j A_j \text{ and}, \forall j, \operatorname{diam}(A_j) < \varepsilon\}$$

where, as usual, diam stands for the diameter. For all $\varepsilon, h > 0$, $\mathcal{H}_\varepsilon^h$ is an outer measure and is called *the (h,ε)-Hausdorff outer measure*. Moreover, for all $h > 0$, the application $\varepsilon \mapsto \mathcal{H}_\varepsilon^h$ is decreasing and it follows that the *h-dimensional Hausdorff measure*

$$\mathcal{H}^h : \wp(X) \to [0, +\infty] : A \mapsto \lim_{\varepsilon \to 0^+} \mathcal{H}_\varepsilon^h(A)$$

is well-defined. Again, $\mathcal{H}^h$ is an outer measure and, once restricted to the $\mathcal{H}^h$-measurable set, the h-dimensional Hausdorff measure is invariant by translation. Therefore, if $X = \mathbb{R}^d$, the d-dimensional Hausdorff measure is related to $\mathcal{L}^d$, the Lebesgue measure[1] (in $\mathbb{R}^d$), by

$$\mathcal{L}^d = \frac{\pi^{\frac{d}{2}}}{2^d \Gamma(\frac{d}{2} + 1)} \mathcal{H}^d.$$

[1]When the (topological) dimension d is clear, we will simply note $\mathcal{L}$, for the sake of simpleness.

The crucial property of Hausdorff measures is that, for any set A, there exists a critical value h_0 such that

$$\mathcal{H}^h(A) = \infty \ \forall\, h < h_0 \quad \text{and} \quad \mathcal{H}^h(A) = 0 \ \forall\, h > h_0.$$

This pivot is used to define the Hausdorff dimension of a set: if A is a non-empty set, *the Hausdorff dimension of A* is

$$\dim_{\mathcal{H}}(A) = \sup\{h > 0 : \mathcal{H}^h(A) = \infty\} = \inf\{h > 0 : \mathcal{H}^h(A) = 0\},$$

while, by convention, $\dim_{\mathcal{H}}(\emptyset) = -\infty$. The main properties are listed in the proposition below.

Proposition 1.5.1.

- *If $A \subset B$, then* $\dim_{\mathcal{H}}(A) \leq \dim_{\mathcal{H}}(B)$,
- *If A is countable, then* $\dim_{\mathcal{H}}(A) = 0$,
- *If $(A_j)_j$ is a sequence of sets,*

$$\dim_{\mathcal{H}}(\bigcup_j A_j) = \sup_j \dim_{\mathcal{H}}(A_j),$$

- *For all $d \in \mathbb{N}$, if $X = \mathbb{R}^d$ and $A \in \wp(\mathbb{R}^d)$,* $\dim_{\mathcal{H}}(A) \leq d$*. Moreover, if $\mathcal{L}^d(A) > 0$, then* $\dim_{\mathcal{H}}(A) = d$.

The Hausdorff dimension is therefore a "continuous extension" of the topological dimension, as, for all $d \in \mathbb{N}$ and any open subset Ω of $\mathbb{R}^d$, $\dim_{\mathcal{H}}(\Omega) = d$.

1.6 Prevalence

Now, we very briefly introduce the notion of prevalence (see [27, 66, 65] for more details).

In $\mathbb{R}^d$, it is well known that if one can associate a probability measure μ to a Borel set B such that $\mu(B + x)$ vanishes for very $x \in \mathbb{R}^d$, then the Lebesgue measure $\mathcal{L}(B)$ of B also vanishes. For the notion of prevalence, this property is turned into a definition in the context of infinite-dimensional spaces.

Definition 1.6.1. Let E be a complete metric vector space; a Borel set B of E is *Haar-null* if there exists a compactly-supported probability measure μ such that $\mu(B+x) = 0$, for every $x \in E$. A subset of E is *Haar-null* if it is contained in a Haar-null Borel set; the complement of a Haar-null set is a *prevalent set*.

If E is finite-dimensional, B is Haar-null if and only if $\mathcal{L}(B) = 0$; if E is infinite-dimensional, the compact sets of E are Haar-null. Moreover, it can be shown that a translated of a Haar-null set is Haar-null and that a prevalent set is dense in E. Finally, the intersection of a countable collection of prevalent sets is prevalent.

Let us make some remarks about how to show that a set is Haar-null. A common choice for the measure in Definition 1.6.1 is the Lebesgue measure on the unit ball of a finite-dimensional subset E' of E. For such a choice, one has to show that $\mathcal{L}(B \cap (E' + x))$ vanishes for every x. Such a subspace is called a probe. If E is a function space, one can choose a random process X whose sample paths almost surely belong to E. In this case, one can show that a property only holds on a Haar-null set by showing that the sample path X is such that, for any $f \in E$, $X_t + f$ almost surely does not satisfy the property.

If a property holds on a prevalent set, we will say that it holds almost everywhere from the prevalence point of view.

1.7 Some fundamental theorems in functional analysis

In this section, for the sake of completeness, we gather some fundamental theorems that we use all along the thesis, together with a reference to a proof.

The first theorem, established by Lebesgue, discusses the reciprocity between integration and differentiation.

Theorem 1.7.1 (Lebesgue differentiation theorem, [15]). *Let f be a Lebesgue-integrable function defined on $\mathbb{R}^d$. For almost every point $x_0 \in \mathbb{R}^d$, the limit*

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}(B(x_0, r))} \int_{B(x_0,r)} f(x)\, dx$$

exists and is equal to $f(x_0)$. Such a point is called a Lebesgue point of f.

To state the next theorem, one first has to introduce the Taylor chain condition.

Definition 1.7.2. If $k \in \mathbb{N}$ and if $(f_\alpha)_{|\alpha| \leq k}$ is a k-jet defined on a closed set F, we say that $(f_\alpha)_{|\alpha| \leq k}$ satisfies the *Taylor chain condition of order k* if the functions

$$(x, y) \mapsto \frac{\left| f_\alpha(x) - \sum_{|\beta| \leq k - \alpha} f_{\alpha+\beta}(y) \frac{(x-y)^\beta}{\beta!} \right|}{|x - y|^{k - |\alpha|}} \tag{1.13}$$

are continuous on $(F \times F) \setminus \{(x, y) \in F \times F : x = y\}$ and can be continuously extended by 0 to the whole of $F \times F$.

Obviously, if $k \in \mathbb{N}$, $f \in C^k(\mathbb{R}^d)$ and F is a closed set, then f and its derivatives up to order k restricted to F define a k-jet which satisfies the Taylor chain condition of order k. Whitney extension theorem establishes the reverse.

Theorem 1.7.3 (**Whitney extension theorem, [19]**). *Let F be a closed set in $\mathbb{R}^d$ and $k \in \mathbb{N}$. A k-jet $(f_\alpha)_{|\alpha|\leq k}$ is obtained by restriction of a function in $C^k(\mathbb{R}^d)$ and its derivatives up to order k if and only if it satisfies the Taylor chain condition of order k.*

Before stating the next theorem, let us recall that, if $\varphi \in C^1(\mathbb{R}^d)$, then

- The *gradient* of φ is the vector $\nabla\varphi = (D_1\varphi, \cdots, D_d\varphi)$,
- The *divergent* of φ is the scalar $\operatorname{div}\varphi = \sum_{j=1}^d D_j\varphi$.

Moreover, if $\varphi \in C^2(\mathbb{R}^d)$, one can apply the Laplace operator Δ to φ:

$$\Delta\varphi = \sum_{j=1}^{d} D_j^2\varphi.$$

Theorem 1.7.4 (**Green's first identity,[120]**). *Let D be a domain in $\mathbb{R}^d$, $\varphi \in C^2(\mathbb{R}^d)$, $\psi \in C^1(\mathbb{R}^d)$; we have*

$$\int_D (\psi\Delta\varphi + \nabla\psi\cdot\nabla\varphi)\,dx = \int_{\partial D} \psi\frac{\partial\varphi}{\partial n}d\sigma,$$

where $\frac{\partial\varphi}{\partial n} = n\cdot\nabla\varphi$ is the directional derivative in the outward normal direction and $d\sigma$ in the surface area on ∂U.

The Banach-Steinhaus theorem is a very strong result to find a uniform bound for a family of continuous linear maps.

Theorem 1.7.5 (**Banach-Steinhaus theorem - Uniform Boundedness Principle,[112]**). *Let X be a Banach space, Y a normed linear space and $(T_\lambda)_{\lambda\in\Lambda}$ a collection of continuous linear transformations of X into Y. If, for any $x \in X$,*

$$\sup_{\lambda\in\Lambda} \|T_\lambda(x)\|_Y < \infty,$$

then

$$\sup_{\lambda\in\Lambda, \|x\|_X=1} \|T_\lambda(x)\|_Y < \infty.$$

The next theorem deals with equicontinuous sequences of functions; let us first recall this notion.

Definition 1.7.6. A sequence of functions $(f_j)_j$ defined on a metric space (X,d) is

- *pointwise bounded* if, for any $x \in X$, the sequence $(f_j(x))_j$ is bounded.
- *equicontinuous* if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever $x, y \in X$ and $d(x,y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Theorem 1.7.7 (**Arzelà-Ascoli theorem,[111]**). *Let K be a compact metric space and $(f_j)_j$ a sequence of continuous functions on K. If $(f_j)_j$ is pointwise bounded and equicontinuous on K, then $(f_j)_j$ is uniformly bounded and contains a uniformly convergent subsequence.*

Hahn-Banach theorems are a family of fundamental results which link a locally convex topological vector space X with its dual X^*, whose elements are the continuous linear functional on X, and its subspaces by the mean of semi-norms on X. Among them, we will need the following.

Theorem 1.7.8 (**Hahn-Banach theorem,[113]**). *Suppose that M is a subspace of a locally convex space X, and $x_0 \in X$. If x_0 is not in the closure of M, then there exists $\Lambda \in X^*$ such that $\Lambda x_0 = 1$ but $\Lambda x = 0$ for every $x \in M$.*

As stated in [113], a consequence of this theorem is that, in order to show that $x_0 \in X$ belongs to the closure of a subspace M of X, it suffices to show that, for every continuous linear functional Λ on X such that Λ vanishes on M, $\Lambda x_0 = 0$.

Now, we recall that if μ is a complex measure on a measurable space $(X, \mathscr{A})$, the variation measure $\|\mu\|$ is the positive measure defined on $\mathscr{A}$ by

$$\|\mu\|(A) = \sup\left\{\sum_j |\mu(A_j)| : (A_j)_j \text{ partition of } A \text{ in } \mathscr{A}\right\}.$$

The *total variation* is then defined by $\mathrm{Var}(\mu) = \|\mu\|(X)$.

Definition 1.7.9. Let X be a locally compact Hausdorff space and $\mathscr{B}$ the Borel σ-algebra on X. A measure μ on $(X, \mathscr{B})$ is *regular* if

- μ is outer regular: for every Borel set A,

$$\mu(A) = \inf\{\mu(\Omega) : A \subseteq \Omega, \Omega \text{ open}\},$$

- μ is inner regular: for every Borel set A with finite measure,

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ compact}\}.$$

If, in addition, μ is finite on compact sets, μ is said to be a *Radon measure*.

A complex measure on $(X, \mathscr{B})$ is regular (resp. a Radon measure) if $\|\mu\|$ is regular (resp. a Radon measure).

In this last theorem, $C_0(X)$ stands for the space of continuous function on X which are vanishing at infinity.

Theorem 1.7.10 (**Riesz representation theorem, [15, 112]**). *If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique Radon measure μ, in the sense that, for every $f \in C_0(X)$,*

$$\Phi(f) = \int f \, d\mu.$$

Moreover, the norm of Φ is the total variation of μ.

2 Generalized Besov spaces

The Besov spaces $B^s_{p,q}$ (with $s \in \mathbb{R}$ and $1 \leq p,q \leq \infty$) were introduced about sixty years ago [8, 9] and many studies have been since devoted to such spaces (see e.g. [123, 10, 124, 11, 125, 126]). They were generalized in the middle of the seventies by several authors in different contexts starting from different points of view. The variant we will present here has been largely considered in [43, 23, 2, 44, 105] for example. Besov spaces are still considered nowadays in connection with embeddings, limiting embedding, entropy numbers, probability theory and theory of stochastic processes for instance (see e.g. [80, 39, 21, 40, 95, 108] and references therein). More recently, such generalizations have been used to numerically detect the law of the iterated logarithm in signals [89, 84, 90].

A classical generalization of the usual Besov spaces was introduced in [101, 31] using interpolation theory. To obtain these spaces $B^f_{p,q}$, s is replaced by a function parameter g such that $f(x) = x^t/g(x^{t-u})$ in the interpolation formula (2.8). These spaces can themselves be generalized using the Littlewood-Paley decomposition instead of the interpolation theory to define the spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$, where σ and γ are two admissible sequences [44]. One has $B^f_{p,q} = B^{\sigma,\gamma}_{p,q}$, with $\gamma = (2^j)_j$ and $\sigma = (f(2^j))_j$. In a way, these spaces provide a very general definition of the spaces of generalized smoothness [105]. This work can be seen as an intent to "close the circle" by defining a generalized interpolation method that allows to define the spaces $B^{\sigma,\gamma}_{p,q}$ starting from the usual Sobolev spaces. This interpolation method is quite different from the one introduced in [31].

In this chapter, in the same spirit as in [87, 88], we propose equivalent definitions of the spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$. The first section is devoted to standard definitions and a brief review of the background material needed for them. Next, we give some preliminary results before looking at the links between these spaces and the weak derivatives of the elements of the historical Sobolev spaces W^k_p. We also give definitions involving Taylor expansion and polynomials before investigating how the generalized Besov spaces can be characterized in terms of convolution. Finally, we show that these spaces can be introduced using generalized interpolation of fractional or historical Sobolev spaces, as were the spaces $B^f_{p,q}$.

Our aim, in this chapter, is to better understand the generalized Besov spaces, using the alternative definitions we obtain, in order to know, with more details, the information we can extract (about uniform regularity) from the fact that a function belongs to a generalized Besov space. In the next chapter, we show that these spaces provide a natural framework for a general multifractal formalism.

Results established in this chapter were published in [97].

Contents

2.1 Definition

Let us start by introducing the Besov spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$ that we will consider (see e.g. [44, 105] and references therein). Let us recall some kind of generalization of the Littlewood-Paley decomposition (more details can be found in [44]). For a strongly increasing admissible sequence $\boldsymbol{\gamma} = (\gamma_j)_j$, let $\rho \in \mathcal{D}$ be a positive function such that $\rho(t) = 1$ for all $|t| \leq 1$, ρ is decreasing for $t \geq 0$ and $\operatorname{supp}(\rho) \subset \{t \in \mathbb{R} : |t| \leq 2\}$ (where, as usual, $\operatorname{supp}(\rho)$ denotes the support of ρ). Given $J \in \mathbb{N}$, let us set

$$\varphi^{\gamma,J}_j := \rho(\gamma_j^{-1}|\cdot|) \quad \text{for } j \in \{0,\dots,Jk_0-1\}$$

and

$$\varphi^{\gamma,J}_j := \rho(\gamma_j^{-1}|\cdot|) - \rho(\gamma_{j-Jk_0}^{-1}|\cdot|) \quad \text{for } j \geq Jk_0.$$

Let us define $c_\varphi := Jk_0$ and

$$\widetilde{\varphi^{\gamma,J}_j} := \sum_{k=-(2J+1)k_0}^{(2J+1)k_0} \varphi^{\gamma,J}_{j+k} \quad \text{for } j \in \mathbb{N}_0,$$

where $\varphi^{\gamma,J}_{-1} = \cdots = \varphi^{\gamma,J}_{-(2J+1)k_0} = 0$. With such a system one has, for all $j \in \mathbb{N}_0$,

$$\widetilde{\varphi^{\gamma,J}_j} = c_\varphi \quad \text{on } \operatorname{supp}(\varphi^{\gamma,J}_j).$$

As a consequence, if we set, for any $f \in \mathcal{S}'$,

$$\Delta_j^{\gamma,J} f := \mathcal{F}^{-1}(\varphi_j^{\gamma,J} \mathcal{F} f) \qquad \text{and} \qquad \widetilde{\Delta_j^{\gamma,J}} f := \mathcal{F}^{-1}(\widetilde{\varphi_j^{\gamma,J}} \mathcal{F} f), \tag{2.1}$$

then

$$\widetilde{\Delta_j^{\gamma,J}} \Delta_j^{\gamma,J} f = c_\varphi \Delta_j^{\gamma,J} f.$$

Remark 2.1.1. From the Littlewood-Paley theory [119, 41], $\Delta_j^{\gamma,J} f$ belongs to the space $C^\infty(\mathbb{R}^d)$.

Definition 2.1.2. Let $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ be two admissible sequences such that $\boldsymbol{\gamma}$ is strongly increasing (most often we will require $\underline{\gamma}_1 > 1$) and $p, q \in [1, \infty]$; the generalized Besov space $B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}}$ is defined as

$$B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}} := \{f \in \mathcal{S}' : \|f\|_{B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}}} = \|(\sigma_j \|\Delta_j^{\gamma,J} f\|_{L^p})_j\|_{\ell^q} < \infty\}.$$

Remark 2.1.3. The space $B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}}$ defined above does not depend on the particular decomposition chosen to represent the functions, in the sense of equivalent norms: two decompositions give rise to the same space (see Remark 3.1.3 in [44]). Indeed, such spaces can be defined with a general representation method which must satisfy conditions that are met by the decomposition given here, see again [44].

If the admissible sequence $\boldsymbol{\gamma}$ is the usual sequence $(2^j)_j$, we prefer to denote $B_{p,q}^{\boldsymbol{\sigma},\mathbf{1}}$ by $B_{p,q}^{\boldsymbol{\sigma}}$, in order to simplify the notation.

Remark 2.1.4. In [23, 22], the authors highlight the fact that, if $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ are admissible sequences such that $\underline{\gamma}_1 > 1$ then, if k_0 is such that $\underline{\gamma}_1^{k_0} \geq 2$, the sequence $\boldsymbol{\beta} = (\beta_j)_j$ defined by

$$\beta_j := \sigma_{k(j)}, \quad \text{with } k(j) := \min\{k \in \mathbb{N}_0 : 2^{j-1} \leq \gamma_{k+k_0}\},$$

for all $j \in \mathbb{N}_0$, is admissible and $B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}} = B_{p,q}^{\boldsymbol{\beta}}$. This allows us to work with one or two non-dyadic sequences, depending on the context. In this chapter, it is more appropriate to work with $B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}}$ as the Boyd indices of $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ are easier to compute separately than the ones of $\boldsymbol{\beta}$. In the next chapter, since we will make use of wavelets, which are heavily connected to the $(2^j)_j$ sequence, we will favor the other option and work with only one admissible sequence.

The following characterization is given in [105]: let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and $0 < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1}$. For any $n \in \mathbb{N}$ such that $\overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n$, we have

$$B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}} = \{f \in L^p : (\sigma_j \sup_{|h| \leq \gamma_j^{-1}} \|\Delta_h^n f\|_{L^p})_j \in \ell^q\}. \tag{2.2}$$

In this framework, Besov spaces of generalized smoothness provide an obvious generalization of the usual Hölder spaces: if $\boldsymbol{\sigma} = (2^{sj})_j$ ($s > 0$) and $\boldsymbol{\gamma} = (2^j)_j$, the space $B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q}$ so defined is the usual Besov space $B^s_{p,q}$ and if $p = q = \infty$, we get the usual Hölder space $\Lambda^s(\mathbb{R}^d)$. One can therefore wonder if polynomials can play a role in the definition of the spaces $B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q}$ (as it is the case for Hölder spaces and their generalized version for example [86, 87, 88]). A theorem due to Whitney (see [20]) states that for $f \in L^p$, $r > 0$ and $n \in \mathbb{N}$, there exists a constant $C > 0$ (which only depends on n and d) such that

$$\inf_{P \in \mathbb{P}^d_{n-1}} \|f - P\|_{L^p(B(x_0,r))} \leq C \sup_{|h| \leq r} \|\Delta^n_h f\|_{L^p(B_{nh}(x_0,r))},$$

where

$$B_{nh}(x_0, r) := \{x \in B(x_0, r) : [x, x + nh] \subset B(x_0, r)\}.$$

It follows that, if $f \in B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{p,q}$, then, for n sufficiently large, there exists $(\varepsilon_j)_j \in \ell^q$ such that for all $x_0 \in \mathbb{R}^d$,

$$\sigma_j \inf_{P \in \mathbb{P}^d_{n-1}} \|f - P\|_{L^p(B(x_0,\gamma_j^{-1}))} \leq \varepsilon_j \quad \forall j \in \mathbb{N}_0. \tag{2.3}$$

The converse is not true unless $p = \infty$, as explained in Remark 2.1.5 below.

Remark 2.1.5. Now, suppose that $f \in L^\infty$ and let $\boldsymbol{\sigma} = (\sigma_j)_j, \boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences satisfying $\overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n$, with $n \in \mathbb{N}$. If $\boldsymbol{\gamma}$ is strongly increasing, given $x_0 \in \mathbb{R}^d$, we can claim that there exists $k_1 \in \mathbb{N}$ such that $n\gamma_k^{-1} \leq \gamma_j^{-1}$ if $j + k_1 \leq k$, which implies that if $|h| \leq \gamma^{-1}_{j+k_1}$, then $x_0 + lh \in B(x_0, \gamma_j^{-1})$ for all $l \in \{0, \ldots, n\}$. Since, for any $P \in \mathbb{P}^d_{n-1}$, the formula

$$\Delta^n_h f(x_0) = \Delta^n_h (f - P)(x_0) = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} (f - P)(x_0 + lh),$$

holds, we have

$$|\Delta^n_h f(x_0)| \leq 2^n \inf_{P \in \mathbb{P}^d_{n-1}} \|f - P\|_{L^\infty(B(x_0,\gamma_j^{-1}))}, \tag{2.4}$$

if $|h| \leq \gamma^{-1}_{j+k_1}$, for almost every x_0. Therefore, if (2.3) holds with $p = \infty$, we have

$$\sigma_j \sup_{|h| \leq \gamma_j^{-1}} \|\Delta^n_h f\|_{L^\infty} \leq C\varepsilon_j \quad \forall j \in \mathbb{N}_0$$

and thus $f \in B^{\boldsymbol{\sigma},\boldsymbol{\gamma}}_{\infty,q}$. Inequality (2.4) is not sufficient to get such a conclusion in L^p with $p < \infty$. Nevertheless, we will see that a approximation by a sequence of "locally" polynomial functions is still possible, in any case.

2.2 Preliminary results involving convolutions

Let us first introduce some results about the convolution product that will be used in the sequel. They are obtained using very classical proofs but we give them for the sake of completeness. If ϕ is a function defined on $\mathbb{R}^d$, given $\varepsilon \neq 0$ one sets

$$\phi_\varepsilon = \frac{1}{|\varepsilon|^d}\,\phi(\frac{\cdot}{\varepsilon}).$$

Proposition 2.2.1. *Let $n \in \mathbb{N}$, $p,q \in [1,\infty]$ and $(\sigma_j)_j$, $(\gamma_j)_j$ be two sequences of positive real numbers. If $f \in L^p$ is such that*

$$(\sigma_j \sup_{|h|\leq \gamma_j^{-1}} \|\Delta_h^n f\|_{L^p})_j \in \ell^q,$$

then there exists $\phi \in \mathcal{D}$ such that

$$(\sigma_j \sup_{0<\varepsilon\leq \gamma_j^{-1}} \|f * \phi_\varepsilon - f\|_{L^p})_j \in \ell^q.$$

Proof. Without loss of generality, we can suppose that $n = 2m$ where m is an odd integer. Let $\rho \in \mathcal{D}$ be a radial function such that

- $\mathrm{supp}(\rho) \subseteq \overline{B(0,1)}$,
- $0 \leq \rho \leq 1$,
- $\|\rho\|_{L^1} = 1$

and set

$$\tilde{\phi} := \sum_{j=0}^{m-1} (-1)^j \binom{n}{j} \rho_{2j-n},$$

$c_n := \sum_{j=0}^{m-1} (-1)^j \binom{n}{j} = \binom{n}{m}/2$ and finally $\phi := \tilde{\phi}/c_n$. Obviously, $\phi \in \mathcal{D}$ and for all $x \in \mathbb{R}^d$ and $\varepsilon > 0$, we have

$$\begin{aligned} f * \phi_\varepsilon(x) - f(x) &= \int f(x-\varepsilon y)\phi(y)\,dy - f(x) \\ &= \frac{1}{c_n}\sum_{j=0}^{m-1} (-1)^j \binom{n}{j} \int f(x-\varepsilon y)\rho_{2j-n}(y)\,dy - f(x) \\ &= \frac{1}{c_n}\sum_{j=0}^{m-1} (-1)^j \binom{n}{j} \int f(x-\varepsilon(2j-n)t)\rho(t)\,dt - f(x). \end{aligned}$$

We get

$$\begin{aligned}
&\sum_{j=m+1}^{n}(-1)^j\binom{n}{j}\int f(x-\varepsilon(2j-n)t)\rho(t)\,dt \\
&\quad=\sum_{j=m+1}^{n}(-1)^j\binom{n}{j}\int f(x-\varepsilon(n-2j)y)\rho(y)\,dy \\
&\quad=\sum_{j=0}^{m-1}(-1)^j\binom{n}{j}\int f(x-\varepsilon(2j-n)y)\rho(y)\,dy
\end{aligned}$$

and

$$\begin{aligned}
&f*\phi_\varepsilon(x)-f(x) \\
&\quad=\frac{1}{2c_n}\Big(\sum_{\substack{j=0\\ j\neq m}}^{n}(-1)^j\binom{n}{j}\int f(x-\varepsilon(2j-n)t)\rho(t)\,dt-2c_nf(x)\Big) \\
&\quad=\frac{1}{2c_n}\int\Big(\sum_{j=0}^{n}(-1)^j\binom{n}{j}f\Big(x-2\varepsilon t(j-\frac{n}{2})\Big)\Big)\rho(t)\,dt \\
&\quad=\frac{1}{2c_n}\int_{\overline{B(0,1)}}\delta^n_{2\varepsilon t}f(x)\rho(t)\,dt.
\end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned}
|f*\phi_\varepsilon(x)-f(x)| &\le C\|\rho\|_{L^q(\overline{B(0,1)})}\|\delta^n_{2\varepsilon\cdot}f(x)\|_{L^p(\overline{B(0,1)})} \\
&\le C\|\Delta^n_{2\varepsilon\cdot}f(x)\|_{L^p(\overline{B(0,1)})},
\end{aligned}$$

where q is the conjugate exponent of p. It follows, with the usual modification if $p=\infty$, that

$$\begin{aligned}
\|f*\phi_\varepsilon-f\|_{L^p} &\le C\Big(\int\int_{\overline{B(0,1)}}|\Delta^n_{2\varepsilon t}f(x)|^p\,dt\,dx\Big)^{1/p} \\
&=C\Big(\int_{\overline{B(0,1)}}\int|\Delta^n_{2\varepsilon t}f(x)|^p\,dx\,dt\Big)^{1/p} \\
&=C\Big(\int_{\overline{B(0,1)}}\|\Delta^n_{2\varepsilon t}f\|^p_{L^p}\,dt\Big)^{1/p} \\
&\le C\sup_{t\in\overline{B(0,1)}}\|\Delta^n_{2\varepsilon t}f\|_{L^p}
\end{aligned}$$

and finally, using a classical result for the last inequality [37, p. 45, formula (7.6)],

$$\sup_{0<\varepsilon\le\gamma_j^{-1}}\|f*\phi_\varepsilon-f\|_{L^p}\le C\sup_{|h|\le\gamma_j^{-1}}\|\Delta^n_{2h}f\|_{L^p}\le C\sup_{|h|\le\gamma_j^{-1}}\|\Delta^n_hf\|_{L^p},$$

as desired. □

Proposition 2.2.2. *Let $p, q \in [1, \infty]$, σ be an admissible sequence and $(\gamma_j)_j$ be a sequence of positive real numbers such that there exists $d_0 > 0$ satisfying*

$$d_0\gamma_j \leq \gamma_{j+1} \quad \forall j \in \mathbb{N}_0.$$

Let also $\phi \in \mathcal{D}$ and $f \in L^p$ satisfying

$$(\sigma_j \|f * \phi_{\gamma_j^{-1}} - f\|_{L^p})_j \in \ell^q.$$

Then, for all $\alpha \in \mathbb{N}_0^d$,

$$(\sigma_j \gamma_j^{-|\alpha|} \|D^\alpha (f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}})\|_{L^p})_{j\in\mathbb{N}} \in \ell^q(\mathbb{N}).$$

Proof. Let us write

$$\begin{aligned} f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}} = \phi_{\gamma_j^{-1}} * (f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}}) \\ + \phi_{\gamma_j^{-1}} * (f - f * \phi_{\gamma_j^{-1}}) \\ - \phi_{\gamma_{j-1}^{-1}} * (f - f * \phi_{\gamma_j^{-1}}). \end{aligned} \tag{2.5}$$

Considering the first term on the right-hand side of this equality, we have, by Young's inequality,

$$\begin{aligned} &\sigma_j \gamma_j^{-|\alpha|} \|D^\alpha \Big(\phi_{\gamma_j^{-1}} * (f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}})\Big)\|_{L^p} \\ &\quad = \sigma_j \gamma_j^{-|\alpha|} \|D^\alpha \phi_{\gamma_j^{-1}} * (f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}})\|_{L^p} \\ &\quad \leq \sigma_j \gamma_j^{-|\alpha|} \|D^\alpha \phi_{\gamma_j^{-1}}\|_{L^1} \|f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}}\|_{L^p}, \end{aligned}$$

but, as $\|D^\alpha \phi_{\gamma_j^{-1}}\|_{L^1} = \gamma_j^{|\alpha|} \int |D^\alpha \phi(y)|\, dy$, we obtain, since σ is admissible,

$$\begin{aligned} &\sigma_j \gamma_j^{-|\alpha|} \|D^\alpha \Big(\phi_{\gamma_j^{-1}} * (f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}})\Big)\|_{L^p} \\ &\quad \leq C\sigma_j \|f * \phi_{\gamma_j^{-1}} - f\|_{L^p} + C\sigma_j \|f - f * \phi_{\gamma_{j-1}^{-1}}\|_{L^p} \\ &\quad \leq C\sigma_j \|f * \phi_{\gamma_j^{-1}} - f\|_{L^p} + C'\sigma_{j-1} \|f - f * \phi_{\gamma_{j-1}^{-1}}\|_{L^p}. \end{aligned}$$

The conclusion comes by applying the same reasoning to the other terms of (2.5). □

2.3 Generalized Besov spaces and weak derivatives

The spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$ can be characterized using weak derivatives and finite differences.

We will need the following condition for a function to belong to W^k_p.

Proposition 2.3.1. *Let $k \in \mathbb{N}$, $p,q \in [1,\infty]$, σ be an admissible sequence and $(\gamma_j)_j$ be a sequence of positive real numbers such that there exists $d_0 > 0$ satisfying*

$$d_0\gamma_j \leq \gamma_{j+1} \quad \forall j \in \mathbb{N}_0.$$

Let us also suppose that the series

$$\sum_{j\in\mathbb{N}_0} \gamma_j^l \sigma_j^{-1} \tag{2.6}$$

converges for all $0 \leq l \leq k$. If $f \in L^p$ is a function satisfying

$$(\sigma_j \sup_{|h|\leq\gamma_j^{-1}} \|\Delta_h^k f\|_{L^p})_j \in \ell^q,$$

then $f \in W_p^k$.

Proof. Let $\phi \in \mathcal{D}$ be a function as constructed in the proof of Proposition 2.2.1. Let us set

$$\psi_0 = \phi_{\gamma_0^{-1}}, \qquad \psi_j = \phi_{\gamma_j^{-1}} - \phi_{\gamma_{j-1}^{-1}} \quad \forall j \in \mathbb{N},$$

and finally define

$$f_j = f * \psi_j \quad \forall j \in \mathbb{N}_0.$$

It follows from Proposition 2.2.2 that for all $\alpha \in \mathbb{N}_0^d$ satisfying $|\alpha| \leq k$, there exists a constant $C_\alpha > 0$ such that for all $j \in \mathbb{N}_0$, we have

$$\|D^\alpha f_j\|_{L^p} \leq C_\alpha \gamma_j^{|\alpha|} \sigma_j^{-1}.$$

As a consequence, since (2.6) converges, the series $\sum_{j\in\mathbb{N}_0} D^\alpha f_j$ converges in L^p for all $|\alpha| \leq k$. Let us denote its limit by f_α and show that $f_\alpha = D^\alpha f$ (with the derivative taken in the weak sense). It is clear that $f_0 = f$ since, by Proposition 2.2.1,

$$\|\sum_{j=0}^{J} f_j - f\|_{L^p} = \|f * \phi_{\gamma_J^{-1}} - f\|_{L^p} \leq C\sigma_J^{-1} \to 0 \quad \text{as } J \to \infty.$$

Finally, for all $\varphi \in \mathcal{D}$ and $|\alpha| \leq k$, we have

$$\begin{aligned}
\int f(x) D^\alpha \varphi(x)\,dx &= \lim_{J\to\infty} \int \sum_{j=0}^{J} f_j(x) D^\alpha \varphi(x)\,dx \\
&= \lim_{J\to\infty} (-1)^{|\alpha|} \int \sum_{j=0}^{J} D^\alpha f_j(x) \varphi(x)\,dx \\
&= (-1)^{|\alpha|} \int f_\alpha(x) \varphi(x)\,dx,
\end{aligned}$$

which is sufficient to conclude. □

We can now give necessary and sufficient conditions for a function to belong to $B^{\sigma,\gamma}_{p,q}$.

Theorem 2.3.2. *Let $p,q \in [1,\infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. Let the numbers $k, n \in \mathbb{N}_0$ be such that*

$$k < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1} \leq \overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n.$$

If $f \in B^{\sigma,\gamma}_{p,q}$, then $f \in W^k_p$ and for all $|\alpha| \leq k$,

$$(\gamma_j^{-|\alpha|}\sigma_j \sup_{|h|\leq\gamma_j^{-1}} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p})_{j\in\mathbb{N}} \in \ell^q,$$

which means that $D^\alpha f \in B^{\gamma^{-|\alpha|}\sigma,\gamma}_{p,q}$.

Conversely, if $f \in W^k_p$ satisfies

$$(\gamma_j^{-|\alpha|}\sigma_j \sup_{|h|\leq\gamma_j^{-1}} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p})_j \in \ell^q \quad \forall |\alpha| = k,$$

then $f \in B^{\sigma,\gamma}_{p,q}$.

Proof. Assume first that $f \in B^{\sigma,\gamma}_{p,q}$; using (2.2) and the convergence of (2.6) for $0 \leq l \leq k$, which follows from the hypothesis on k and n, it is clear from Proposition 2.3.1 that we have $f \in W^k_p$. Keeping the same notations used in the proof of Proposition 2.3.1, let us fix $J \in \mathbb{N}$; for all $|h| \leq \gamma_J^{-1}$, we have

$$\begin{aligned}
&\|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p} \\
&\leq \sum_{j=0}^{J} C|h|^{n-|\alpha|} \sup_{|\beta|=n} \|D^\beta f_j\|_{L^p} + \sum_{j=J+1}^{\infty} C|h|^{k-|\alpha|} \sup_{|\beta|=k} \|D^\beta f_j\|_{L^p} \\
&\leq \sum_{j=0}^{J} C\gamma_J^{|\alpha|-n} \sup_{|\beta|=n} \|D^\beta f_j\|_{L^p} + \sum_{j=J+1}^{\infty} C\gamma_J^{|\alpha|-k} \sup_{|\beta|=k} \|D^\beta f_j\|_{L^p}.
\end{aligned}$$

For all $|\beta| = k$, we also have, as $k < \underline{s}(\sigma)\overline{s}(\gamma)^{-1}$,

$$\begin{aligned}
&\|(\sum_{j=J+1}^{\infty} \gamma_J^{-k}\sigma_J \|D^\beta f_j\|_{L^p})_J\|_{\ell^q} \\
&\quad \leq \|(\sum_{j=J+1}^{\infty} \overline{\gamma}_{j-J}^k \underline{\sigma}_{j-J}^{-1} \gamma_j^{-k}\sigma_j \|D^\beta f_j\|_{L^p})_J\|_{\ell^q} \\
&\quad \leq \sum_{j=1}^{\infty} \overline{\gamma}_j^k \underline{\sigma}_j^{-1} \|(\gamma_{j+J}^{-k}\sigma_{j+J}\|D^\beta f_{j+J}\|_{L^p})_J\|_{\ell^q} \\
&\quad \leq C\sum_{j=1}^{\infty} \overline{\gamma}_j^k \underline{\sigma}_j^{-1} < \infty.
\end{aligned}$$

Similarly, as $\overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n$, for all $|\beta| = n$, we get

$$\|(\sum_{j=0}^{J} \gamma_J^{-n} \sigma_J \|D^\beta f_j\|_{L^p})_J\|_{\ell^q} < \infty,$$

which allows us to conclude that

$$\|(\gamma_J^{-|\alpha|} \sigma_J \sup_{|h| \leq \gamma_J^{-1}} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p})_J\|_{\ell^q} < \infty.$$

For the converse, assume $f \in W_p^k$; the desired conclusion follows directly from (2.2) and the fact that for all $|h| \leq \gamma_j^{-1}$, we have, using classical inequalities (see [16] for example),

$$\begin{aligned}\|\Delta_h^n f\|_{L^p} &\leq C|h|^k \sup_{|\alpha|=k} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p} \\ &\leq C\gamma_j^{-k} \sup_{|\alpha|=k} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p}.\end{aligned}$$

□

As a corollary, we have the following alternative definition of $B_{p,q}^{\sigma,\gamma}$.

Corollary 2.3.3. *Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. Let the numbers $k, n \in \mathbb{N}_0$ be such that*

$$k < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1} \leq \overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n.$$

We have

$$B_{p,q}^{\sigma,\gamma} = \{f \in W_p^k : (\gamma_j^{-|\alpha|} \sigma_j \sup_{|h| \leq \gamma_j^{-1}} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p})_j \in \ell^q \quad \forall |\alpha| = k\}.$$

2.4 Generalized Besov spaces and polynomials

The following characterization is inspired by [78], where links between classical Besov spaces and related spaces are explored.

Theorem 2.4.1. *Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. Let the number $n \in \mathbb{N}$ be such that*

$$n < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1} \leq \overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n + 1;$$

the following assertions are equivalent:

1. *The function f belongs to $B_{p,q}^{\sigma,\gamma}$;*

2. *The function f belongs to W_p^n and, for all $h \in \mathbb{R}^d$ and almost every $x \in \mathbb{R}^d$, we have*

$$f(x+h) = \sum_{|\alpha|\leq n} D^\alpha f(x) \frac{h^\alpha}{|\alpha|!} + R_n(x,h)\frac{|h|^n}{n!},$$

where

$$(\sigma_j \gamma_j^{-n} \sup_{|h|\leq \gamma_j^{-1}} \|R_n(\cdot,h)\|_{L^p})_j \in \ell^q;$$

3. *If, given $j \in \mathbb{N}_0$, π_j is a net of $\mathbb{R}^d$ made of cubes of diagonal γ_j^{-1}, then for all $j \in \mathbb{N}_0$, there exists g_{π_j} such that*
   - *the trace of g_{π_j} in each cube of π_j is a polynomial of degree at most n,*
   - *one has $(\sigma_j \|f - g_{\pi_j}\|_{L^p})_j \in \ell^q$.*

Proof. Let us first show that assertion 1 implies assertion 2. We know from Corollary 2.3.3 that $f \in W_p^n$; using the Taylor expansion with weak derivatives, we get

$$f(x+h) = \sum_{|\alpha|\leq n-1} D^\alpha f(x)\frac{h^\alpha}{|\alpha|!} + \sum_{|\alpha|=n} h^\alpha \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} D^\alpha f(x+th)\,dt,$$

for all $h \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$. Of course, we have

$$\begin{aligned}
&\int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} D^\alpha f(x+th)\,dt \\
&\quad = \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} \Delta^1_{th} D^\alpha f(x)\,dt + \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} D^\alpha f(x)\,dt \\
&\quad = \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} \Delta^1_{th} D^\alpha f(x)\,dt + \frac{D^\alpha f(x)}{n!}.
\end{aligned}$$

Let us set

$$R_n(x,h) := \begin{cases} \dfrac{n!}{|h|^n} \displaystyle\sum_{|\alpha|=n} h^\alpha \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} \Delta^1_{th} D^\alpha f(x)\,dt & \text{if } h \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for all $h \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$,

$$f(x+h) = \sum_{|\alpha|\leq n} D^\alpha f(x)\frac{h^\alpha}{|\alpha|!} + R_n(x,h)\frac{|h|^n}{n!}.$$

Moreover, for any $h \neq 0$ such that $|h| \leq \gamma_j^{-1}$, Hölder's inequality allows us to write

$$|R_n(x,h)| \leq C \sum_{|\alpha|=n} \|\Delta^1_{\cdot h} D^\alpha f(x)\|_{L^p([0,1])}$$

and it follows that

$$\begin{aligned}
\|R_n(\cdot,h)\|_{L^p} &\le C\sum_{|\alpha|=n}\Big(\int_{\mathbb{R}^d}\int_0^1 |\Delta^1_{th}D^\alpha f(x)|^p\,dt\,dx\Big)^{1/p}\\
&= C\sum_{|\alpha|=n}\Big(\int_0^1\int_{\mathbb{R}^d} |\Delta^1_{th}D^\alpha f(x)|^p\,dx\,dt\Big)^{1/p}\\
&= C\sum_{|\alpha|=n}\Big(\int_0^1 \|\Delta^1_{th}D^\alpha f\|^p_{L^p}\,dt\Big)^{1/p}\\
&\le C\sum_{|\alpha|=n}\sup_{|h|\le\gamma_j^{-1}}\|\Delta^1_h D^\alpha f\|_{L^p}.
\end{aligned}$$

We can conclude this second point, using Corollary 2.3.3.

Now, let us show that assertion 2 implies 3. We fix $j\in\mathbb{N}_0$ and let $\pi_j=(A_k)_k$ be a net of $\mathbb{R}^d$ with cubes of diagonal γ_j^{-1}. Set for all $k\in\mathbb{N}_0$,

$$P_k(x)=\frac{1}{\mathcal{L}(A_k)}\int_{A_k}\sum_{|\alpha|\le n}D^\alpha f(y)\frac{(x-y)^\alpha}{|\alpha|!}\,dy.$$

Of course, P_k is a polynomial of degree less or equal than n. Let us then define

$$g_{\pi_j}:\mathbb{R}^d\to\mathbb{R}^d \quad x\mapsto P_k(x)\text{ if } x\in A_k \quad (k\in\mathbb{N}_0);$$

the trace of g_{π_j} in each cube of π_j is a polynomial of degree at most n and if $x\in A_k$, then $A_k\subset B(x,\gamma_j^{-1})$. Moreover, if q is the conjugate exponent of p, using Hölder's inequality, we get, for $x\in A_k$,

$$\begin{aligned}
|f(x)-g_{\pi_j}(x)| &\le \frac{1}{\mathcal{L}(A_k)}\int_{B(x,\gamma_j^{-1})}|f(x)-\sum_{|\alpha|\le n}D^\alpha f(y)\frac{(x-y)^\alpha}{|\alpha|!}|\,dy\\
&\le C\gamma_j^d\int_{B(0,\gamma_j^{-1})}|f(x)-\sum_{|\alpha|\le n}D^\alpha f(x-h)\frac{h^\alpha}{|\alpha|!}|\,dh\\
&\le C\gamma_j^d\gamma_j^{-d/q}\|f(x)-\sum_{|\alpha|\le n}D^\alpha f(x-\cdot)\frac{\cdot^\alpha}{|\alpha|!}\|_{L^p(B(0,\gamma_j^{-1}))}.
\end{aligned}$$

We thus can write

$$\begin{aligned}
&\|f - g_{\pi_j}\|_{L^p} \\
&\quad \leq C\gamma_j^d \gamma_j^{-d/q} \Big(\int_{B(0,\gamma_j^{-1})} \int_{\mathbb{R}^d} |f(x) - \sum_{|\alpha|\leq n} D^\alpha f(x-h)\frac{h^\alpha}{|\alpha|!}|^p \, dx\, dh\Big)^{1/p} \\
&\quad = C\gamma_j^d \gamma_j^{-d/q} \Big(\int_{B(0,\gamma_j^{-1})} \frac{|h|^{np}}{(n!)^p} \int_{\mathbb{R}^d} |R_n(x-h,h)|^p \, dx\, dh\Big)^{1/p} \\
&\quad \leq C\gamma_j^d \gamma_j^{-d(\frac{1}{p}+\frac{1}{q})} \gamma_j^{-n} \sup_{|h|\leq \gamma_j^{-1}} \|R_n(\cdot,h)\|_{L^p} \\
&\quad = C\gamma_j^{-n} \sup_{|h|\leq \gamma_j^{-1}} \|R_n(\cdot,h)\|_{L^p},
\end{aligned}$$

which procures the desired membership.

Finally, let us show that assertion 3 implies 1. As $\underline{\gamma}_1 > 1$, there exists $k_1 \in \mathbb{N}$ such that for any $x_0 \in \mathbb{R}^d$ and any $|h| \leq \gamma_{j+k_1}^{-1}$, we have $x_0 + kh \in B(x_0, \gamma_j^{-1}/3\sqrt{d})$ for all $k \in \{0,\ldots,n+1\}$. Let us fix $|h| \leq \gamma_{j+k_1}^{-1}$ and let $\pi_j = (A_k)_k$ be a net of $\mathbb{R}^d$ made of cubes of diagonal $\gamma_j^{-1}/3$ such that each vertex is the vertex of 2^d distinct cubes.

If $l \in \mathbb{N}$, then for all $x \in A_k$ and for all $l \in \{0,\ldots,n+1\}$, $x + lh \in C_k$, where C_k is the cube of diagonal γ_j^{-1} whose center coincides with the center of A_k. Let π_j' be a net of $\mathbb{R}^d$ defined in the same way as above but made of cubes of diagonal γ_j^{-1} that contain C_k and let P_k be the polynomial that is the trace of $g_{\pi_j'}$ on C_k. As the degree of P_k is at most n, we have

$$\begin{aligned}
\|\Delta_h^{n+1} f\|_{L^p}^p = \sum_{k\in\mathbb{N}_0} \|\Delta_h^{n+1} f\|_{L^p(A_k)}^p &= \sum_{k\in\mathbb{N}_0} \|\Delta_h^{n+1}(f - P_k)\|_{L^p(A_k)}^p \\
&= \sum_{k\in\mathbb{N}_0} \|\Delta_h^{n+1}(f - g_{\pi_j'})\|_{L^p(A_k)}^p,
\end{aligned}$$

with the usual modification if $p = \infty$. Let us remark that there exist $m := 3^d$ such nets with cubes of diagonal γ_j^{-1} whose centers are also center of some cube in π_j; let us denote by $\pi'_{j,1},\dots,\pi'_{j,m}$ those nets. We have

$$\begin{aligned}\|\Delta_h^{n+1} f\|_{L^p}^p &\leq \sum_{k\in\mathbb{N}_0}\sum_{l=1}^m \|\Delta_h^{n+1}(f-g_{\pi'_{j,l}})\|_{L^p(A_k)}^p\\ &= \sum_{l=1}^m \|\Delta_h^{n+1}(f-g_{\pi'_{j,l}})\|_{L^p}^p\\ &\leq C\sum_{l=1}^m \|(f-g_{\pi'_{j,l}})\|_{L^p}^p.\end{aligned}$$

Since we have

$$(\sigma_j\|(f-g_{\pi'_{j,l}})\|_{L^p})_j \in \ell^q,$$

for all $l \in \{1,\dots,m\}$ by hypothesis, we can write

$$(\sigma_j \sup_{|h|\leq\gamma_j^{-1}} \|\Delta_h^{n+1} f\|_{L^p})_{j\in\mathbb{N}} \in \ell^q,$$

as desired. □

2.5 Generalized Besov spaces and convolution

The spaces of generalized smoothness $B_{p,q}^{\sigma,\gamma}$ can be defined in terms of convolutions.

The characterization relies on the following condition for a function to belong to $B_{p,q}^{\sigma,\gamma}$.

Proposition 2.5.1. *Let $p,q \in [1,\infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and $\underline{s}(\boldsymbol{\sigma}) > 0$. If $f \in L^p$ is such that there exists $\phi \in \mathcal{D}$ for which*

$$(\sigma_j\|f * \phi_{\gamma_j^{-1}} - f\|_{L^p})_j \in \ell^q, \tag{2.7}$$

then $f \in B_{p,q}^{\sigma,\gamma}$.

Proof. Let $n \in \mathbb{N}$ be such that

$$0 < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1} \leq \overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n.$$

As done before in the proof of Proposition 2.3.1 and having in mind that $\underline{s}(\boldsymbol{\sigma}) > 0$, if (2.7) holds, we can build a sequence $(f_j)_j$ of infinitely differentiable functions belonging to L^p such that

$$f = \sum_{j\in\mathbb{N}_0} f_j$$

in L^p. It follows that, for any $J \in \mathbb{N}_0$ and any $|h| \leq \gamma_J^{-1}$, we have

$$\begin{aligned}\|\Delta_h^n f\|_{L^p} &\leq \sum_{j=0}^{J} \|\Delta_h^n f_j\|_{L^p} + \sum_{j=J+1}^{\infty} \|\Delta_h^n f_j\|_{L^p} \\ &\leq C \sum_{j=0}^{J} \gamma_J^{-n} \sup_{|\alpha|=n} \|D^\alpha f_j\|_{L^p} + C \sum_{j=J+1}^{\infty} \|f_j\|_{L^p}.\end{aligned}$$

Since, by Proposition 2.2.2, we also know that $(\sigma_j \gamma_j^{-|\alpha|} \|D^\alpha f_j\|_{L^p})_j \in \ell^q$ for all $|\alpha| \leq n$, we can proceed as in the proof of Theorem 2.3.2, using the fact that $\underline{s}(\boldsymbol{\sigma}) > 0$ and $\bar{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n$, to conclude that the sequence $(\sigma_j \sup_{|h| \leq \gamma_j^{-1}} \|\Delta_h^n f\|_{L^p})_j$ belongs to ℓ^q, and hence, by (2.2), $f \in B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}}$. □

From Propositions 2.2.1 and 2.5.1, we have the following corollary.

Corollary 2.5.2. *Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ and $\boldsymbol{\gamma} = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and $\underline{s}(\boldsymbol{\sigma}) > 0$; we have*

$$B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}} = \{f \in L^p : \exists\, \phi \in \mathcal{D} \text{ such that } (\sigma_j \|f * \phi_{\gamma_j^{-1}} - f\|_{L^p})_j \in \ell^q\}.$$

2.6 Generalized real interpolation methods

Let us first recall some notions of real interpolation, more details can be found in [7, 96, 121]. In the sequel, we will consider two normed vector spaces A_0 and A_1 which are continuously embedded in a Hausdorff topological vector space V. As a consequence, the spaces $A_0 \cap A_1$ and $A_0 + A_1$ are also normed vector spaces. The J-operator of interpolation is defined for $t > 0$ and $a \in A_0 \cap A_1$ by

$$J(t, a) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}.$$

If $0 < \alpha < 1$ and $q \in [1, \infty]$, we say that a belongs to the interpolation space $[A_0, A_1]_{\alpha,q,J}$ if there exists $(u_j)_{j\in\mathbb{Z}} \in A_0 \cap A_1$ such that $a = \sum_{j\in\mathbb{Z}} u_j$, with convergence in $A_0 + A_1$ and $(2^{-\alpha j} J(2^j, u_j))_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$.

On the other hand, the K-operator of interpolation is defined for $t > 0$ and $a \in A_0 + A_1$ by

$$K(t, a) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

Similarly, if $0 < \alpha < 1$ and $q \in [1, \infty]$, we say that a belongs to the interpolation space $[A_0, A_1]_{\alpha,q,K}$ if $a \in A_0 + A_1$ and $(2^{-\alpha j} K(2^j, a))_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$.

One can then show that, for all $0 < \alpha < 1$ and $q \in [1,\infty]$, these two spaces are identical, and the resulting

$$[A_0, A_1]_{\alpha,q} := [A_0, A_1]_{\alpha,q,J} = [A_0, A_1]_{\alpha,q,K}$$

"lies" in between $A_0 \cap A_1$ and $A_0 + A_1$.

The Besov space $B^s_{p,q}$ is an interpolation space between the Sobolev spaces H^t_p and H^u_p with $s = (1-\alpha)t + \alpha u$: we have

$$B^s_{p,q} = [H^t_p, H^u_p]_{\alpha,q} \tag{2.8}$$

(see e.g. [7, 32, 125, 121]). In an aim to obtain such a characterization in the context of generalized Besov spaces, we first have to define a generalized real interpolation method in which admissible sequences play a role. Let us first introduce this method before applying it to the generalized Besov spaces.

Definition 2.6.1. Let $\theta = (\theta_j)_{j\in\mathbb{Z}}$ and $\psi = (\psi_j)_{j\in\mathbb{Z}}$ be two sequences and let $q \geq 1$. We say that a belongs to the (J,q)-generalized interpolation space $[A_0, A_1]^{\theta,\psi}_{J,q}$ if there exists $(u_j)_{j\in\mathbb{Z}} \in A_0 \cap A_1$ such that $a = \sum_{j\in\mathbb{Z}} u_j$, with convergence in $A_0 + A_1$ and

$$\Big(\theta_j J(\psi_j, u_j)\Big)_j \in \ell^q(\mathbb{Z}).$$

Definition 2.6.2. Let $\theta = (\theta_j)_{j\in\mathbb{Z}}$ and $\psi = (\psi_j)_{j\in\mathbb{Z}}$ be two sequences and let $q \geq 1$. We say that a belongs to the (K,q)-generalized interpolation space $[A_0, A_1]^{\theta,\psi}_{K,q}$ if $a \in A_0 + A_1$ and

$$\Big(\theta_j K(\psi_j, a)\Big)_j \in \ell^q(\mathbb{Z}).$$

Remark 2.6.3. If one considers the admissible sequences $(\theta_j = 2^{-\alpha j})_{j\in\mathbb{Z}}$ and $(\psi_j = 2^j)_{j\in\mathbb{Z}}$, the two preceding definitions correspond to the classical interpolation spaces $[A_0, A_1]_{\alpha,J,q}$ and $[A_0, A_1]_{\alpha,K,q}$ respectively.

As for the usual case, such interpolation methods often coincide; this result is a generalization of Proposition 11 in [88].

Theorem 2.6.4. *Let $r, s \in \mathbb{R}$ and $\boldsymbol{\sigma}$, $\boldsymbol{\gamma}$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and*

$$r < \min\{\underline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1}, \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1}\} \leq \max\{\overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1}, \overline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1}\} < s. \tag{2.9}$$

We have

$$[A_0, A_1]^{\theta,\psi}_{J,q} = [A_0, A_1]^{\theta,\psi}_{K,q},$$

where sequences $\theta = (\theta_j)_{j\in\mathbb{Z}}$ and $\psi = (\psi_j)_{j\in\mathbb{Z}}$ are defined by

$$\theta_j := \begin{cases} \gamma_{-j}^{-r}\sigma_{-j} & \text{if } -j \in \mathbb{N}_0 \\ \gamma_j^r \sigma_j^{-1} & \text{if } j \in \mathbb{N} \end{cases}$$

and

$$\psi_j := \begin{cases} \gamma_{-j}^{-(s-r)} & \text{if } -j \in \mathbb{N}_0 \\ \gamma_j^{(s-r)} & \text{if } j \in \mathbb{N}\,. \end{cases}$$

Proof. Consider $f \in [A_0, A_1]_{J,q}^{\theta,\psi}$; we know that there exists $(f_l)_{l\in\mathbb{Z}} \in A_0 \cap A_1$ such that

$$f = \sum_{l\in\mathbb{Z}} f_l,$$

with convergence in $A_0 + A_1$ and

$$\|(\theta_l \max\{\|f_l\|_{A_0}, \psi_l \|f_l\|_{A_1}\})_l\|_{\ell^q(\mathbb{Z})} < \infty. \tag{2.10}$$

Set, for any $j \in \mathbb{Z}$,

$$b_j = \sum_{l=-\infty}^{j-1} f_l \qquad \text{and} \qquad c_j = \sum_{l=j}^{\infty} f_l.$$

Because of (2.9) and (2.10), we have $b_j \in A_0$, $c_j \in A_1$ and $f = b_j + c_j$. Let us prove that

$$\|\Big(\theta_j(\|b_j\|_{A_0} + \psi_j\|c_j\|_{A_1})\Big)_j\|_{\ell^q(\mathbb{Z})} < \infty.$$

We have

$$\begin{aligned} &\|\Big(\theta_j(\|b_j\|_{A_0} + \psi_j\|c_j\|_{A_1})\Big)_j\|_{\ell^q} \\ &\le \underbrace{\|(\theta_j \sum_{l=-\infty}^{j-1} \|f_l\|_{A_0})_j\|_{\ell^q}}_{(A)} + \underbrace{\|\Big(\theta_j(\sum_{l=j}^{\infty} \psi_j \|f_l\|_{A_1})\Big)_j\|_{\ell^q}}_{(B)}. \end{aligned}$$

Using triangle inequality, we obtain

$$\begin{aligned} (A) &\le \sum_{l=-\infty}^{0} \|(\theta_j \theta_{j+l-1}^{-1} \theta_{j+l-1} \|f_{l+j-1}\|_{A_0})_j\|_{\ell^q} \\ &= \sum_{l=-\infty}^{0} \Big(\sum_{j=-\infty}^{0} \Big((\frac{\gamma_{-j-l+1}}{\gamma_{-j}})^r (\frac{\sigma_{-j-l+1}}{\sigma_{-j}})^{-1} \theta_{j+l-1} \|f_{l+j-1}\|_{A_0} \Big)^q \\ &\quad + \sum_{j=1}^{1-l} (\gamma_j^r \sigma_j^{-1} \gamma_{-j-l+1}^r \sigma_{-j-l+1}^{-1} \theta_{j+l-1} \|f_{l+j-1}\|_{A_0})^q \\ &\quad + \sum_{j=2-l}^{\infty} \Big((\frac{\gamma_j}{\gamma_{j+l-1}})^r (\frac{\sigma_j}{\sigma_{j+l-1}})^{-1} \theta_{j+l-1} \|f_{l+j-1}\|_{A_0} \Big)^q \Big)^{\frac{1}{q}}, \end{aligned}$$

with the usual modification if $q=\infty$. If $r\geq 0$, there exists $\varepsilon>0$ such that $r\overline{s}(\gamma)-\underline{s}(\sigma)+(r+1)\varepsilon<0$ and (1.2) implies the existence of a constant $C>0$ such that

$$(\frac{\gamma_{-j-l+1}}{\gamma_{-j}})^r(\frac{\sigma_{-j-l+1}}{\sigma_{-j}})^{-1}\leq \overline{\gamma}_{1-l}^r\underline{\sigma}_{1-l}^{-1}\leq C2^{(1-l)(r\overline{s}(\gamma)-\underline{s}(\sigma)+(r+1)\varepsilon)}.$$

If $r<0$, we can choose $\varepsilon>0$ such that $r\underline{s}(\gamma)-\underline{s}(\sigma)+(1-r)\varepsilon<0$ and find $C>0$ such that

$$(\frac{\gamma_{-j-l+1}}{\gamma_{-j}})^r(\frac{\sigma_{-j-l+1}}{\sigma_{-j}})^{-1}\leq \underline{\gamma}_{1-l}^r\underline{\sigma}_{1-l}^{-1}\leq C2^{(1-l)(r\underline{s}(\gamma)-\underline{s}(\sigma)+(1-r)\varepsilon)}.$$

Adapting this reasoning for the other terms, we can claim that there exists $\alpha<0$ such that

$$(A)\leq C\sum_{l=-\infty}^{0}2^{\alpha(1-l)}\|(\theta_{j+l-1}\|f_{l+j-1}\|_{A_0})_j\|_{\ell^q}<\infty.$$

Similarly, there exists $\beta>0$ such that

$$(B)\leq C\sum_{l=-\infty}^{0}2^{\beta l}\|(\theta_{j-l}\psi_{j-l}\|f_{j-l}\|_{A_1})_j\|_{\ell^q}<\infty.$$

Reciprocally, let us consider $f\in[A_0,A_1]_{K,q}^{\theta,\psi}$; for any $j\in\mathbb{Z}$ there exists $b_j\in A_0$ and $c_j\in A_1$ such that $f=b_j+c_j$ and

$$\|\Big(\theta_j(\|b_j\|_{A_0}+\psi_j\|c_j\|_{A_1})\Big)_j\|_{\ell^q}<\infty. \tag{2.11}$$

Let us remark that, because of (2.9) and (2.11) , $b_0=\sum_{j=-\infty}^{-1}(b_{j+1}-b_j)$ with convergence in A_0 and $c_0=\sum_{j=0}^{\infty}(c_j-c_{j+1})$ with convergence in A_1. Now, let us set, for any $j\in\mathbb{Z}$,

$$f_j=b_{j+1}-b_j=c_j-c_{j+1}.$$

Clearly, $f_j\in A_0\cap A_1$ for any $j\in\mathbb{Z}$ and $f=b_0+c_0=\sum_{j\in\mathbb{Z}}f_j$, with convergence in A_0+A_1. Let us prove that

$$\|(\theta_j\max\{\|f_j\|_{A_0},\psi_j\|f_j\|_{A_1}\})_j\|_{\ell^q}<\infty.$$

We have, as σ and γ are admissible,

$$\begin{aligned}\|(\theta_j\max\{\|f_j\|_{A_0},\psi_j\|f_j\|_{A_1}\})_j\|_{\ell^q}&\leq\|(\theta_j(\|f_j\|_{A_0}+\psi_j\|f_j\|_{A_1})_j\|_{l^q}\\&=\|(\theta_j(\|b_{j+1}-b_j\|_{A_0}+\psi_j\|c_j-c_{j+1}\|_{A_1})_j\|_{\ell^q}\\&\leq C\|\Big(\theta_j(\|b_j\|_{A_0}+\psi_j\|c_j\|_{A_1})\Big)_j\|_{\ell^q}\\&<\infty,\end{aligned}$$

which allows to conclude. □

Definition 2.6.5. Given two admissible sequences σ and γ with $\underline{\gamma}_1>1$, let θ and ψ be the sequences defined as in Theorem 2.6.4 for some r,s as in (2.9); we define the space $[A_0,A_1]_q^{\sigma,\gamma}$ as follows:

$$[A_0,A_1]_q^{\sigma,\gamma}:=[A_0,A_1]_{J,q}^{\theta,\psi}=[A_0,A_1]_{K,q}^{\theta,\psi}.$$

2.7 Generalized interpolation of Sobolev spaces

Let us show that the generalized Besov spaces $B^{\sigma,\gamma}_{p,q}$ can be defined from the usual Sobolev spaces W^s_p or H^s_p as generalized interpolation spaces, as it is the case with the usual Besov spaces $B^{\mathbf{s}}_{p,q}$ and the classical real interpolation theory. Let us recall that u_s is the tempered distribution defined by

$$\mathcal{F}u_s = (1+|\cdot|^2)^{s/2}$$

and that $\|\cdot * u_s\|_p$ is a norm on H^s_p.

We need some auxiliary results. Roughly speaking, we aim at showing that there exists a constant $C>0$ (depending on s) for which, for any j,

$$C^{-1}\|\Delta^{\gamma,J}_j f\|_{H^s_p} \leq \gamma^s_j \|\Delta^{\gamma,J}_j f\|_{L^p} \leq C\|f\|_{H^s_p}.$$

Lemma 2.7.1. *Let γ be an admissible sequence such that $\underline{\gamma}_1 > 1$; given $s\in\mathbb{R}$ and $N\in\mathbb{N}_0$, there exists a constant $C_{s,N}>0$ such that for all $j\in\mathbb{N}_0$,*

$$|\Delta^{\gamma,J}_j u_s| \leq C_{s,N}\gamma_j^{d+s}(1+\gamma_j|\cdot|)^{-N}.$$

Proof. Let us fix $j \geq Jk_0$, the proof being similar for $0\leq j\leq Jk_0-1$. As $\Delta^{\gamma,J}_j u_s = \mathcal{F}^{-1}(\varphi^{\gamma,J}_j \mathcal{F}u_s)$, we get that

$$\begin{aligned}(2\pi)^d|\Delta^{\gamma,J}_j u_s| &\leq \int_{\mathbb{R}^d} |\rho(\gamma_j^{-1}|\xi|) - \rho(\gamma^{-1}_{j-Jk_0}|\xi|)|(1+|\xi|^2)^{s/2}\,d\xi \\ &= \gamma_j^d \int_{\mathbb{R}^d} |\rho(|y|) - \rho(\gamma^{-1}_{j-Jk_0}\gamma_j|y|)|(1+\gamma_j^2|y|^2)^{s/2}\,dy.\end{aligned}$$

Since the support of $\rho(|\cdot|)-\rho(\gamma^{-1}_{j-Jk_0}\gamma_j|\cdot|)$ is included in $\overline{\Omega}$ defined by

$$\Omega = B(0,2)\setminus B(0,\underline{\gamma}^{-1}_{Jk_0}),$$

we have

$$(2\pi)^d|\Delta^{\gamma,J}_j u_s| \leq \gamma_j^{d+s}\int_\Omega (|\rho(|y|)| + |\rho(\underline{\gamma}_{Jk_0}|y|)|)(\frac{1}{\gamma_j^2}+|y|^2)^{s/2}\,dy$$

and, as $0<1/\gamma_j^2 \leq 1/\gamma_0^2$, this implies the existence of a constant $C_{s,0}>0$ such that

$$|\Delta^{\gamma,J}_j u_s(x)| \leq C_{s,0}\gamma_j^{d+s}.$$

Now, if $\alpha\in\mathbb{N}^d_0$ is a multi-index such that $|\alpha|\geq 1$, then

$$\begin{aligned}(2\pi)^d|x^\alpha\Delta^{\gamma,J}_j u_s(x)| &\leq (2\pi)^d\sqrt{d}\max_{1\leq k\leq d}|x_k^{|\alpha|}\Delta^{\gamma,J}_j u_s(x)| \\ &\leq \sqrt{d}\max_{1\leq k\leq d}\int_{\mathbb{R}^d}|D_k^{|\alpha|}(\varphi^{\gamma,J}_j(\xi)(1+|\xi|^2)^{s/2})|\,d\xi\end{aligned}$$

and similarly we get

$$|x^\alpha \Delta_j^{\gamma,J} u_s(x)| \leq C_{s,|\alpha|} \gamma_j^{d+s-|\alpha|},$$

for such an α, which is sufficient to conclude. $\square$

Remark 2.7.2. Using the same proof as in Lemma 2.7.1, one can obtain the following result: for all $s \in \mathbb{R}$ and $N \in \mathbb{N}_0$ there exists a constant $\widetilde{C_{s,N}} > 0$ such that for all $j \in \mathbb{N}_0$,

$$|\widetilde{\Delta_j^{\gamma,J} u_s}| \leq \widetilde{C_{s,N}} \gamma_j^{d+s} (1 + \gamma_j |\cdot|)^{-N}.$$

Proposition 2.7.3. *Let $s \in \mathbb{R}$ and $p \in [1,\infty]$; if $f \in H_p^s$ then there exists a constant $C_s > 0$ such that*

$$\|\Delta_j^{\gamma,J} f\|_{L^p} \leq C_s \gamma_j^{-s} \|f\|_{H_p^s} \quad (j \in \mathbb{N}_0),$$

where the notations defined by (2.1) have been used.

Proof. As $f = u_{-s} * u_s * f$, we get

$$\Delta_j^{\gamma,J} f = (\Delta_j^{\gamma,J} u_{-s}) * (u_s * f).$$

It follows from the Young's inequality and Lemma 2.7.1 that

$$\begin{aligned} \|\Delta_j^{\gamma,J} f\|_{L^p} &\leq \|\Delta_j^{\gamma,J} u_{-s}\|_1 \|u_s * f\|_{L^p} \\ &\leq C_s \gamma_j^{-s} \|f\|_{H_p^s}, \end{aligned}$$

for some constant C_s, which is the desired result. $\square$

Proposition 2.7.4. *Let $s \in \mathbb{R}$, $p \in [1,\infty]$ and $f \in \mathcal{S}'$. If, using the notations defined by (2.1), $\Delta_j^{\gamma,J} f \in L^p$ (for some $j \in \mathbb{N}_0$), then there exists a constant $C_s > 0$ such that*

$$\|\Delta_j^{\gamma,J} f\|_{H_p^s} \leq C_s \gamma_j^s \|\Delta_j^{\gamma,J} f\|_{L^p}.$$

Proof. From $\widetilde{\Delta_j^{\gamma,J}} \Delta_j^{\gamma,J} f = c_\varphi \Delta_j^{\gamma,J} f$, we get

$$\begin{aligned} \mathcal{F}^{-1}(\mathcal{F} u_s \mathcal{F} \Delta_j^{\gamma,J} f) &= \frac{1}{c_\varphi} \mathcal{F}^{-1}\Big(\mathcal{F} u_s \mathcal{F}(\widetilde{\Delta_j^{\gamma,J}} \Delta_j^{\gamma,J} f)\Big) \\ &= \frac{1}{c_\varphi} \mathcal{F}^{-1}(\widetilde{\varphi_j^{\gamma,J}} \mathcal{F} u_s \mathcal{F} \Delta_j^{\gamma,J} f) \\ &= \frac{1}{c_\varphi} \widetilde{\Delta_j^{\gamma,J} u_s} * \Delta_j^{\gamma,J} f. \end{aligned}$$

It follows here again from the Young's inequality and Remark 2.7.2 that

$$\begin{aligned} \|\Delta_j^{\gamma,J} f\|_{H_p^s} &\leq \frac{1}{c_\varphi} \|\widetilde{\Delta_j^{\gamma,J} u_s}\|_1 \|\Delta_j^{\gamma,J} f\|_{L^p} \\ &\leq C_s \gamma_j^s \|\Delta_j^{\gamma,J} f\|_{L^p}, \end{aligned}$$

for a constant $C_s > 0$. $\square$

We are now able to define the generalized Besov spaces $B^{\sigma,\gamma}_{p,q}$ from the Sobolev spaces using generalized interpolation.

Theorem 2.7.5. *Let $p,q \in [1,\infty]$, $r,s \in \mathbb{R}$, and $\boldsymbol{\sigma}$, $\boldsymbol{\gamma}$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and*

$$r < \min\{\underline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1}, \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1}\} \leq \max\{\overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1}, \overline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1}\} < s; \tag{2.12}$$

we have

$$B^{\sigma,\gamma}_{p,q} = [H^r_p, H^s_p]^{\sigma,\gamma}_q.$$

Proof. Let θ and ψ be the sequences defined as in Theorem 2.6.4.

Let us first suppose that $f \in B^{\sigma,\gamma}_{p,q}$ and set

$$u_j = \begin{cases} \Delta^{\gamma,J}_{-j} f & \text{if } -j \in \mathbb{N}_0 \\ 0 & \text{if } j \in \mathbb{N}. \end{cases}$$

From Proposition 2.7.4, for any $t \in \{r,s\}$ and $j \in -\mathbb{N}_0$, there exists a constant C_t such that

$$\|u_j\|_{H^t_p} \leq C_t \gamma^t_{-j} \|\Delta^{\gamma,J}_{-j} f\|_{L^p},$$

which implies $u_j \in H^s_p$. Now, since $(\sigma_k \|\Delta^{\gamma,J}_k f\|_{L^p})_{k\in\mathbb{N}_0}$ belongs to ℓ^q and (2.12) holds, we have $f = \sum_{j\in\mathbb{Z}} u_j$, with convergence in H^r_p. Moreover, for all j belonging to $-\mathbb{N}_0$, we get

$$\theta_j \|u_j\|_{H^r_p} \leq C_r \sigma_{-j} \|\Delta^{\gamma,J}_{-j} f\|_{L^p} \qquad \text{and} \qquad \theta_j \psi_j \|u_j\|_{H^s_p} \leq C_s \sigma_{-j} \|\Delta^{\gamma,J}_{-j} f\|_{L^p}.$$

From this, we can conclude that $\left(\theta_j J(\psi_j, u_j)\right)_{j\in\mathbb{Z}}$ belongs to $\ell^q(\mathbb{Z})$ and thus $f \in [H^r_p, H^s_p]^{\theta,\psi}_{J,q}$.

Let us now consider $f \in [H^r_p, H^s_p]^{\theta,\psi}_{J,q}$; there exists $(f_l)_{l\in\mathbb{Z}} \in H^s_p$ such that $f = \sum_{l\in\mathbb{Z}} f_l$ in H^r_p and $\left(\theta_l J(\psi_l, f_l)\right)_{l\in\mathbb{Z}}$ belongs to $\ell^q(\mathbb{Z})$. Now, for all $j \in \mathbb{N}_0$, Proposition 2.7.3 allows us to write

$$\begin{aligned} \|\Delta^{\gamma,J}_j f\|_{L^p} &\leq \sum_{l\in\mathbb{Z}} \|\Delta^{\gamma,J}_j f_l\|_{L^p} \\ &\leq C_r \sum_{l=-\infty}^{-j-1} \gamma_j^{-r} \|f_l\|_{H^r_p} + C_s \sum_{l=-j}^{0} \gamma_j^{-s} \|f_l\|_{H^s_p} + C_s \sum_{l=1}^{\infty} \gamma_j^{-s} \|f_l\|_{H^s_p}. \end{aligned}$$

It follows that

$$\begin{aligned}\|(\sigma_j\|\Delta_j^{\gamma,J} f\|_{L^p})_j\|_{\ell^q} &\leq C\|(\sum_{l=-\infty}^{-j-1}\sigma_j\gamma_j^{-r}\gamma_{-l}^r\sigma_{-l}^{-1}\theta_l\|f_l\|_{H_p^r})_j\|_{\ell^q}\\ &+ C\|(\sum_{l=-j}^{0}\sigma_j\gamma_j^{-s}\gamma_{-l}^s\sigma_{-l}^{-1}\theta_l\psi_l\|f_l\|_{H_p^s})_j\|_{\ell^q}\\ &+ C\|(\sum_{l=1}^{\infty}\sigma_j\gamma_j^{-s}\gamma_l^{-s}\sigma_l\theta_l\psi_l\|f_l\|_{H_p^s})_j\|_{\ell^q}.\end{aligned}$$

If $r \geq 0$, there exists $\varepsilon > 0$ such that $\alpha = r\overline{s}(\gamma) - \underline{s}(\sigma) + (r+1)\varepsilon < 0$ and (1.2) implies the existence of a constant $C > 0$ such that

$$(\frac{\gamma_{-l}}{\gamma_j})^r\frac{\sigma_j}{\sigma_{-l}} \leq \overline{\gamma}^r_{-l-j}\underline{\sigma}^{-1}_{-l-j} \leq C2^{(-l-j)\alpha}.$$

Using the triangle inequality, we get

$$\begin{aligned}\|(\sum_{l=-\infty}^{-j-1}\sigma_j\gamma_j^{-r}\gamma_{-l}^r\sigma_{-l}^{-1}\theta_l\|f_l\|_{H_p^r})_j\|_{\ell^q} &\leq C\sum_{l=-\infty}^{-1}2^{-\alpha l}\|(\theta_{l-j}\|f_{l-j}\|_{H_p^r})_j\|_{\ell^q}\\ &< \infty.\end{aligned}$$

If $r < 0$, we can choose $\varepsilon > 0$ such that $\beta = r\underline{s}(\gamma) - \underline{s}(\sigma) + (1-r)\varepsilon < 0$ and find $C > 0$ such that

$$(\frac{\gamma_{-l}}{\gamma_j})^r\frac{\sigma_j}{\sigma_{-l}} \leq \underline{\gamma}^r_{-l-j}\underline{\sigma}^{-1}_{-l-j} \leq C2^{(-l-j)\beta}.$$

Again, we have

$$\|(\sum_{l=-\infty}^{-j-1}\sigma_j\gamma_j^{-r}\gamma_{-l}^r\sigma_{-l}^{-1}\theta_l\|f_l\|_{H_p^r})_j\|_{\ell^q} < \infty.$$

The same reasoning can be applied to the other terms in order to obtain

$$\|(\sigma_j\|\Delta_j^{\gamma,J} f\|_{L^p})_j\|_{\ell^q} < \infty,$$

which means that f belongs to $B^{\sigma,\gamma}_{p,q}$. □

If the admissible sequence $\boldsymbol{\gamma}$ is the usual sequence $(2^j)_j$, (2.12) can be written in a simpler way, which is given by Corollary 2.7.6.

Corollary 2.7.6. *Let $p,q \in [1,\infty]$, $r,s \in \mathbb{R}$ and $\boldsymbol{\sigma}$ be an admissible sequence such that*

$$r < \underline{s}(\boldsymbol{\sigma}) \leq \overline{s}(\boldsymbol{\sigma}) < s; \tag{2.13}$$

we have

$$B^{\sigma}_{p,q} = [H^r_p, H^s_p]^{\sigma,\mathbf{1}}_q.$$

The classical Besov spaces can be defined by interpolating the Sobolev spaces W_p^s even when $p = 1$ or $p = \infty$. Let us show that it is also the case in the generalized version.

Theorem 2.7.7. *Let $p, q \in [1, \infty]$, $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. If $k, n \in \mathbb{N}_0$ are two numbers such that*

$$k < \underline{s}(\boldsymbol{\sigma})\overline{s}(\boldsymbol{\gamma})^{-1} \leq \overline{s}(\boldsymbol{\sigma})\underline{s}(\boldsymbol{\gamma})^{-1} < n,$$

we have

$$B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}} = [W_p^k, W_p^n]_q^{\boldsymbol{\sigma},\boldsymbol{\gamma}}.$$

Proof. Let us first suppose that $f \in B_{p,q}^{\boldsymbol{\sigma},\boldsymbol{\gamma}}$; again, as in the proof of Proposition 2.3.1, there exists a sequence $(f_j)_j$ of infinitely differentiable functions belonging to L^p such that

$$D^\alpha f = \sum_{j \in \mathbb{N}_0} D^\alpha f_j$$

in L^p for all $|\alpha| \leq k$. Moreover, we have $(\sigma_j \gamma_j^{-|\alpha|} \|D^\alpha f_j\|_{L^p})_j \in \ell^q$ for all $|\alpha| \leq n$. Let us then define the sequence

$$u_j := \begin{cases} f_{-j} & \text{if } -j \in \mathbb{N}_0 \\ 0 & \text{if } j \in \mathbb{N}. \end{cases}$$

We can write $f = \sum_{j \in \mathbb{Z}} u_j$ (with convergence in W_p^k); moreover, for all $j \in -\mathbb{N}_0$, we have

$$\begin{aligned} \theta_j \|u_j\|_{W_p^k} &= \sum_{|\alpha| \leq k} \gamma_{-j}^{-k} \sigma_{-j} \|D^\alpha f_{-j}\|_{L^p} \\ &\leq C \sum_{|\alpha| \leq k} \gamma_{-j}^{-|\alpha|} \sigma_{-j} \|D^\alpha f_{-j}\|_{L^p} \end{aligned}$$

and

$$\begin{aligned} \theta_j \psi_j \|u_j\|_{W_p^n} &= \sum_{|\alpha| \leq n} \gamma_{-j}^{-n} \sigma_{-j} \|D^\alpha f_{-j}\|_{L^p} \\ &\leq C \sum_{|\alpha| \leq n} \gamma_{-j}^{-|\alpha|} \sigma_{-j} \|D^\alpha f_{-j}\|_{L^p}, \end{aligned}$$

which implies $(\theta_j J(u_j, \psi_j))_j \in \ell^q$ and thus $f \in [W_p^k, W_p^n]_q^{\boldsymbol{\sigma},\boldsymbol{\gamma}}$.

Now, let $f \in [W_p^k, W_p^n]_q^{\boldsymbol{\sigma},\boldsymbol{\gamma}}$; there exists a sequence of functions $(u_l)_{l \in \mathbb{Z}}$ in W_p^n such that $f = \sum_{l \in \mathbb{Z}} u_j$ in W_p^k and $(\theta_l J(u_j, \psi_l))_l \in \ell^q$. It follows that $D^\alpha f = \sum_{l \in \mathbb{Z}} D^\alpha u_l$ in L^p for all $|\alpha| \leq k$. Let us fix $|h| \leq \gamma_j^{-1}$ and $|\alpha| = k$; we have

$$\begin{aligned}
\|\Delta_h^{n-k} D^\alpha f\|_{L^p} &\leq \sum_{l\in\mathbb{Z}} \|\Delta_h^{n-k} D^\alpha u_l\|_{L^p} \\
&\leq C \sum_{l=-\infty}^{-j-1} \|D^\alpha u_l\|_{L^p} + C \sum_{l=-j}^{\infty} \gamma_j^{k-n} \sup_{|\beta|=n} \|D^\beta u_l\|_{L^p} \\
&\leq C \sum_{l=-\infty}^{-j-1} \|u_l\|_{W_p^k} + C \sum_{l=-j}^{\infty} \gamma_j^{k-n} \|u_l\|_{W_p^n}.
\end{aligned}$$

It follows, using the same arguments as before, that

$$(\gamma_j^{-k} \sigma_j \sup_{|h|\leq\gamma_j^{-1}} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p})_j \in \ell^q \quad \forall |\alpha| = k,$$

which implies $f \in B_{p,q}^{\sigma,\gamma}$, by Corollary 2.3.3. □

3 To a general framework for the WLM

The Hölderian regularity can be seen as a notion that fills gaps between being "n times continuously differentiable" and "$n+1$ times continuously differentiable". More precisely, a function f from $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to the space $T^p_u(x_0)$ (with $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and $u > 0$) if there exist a polynomial P_{x_0} of degree strictly less than u and a positive constant C such that

$$r^{-u}\|f - P_{x_0}\|_{L^p(B(x_0,r))} \leq C, \tag{3.1}$$

for $r > 0$, (see [26]); $T^\infty_u(x_0)$ is called a Hölder space (and is usually denoted by $\Lambda^u(x_0)$ [86]). These spaces are embedded and the Hölder exponent of f at x_0 is defined as

$$h_\infty(x_0) := \sup\{u > 0 : f \in T^\infty_u(x_0)\}. \tag{3.2}$$

The discrete wavelet transform provides a useful tool for studying Hölder spaces, since the condition on f at x_0 can be transposed to a condition on some wavelet coefficients near x_0 (for more details, see [70, 77] for example), the so-called wavelet leaders (see Definition 3.3.1 with $p = \infty$). Indeed, if a function belongs to a space $T^\infty_u(x_0)$, the wavelet leaders of x_0 satisfy an inequality somehow similar to (3.1). Conversely, if this condition on the wavelet leaders is met, the corresponding function belongs to a space close to $T^\infty_u(x_0)$. More precisely, in this case one has

$$\theta_u^{-1}(r)\|f - P_{x_0}\|_{L^\infty(B(x_0,r))} \leq C, \tag{3.3}$$

with $\theta_u(r) = r^u|\ln(r)|$. In other words, f belongs to $T^\infty_u(x_0)$ up to a logarithmic correction. If such results hold, we will say that we have a quasi-characterization of the space ($T^\infty_u(x_0)$ in this case). Such a quasi-characterization provides an exact characterization of the Hölderian regularity, i.e. of the Hölder exponent $h_\infty(x_0)$.

This notion of regularity can be generalized in several ways. First, in the same spirit as what has been done for Besov spaces in the previous chapter, one can replace the expression r^{-u} appearing in (3.1) with a general function $\theta_u(r)$ satisfying some requirements, as in inequality (3.3). By doing so, one defines spaces that are able to make subtle distinctions between functions associated to the same Hölder exponent, giving tools for detecting the presence of a Brownian motion in the signal. Such spaces have been studied in [90], where a quasi-characterization is obtained. Another idea consists in replacing the Hölder space appearing in (3.2) with a general T^p_u space, in order to study non-locally bounded functions (see [73] for such an application). This approach has been undertaken in [71], where generalized wavelet leaders, called p-leaders, are introduced. However, this definition is not a direct generalization of the usual leaders and fails to quasi-characterize the $T_u^p(x_0)$ spaces, although they still can be used to study the corresponding generalized Hölder exponent.

The first part of this chapter consists in combining these two points of view, by considering the spaces of functions satisfying the condition

$$\theta_u^{-1}(r)\|f - P_{x_0}\|_{L^p(B(x_0,r))} \leq C. \tag{3.4}$$

Indeed, we consider an even larger class of spaces called here spaces of generalized pointwise smoothness (see Definition 3.1.1). Their functional properties, up to slightly different definitions (see Remark 5.1.3), will be studied in Chapter 5 while links with partial differential equations will be explored in Chapter 6. They correspond in some way to a pointwise version of the generalized Besov spaces introduced in the preceding chapter. We obtain a quasi-characterization of such spaces by introducing a variant definition of the p-leaders that naturally extends the classical case where $p = \infty$.

The second part of this chapter aims at providing a multifractal formalism suited for the spaces introduced here. A multifractal formalism is an empirical method that allows to estimate the quantity

$$\dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_p(x_0) = h\},$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension, see Section 1.5 for more details, and $h_p(x_0)$ is the generalized Hölder exponent obtained by replacing $T_u^\infty(x_0)$ with $T_u^p(x_0)$ in (3.2). Usually, one requires such a method to be valid for a large class of functions. Such a multifractal formalism was first presented in [109] in the context of the analysis of fully developed turbulence velocity data and it can be shown that, from a prevalence point of view (see Section 1.6), almost every function belonging to a given Besov space satisfies this formalism. We aim at providing here a multifractal formalism for the exponents defined from the $T_{p,q}^\sigma$ spaces (see (3.16)), thus generalizing the wavelet leaders method [70, 75]. We show here that, from the prevalence point of view, almost every function belonging to a space of generalized smoothness satisfies a multifractal formalism derived from the formalism relying on the p-wavelet leaders. By doing so, we show that the generalized Besov spaces provide a natural framework for supporting this theory, reinforcing the idea that the spaces of generalized smoothness are a natural pointwise version of these spaces. To achieve this goal, we will mainly use the wavelet representation of generalized Besov spaces (see [2]): if σ is an admissible sequence and $p, q \in [1, \infty]$, a tempered distribution f belongs to $B_{p,q}^\sigma$ if and on only if the sequence $(C_k)_k$ defined by (1.11) belongs to ℓ^q and if

$$(\sigma_j 2^{-j\frac{d}{p}}\|(c_\lambda)_{\lambda \in \Lambda_j}\|_{\ell^p})_j \in \ell^q. \tag{3.5}$$

This chapter can be seen as a generalization of the ideas and techniques employed in [70, 75, 50, 90]. Results obtained here have been submitted for publication in [98].

Contents

3.1 Generalized spaces of pointwise smoothness

Definition 3.1.1. Let $p,q \in [1,\infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -\frac{d}{p}$, $f \in L^p_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}^d$; f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$ whenever[1]

$$(\sigma_j 2^{jd/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))})_j \in \ell^q,$$

where, given $r>0$, if $\overline{s}(\boldsymbol{\sigma})>0$, we recall that

$$B_h(x_0,r) = \{x : [x, x+(\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor + 1)h] \subset B(x_0,r)\},$$

and $B_h(x_0,r) = B(x_0,r)$ otherwise.

It is easy to check that $T^{\boldsymbol{\sigma}}_{\infty,\infty}(x_0)$ is the generalized Hölder space $\Lambda^{\boldsymbol{\sigma}}(x_0)$ introduced in [90]. These spaces can also be seen as a generalization of the spaces $T^p_u(x_0)$ introduced by Calderón and Zygmund in [26]. This aspect will be studied in details in Chapters 5 and 6. Let us also mention that we can equip $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$ with the natural norm

$$\|\cdot\|_{T^{\boldsymbol{\sigma}}_{p,q}(x_0)} : f \mapsto \|f\|_{L^p(B(x_0,1))} + \|(\sigma_j 2^{jd/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))})_j\|_{\ell^q}$$

and, from the completeness of L^p spaces, $(T^{\boldsymbol{\sigma}}_{p,q}(x_0), \|\cdot\|_{T^{\boldsymbol{\sigma}}_{p,q}(x_0)})$ is a Banach space. Other functional properties of the pointwise spaces of generalized smoothness will be explored in Chapter 5.

A comparison between Definitions 3.1.1 and (2.2), taking into account Remark 2.1.4 allows to declare the spaces $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$ as the pointwise Besov spaces, as the L^p norm is now taken on some balls around the point x_0 instead of the whole space $\mathbb{R}^d$. Only

[1] We recall that, if $n \leq 0$, $\Delta_h^n f = f$.

the factor $2^{jd/p}$ differs, it corresponds to the inverse of the L^p norm of the characteristic function of $B(x_0, 2^{-j})$. It is introduced in order that the measure of the ball does not interfere in the regularity measurement.

We refer to Section 4.3 for examples of functions belonging to $T^{\sigma}_{p,q}(x_0)$ spaces, once all the necessary tools to discuss them properly will be available.

Let us give an alternative definition of $T^{\sigma}_{p,q}(x_0)$.

Proposition 3.1.2. *Let $p,q \in [1,\infty]$, $f \in L^p_{loc}$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that $\overline{s}(\boldsymbol{\sigma}) > 0$. We have $f \in T^{\sigma}_{p,q}(x_0)$ if and only if there exists a sequence of polynomials $(P_{j,x_0})_j$ of degree less than or equal to $\lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor$ such that*

$$(\sigma_j 2^{jd/p} \|f - P_{j,x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^q. \tag{3.6}$$

Proof. The necessity of the condition being a consequence of the Whitney's theorem, let us check the sufficiency. Let $j \in \mathbb{N}$; for any polynomial P of degree less than or equal to $n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor$, we have, given $x, h \in \mathbb{R}^d$,

$$|\Delta_h^{n+1} f(x)| \leq |\Delta_h^{n+1}\big(f(x) - P(x)\big)| \leq C_n \sum_{k=0}^{n+1} |f(x+kh) - P(x+kh)|,$$

for a constant C_n. Therefore, for $|h| \leq 2^{-j}$ and $x \in B_h(x_0, 2^{-j})$, we get

$$\|\Delta_h^{n+1} f\|_{L^p(B_h(x_0,2^{-j}))} \leq C_n(n+2)\|f - P\|_{L^p(B(x_0,2^{-j}))},$$

hence the conclusion. □

3.2 Independence of the polynomial from the scale

Under some additional assumptions on the admissible sequence $\boldsymbol{\sigma}$, the sequence of polynomials $(P_{j,x_0})_j$ appearing in inequality (3.6) can be replaced by a unique polynomial P_{x_0} independent from the scale j: $P_{x_0} = P_{j,x_0}$.

We first need some preliminary results. Let us first state a somehow standard result about inequalities on polynomials; we sketch a proof for the sake of completeness.

Lemma 3.2.1. *Given $x_0 \in \mathbb{R}^d$, a radius $r > 0$, $p \in (0,\infty]$ and a maximum degree n, there exist two constants $C, C' > 0$ only depending on p, d and n such that, for any polynomial P of degree less than or equal to n,*

$$\|D^\alpha P\|_{L^p(B(x_0,r))} \leq C r^{-|\alpha|} \|P\|_{L^p(B(x_0,r))},$$

for any multi-index α and

$$\sup_{x \in B(x_0,r)} |P(x)| \leq C' r^{d/p} \|P\|_{L^p(B(x_0,r))}.$$

Proof. For the first inequality, let us recall that the Markov inequality (see e.g. [38]) affirms that, given a convex bounded set E of $\mathbb{R}^d$, there exists a constant $C_{E,p} > 0$ such that for any $n \in \mathbb{N}$ and $k \in \{1,\dots,d\}$, we have

$$\|D_k P\|_{L^p(E)} \leq C_{E,p}(n+1)^2 \|P\|_{L^p(E)},$$

for any polynomial P of degree less than or equal to n. As a consequence, given $r > 0$, there exists a constant $C > 0$ depending on n and p such that, for any multi-index α, we have

$$\|D^\alpha P\|_{L^p(B(x_0,r))} \leq C r^{-|\alpha|} \|P\|_{L^p(B(x_0,r))}.$$

That being done, using Sobolev's inequality, we can now write

$$\sup_{x \in B(x_0,r)} |P(x)| \leq C' r^{d/p} \|P\|_{L^p(B(x_0,r))},$$

for a constant $C' > 0$ which only depends on n and p. □

The main theorem of this section relies on the following lemma.

Lemma 3.2.2. *Let $p,q \in [1,\infty]$, $f \in L^p_{loc}$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that $0 \leq n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor < \underline{s}(\boldsymbol{\sigma})$. If f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$, the sequence of polynomials $(P_{j,x_0})_j$ satisfying (3.6) is such that, given a multi-index α for which $|\alpha| \leq n$, there exists a sequence $\boldsymbol{\xi} \in \ell^q$ satisfying*

$$2^{-|\alpha| j} \sigma_j |D^\alpha (P_{j,x_0} - P_{k,x_0})(x_0)| \leq \xi_j,$$

whenever $j < k$.

In particular, under the same hypothesis, the sequence $(D^\alpha P_{j,x_0}(x_0))_j$ is Cauchy and its limit does not depend on the chosen sequence of polynomials satisfying (3.6).

Proof. Let $\boldsymbol{\varepsilon} \in \ell^q$ be such that

$$\sigma_j 2^{jd/p} \|f - P_{j,x_0}\|_{L^p(B(x_0,2^{-j}))} \leq \varepsilon_j,$$

for any $j \in \mathbb{N}$. Given a multi-index α satisfying the hypothesis and $j \in \mathbb{N}$, we know that there exists a constant $C > 0$ such that

$$\begin{aligned}
&\|D^\alpha (P_{j,x_0} - P_{j+1,x_0})\|_{L^p(B(x_0,2^{-(j+1)}))} \\
&\leq C 2^{|\alpha|(j+1)} \|P_{j,x_0} - P_{j+1,x_0}\|_{L^p(B(x_0,2^{-(j+1)}))} \\
&\leq C 2^{|\alpha|(j+1)} \|P_{j,x_0} - f\|_{L^p(B(x_0,2^{-(j+1)}))} + \|f - P_{j+1,x_0}\|_{L^p(B(x_0,2^{-(j+1)}))} \\
&\leq C 2^{|\alpha|(j+1)} (\varepsilon_j 2^{jd/p} \sigma_j^{-1} + \varepsilon_{j+1} 2^{(j+1)d/p} \sigma_{j+1}^{-1}),
\end{aligned}$$

which implies, from what we have obtained so far,

$$|D^\alpha (P_{j,x_0} - P_{j+1,x_0})(x_0)| \leq C' (\varepsilon_j + \varepsilon_{j+1}) 2^{|\alpha| j} \sigma_j^{-1}.$$

For $j < k$, Lemma 1.2.2 then implies

$$|D^\alpha(P_{j,x_0} - P_{k,x_0})(x_0)| \leq \xi_j 2^{|\alpha| j} \sigma_j^{-1},$$

for the appropriate sequence $\boldsymbol{\xi} \in \ell^q$.

It remains to show that the limit $\mathscr{D}^\alpha f(x_0)$ of the sequence $(D^\alpha P_{j,x_0}(x_0))_j$ is independent of the peculiar choice of the sequence $(D^\alpha P_{j,x_0}(x_0))_j$; let $(Q_{j,x_0})_j$ be another sequence of polynomials satisfying (3.6). With the same reasoning as before, we get

$$|D^\alpha(P_{j,x_0} - Q_{j,x_0})(x_0)| \leq C 2^{|\alpha| j} \sigma_j^{-1},$$

for j large enough, which is sufficient to assert that

$$|D^\alpha Q_{j,x_0}(x_0) - \mathscr{D}^\alpha f(x_0)|$$

tends to zero as j tends to infinity. □

We are now able to show the existence of the unique polynomial P_{x_0} introduced in the beginning of this section.

Theorem 3.2.3. *Let $p, q \in [1, \infty]$, $f \in L^p_{loc}$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that $0 \leq n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor < \underline{s}(\boldsymbol{\sigma})$. The following assertions are equivalent:*

- *f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$;*
- *there exists a unique polynomial P_{x_0} of degree less than or equal to n such that*

$$(\sigma_j 2^{jd/p} \|f - P_{x_0}\|_{L^p(B(x_0, 2^{-j}))})_j \in \ell^q. \tag{3.7}$$

Proof. We need to prove that the first assertion implies the second one. As f belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$, there exists a sequence of polynomials $(P_{j,x_0})_j$ of degree less than or equal to n such that

$$(\sigma_j 2^{jd/p} \|f - P_{j,x_0}\|_{L^p(B(x_0, 2^{-j}))})_j \in \ell^q.$$

Given a multi-index α satisfying $|\alpha| \leq n$, let us set

$$\mathscr{D}^\alpha f(x_0) := \lim_j D^\alpha P_{j,x_0}(x_0)$$

and define the polynomial

$$P_{x_0} : x \mapsto \sum_{|\alpha| \leq n} \mathscr{D}^\alpha f(x_0) \frac{(x - x_0)^\alpha}{|\alpha|!}. \tag{3.8}$$

One directly gets

$$\|P_{j,x_0} - P_{x_0}\|_{L^p(B(x_0, 2^{-j}))} \leq \sum_{|\alpha| \leq n} |D^\alpha P_{j,x_0}(x_0) - \mathscr{D}^\alpha f(x_0)| \, 2^{-j(|\alpha| + d/p)}.$$

That being said, we know from the previous lemma that, given α, there exists a sequence $\boldsymbol{\xi}^{(\alpha)} \in \ell^q$ such that

$$|D^\alpha P_{j,x_0}(x_0) - \mathscr{D}^\alpha f(x_0)| \leq \xi_j^{(\alpha)} 2^{|\alpha| j} \sigma_j^{-1}.$$

We thus have

$$(\sigma_j 2^{jd/p} \|P_{j,x_0} - P_{x_0}\|_{L^p(B(x_0,2^{-j}))})_j \in \ell^q,$$

which proves the first part of the theorem.

Concerning the uniqueness of the polynomial, the idea of the proof is the same as the one given in [26] for the spaces $T_u^p(x_0)$. Let P and Q be two polynomials satisfying a relation of type (3.7); one directly gets $P(x_0) = Q(x_0)$. That being said, let us define

$$L := \sum_{|\alpha|=m} c_\alpha (\cdot - x_0)^\alpha,$$

where m is the lowest degree of $P - Q$, with

$$c_\alpha := \frac{D^\alpha (P-Q)(x_0)}{|\alpha|!}.$$

If $m < \sup\{l \in \mathbb{Z} : l < \underline{s}(\sigma)\}$, one can write

$$\|L\|_{L^1(B(x_0,1))} \leq C(2^{-mj}\sigma_j^{-1} + 2^{-j}),$$

for a constant C, which means $L = 0$. For $m = \sup\{l \in \mathbb{Z} : l < \underline{s}(\sigma)\}$, we simply get $\|L\|_{L^1(B(x_0,1))} \leq C 2^{-mj}\sigma_j^{-1}$, which implies $L = P - Q = 0$. □

Remark 3.2.4. In the previous result, if σ is the usual sequence $\boldsymbol{u}$ with $u \in \mathbb{N}$, it is easy to check that the polynomial P_{x_0} is unique if one requires its degree to be strictly smaller than u. This requirement does not modify the functional spaces as, if $u \in \mathbb{N}$, for any $j \in \mathbb{N}$

$$2^{j\frac{d}{p}} \|(x - x_0)^u\|_{L^p(B(x_0,2^{-j}))} = 2^{-ju}.$$

3.3 Spaces of generalized smoothness and wavelets

Let now us focus on the quasi-characterization of $T_{p,q}^\sigma(x_0)$ spaces. As for the wavelet-based study of pointwise Hölder spaces, we will use wavelet leaders [70]. However, as we work here with L^p norms, we need to introduce a generalized version.

Definition 3.3.1. Given a dyadic cube $\lambda \in \Lambda_j$ at scale j, the p-wavelet leader of λ ($p \in [1,\infty]$) is defined by

$$d_\lambda^p = \sup_{j' \geq j} \Big(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} (2^{(j-j')d/p} |c_{\lambda'}|)^p \Big)^{1/p}.$$

Given $x_0 \in \mathbb{R}^d$, we set

$$d_j^p(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda^p.$$

Remark 3.3.2. The definition of the wavelet leaders given in this thesis is different from the one presented in [94]. The quantities introduced here are easier to work with and naturally generalize the usual wavelet leaders $d_j(x_0)$ introduced in [70], since we have $d_j(x_0) = d_j^{\infty}(x_0)$.

Compactly supported wavelets

In this section, we work with compactly supported wavelets of regularity $r > \overline{s}(\sigma)$ (see [36]). In this context, j_0 is a natural number such that the support of each wavelet is contained in $B(0, 2^{j_0})$. We will need the following definition (see [103]), ensuring a minimum regularity condition for a function.

Definition 3.3.3. Given $x_0 \in \mathbb{R}^d$, a function f defined on $\mathbb{R}^d$ belongs to the Xu space $\dot{X}^s_{p,q}(x_0)$ ($s \in \mathbb{R}$, $p, q \in [1, \infty]$) if there exists a constant $C_* > 0$ such that

$$\Big(\sum_{|k - 2^j x_0| < C_* 2^j} (2^{(s-d/p)j} |c_{\lambda^{(i)}_{j,k}}|)^p \Big)^{1/p} \in \ell^q.$$

Theorem 3.3.4. *If f belongs to the space $T^{\sigma}_{p,q}(x_0)$, then*

$$(\sigma_j d_j^p(x_0))_j \in \ell^q. \tag{3.9}$$

Proof. Let $\varepsilon \in \ell^q$ and $(P_j)_j$ be a sequence of polynomials of degree less than or equal to $\overline{s}(\sigma)$ such that

$$\sigma_j 2^{jd/p} \|f - P_j\|_{L^p(B(x_0, 2^{-j}))} \leq \varepsilon_j,$$

for all $j \in \mathbb{N}$. Let us choose $j_1 \in \mathbb{N}$ such that $2\sqrt{d} + 2^{j_0} \leq 2^{j_1}$ and fix $n \geq j_1$. For $\lambda^{(i)}_{j,k} \subset 3\lambda_n(x_0)$, we have

$$\Big|\frac{k}{2^j} - x_0\Big| \leq 2\sqrt{d}\, 2^{-n}.$$

By setting

$$\Lambda_{j,n} := \{\lambda^{(i)}_{j,k} \in \Lambda_j : |k - 2^j x_0| \leq 2\sqrt{d}\, 2^{j-n}\},$$

for $\lambda \in 3\lambda_n(x_0)$, we can write

$$\sum_{\lambda' \in \Lambda_j, \lambda' \subset \lambda} 2^{(n-j)d} |c_{\lambda'}|^p \leq \sum_{\lambda' \in \Lambda_{j,n}} 2^{(n-j)d} |c_{\lambda'}|^p,$$

whenever $p \neq \infty$. In this case, let us set

$$s_{n,j} := \sum_{\lambda' \in \Lambda_{j,n}} |c_{\lambda'}|^p$$

and define

$$g_{n,j} := \sum_{\lambda' \in \Lambda_{j,n}} |c_{\lambda'}|^{p-1} \operatorname{sign}(c_{\lambda'}) \psi_{\lambda'}.$$

One easily checks that the support of $g_{n,j}$ is contained in $B(x_0, 2^{j_1-n})$ and

$$s_{n,j} = 2^{jd}\langle f, g_{n,j}\rangle = 2^{jd}\int_{B(x_0,2^{j_1-n})} (f(x) - P_{n-j_1}(x))\overline{g_{n,j}(x)}\,dx,$$

so that, if we denote by q the conjugate exponent of p,

$$s_{n,j} \leq 2^{jd}\|f - P_{n-j_1}\|_{L^p(B(x_0,2^{j_1-n}))}\|g_{n,j}\|_{L^q}.$$

To estimate $\|g_{n,j}\|_{L^q}$, let us remark that there exists a constant $C_* \in \mathbb{N}$ that depends neither on λ nor on the scale j such that the cardinal of

$$\{\lambda' \in \Lambda_j : \mathrm{supp}(\psi_\lambda) \cap \mathrm{supp}(\psi_{\lambda'}) \neq \emptyset\}$$

is bounded by C_*. Therefore, given $j \in \mathbb{N}$, we can choose a partition $E_1, \ldots, E_{C_*}$ of Λ_j such that $\lambda', \lambda'' \in E_m$ $(1 \leq m \leq C_*)$ and

$$\mathrm{supp}(\psi_{\lambda'}) \cap \mathrm{supp}(\psi_{\lambda''}) \neq \emptyset$$

implies $\lambda' = \lambda''$. For $p \neq 1$, we easily get

$$|g_{n,j}|^q \leq C_*^q \sum_{\lambda' \in \Lambda_{j,n}} |c_{\lambda'}|^p |\psi_{\lambda'}|^q$$

and thus

$$\|g_{n,j}\|_{L^q} \leq C_* 2^{-jd/q} s_{n,j}^{1/q} \max_{1\leq i<2^d} \|\psi^{(i)}\|_{L^q}. \tag{3.10}$$

If $p = 1$, one easily checks that

$$\|g_{n,j}\|_{L^\infty} \leq C_* 2^{-jd/q} \max_{1\leq i<2^d} \|\psi^{(i)}\|_{L^\infty},$$

so that (3.10) is still satisfied in this case.

That being done, since we have

$$s_{n,j}^{1/p} \leq C\varepsilon_{n-j_1} 2^{(j-n)d/p} \sigma_n^{-1},$$

for a constant $C > 0$, we get

$$\sum_{\lambda' \in \Lambda_j, \lambda' \subset \lambda} 2^{(n-j)d} |c_{\lambda'}|^p \leq 2^{(n-j)d} s_{n,j} \leq C\varepsilon_{n-j_1}^p \sigma_n^{-p},$$

which is sufficient to conclude in the case $p \neq \infty$.

Finally, let us consider the case $p = \infty$. The conclusion is straightforward since, given $\lambda \subset 3\lambda_n(x_0)$, one easily checks that, using an analogous reasoning, we can write

$$|c_\lambda| \leq C\varepsilon_{n-j_1}\sigma_n,$$

for a constant $C > 0$. □

For the sufficient condition, we need the following definition.

Definition 3.3.5. Let $p,q \in [1,\infty]$, $x_0 \in \mathbb{R}^d$ and f be a function from L^p_{loc}; if σ is an admissible sequence such that $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to ∞, we say that f belongs to $T^{\sigma}_{p,q,\log}(x_0)$ if there exists $J \in \mathbb{N}$ for which

$$(\frac{2^{jd/p}\sigma_j}{|\log_2(2^{-jd/p}\sigma_j^{-1})|}\sup_{|h|\leq 2^{-j}}\|\Delta_h^{\lfloor \bar{s}(\sigma)\rfloor+1}f\|_{L^p(B_h(x_0,2^{-j}))})_{j\geq J} \in \ell^q.$$

Theorem 3.3.6. *Let $p,q \in [1,\infty]$, $x_0 \in \mathbb{R}^d$ and f be a function from L^p_{loc}; let also σ be an admissible sequence such that $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to ∞ and $\underline{\sigma}_1 > 2^{-d/p}$. If f belongs to $\dot{X}^s_{p,q}(x_0)$ for some $s>0$, then $(\sigma_j d_j^p(x_0))_j \in \ell^q$ implies $f \in T^{\sigma}_{p,q,\log}(x_0)$.*

Proof. Let us first suppose that $\bar{s}(\sigma) \geq 0$ and set $n := \lfloor \bar{s}(\sigma)\rfloor$. We need to define some quantities. First, choose $m \in \mathbb{N}$ such that $k/2^j \in B(x,r)$ implies $\lambda^{(i)}_{j,k} \subset B(x,2^m r)$, for any $x \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$, $j \in \mathbb{N}$ and $r \geq 2^{-j}$. Let also $m' \in \mathbb{N}$ be such that, for any $x \in \mathbb{R}^d$ and any $j \in \mathbb{N}$, $B(x,2^{-j})$ is included in some dyadic cube of side length $2^{m'-j}$ and define $J_0 := j_0 + m + m'$. Let $C_* > 0$ be such that

$$(\sum_{|k-2^j x_0|\leq C_* 2^j}(2^{s-d/p)j}|c_{\lambda^{(i)}_{j,k}}|)^p)^{1/p} \in \ell^q$$

and choose a number $J_1 \in \mathbb{N}$ for which we have $(1+2^{j_0}) \leq C_* 2^{J_1}$. We also need a sequence $\varepsilon \in \ell^q$ satisfying $\sigma_j d_j^p(x_0) \leq \varepsilon_j$, for all $j \in \mathbb{N}$. Finally, given $J \geq \max\{J_0,J_1\}$, define

$$P_J := \sum_{|\alpha|\leq n}(\frac{(\cdot - x_0)^\alpha}{|\alpha|!}\sum_{j=-1}^{J} D^\alpha f_j(x_0)),$$

where

$$f_{-1} := \sum_{k\in\mathbb{Z}^d} C_k\varphi_k \qquad \text{and} \qquad f_j := \sum_{\lambda\in\Lambda_j} c_\lambda\psi_\lambda,$$

for $j \geq 0$. We have

$$\begin{aligned} &2^{Jd/p}\|f - P_J\|_{L^p(B(x_0,2^{-J}))} \\ &\leq \sum_{j=-1}^{J} 2^{Jd/p}\|f_j - \sum_{|\alpha|\leq n}\frac{(\cdot - x_0)^\alpha}{|\alpha|!}D^\alpha f_j(x_0)\|_{L^p(B(x_0,2^{-J}))} \qquad (3.11)\\ &\quad + \sum_{j=J+1}^{\infty} 2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))}. \qquad (3.12)\end{aligned}$$

Let us fix $y \in B(x_0,2^{-J})$ and $|\alpha| = n+1$. We will first consider the case $p \neq \infty$. We have $D^\alpha\psi_{\lambda^{(i)}_{j,k}}(y) \neq 0$ only if $k/2^j$ belongs to $B(y,2^{j_0-j})$; for $J_0 \leq j \leq J$, we have

$$\lambda^{(i)}_{j,k} \subset B(y,2^{m-j-j_0}) \subset \lambda_{j-J_0}(x_0),$$

so that we can write, using the same reasoning as in the previous proof,

$$\begin{aligned}|D^\alpha f_j(y)| &\leq C 2^{jp(n+1)} \sum_{\lambda\in\Lambda_j} |c_\lambda|^p |D^\alpha \psi_\lambda(y)|^p \\ &\leq C 2^{jp(n+1)} \sum_{\lambda\in\Lambda_j, \lambda\subset\lambda_{j-J_0}(x_0)} |c_\lambda|^p |D^\alpha \psi_\lambda(y)|^p \\ &\leq C 2^{jp(n+1)} \varepsilon_{j-J_0}^p \sigma_j^{-p},\end{aligned}$$

since $\boldsymbol{\sigma}$ is an admissible sequence. Moreover, as the wavelet coefficients are finite and there exists a constant C_d which only depends on d such that

$$\#\{k \in \mathbb{Z}^d : k \in B(y, 2^{j_0})\} \leq C_d, \quad \#\{k \in \mathbb{Z}^d : k/2^j \in B(y, 2^{j_0-j})\} \leq C_d,$$

we also have

$$|D^\alpha f_j(y)|^p \leq C 2^{jp(n+1)} \sigma_j^{-p},$$

for all $j \in \{-1, \ldots, J_0 - 1\}$. As a consequence, we can write, for any $j \in \{-1, \ldots, J\}$,

$$\|f_j - \sum_{|\alpha|\leq n} \frac{(\cdot - x_0)^\alpha}{|\alpha|!} D^\alpha f_j(x_0)\|_{L^p(B(x_0,2^{-J}))} \leq \theta_j 2^{-J(n+1+d/p)} 2^{j(n+1)} \sigma_j^{-1},$$

for some sequence $\boldsymbol{\theta} \in \ell^q$. A similar reasoning gives the same inequality for $p = \infty$. Now, since $\overline{s}(\boldsymbol{\sigma}) < n + 1$, (3.11) is upper bounded by

$$C' 2^{-J(n+1)} \sum_{j=-1}^{J} \theta_j 2^{j(n+1)} \sigma_j^{-1} \leq C' \xi_J \sigma_J^{-1},$$

for some constant $C' > 0$, where the sequence $\boldsymbol{\xi}$ is given by Lemma 1.2.2.

For the second term, let us fix $j \geq J + 1$ and $p \neq \infty$ to define

$$\Lambda_{j,J} := \{\lambda_{j,k}^{(i)} \in \Lambda_j : B(k/2^j, 2^{j_0}/2^j) \cap B(x_0, 2^{-J}) \neq \emptyset\}.$$

By proceeding as before for $x \in B(x_0, 2^{-J})$, we get

$$\|f_j(x)\|^p_{L^p(B(x_0,2^{-J}))} \leq C \sum_{\lambda\in\Lambda_{j,J}} 2^{-dj} |c_\lambda|^p, \tag{3.13}$$

for some constant C, which gives

$$2^{Jd/p} \|f_j(x)\|_{L^p(B(x_0,2^{-J}))} \leq C \varepsilon_{J-J_0} \sigma_J^{-1}.$$

Moreover, since the coefficient $c_{\lambda_{j,k}^{(i)}}$ in the sum (3.13) vanishes always but when $|k - 2^j x_0| \leq C_* 2^j$, we also have

$$\|f_j(x)\|^p_{L^p(B(x_0,2^{-J}))} \leq \delta_j^p 2^{-spj},$$

for a sequence $\delta \in \ell^q$, as f belongs to the space $\dot{X}^s_{p,q}(x_0)$. Let us obtain upper bounds for the case $p = \infty$; for $x \in B(x_0, 2^{-J})$, $k/2^j \in B(x, 2^{j_0 - j})$ implies $\lambda^{(i)}_{j,k} \subset \lambda_{J-j_0}(x_0)$, so that we have $|c_{\lambda^{(k)}_{j,l}}| \leq C\varepsilon_{J-J_0}\sigma_N$. The same reasoning as before leads to

$$\|f_j(x)\|_{L^\infty(B(x_0,2^{-J}))} \leq C\delta_j 2^{-sj}.$$

Let us now set $j_*(J) := \lceil |\log_2(2^{-Jd/p}\sigma_J^{-1})|/s\rceil$ and choose s small enough in order to ensure that we have $\log_2(2^{d/p}\underline{\sigma}_1)/s > 1$. With such a definition, we have $j_*(J) = j_*(J')$ if and only if $J = J'$ and we can write

$$\begin{aligned}
&\sum_{j=J+1}^{\infty} 2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))} \\
&= \sum_{j=J+1}^{j_*(J)} 2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))} + \sum_{j=j_*(J)+1}^{\infty} 2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))} \\
&\leq C\sum_{j=J+1}^{j_*(J)} \varepsilon_{J-J_0}\sigma_J^{-1} + C2^{Jd/p}\sum_{j=j_*(J)+1}^{\infty} \delta_j 2^{-sj} \\
&\leq C(\varepsilon_{J-J_0} + \xi_{j_*(J)})|\log_2(2^{-Jd/p}\sigma_J^{-1})|\sigma_J^{-1},
\end{aligned}$$

for J large enough, where the sequence $(\xi_{j_*(J)})_J$ belongs to ℓ^q.

It only remains to consider the situation where $\underline{s}(\sigma) < 0$. In this case, let us set $P_J = 0$ whenever $J \geq \max\{J_0, J_1\}$. Once again, there exists a sequence $\boldsymbol{\xi} \in \ell^q$ such that

$$|f_j(y)| \leq \xi_j\sigma_J^{-1},$$

for $y \in B(x_0, 2^{-J})$, any $J \geq \max\{J_0, J_1\}$ and any $j \in \{-1, \ldots, J\}$. As done previously, we get

$$\begin{aligned}
&2^{Jd/p}\|f - P_J\|_{L^p(B(x_0,2^{-J}))} \\
&\leq C\sum_{j=-1}^{J} 2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))} + \sum_{j=J+1}^{\infty} 2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))} \\
&\leq \delta_J|\log_2(2^{-Jd/p}\sigma_J^{-1})|\sigma_J^{-1},
\end{aligned}$$

with $\boldsymbol{\delta} \in \ell^q$. □

Schwartz wavelets

In practise, compactly supported wavelets are used most of the time; however, for theoretical applications, it can be handy to have similar results concerning wavelets in the Schwarz class [102]. We will thus consider such wavelets in this section.

Lemma 3.3.7. *Let $p,q \in [1,\infty]$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that either $\overline{s}(\boldsymbol{\sigma}) > -d/p$, $\overline{s}(\boldsymbol{\sigma}) < 0$, or $0 \leq n := \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor < \underline{s}(\boldsymbol{\sigma})$; if $f \in L^p$ belongs to $T^{\sigma}_{p,q}(x_0)$, then we have*

$$(\sigma_j 2^{j(d-u)} \int_{\mathbb{R}^d \setminus B(x_0,2^{-j})} \frac{|f(x) - P(x)|}{|x_0 - x|^u} \, dx)_j \in \ell^q,$$

for any $u > \overline{s}(\boldsymbol{\sigma}) + d$, where P is the polynomial given by Theorem 3.2.3.

Proof. Let us set $R := f - P$; without loss of generality, we can assume $x_0 = 0$. Let us define, for $r > 0$,

$$\varphi(r) := \int_{B(0,r)} |R(x)| \, dx;$$

we know that there exists a sequence $\boldsymbol{\varepsilon} \in \ell^q$ such that

$$\varphi(2^{-j}) \leq 2^{-jd} \varepsilon_j \sigma_j^{-1},$$

for all $j \in \mathbb{N}$. Moreover, for $r \geq 1$, we have

$$\varphi(r) \leq C r^{d(1-1/p)} \|f\|_{L^p} + c r^{n+d} \leq C r^{d+\overline{s}(\sigma)}.$$

Using spherical coordinates, we can write

$$\varphi(r) = \int_0^r \psi(\rho) \, d\rho,$$

with

$$\psi(\rho) := \rho^{d-1} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} |R(x(\rho, \theta_1, \ldots, \theta_{d-1}))| \, d\Omega_d,$$

where $d\Omega_d$ stands for

$$\sin^{d-2}(\theta_1) \cdots \sin(\theta_{d-2}) \, d\theta_1 \cdots d\theta_{d-1}.$$

Since, for all $r > 0$, we have

$$\frac{\varphi(r)}{r^u} - \phi(2^{-j}) 2^{ju} = \int_{B(0,2^{-j})} \frac{|R(x)|}{|x|^u} \, dx - \int_{2^{-j}}^r \frac{u}{\rho^{u+1}} \varphi(\rho) \, d\rho,$$

we get

$$\begin{aligned} &\int_{B(0,r) \setminus B(0,2^{-j})} \frac{|R(x)|}{|x|^u} \, dx \\ &\leq \frac{\varphi(r)}{r^u} + \int_1^r \frac{u}{\rho^{u+1}} \varphi(\rho) \, d\rho + \sum_{k=1}^{j} \int_{2^{-k}}^{2^{1-k}} \frac{u}{\rho^{u+1}} \varphi(\rho) \, d\rho. \end{aligned}$$

Since

$$\varphi(r)/r^u \leq C \leq C 2^{j(u-d)} 2^{-\delta j} \sigma_j^{-1},$$

where $\delta > 0$ has been chosen such that $\delta < u - d - \overline{s}(\sigma)$, we can write

$$\int_1^r \frac{u}{\rho^{u+1}} \varphi(\rho)\, d\rho \leq C 2^{j(u-d)} 2^{-j\delta} \sigma_j^{-1}.$$

Finally, as $\boldsymbol{\sigma}$ is admissible and $u > \overline{s}(\boldsymbol{\sigma}) + d$, we have

$$\sum_{k=1}^{j} \int_{2^{-k}}^{2^{1-k}} \frac{u}{\rho^{u+1}} \varphi(\rho)\, d\rho \leq 2^{j(u-d)} \xi_j \sigma_j^{-1},$$

where $\boldsymbol{\xi} \in \ell^q$ is given by Lemma 1.2.2. Putting all these together, we can claim that there exists a sequence $\boldsymbol{\theta} \in \ell^q$ such that the inequality

$$\int_{B(0,r)\setminus B(0,2^{-j})} \frac{|R(x)|}{|x|^u}\, dx \leq 2^{j(u-d)} \theta_j \sigma_j^{-1}$$

holds for $r \geq 1$. □

Theorem 3.3.8. *Let $p, q \in [1, \infty]$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that either $\overline{s}(\boldsymbol{\sigma}) > -d/p$, $\overline{s}(\boldsymbol{\sigma}) < 0$, or $0 \leq \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor < \underline{s}(\boldsymbol{\sigma})$; if $f \in L^p$ belongs to $T^{\boldsymbol{\sigma}}_{p,q}(x_0)$, then we have*

$$(\sigma_j d_j^p(x_0))_j \in \ell^q.$$

Proof. Let $\boldsymbol{\varepsilon} \in \ell^q$ be such that

$$\sigma_j 2^{jd/p} \|f - P_j\|_{L^p(B(x_0, 2^{-j}))} \leq \varepsilon_j,$$

for any $j \in \mathbb{N}$, choose $j_1 \in \mathbb{N}$ such that $2\sqrt{d} \leq 2^{j_1}$ and fix $n \geq j_1 + 1$.

Let us first suppose that $p \in (1, \infty)$; define

$$\Lambda_{j,n} := \{\lambda^{(l)}_{j,k} \in \Lambda_j : |k - 2^j x_0| \leq \sqrt{d} 2^{j+1-n}\},$$

so that $\lambda \in 3\lambda_n(x_0)$ and $\lambda \in \Lambda_j$ implies $\lambda \in \Lambda_{j,n}$,

$$s_{j,n} := \sum_{\lambda' \in \Lambda_{j,n}} |c_{\lambda'}|^p$$

and

$$g_{j,n} := \sum_{\lambda' \in \Lambda_{j,n}} |c_{\lambda'}|^{p-1} \operatorname{sign}(c_{\lambda'}) \psi_{\lambda'}.$$

We have

$$\begin{aligned} s_{j,n} = {} & 2^{jd} \int_{B(x_0, 2^{j_1 - n+1})} (f(x) - P(x)) \overline{g_{j,n}(x)}\, dx \\ & + 2^{jd} \int_{\mathbb{R}^d \setminus B(x_0, 2^{j_1 - n+1})} (f(x) - P(x)) \overline{g_{j,n}(x)}\, dx. \end{aligned}$$

Using Hölder's inequality, we can write

$$\begin{aligned}
&2^{jd}\int_{B(x_0,2^{j_1-n+1})}(f(x)-P(x))\overline{g_{j,n}(x)}\,dx\\
&\leq C\varepsilon_{n-j_1-1}2^{jd}2^{-nd/p}\|g_{j,n}\|_{L^{p'}}\sigma_n^{-1},
\end{aligned}$$

where p' is the conjugate exponent of p, with

$$\|g_{j,n}\|_{L^{p'}}\leq C2^{-jd/p'}s_{j,n}^{1/p'},$$

for a constant $C>0$, thanks to the wavelet characterization of L^p spaces (see e.g. [102]). Now, for all $u>d+\overline{s}(\sigma)$, it is easy to check, using the fast decay of the wavelets, that there exists a constant $C_{d,u}>0$ such that, for all $x\in\mathbb{R}^d\setminus B(x_0,2^{j_1-n+1})$,

$$\Big(\sum_{\lambda'\in\Lambda_{j,n}}|\psi_{\lambda'}|^p\Big)^{1/p}\leq C_{d,u}/(2^j|x-x_0|)^u.$$

Using the previous lemma, we can claim that there exists a sequence $\boldsymbol{\theta}\in\ell^q$ for which

$$2^{jd}\int_{\mathbb{R}^d\setminus B(x_0,2^{j_1-n+1})}(f(x)-P(x))\overline{g_{j,n}(x)}\,dx\leq\theta_n s_{j,n}^{1/p'}2^{(j-n)d/p}\sigma_n^{-1}.$$

As a consequence, there exists a sequence $\boldsymbol{\xi}\in\ell^q$ such that

$$s_{j,n}^{1/p}\leq\xi_n2^{(j-n)d/p}\sigma_n^{-1}.$$

If $p=1$, keeping the same notations, we have

$$\begin{aligned}
s_{j,n}\leq\;&2^{jd}\int_{B(x_0,2^{j_1-n+1})}|f(x)-P(x)|\sum_{\lambda'\in\Lambda_{j,n}}|\psi_{\lambda'}|\,dx\\
&+2^{jd}\int_{\mathbb{R}^d\setminus B(x_0,2^{j_1-n+1})}|f(x)-P(x)|\sum_{\lambda'\in\Lambda_{j,n}}|\psi_{\lambda'}|\,dx.
\end{aligned}$$

To bound the first integral, remark that $\sum_{\lambda'\in\Lambda_j}|\psi_{\lambda'}|$ is bounded and

$$\|f-P\|_{L^1(B(x_0,2^{j_1-n+1}))}\leq C\varepsilon_{n-j_1-1}2^{-nd}\sigma_n^{-1}.$$

The second integral can be treated as in the case $p\in(1,\infty)$.

Finally, assume that $p=\infty$, fix $j\geq n$ and suppose that $\lambda\in\Lambda_j$ satisfies $\lambda\in3\lambda_n(x_0)$. We have

$$\begin{aligned}
|c_\lambda|\leq\;&2^{jd}\int_{B(x_0,2^{j_1-n+1})}|f(x)-P(x)||\psi_\lambda|\,dx\\
&+2^{jd}\int_{\mathbb{R}^d\setminus B(x_0,2^{j_1-n+1})}|f(x)-P(x)||\psi_\lambda|\,dx.
\end{aligned}$$

Once again, it is sufficient to bound the first integral, which is easy since we have

$$2^{jd}\int_{B(x_0,2^{j_1-n+1})}|f(x)-P(x)|\,dx \leq C\varepsilon_{n-j_1-1}\sigma_n^{-1},$$

for some constant $C>0$. □

Theorem 3.3.9. *Let $p,q\in[1,\infty]$, $f\in L^p_{loc}$, $x_0\in\mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma})>-d/p$ and $\underline{\sigma}_1>2^{-d/p}$. If there exists $s>0$ such that $f\in B^s_{p,q}$, then $(\sigma_j d^p_j(x_0))_j\in\ell^q$ implies $f\in T^{\boldsymbol{\sigma}}_{p,q,\log}(x_0)$.*

Proof. Let us use the definitions of n, m, J_0, J_1, J, ε, P_J and f_j introduced in the proof of Theorem 3.3.6. Of course, we have

$$\begin{aligned}
&2^{Jd/p}\|f-P_J\|_{L^p(B(x_0,2^{-J}))}\\
&\leq \sum_{j=-1}^{J}2^{Jd/p}\|f_j-\sum_{|\alpha|\leq n}\frac{(\cdot-x_0)^\alpha}{|\alpha|!}D^\alpha f_j(x_0)\|_{L^p(B(x_0,2^{-J}))} &(3.14)\\
&\quad+\sum_{j=J+1}^{\infty}2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))}. &(3.15)
\end{aligned}$$

Let us first consider the term (3.14) of the last bound. Let α be a multi-index such that $|\alpha|=n+1$; from Taylor's formula, we need to bound $|D^\alpha f_j(x)|$ for $x\in B(x_0,2^{-J})$. Assume now that j is such that $j-\lceil j/2\rceil\geq J_0$ and define

$$\Lambda_{j,0}:=\{\lambda^{(i)}_{j,k}\in\Lambda_j : |2^jx_0-k|\leq 1\},$$

for l such that $1\leq l\leq\lceil j/2\rceil$,

$$\Lambda_{j,l}:=\{\lambda^{(i)}_{j,k}\in\Lambda_j : 2^{l-1}<|2^jx_0-k|\leq 2^l\}$$

and

$$\Lambda_{j,*}:=\{\lambda^{(i)}_{j,k}\in\Lambda_j : |2^jx_0-k|\geq 2^{\lceil j/2\rceil}\}.$$

A sum over Λ_j can be decomposed into a sum over the sets $\Lambda_{j,l}$ (with $l\in\{0,\dots,\lceil j/2\rceil\}$) and $\Lambda_{j,*}$. For $1\leq l\leq\lceil j/2\rceil$, we have, by Hölder's inequality,

$$\begin{aligned}
&\sum_{\lambda\in\Lambda_{j,l}}|c_\lambda||D^\alpha\psi_\lambda(x)|\\
&\leq(\sum_{\lambda\in\Lambda_{j,l}}|c_\lambda|^p)^{1/p}(\sum_{\lambda\in\Lambda_{j,l}}|D^\alpha\psi_\lambda(y)|^{p'})^{1/p'}\\
&\leq C(\varepsilon_{j-l-J_0}2^{ld/p}\sigma_{j-l}^{-1})(\sum_{\lambda\in\Lambda_{j,l}}(\frac{1}{(1+|2^jx-k|)^{2^{d+1}+u+d/p)p'}})^{1/p'}\\
&\leq C\varepsilon_{j-l-J_0}2^{-ul}\sigma_{j-l}^{-1},
\end{aligned}$$

where u is such that $u > \overline{s}(\sigma)$ and p' is the conjugate exponent of p; for $l = 0$, we can write

$$\sum_{\lambda\in\Lambda_{j,0}} |c_\lambda||D^\alpha \psi_\lambda(x)| \leq \varepsilon_{j-J-J_0}\sigma_j^{-1}.$$

For the last set, we get

$$\sum_{\lambda\in\Lambda_{j,*}} |c_\lambda||D^\alpha \psi_\lambda(x)| \leq \delta_j\sigma_j^{-1},$$

for a sequence $\boldsymbol{\delta} \in \ell^q$, as $f \in B^s_{p,q}$. Using these results, we obtain

$$\sum_{\lambda\in\Lambda_j} |c_\lambda||D^\alpha \psi_\lambda(x)| \leq \delta_j\sigma_j^{-1} + \sum_{l=0}^{\lceil j/2\rceil} \varepsilon_{j-l-J0}2^{-ul}\sigma_{j-l}^{-1} \leq (\delta_j + \xi_j)\sigma_j^{-1},$$

where $\boldsymbol{\xi} \in \ell^q$ is defined as in the proof of Lemma 1.2.2. For the first term (3.14), we still have to consider the case $j - \lceil j/2\rceil < J_0$; since $f \in B^s_{p,q}$, we can write

$$\sum_{\lambda\in\Lambda_j} |c_\lambda||D^\alpha \psi_\lambda(x)| \leq \delta_j 2^{-sj}2^{jd/p} \leq C\delta_j\sigma_j^{-1},$$

so that $|D^\alpha f_j(x)|$ is bounded by $C'(\delta_j + \xi_j)2^{n+1}\sigma_j^{-1}$, for any $j \leq J$; we thus have

$$\begin{aligned}\|f_j - \sum_{|\alpha|\leq n} \frac{(\cdot - x_0)^\alpha}{|\alpha|!} D^\alpha f_j(x_0)\|_{L^p(B(x_0,2^{-J}))} \\ \leq C(\delta_j + \xi_j)2^{-(n+1+d/p)J}2^{(n+1)j}\sigma_j^{-1}.\end{aligned}$$

Finally, as $\overline{s}(\boldsymbol{\sigma}) < n+1$, (3.14) is bounded by

$$C2^{-(n+1)J}\sum_{j=-1}^{J}(\delta_j + \xi_j)2^{(n+1)j}\sigma_j^{-1} \leq \theta_J\sigma_J^{-1},$$

where $\boldsymbol{\theta} \in \ell^q$ is given by Lemma 1.2.3.

Let us now consider the second term (3.15); we actually need to bound the L^p norm of f_j for $j \geq J$. Let us, in the same spirit as before, define

$$\Lambda'_{j,0} := \{\lambda^{(i)}_{j,k} \in \Lambda_j : |2^j x_0 - k| \leq 2^{j+J_0-J}\},$$

for l such that $1 \leq l \leq J$,

$$\Lambda'_{j,l} := \{\lambda^{(i)}_{j,k} \in \Lambda_j : 2^{j+J_0-J+l-1} < |2^j x_0 - k| \leq 2^{j+J_0-J+l}\}$$

and

$$\Lambda'_{j,*} := \{\lambda^{(i)}_{j,k} \in \Lambda_j : 2^j < |2^j x_0 - k|\}.$$

Using the wavelet characterization of L^p spaces (see [102]), we can write

$$\begin{aligned}\|\sum_{\lambda\in\Lambda'_{j,0}} c_\lambda\psi_\lambda\|_{L^p(B(x_0,2^{-J}))} &\leq C(\sum_{\lambda\in\Lambda_j,\lambda\subset\lambda_J(x_0)} 2^{-dj}|c_\lambda|^p)^{1/p}\\ &\leq C\varepsilon_J 2^{-Jd/p}\sigma_J^{-1}.\end{aligned}$$

For $l\in\{1,\ldots,J\}$, we get this time

$$\sum_{\lambda\in\Lambda'_{j,l}}|c_\lambda||\psi_\lambda(x)|\leq C\varepsilon_{J-l}2^{-(j-J+l)u}\sigma_{J-l}^{-1}\leq C2^{-ul}\varepsilon_{J-l}\overline{\sigma}_l\sigma_J^{-1},$$

for $x\in B(x_0,2^{-J})$ and

$$\sum_{\lambda\in\Lambda'_{j,*}}|c_\lambda||\psi_\lambda(x)|\leq C2^{-\delta J}\sigma_J^{-1},$$

for some $\delta>0$. As previously, we get that there exists a sequence $\boldsymbol{\rho}\in\ell^q$ such that

$$2^{Jd/p}\|f_j\|_{L^p(B(x_0,2^{-J}))}\leq\rho_J\sigma_J^{-1},$$

so that we can conclude using the same arguments as in the compactly supported case. □

3.4 A multifractal formalism associated to the generalized Besov spaces

We show in this section that the generalized Besov spaces, studied in the preceding chapter, provide a natural framework for the multifractal formalism based on the $T^\sigma_{p,q}$ spaces.

As the wavelet leaders method (WLM) involves the oscillation spaces $\mathcal{O}_p^{s,s'}$ (see [70, 75]), we will temporarily use them in our general framework.

Definition 3.4.1. Let $p,q,r\in[1,\infty]$; a function f belongs to the *generalized oscillation space* $\mathcal{O}^\sigma_{p,r,q}$ if the sequence $(C_k)_k$ defined by (1.11) belongs to ℓ^q and if

$$(\sum_{j\in\mathbb{N}}(\sum_{\lambda\in\Lambda_j}(\sigma_j 2^{-dj/r}d_\lambda^p)^r)^{q/r})^{1/q}\leq C,$$

for some positive constant C.

We will show that these spaces are closely related to generalized Besov spaces. We first need the following definition to introduce a multifractal formalism.

Definition 3.4.2. Let $p,q \in [1,\infty]$; if, given $h > -d/p$, $\gamma^{(h)}$ is an admissible sequence, the family of admissible sequences $h \mapsto \gamma^{(h)}$ is (p,q)-decreasing if it satisfies $\underline{s}(\gamma^{(h)}) > -d/p$, $\underline{\gamma}_1^{(h)} > 2^{-d/p}$ for any $h > -d/p$ and if $-d/p < h < h'$ implies

$$T_{p,q}^{\gamma^{(h')}}(x_0) \subset T_{p,q}^{\gamma^{(h)}}(x_0).$$

In the sequel, we will only consider families of admissible sequences $\gamma^{(\cdot)}$ that are implicitly (p,q)-decreasing. This notion was introduced in [90], where criteria to obtain such families are presented. The idea is to work with a familly of sequences of the form $(\sigma^{(h)} = (2^{jg(h)}\delta_j^{(h)})_j)_{h>-\frac{d}{p}}$ where g is an increasing function and, for any h, $(\delta_j^{(h)})_j$ is the non-dyadic part of $\sigma^{(h)}$. To summarize, if the Boyd indices of the sequences $(\delta_j^{(h)})_j$ vary sufficiently slowly compared to $g(h)$, then the family is decreasing. This can be done, following Corollary 1.2.12, using slowly varying functions.

Definition 3.4.3. Given $p,q \in [1,\infty]$ and a family of admissible sequences $\gamma^{(\cdot)}$, the generalized (p,q)-Hölder exponent associated to $f \in L^p_{\text{loc}}$ and $\gamma^{(\cdot)}$ at $x_0 \in \mathbb{R}^d$ is defined by

$$h_{p,q}(x_0) := \sup\{h > -d/p : f \in T_{p,q}^{\gamma^{(h)}}(x_0)\}. \tag{3.16}$$

The most natural family of admissible sequences is $h \mapsto (2^{jh})_j$. In this case, $h_{\infty,\infty}(x_0)$ is the usual Hölder exponent [70], while $h_{p,\infty}(x_0)$ is the p-exponent considered in [75].

Given $p,q \in [1,\infty]$, a family of admissible sequences $\gamma^{(\cdot)}$ and a function $f \in L^p_{\text{loc}}$, we set

$$D_{p,q}(h) := \dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) = h\}).$$

In the following, we will implicitly work with indices $p,q,r \in [1,\infty]$, a function f that belongs to L^p_{loc}, a point $x_0 \in \mathbb{R}^d$, a family of admissible sequences $\gamma^{(\cdot)}$ and an admissible sequence σ.

Lemma 3.4.4. *If*

$$\gamma_j^{(h)} 2^{\eta j} d_j^p(x_0) \in \ell^q,$$

for some $\eta > 0$ such that $\lfloor \bar{s}(\gamma^{(h)}) + \eta \rfloor = \lfloor \bar{s}(\gamma^{(h)}) \rfloor$, then $h_{p,q}(x_0) \geq h$.

Proof. We know that there exist a sequence of polynomials $(P_j)_j$ of degree at most $\bar{s}(\gamma^{(h)})$ and a sequence $\varepsilon \in \ell^q$ such that

$$\gamma_j^{(h)} 2^{jd/p} \|f - P_j\|_{L^p(B(x_0,2^{-j}))} \leq C\varepsilon_j 2^{-\eta j} |\log_2(2^{-\eta j - jd/p}/\gamma_j^{(h)})|,$$

for j large enough, which implies $f \in T_{p,q}^{\gamma^{(h)}}(x_0)$. □

Proposition 3.4.5. *If the function f belongs to both $B^{\eta}_{p,q}$, for some $\eta > 0$, and $\mathcal{O}^{\sigma}_{p,r,q}$, then*

$$\dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\}) \leq d + r\overline{s}(\frac{\boldsymbol{\gamma}^{(h)}}{\boldsymbol{\sigma}}).$$

Proof. Let $\boldsymbol{\varepsilon} \in \ell^q$ be such that $\varepsilon_j \neq 0$ and

$$(\sum_{\lambda \in \Lambda_j} (\sigma_j 2^{-jd/r} d^p_{\lambda})^r)^{1/r} \leq \varepsilon_j,$$

for all $j \in \mathbb{N}$. Let us first consider the case $r = \infty$; if $\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}) < 0$, there exists $\delta > 0$ such that $\gamma^{(h)}_j 2^{\delta j} d^p_j(x_0) \leq C\varepsilon_j$ for any j and $h_{p,q}(x_0) \geq h$ for all $x_0 \in \mathbb{R}^d$. As a consequence, we have

$$\dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\}) = -\infty = d + r\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}).$$

On the other hand, if $\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}) \geq 0$,

$$\dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\}) \leq d \leq d + r\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}).$$

Now, suppose $r < \infty$, fix $h > -d/p$ and define, given $j \in \mathbb{N}$ and $\delta > 0$ sufficiently small,

$$E^h_{j,\delta} := \{\lambda \in \Lambda_j : d^p_{\lambda} \geq \varepsilon_j 2^{-\delta j}/\gamma^{(h)}_j\}$$

and set $n = \# E^h_{j,\delta}$. As $f \in \mathcal{O}^{\sigma}_{p,r,q}$, we have

$$\sigma^r_j 2^{-jd} n (2^{-\delta j}/\gamma^{(h)}_j)^r \leq \varepsilon^{-r}_j \sigma^r_j 2^{-jd} \sum_{\lambda \in E^h_{j,\delta}} (d^p_{\lambda})^r \leq 1,$$

so that

$$n \leq 2^{jd} (2^{-\delta j}/\gamma^{(h)}_j)^{-r}/\sigma^r_j.$$

Now, define $\Lambda^h_{j,\delta}$ as the set of the dyadic cubes $\lambda \in \Lambda_j$ such that there exists a neighbor $\lambda' \in 3\lambda$ that belongs to $E^h_{j,\delta}$. Finally, define

$$F^h_{\delta} := \limsup_j \{x_0 \in \mathbb{R}^d : \lambda_j(x_0) \in \Lambda^h_{j,\delta}\}.$$

If x_0 does not belong to F^h_{δ}, then there exists $J \in \mathbb{N}$ such that $j \geq J$ implies $\lambda_j(x_0) \notin \Lambda^h_{j,\delta}$ and, from what we have obtained for n, there exists a constant $C > 0$ for which $j \geq J$ implies

$$2^{\delta j} \gamma^{(h)}_j d^p_j(x_0) \leq C\varepsilon_j$$

and therefore

$$\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) < h\} \subset F^h_{\delta}. \tag{3.17}$$

Let $\alpha > 0$, set $j_1 := \inf\{j : \sqrt{d} 2^{-j} < \alpha\}$ and

$$E_{\delta} := \{\lambda \in \Lambda^h_{j,\delta} : j \geq j_1\}.$$

It is easy to check that E_δ is an α-covering of F_δ^h; given $s, \eta > 0$, we have

$$\begin{aligned}\sum_{\lambda \in E_\delta} \operatorname{diam}(\lambda)^s &\leq \sum_{j \geq j_0} \# F_j^h (\sqrt{d} 2^{-j})^s \\ &\leq C \sum_{j \geq j_0} 2^{(d-s)j} (2^{-\delta j} / \gamma_j^{(h)})^{-r} / \sigma_j^r \\ &\leq C' \sum_{j \in \mathbb{N}} 2^{rj(\overline{s}(\gamma^{(h)}/\sigma) + \delta + \eta} 2^{(d-s)j}.\end{aligned}$$

As a consequence, we have

$$\dim_{\mathcal{H}}(F_\delta^h) \leq d + r(\overline{s}(\frac{\gamma^{(h)}}{\sigma}) + \delta + \eta),$$

for any $\eta > 0$ and we can conclude thanks to (3.17). □

Of course, for classic examples of families of admissible sequences, the application $h \mapsto \overline{s}(\gamma^{(h)})$ is continuous (see [90]); in such a case, the previous result can be improved.

Remark 3.4.6. If there exists a sequence ε converging to 0^+ such that

$$\overline{s}(\frac{\gamma^{(h+\varepsilon_j)}}{\sigma}) \to \overline{s}(\frac{\gamma^{(h)}}{\sigma}),$$

we have

$$\dim_{\mathcal{H}}\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) \leq h\} \leq d + r\overline{s}(\frac{\gamma^{(h)}}{\sigma}).$$

Proposition 3.4.7. *If σ is an admissible sequence such that $\underline{s}(\sigma) > 0$ and $\underline{s}(\sigma) - d/r > -d/p$, we have $\mathcal{O}_{p,r,q}^\sigma = B_{r,q}^\sigma$.*

Proof. We obviously have $\mathcal{O}_{p,r,q}^\sigma \hookrightarrow B_{r,q}^\sigma$. If f belongs to $B_{r,q}^\sigma$, we have

$$\begin{aligned}&(\sum_{\lambda \in \Lambda_j} (\sigma_j 2^{-jd/r} d_\lambda^p)^r)^{q/r} \\ &\leq \Big(\sum_{\lambda \in \Lambda_j} (\sigma_j 2^{-jd/r})^q \sum_{j' \geq j} \Big(\sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} (2^{(j-j')d/p} |c_{\lambda'}|)^p \Big)^{r/p} \Big)^{q/r}, \qquad (3.18)\end{aligned}$$

for any $j \in \mathbb{N}$.

Let us first suppose that $r \leq p$; in this case, (3.18) is bounded by

$$\left(\sum_{j' \geq j} (\sigma_j \sigma_{j'}^{-1} 2^{(j-j')d/p} 2^{(j'-j)d/r})^r \sum_{\lambda' \in \Lambda_{j'}} (\sigma_{j'} 2^{-j'd/r} |c_{\lambda'}|)^r \right)^{q/r}.$$

Let $\varepsilon > 0$ be such that $\underline{s}(\sigma) - \varepsilon - d/r > -d/p$; there exists a constant $C_\varepsilon > 0$ such that

$$\sigma_j \sigma_{j'}^{-1} < C_\varepsilon 2^{(\underline{s}(\sigma)-\varepsilon)(j-j')}.$$

If $q \leq r$, (3.18) is bounded by

$$C\left(\sum_{j' \geq j} (2^{(\underline{s}(\sigma)+\varepsilon-d/r-d/r)(j-j')})^q \Big(\sum_{\lambda' \in \Lambda_{j'}} (\sigma_{j'} 2^{-j'd/r} |c_{\lambda'}|)^r\Big)^{q/r}\right).$$

As f belongs to $B^\sigma_{r,q}$, we can write

$$\Big(\sum_{j \in \mathbb{N}} \Big(\sum_{\lambda \in \Lambda_j} (\sigma_j 2^{-jd/r} d^p_\lambda)^r\Big)^{q/r}\Big)^{1/q} \leq C\Big(\sum_{j' \in \mathbb{N}} \Big(\sum_{\lambda' \in \Lambda_{j'}} (\sigma_{j'} 2^{-j'd/r} |c_{\lambda'}|)^r\Big)^{q/r}\Big)^{1/q},$$

which implies $f \in \mathcal{O}^\sigma_{p,r,q}$. If $r < q$, by denoting s the conjugate exponent of q/r, we can use Hölder's inequality to bound (3.18) by

$$\begin{aligned}
&C\Big(\sum_{j' \geq j} (2^{-\underline{s}(\sigma)+\varepsilon-d/p-d/r)(j'-j)})^{rs/2}\Big)^{q/(rs)} \\
&\quad \left(\sum_{j' \geq j} (2^{-\underline{s}(\sigma)+\varepsilon-d/p-d/r)(j'-j)})^{q/(2r)} \Big(\sum_{\lambda' \in \Lambda_{j'}} (\sigma_{j'} 2^{-j'd/r} |c_{\lambda'}|)^r\Big)^{q/r}\right) \\
&\leq C\left(\sum_{j' \geq j} (2^{-\underline{s}(\sigma)+\varepsilon-d/p-d/r)(j'-j)})^{q/(2r)} \Big(\sum_{\lambda' \in \Lambda_{j'}} (\sigma_{j'} 2^{-j'd/r} |c_{\lambda'}|)^r\Big)^{q/r}\right),
\end{aligned}$$

so that f belongs to $\mathcal{O}^\sigma_{p,r,q}$, as in the previous case.

We still have to consider the case $p < r$; by Jensen's inequality, we can bound (3.18) by

$$\begin{aligned}
&\left(\sum_{\lambda \in \Lambda_j} (\sigma_j 2^{-jd/r})^r \sum_{j \geq j'} \sum_{\lambda' \in \Lambda_{j'}, \lambda' \subset \lambda} 2^{(j-j')d} |c_{\lambda'}|^r\right)^{q/r} \\
&\leq \left(\sum_{j' \geq j} (\sigma_j/\sigma_{j'})^r \sum_{\lambda' \in \Lambda_{j'}} (\sigma_{j'} 2^{-j'd/r} |c_{\lambda'}|)^r\right)^{q/r},
\end{aligned}$$

so that we can conclude as in the other cases. □

We propose the following formula to estimate the spectrum $D_{p,q}$ related to a function $f \in B^\sigma_{r,s}$:

$$D_{p,q}(h) = d + r\overline{s}\Big(\frac{\gamma^{(h)}}{\sigma}\Big)$$

and we show that, under natural smooth conditions, this equality is satisfied almost everywhere from a prevalence point of view.

Definition 3.4.8. An admissible sequence $\boldsymbol{\sigma}$ and a family of admissible sequences $\boldsymbol{\gamma}^{(\cdot)}$ are compatible for $p,q,r,s \in [1,\infty]$ with $s \leq q$ if

- $\underline{s}(\boldsymbol{\sigma}) > 0$,
- $\underline{s}(\boldsymbol{\sigma}) - d/r > -d/p$,
- the function ζ defined on $(-d/p,\infty)$ by
$$\zeta(h) := \underline{s}(\frac{\boldsymbol{\gamma}^{(h)}}{\boldsymbol{\sigma}}) = \overline{s}(\frac{\boldsymbol{\gamma}^{(h)}}{\boldsymbol{\sigma}})$$
is non decreasing, continuous and such that
$$\{h > -d/p : \zeta(h) < -d/r\} \neq \emptyset.$$

We call ζ the ratio function. We will also frequently use the quantity
$$h_{\min}(r) := \sup\{h > -d/p : \zeta(h) < -d/r\}.$$

The following remark stresses the importance of $h_{\min}$.

Remark 3.4.9. Suppose that $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}^{(\cdot)}$ are compatible as in the previous definition. If f belongs to $B^{\boldsymbol{\sigma}}_{p,q}$, there exists $\eta > 0$ such that $B^{\boldsymbol{\sigma}}_{p,q} \hookrightarrow B^{\eta}_{p,q}$. For $\lambda \in \Lambda_j$ and $j' \geq j$, we have
$$\begin{aligned}\left(\sum_{\lambda'\in\Lambda_{j'},\lambda'\subset\lambda} (2^{(j-j')d/p}|c_{\lambda'}|)^p\right)^{1/p} &\leq 2^{jd/p}\left(\sum_{\lambda'\in\Lambda_{j'}} (\sigma_{j'}2^{-j'd/p}|c_{\lambda'}|)^p\right)^{1/p} \sigma_{j'}^{-1}\\ &\leq 2^{jd/p}\varepsilon_{j'}\sigma_{j'}^{-1},\end{aligned}$$
for a sequence $\boldsymbol{\varepsilon} \in \ell^q$. As a consequence, there exist $\eta > 0$ and a sequence $\boldsymbol{\xi} \in \ell^q$ given by Lemma 1.2.2 such that, for $\lambda \in \Lambda_j$,
$$d_\lambda^p \leq C\sum_{j'\geq j} 2^{jd/p}\varepsilon_{j'}\sigma_{j'}^{-1} \leq C\xi_j 2^{-\eta j}/\gamma_j^{(h)},$$
for all $h > -d/p$ such that $\overline{s}(\boldsymbol{\gamma}^{(h)}/\boldsymbol{\sigma}) < -d/p$. Therefore, one has $h_{p,q}(x_0) \geq h_{\min}(p)$, for any $x_0 \in \mathbb{R}^d$.

In the same spirit, for $r \leq p$, one has $B^{\boldsymbol{\sigma}}_{r,q} \hookrightarrow B^{\boldsymbol{\theta}}_{p,q}$, where $\boldsymbol{\theta}$ is the admissible sequence defined by $\theta_j := 2^{(d/p-d/r)j}\sigma_j$ ($j \in \mathbb{N}$). As $\underline{s}(\boldsymbol{\sigma}) - d/r > -d/p$ implies $\underline{s}(\boldsymbol{\theta}) > 0$, there exists $\eta > 0$ such that $B^{\boldsymbol{\sigma}}_{r,q} \hookrightarrow B^{\eta}_{p,q}$ and $h_{p,q}(x_0) \geq h_{\min}(r)$, for any $x_0 \in \mathbb{R}^d$.

That being done, if $p < r$ then, for any $f \in B^{\boldsymbol{\sigma}}_{r,q}$,
$$h_{p,q}(x_0) \geq h_{r,q}(x_0) \geq h_{\min}(r).$$
Thus, if $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have $f \in B^{\boldsymbol{\sigma}}_{r,q}$ and $h_{p,q}(x_0) \geq h_{\min}(r)$.

From what we have done so far, we get the following corollary.

Corollary 3.4.10. *Let $p,q,r,s \in [1,\infty]$, $\boldsymbol{\sigma}$ be an admissible sequence and $\boldsymbol{\gamma}^{(\cdot)}$ be a family of admissible sequences such that $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}^{(\cdot)}$ are compatible. If f belongs to $B^{\sigma}_{r,s}$, then*

- $\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) \leq h\} = \emptyset$ *for any $h < h_{\min}(r)$,*
- $\dim_{\mathcal{H}}(\{x_0 \in \mathbb{R}^d : h_{p,q}(x_0) \leq h\}) \leq d + r\zeta(h)$ *for any $h \geq h_{\min}(r)$.*

To show that, under some general hypothesis, the last upper bound is optimal for a prevalent set of functions in $B^{\sigma}_{r,s}$, we need the following definition.

Definition 3.4.11. Let $x_0 \in \mathbb{R}^d$ and $r > 0$; the strict cone of influence above x_0 of width r is

$$\mathcal{C}_{x_0}(r) := \{(j,k) \in \mathbb{N} \times \mathbb{Z}^d : \|\frac{k}{2^j} - x_0\|_\infty < \frac{r}{2^j}\},$$

where $\|x-y\|_\infty$ is the Chebyshev distance between x and y ($x,y \in \mathbb{R}^d$):

$$\|x-y\|_\infty := \max_{1 \leq n \leq d} |x_n - y_n|.$$

This definition is related to the wavelets as follows: in this context, we set

$$\mathcal{K}_{x_0}(r) := \{\lambda^{(i)}_{j,k} \in \Lambda : (j,k) \in \mathcal{C}_{x_0}(r)\}.$$

The following result explains why $\mathcal{K}_{x_0}$ can be seen as a cone of influence for the wavelets.

Proposition 3.4.12. *If f belongs to $T^{\sigma}_{p,q}(x_0)$, then*

$$(\sigma_j \sum_{\lambda \in \Lambda_j \cap \mathcal{K}_{x_0}(r)} |c_\lambda|^p)^{1/p})_j \in \ell^q.$$

Proof. Choose $j_1 \in \mathbb{N}$ such that $\sqrt{d}r + 2^{j_0} \leq 2^{j_1}$; for $j \geq j_1$, if $\lambda \in \Lambda_j$ also belongs to $\mathcal{K}_{x_0}(r)$, then the support of ψ_λ is included in $B(x_0, 2^{j_1-j})$. From the proof of Theorem 3.3.4, we know that there exists a sequence $\varepsilon \in \ell^q$ such that

$$\sigma_j (\sum_{\lambda \in \Lambda_j \cap \mathcal{K}_{x_0}(r)} |c_\lambda|^p)^{1/p} \leq \varepsilon_j,$$

for any $j \geq j_1$. The conclusion then comes from the Archimedean property of the real line. □

Given a dyadic cube $\lambda = \lambda^{(i)}_{j,k}$, let us denote by $k(\lambda)$ and $j(\lambda)$ the numbers such that $k(\lambda)/2^{j(\lambda)}$ is the dyadic irreducible form of $k/2^j$. For $\alpha \in [1,\infty]$, let us set

$$h_*(\alpha) := \zeta^{-1}(\frac{d}{\alpha r} - \frac{d}{r}).$$

We have $h_*(\alpha) \geq h_{\min}(r) = h_*(\infty)$. If $\zeta(h) > d/\alpha r - d/r$, choose $\varepsilon_0 > 0$ such that $\zeta(h) - \varepsilon_0 > d/\alpha r - d/r$ and let $m_0 \in \mathbb{N}$ be such that

$$d - (\frac{d}{\alpha r} - \frac{d}{r} - \zeta(h) + \varepsilon_0)2^{dm_0}\alpha < 0. \tag{3.19}$$

Let us split each cube $\lambda \in \Lambda_j$ into 2^{dm_0} cubes at the scale $j + m_0$ and for each $n \in \{1,\ldots,2^{dm_0}\}$, choose the unique subcube $\lambda^{(n)}$ of λ such that $n \neq n'$ implies $\lambda^{(n)} \neq \lambda^{(n')}$. From this, we can consider a function $g^{(n)}$ such that its wavelet coefficients c_λ satisfy the following conditions:

$$c_{\lambda^{(n)}} := j^{-a_0} 2^{jd/r} 2^{-j(\lambda)d/r} \sigma_j^{-1} \qquad \text{if } \lambda \in \Lambda_j \cap [0,1]^d,$$

with $a_0 := 1 + 1/r + 1/s$ and $c_\lambda := 0$ if λ is not of the form $\lambda^{(n)}$ for some n.

Proposition 3.4.13. *For all $n \in \{1,\ldots,2^{dm_0}\}$, $g^{(n)}$ belongs to $B^\sigma_{r,s}$.*

Proof. For $j \geq 1$, we have

$$\begin{aligned}
&(\sum_{\lambda\in\Lambda_{j+m_0}} (\sigma_{j+m_0} 2^{-(j+m_0)d/r}|c_\lambda|)^r)^{1/r} \\
&= (\sum_{l=0}^{j} \sum_{\substack{\lambda\in\Lambda_j\cap[0,1]^d \\ j(\lambda)=l}} (\sigma_{j+m_0} 2^{-(j+m_0)d/r} j^{-a_0} 2^{jd/r} 2^{-ld/r} \sigma_j^{-1})^r)^{1/r}
\end{aligned}$$

and

$$\begin{aligned}
(\sum_{\lambda\in\Lambda_{j+m_0}} (\sigma_{j+m_0} 2^{-(j+m_0)d/r}|c_\lambda|)^r)^{1/r} &\leq (\sum_{l=0}^{j} (\overline{\sigma}_{m_0} 2^{-(j+m_0)d/r} j^{-a_0})^r)^{1/r} \\
&\leq C j^{-a_0+1/r}.
\end{aligned}$$

As $a_0 > 1/r + 1/s$, we get

$$(\sum_{j\geq 1}(\sum_{\lambda\in\Lambda_{j+m_0}} (\sigma_{j+m_0} 2^{-(j+m_0)d/r}|c_\lambda|)^r)^{s/r})^{1/s} \leq C(\sum_{j\geq 1} j^{-s(a_0-1/r)})^{1/s} < \infty,$$

which is sufficient to conclude. □

Definition 3.4.14. Let $\alpha \geq 1$; a point $x_0 \in [0,1]^d$ is α-approximable by dyadics if there exist two sequences $\boldsymbol{k}$ and $\boldsymbol{j}$ of natural numbers with $k_n < 2^{j_n}$ for any $n \in \mathbb{N}$ such that

$$\|x_0 - \frac{k_n}{j_n}\|_\infty \leq \frac{1}{2^{\alpha j_n}},$$

for any $n \in \mathbb{N}$.

Let us denote the set of points of $[0,1]^d$ which are α-approximable by dyadics by E^α and define

$$E_j^\alpha := \{x_0 \in [0,1]^d : \exists k \in \{0,\ldots,2^j-1\}^d \text{ such that } \|x_0 - \frac{k}{2^j}\|_\infty \leq \frac{1}{2^{\alpha j}}\},$$

so that $E^\alpha = \limsup_j E_j^\alpha$. We also define

$$E_{j,k}^\alpha := \{x_0 \in [0,1]^d : \|x_0 - \frac{k}{2^j}\|_\infty \leq \frac{1}{2^{\alpha j}}\},$$

for $k \in \{0,\ldots,2^j-1\}^d$, in order to have

$$E_j^\alpha = \bigcup_{l \in \{0,\ldots,2^j-1\}^d} E_{j,k}^\alpha.$$

Finally, set $E^\infty = \cap_{\alpha \geq 1} E^\alpha$; this set is non-empty since it contains the dyadic numbers.

Proposition 3.4.15. *Given $C > 0$, $j \in \mathbb{N}$ and $k \in \{0,\ldots,2^j-1\}^d$, the set*

$$F_{j,k}^{\alpha,C}(h) := \{f \in B_{r,s}^\sigma : (\exists x \in E_{j,k}^\alpha : \forall n \in \mathbb{N} \forall \lambda \in \Lambda_n \cap \mathcal{K}_x(2^{m_0+1}), |c_\lambda| \leq C/\gamma_n^{(h)})\}$$

is closed in $B_{r,s}^\sigma$.

Proof. Let $(f_l)_l$ be a sequence of functions of $F_{j,k}^{\alpha,C}$ such that $f_l \to f$ in $B_{r,s}^\sigma$ and denote by $c_\lambda^{(l)}$ (resp. c_λ) the wavelet coefficients of f_l (resp. f). Since

$$B_{r,s}^{\overline{s}(\sigma)+\gamma} \hookrightarrow B_{r,s}^\sigma \hookrightarrow B_{r,s}^{\underline{s}(\sigma)-\gamma},$$

for any $\gamma > 0$ and as the application that associates to a function its wavelet coefficients is continuous on the Besov spaces, we have $c_\lambda^{(l)} \to c_\lambda$ for all $\lambda \in \Lambda$.

For $l \in \mathbb{N}$, let $x_l \in E_{j,k}^\alpha$ be such that, for all $n \in \mathbb{N}$ and $\lambda \in \Lambda_n \cap \mathcal{K}_{x_l}(2^{m_0+1})$, we have $|c_\lambda^{(l)}| \leq C/\gamma_n^{(h)}$. As $E_{j,k}^\alpha$ is compact, we can suppose that the sequence $(x_j)_l$ converges to a point x_0 of $E_{j,k}^\alpha$. Now, let us fix $N \in \mathbb{N}$ and $\delta > 0$; if l is sufficiently large, we have $\mathcal{K}_{x_0}(2^{m_0+1}) \subset \mathcal{K}_{x_l}(2^{m_0+1})$ and, for $n \leq N$, we have, for $\lambda \in \Lambda_n \cap \mathcal{K}_{x_l}(2^{m_0+1})$, $|c_\lambda^{(l)} - c_\lambda| \leq \delta/\gamma_n^{(h)}$ as $c_\lambda^{(l)}$ converges to c_λ. Also, we have $|c_\lambda^{(l)}| \leq C/\gamma_n^{(h)}$ for $\lambda \in \Lambda_n \cap \mathcal{K}_{x_l}(2^{m_0+1})$. As a consequence, $\lambda \in \Lambda_n \cap \mathcal{K}_{x_0}(2^{m_0+1})$ implies

$$|c_\lambda| \leq (C+\delta)/\gamma_n^{(h)},$$

for all $n \leq N$. Taking the limit for $N \to \infty$ and $\delta \to 0^+$ leads to $f \in F_{j,k}^{\alpha,C}(h)$. □

Let us set

$$F_j^{\alpha,C}(h) := \bigcup_{k \in \{0,\ldots,2^j-1\}^d} F_{j,k}^{\alpha,C}(h)$$

and $F^{\alpha,C}(h) := \limsup_j F_j^{\alpha,C}(h)$. All these sets are obviously Borel sets.

Proposition 3.4.16. *The set $F^{\alpha,C}(h)$ is a Haar-null Borel set.*

Proof. Set $m_1 := 2^{m_0 d}$ and let us fix $j \in \mathbb{N}$ and $k \in \{0,\dots,2^j-1\}$; for $f \in B^{\sigma}_{r,s}$, suppose that there exist two points of $\mathbb{R}^{m_1}$, $a^{(1)} = (a^{(1)}_1,\dots,a^{(1)}_{m_1})$ and $a^{(2)} = (a^{(2)}_1,\dots,a^{(2)}_{m_1})$, such that

$$f_l := f + \sum_{m=1}^{m_1} a^{(l)}_m g^{(m)}$$

belongs to $F^{\alpha,C}_{j,k}(h)$ ($l \in \{1,2\}$). For $l \in \{1,2\}$, let us also denote by $c^{(l)}_\lambda$ the wavelet coefficient of f_l associated to the dyadic cube $\lambda \in \Lambda$ and let x_l be a point of $E^{\alpha}_{j,k}$ such that $\lambda \in \Lambda_{\lfloor \alpha j \rfloor} \cap \mathcal{K}_{x_l}(2^{m_0+1})$ implies $|c^{(l)}_\lambda| \leq C/\gamma^{(h)}_{\lfloor \alpha j \rfloor}$.

For $\lambda' \in \Lambda_{\lfloor \alpha j \rfloor + m_0}$ satisfying $\lambda' \subset \lambda^{(i)}_{\lfloor \alpha j \rfloor, k}$, we have

$$|c^{(l)}_{\lambda'}| \leq C/\gamma^{(h)}_{\lfloor \alpha j \rfloor + m_0}.$$

As a consequence, we get, by denoting $c'^{(m)}_\lambda$ the wavelet coefficient of $g^{(m)}$ associated to λ,

$$|a^{(1)}_m - a^{(2)}_m| = |a^{(1)}_m - a^{(2)}_m||c'^{(m)}_{\lambda^{(m)}}|/|c'^{(m)}_{\lambda^{(m)}}| \leq 2C/(\gamma^{(h)}_{\lfloor \alpha j \rfloor + m_0}|c'^{(m)}_{\lambda^{(m)}}|),$$

for any $m \in \{1,\dots,m_1\}$. On the other hand, for $j \geq j(\lambda)$, we have

$$\begin{aligned} |c'^{(m)}_{\lambda^{(n)}}| &= \lfloor \alpha j \rfloor^{-a_0} 2^{\lfloor \alpha j \rfloor d/q} 2^{-j(\lambda) d/q} \sigma^{-1}_{\lfloor \alpha j \rfloor} \\ &\geq C' \lfloor \alpha j \rfloor^{-a_0} 2^{\lfloor \alpha j \rfloor d/q} 2^{-\lfloor \alpha j \rfloor d/\alpha q} \sigma^{-1}_{\lfloor \alpha j \rfloor}, \end{aligned}$$

so that there exists a constant $C'' > 0$ for which

$$\|a^{(1)} - a^{(2)}\|_\infty \leq C'' \lfloor \alpha j \rfloor^{-a_0} 2^{\lfloor \alpha j \rfloor (d/\alpha q - d/q)} \sigma_{\lfloor \alpha j \rfloor}/\gamma^{(h)}_{\lfloor \alpha j \rfloor}. \tag{3.20}$$

That being done, for $f \in B^{\sigma}_{r,s}$, we have

$$\begin{aligned} \{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}(h)\} &\subset \bigcup_{j \geq J} \{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}_j(h)\} \\ &\subset \bigcup_{j \geq J} \bigcup_{k \in \{0,\dots,2^j-1\}^d} \{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}_{j,k}(h)\}, \end{aligned}$$

for any $J \in \mathbb{N}$. Thus, from (3.20), we get

$$\begin{aligned} &\mathcal{L}(\{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}(h)\}) \\ &\leq \sum_{j \geq J} 2^{jd} (C'' \lfloor \alpha j \rfloor^{a_0} 2^{\lfloor \alpha j \rfloor (d/\alpha q - d/q)} \sigma_{\lfloor \alpha j \rfloor}/\gamma^{(h)}_{\lfloor \alpha j \rfloor})^M \\ &\leq C''' \sum_{j \geq J} \lfloor \alpha j \rfloor^{a_0 m_1} 2^{j(d - m_1 \alpha(\zeta(h) - d/\alpha q - d/q - \varepsilon_0))}. \end{aligned}$$

Letting J going to ∞, (3.19) implies

$$\mathcal{L}(\{a \in \mathbb{R}^{m_1} : f + ag \in F^{\alpha,C}(h)\}) = 0,$$

hence the conclusion. □

Theorem 3.4.17. *Let $p,q,r,s \in [1,\infty]$ with $s \leq q$, $\boldsymbol{\sigma}$ be an admissible sequence and $\boldsymbol{\gamma}^{(\cdot)}$ be a family of admissible sequences compatible with $\boldsymbol{\sigma}$. From the prevalence point of view, for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, $D_{p,q}$ is defined on $I = [\zeta^{-1}(-d/r), \zeta^{-1}(0)]$ and*

$$D_{p,q}(h) = d + r\zeta(h),$$

for any $h \in I$.

Moreover, for almost every $x_0 \in \mathbb{R}^d$, we have $h_{p,q}(x_0) = \zeta^{-1}(0)$.

Proof. We know that

$$\{f \in B^{\boldsymbol{\sigma}}_{r,s} : \exists x_0 \in E^{\alpha} : f \in T^{\boldsymbol{\sigma}^{(h)}}_{p,q}(x_0)\} \subset \bigcup_{l \in \mathbb{N}} F^{\alpha,l}(h),$$

so that, for any $\alpha \geq 1$ and any $h > h_*(\alpha)$, for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have $h_{p,q}(x_0) \leq h$ for every $x_0 \in E^{\alpha}$. By countable intersection, we thus get that for almost every $f \in B^{\boldsymbol{\sigma}}_{r,s}$, we have $h_{p,q}(x_0) \leq h(\alpha)$ for every $x_0 \in E^{\alpha}$. Let $f \in B^{\boldsymbol{\sigma}}_{r,s}$ be such that the preceding assertion holds.

First, let us fix $\alpha \in (1,\infty)$; if $\boldsymbol{\alpha}$ is an increasing sequence of rational numbers converging to α, the sequence $(E^{\alpha_n})_n$ is decreasing and $E^{\alpha} \subset \cup_n E^{\alpha_n}$. If x_0 belongs to E^{α_n}, we have $h_{p,q}(x_0) \leq h_*(\alpha_n)$ and thus $h_{p,q}(x_0) \leq h_*(\alpha)$, for every $x_0 \in E^{\alpha}$. Let μ_α be a measure such that

- $\mathrm{supp}(\mu_\alpha) \subset E^{\alpha}$,
- $\mu_\alpha(E^{\alpha}) > 0$,
- $\mu_\alpha(F) = 0$ whenever $\dim_{\mathcal{H}}(F) < d/\alpha$;

let us define

$$F^{\alpha} := \{x_0 \in [0,1]^d : h_{p,q}(x_0) < h_*(\alpha)\}$$

and, for $n \in \mathbb{N}$,

$$F^{\alpha}_n := \{x_0 \in [0,1]^d : h_{p,q}(x_0) < h_*(\alpha) - 1/n\}.$$

For n large enough, we have $h(\alpha) - 1/n \geq -d/p$ and thus $\dim_{\mathcal{H}}(F^{\alpha}_n) < d/\alpha$. Since F^{α} is included in a countable union of μ_α-measurable null sets, we have $\mu_\alpha(F^{\alpha}) = 0$. As a consequence, we have

$$\mu_\alpha(E^{\alpha} \setminus F^{\alpha}) \geq d + r\zeta(h_*(\alpha)).$$

Since

$$E^{\alpha} \setminus F^{\alpha} \subset \{x_0 \in [0,1]^d : h_{p,q}(x_0) = h_*(\alpha)\},$$

we get

$$D(h_*(\alpha)) = d + r\zeta(h_*(\alpha)).$$

If $\alpha = \infty$, we know that $x_0 \in E^\infty$ implies $h_{p,q}(x_0) \leq h_*(\alpha_n)$ for any $n \in \mathbb{N}$ and thus $h_{p,q}(x_0) \leq h_{\min}(r)$. As a consequence, the set

$$\{x_0 \in [0,1]^d : h_{p,q}(x_0) = h_{\min}(r)\}$$

is non-empty.

It remains to consider the case $\alpha = 1$. In this case, $E^1 = [0,1]^d$ and μ_1 can be chosen to be the Lebesgue measure restricted on $[0,1]^d$. For $x_0 \in E^1$, $h_{p,q}(x_0) \leq h_*(1)$ and by the same argument as in the first case, we get

$$\mu_1(\{x_0 \in [0,1]^d : h_{p,q}(x_0) < h_*(1)\}) = 0,$$

so that E^1 is equal to $E^1 \setminus F^1$ almost everywhere.

As the proof can be easily adapted to any translated of $[0,1]^d$, the conclusion follows by countable intersection. □

The next theorem shows that, as usual, there is no Fubini-like theorem in the theory of prevalence.

Theorem 3.4.18. *Let $p,q,r,s \in [1,\infty]$ with $s \leq q$, $\boldsymbol{\sigma}$ be an admissible sequence and $\boldsymbol{\gamma}^{(\cdot)}$ be a family of admissible sequences compatible with $\boldsymbol{\sigma}$. Let x_0 be a point of $\mathbb{R}^d$; from the prevalence point of view, for almost every $f \in B^{\sigma}_{r,s}$, we have $h_{p,q}(x_0) = \zeta^{-1}(-d/r)$.*

Proof. Given $n \in \mathbb{N}$, let us define the admissible sequence $\boldsymbol{\theta}^{(n)}$ by

$$\theta_j^{(n)} := \frac{1}{\gamma_j^{(\zeta^{-1}(-d/r)+1/n)}} \frac{1}{(j+1)^{1+1/s}},$$

$j \in \mathbb{N}$. We can now define the function $g^{(n)}$ which is a function whose wavelet coefficients are

$$c_\lambda^{(n)} := \begin{cases} \theta_j^{(n)} & \text{if } \lambda \in \Lambda_j \text{ and } \lambda = \lambda_j(x_0) \\ 0 & \text{if } \lambda \in \Lambda_j \text{ and } \lambda \neq \lambda_j(x_0). \end{cases}$$

Since, for $n \in \mathbb{N}$, there exists $C_n > 0$ such that

$$\Big(\sum_{\lambda \in \Lambda_j} (\sigma_j 2^{jd/r} |c_\lambda^{(n)}|)^r\Big)^{1/r} \leq C_n/(j+1)^{1+1/s},$$

$g^{(n)}$ belongs to $B^{\sigma}_{r,s}$.

Let us fix $n_0 \in \mathbb{N}$ and define

$$F_{n_0} := \{f \in B^{\sigma}_{r,s} : \forall j \in \mathbb{N} \forall \lambda \in \Lambda_j \cap \mathcal{K}_{x_0}(2), |c_\lambda| \leq n_0 \theta_j^{(n)}/j\}.$$

As shown before, F_{n_0} is a Borel set. For $f \in B^{\sigma}_{r,s}$ and $a, a' \in \mathbb{R}$ satisfying $f + ag^{(n)} \in F_{n_0}$ and $f + a'g^{(n)} \in F_{n_0}$, we get

$$|a - a'| \leq 2n_0/j,$$

so that the Lebesgue measure of $\{a \in \mathbb{R} : f + ag^{(n)} \in F_{n_0}\}$ vanishes, implying that F_{n_0} is Haar-null. As we have

$$\{f \in B^{\sigma}_{r,s} : f \in T^{\boldsymbol{\theta}^{(n)}}_{p,q}(x_0)\} \subset \bigcup_{l \in \mathbb{N}} F_l,$$

for almost every $f \in B^{\sigma}_{r,s}$, we have $h_{p,q}(x_0) \leq \zeta^{-1}(-d/r) + 1/n$, which leads to the conclusion. □

4 Complements with compactly supported wavelets

As mentioned before, the definition of the p-wavelet leaders given in this thesis slightly differs from the one of [75, 94]. This choice has been already justified in the previous chapter by the possibility to obtain a quasi-characterization of the pointwise functional spaces we consider. In this chapter, we show that, using compactly supported wavelets, "our" leaders are relevant in different contexts.

First we consider irregularity spaces which is the counterpart of the pointwise Hölderian regularity, in the spirit of [122, 29, 30]. This gives a complementary information about the pointwise behaviour of a function, estimating its oscillation by below. We give here a quasi-characterization of the irregularity, by the mean of p-wavelet leaders. This generalizes and improves results of [29].

Secondly, we discuss about the logarithm correction which appears in Theorem 3.3.6. We show that, from the prevalence point of view, this correction is necessary for almost every function satisfying the conditions of this last theorem. In some sense, this shows that Theorem 3.3.6 is optimal.

In the third section, we give examples of functions displaying a precise given pointwise regularity and, using the tools presented in the two first sections, we discuss the relevance of pointwise spaces of generalized smoothness.

Finally, we open some perspectives for applications based on the theoretical framework established in this thesis.

In this chapter, we mainly focus on the case where the exponent q in Definition 3.1.1 is equal to ∞. In order to simplify the notation, we will write $T_p^\sigma(x_0)$ instead of $T_{p,\infty}^\sigma(x_0)$

Contents

4.1 Irregularity spaces

Definition 4.1.1. Let $p \in [1,\infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -\frac{d}{p}$, $f \in L^p_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}^d$; *f belongs to $I^{\boldsymbol{\sigma}}_p(x_0)$* if there exist $C > 0$, $J \in \mathbb{N}$ such that

$$2^{j\frac{d}{p}} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))} \geq C\sigma_j^{-1} \quad \forall j \geq J.$$

Note that the previous definition is not a contradiction of Definition 3.1.1 as the inequality is assumed to hold for all values of j (sufficiently large), with an uniform constant C. A particularly interesting situation is when a function belongs to both the irregularity and the regularity space associated to the same admissible sequence.

Definition 4.1.2. Let $p \in [1,\infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -\frac{d}{p}$, $f \in L^p_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}^d$; *f belongs to $\overline{T}^{\boldsymbol{\sigma}}_p(x_0)$* if f belongs to $T^{\boldsymbol{\sigma}}_p(x_0) \cap I^{\boldsymbol{\sigma}}_p(x_0)$.

Interests and applications of such spaces, with dyadic sequences and $p = \infty$, have already been discussed in [28, 29, 30]. Here, we will characterize them with the help of p-wavelet leaders. On this purpose, we introduce the following space, whose definition is obtained by contradicting the $I^{\boldsymbol{\sigma}}_p(x_0)$ condition.

Definition 4.1.3. Let $p \in [1,\infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -\frac{d}{p}$, $f \in L^p_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}^d$; *f belongs to $T^{\boldsymbol{\sigma}_w}_p(x_0)$* if for all $C > 0$ there exists a sequence $(k(j))_j$ with $k(j) \to +\infty$ such that

$$2^{k(j)\frac{d}{p}} \sup_{|h|\leq 2^{-k(j)}} \|\Delta_h^{\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor+1} f\|_{L^p(B_h(x_0,2^{-k(j)}))} \leq C\sigma_{k(j)}^{-1} \quad \forall n \in \mathbb{N}.$$

Similarly to Proposition 3.1.2, we can show that polynomials do characterize the belonging to $T^{\boldsymbol{\sigma}_w}_p(x_0)$.

Proposition 4.1.4. *Let $p \in [1,\infty]$, $f \in L^p_{loc}$, $x_0 \in \mathbb{R}^d$ and $\boldsymbol{\sigma}$ be an admissible sequence such that $\overline{s}(\boldsymbol{\sigma}) > 0$. We have $f \in T^{\boldsymbol{\sigma}_w}_p(x_0)$ if and only if for all $C > 0$ there exist a sequence $(k(j))_j$ with $k(j) \to +\infty$ and a sequence of polynomials $(P_{k(j),x_0})_j$ of degree less than or equal to $\lfloor \overline{s}(\boldsymbol{\sigma})\rfloor$ such that*

$$2^{k(j)d/p}\|f - P_{k(j),x_0}\|_{L^p(B(x_0,2^{-k(j)}))} \leq C\sigma_{k(j)}^{-1}. \tag{4.1}$$

Proposition 4.1.5. *Let $p \in [1,\infty]$, $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence such that $\underline{s}(\boldsymbol{\sigma}) > -\frac{d}{p}$, $f \in L^p_{loc}$ and $x_0 \in \mathbb{R}^d$. If there exist $J \in \mathbb{N}$ and $C > 0$ such that for all $j \geq J$,*

$$d_j^p(x_0) \geq C\sigma_j^{-1}, \tag{4.2}$$

then $f \in I^{\boldsymbol{\sigma}}_p(x_0)$.

Proof. We keep the notation used in the proof of Theorem 3.3.4. Let $d_0 > 0$ be such that, for all $j \in \mathbb{N}$, $d_0\sigma_j^{-1} \le \sigma_{j+1}^{-1}$ and let us set

$$C_p^* := \begin{cases} \dfrac{Cd_0^{j_1}}{2C_* \max_{1\le i<2^d} \|\psi^{(i)}\|_q 2^{dj_1}} & \text{if } p \in [1,\infty) \\ \dfrac{Cd_0^{j_1}}{2\max_{1\le i<2^d} \|\psi^{(i)}\|_1} & \text{if } p = \infty. \end{cases}$$

If $f \in T_p^{\sigma_w}(x_0)$, there exist a sequence $(k(j))_j$ with $k(j) \to +\infty$ and a sequence of polynomials $(P_{k(j),x_0})_j$ of degree less than or equal to $\lfloor \overline{s}(\sigma) \rfloor$ such that, for all j,

$$2^{k(j)d/p}\|f - P_{k(j),x_0}\|_{L^p(B(x_0,2^{-k(j)}))} \le C_p^* \sigma_{k(j)}^{-1}.$$

Following the steps of the proof of Theorem 3.3.4, one can see that, for all j,

$$d_{k(j)+j_1}^p \le \frac{C}{2}\sigma_{k(j)+j_1}^{-1},$$

which is in contradiction with inequality (4.2). □

Theorem 4.1.6. *Let $p \in [1,\infty]$, $\sigma = (\sigma_j)_j$ be an admissible sequence such that $2^{-jd/p}\sigma_j^{-1}$ tends to 0 as j tends to ∞, $f \in L^p_{loc}$ and $x_0 \in \mathbb{R}^d$. If f belongs to $\dot{X}^s_{p,q}(x_0)$ for some $s > 0$, then if $f \in \overline{T}_p^{\sigma}(x_0)$ there exist $C_1, C_2 > 0$ and $J \in \mathbb{N}$ such that*

$$\frac{C_1}{|\log_2(2^{-j\frac{d}{p}}\sigma_j^{-1})|} \le \sigma_j d_j^p(x_0) \le C_2 \quad \forall j \ge J. \tag{4.3}$$

Proof. The inequality $\sigma_j d_j^p(x_0) \le C_2$ coming from the fact that $f \in T_p^{\sigma}(x_0)$ and Theorem 3.3.4, let us prove the other inequality. Let us first assume that $\overline{s}(\sigma) \ge 0$ and set $n := \lfloor \overline{s}(\sigma) \rfloor$. We keep the notations used in the proofs of Theorems 3.3.4 and 3.3.6 and we set

- $C_{B,d} = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$, the volume of the unit ball in $\mathbb{R}^d$,
- $\varepsilon > 0$ such that $\overline{s}(\sigma) + \varepsilon < n+1$ (in particular $1 < 2^{n+1-\overline{s}(\sigma)-\varepsilon}$) and $C_\varepsilon > 0$ such that for all $j \in \mathbb{N}$,

$$C_\varepsilon 2^{j(\underline{s}(\sigma^{-1})-\varepsilon)} \le \underline{\sigma^{-1}}_j, \tag{4.4}$$

- $\xi > 0$ such that $\underline{s}(\sigma) - \xi > -\frac{d}{p}$ and $C_\xi > 0$ such that for all $j \in \mathbb{N}$,

$$\sigma_j^{-1} \le C_\xi 2^{-j(\underline{s}(\sigma)-\frac{\xi}{2})}. \tag{4.5}$$

Without loss of generality, we can assume that s is small enough, so that

$$2(\underline{s}(\sigma) + \frac{d}{p} - \xi) \ge s, \tag{4.6}$$

- $C_\sigma > 0$ such that, for all $J \in \mathbb{N}$,

$$\sum_{j=-1}^{J} 2^{j(n+1)} \sigma_j^{-1} \leq C_\sigma 2^{J(n+1)} \sigma_J^{-1}, \tag{4.7}$$

where we have put $\sigma_{-1} = 1$,

- $d_0, d_1 > 0$ such that

$$d_0 \sigma_j^{-1} \leq \sigma_{j+1}^{-1} \leq d_1 \sigma_j^{-1} \quad \forall j \in \mathbb{N}, \tag{4.8}$$

- $d_0', d_1' > 0$ such that, for all j sufficiently large,

$$d_0' \frac{\sigma_j^{-1}}{|\log_2(2^{-j\frac{d}{p}} \sigma_j^{-1})|} \leq \frac{\sigma_{j+1}^{-1}}{|\log_2(2^{-(j+1)\frac{d}{p}} \sigma_{j+1}^{-1})|} \leq d_1' \frac{\sigma_j^{-1}}{|\log_2(2^{-j\frac{d}{p}} \sigma_j^{-1})|}. \tag{4.9}$$

We know that there exists $C_3 > 0$ such that for all $j \geq -1$, $k \in \mathbb{N}$, if $|h| \leq 2^{-k}$ and if $x \in B_h(x_0, 2^{-k})$,

$$|\Delta_h^{n+1} f_j(x)| \leq C_3 |h|^{n+1} \sup_{\substack{y \in B(x_0, 2^{-m}) \\ |\alpha| = n+1}} |D^\alpha f_j(y)|.$$

Moreover, as $d_j^p(x_0) \leq C_2 \sigma_j$, we know from the proof of Theorem 3.3.6 that there exists $C_4 > 0$ such that for all $k \in \mathbb{N}$, if $j \leq k$,

$$\sup_{\substack{y \in B(x_0, 2^{-k}) \\ |\alpha| = n+1}} |D^\alpha f_j(y)| \leq C_4 2^{j(n+1)} \sigma_j^{-1}.$$

It follows that there exists $C_n > 0$ such that for all $k \in \mathbb{N}$, if $|h| \leq 2^{-k}$ and if $x \in B_h(x_0, 2^{-k})$ and if $-1 \leq j \leq k$,

$$|\Delta_h^{n+1} f_j(x)| \leq C_n 2^{-k(n+1)} 2^{j(n+1)} \sigma_j^{-1}. \tag{4.10}$$

Let $C > 0$, there exists $l \geq \max\{j_0, m'\}$ such that

$$2^{j_0(n+1)} C_{B,d}^{\frac{1}{p}} \frac{1}{C_\varepsilon} d_1^{j_0} C_n C_\sigma \leq (2^{n+1-\overline{s}(\sigma)-\varepsilon})^l \frac{C}{3}. \tag{4.11}$$

If the first inequality of (4.3) is not true, there exists a sequence $(k(r))_r$ with $k(r) \to +\infty$ such that, for all r,

$$d_{k(r)}^p(x_0) \leq C_p^* \frac{\sigma_{k(r)}^{-1}}{|\log_2(2^{-k(r)\frac{d}{p}} \sigma_{k(r)}^{-1})|}, \tag{4.12}$$

where we choose

$$C_p^* := \begin{cases} \dfrac{Cs}{42^{n+1}2^{l\frac{d}{p}}3^{\frac{d}{p}}(d_0')^{-l}3C_* \max_{1\le i<2^d}\|\psi^{(i)}\|_p} & \text{if } p\in[1,\infty) \\ \dfrac{Cs}{42^{n+1}(d_0')^{-l}3C_d \max_{1\le i<2^d}\|\psi^{(i)}\|_\infty} & \text{if } p=\infty. \end{cases}$$

Let us set for all r, $l(r)=k(r)+l$ and $L(r)=\lceil\frac{|\log_2(2^{-l(r)\frac{d}{p}}\sigma_{l(r)}^{-1})|}{s}\rceil$. If $|h|\le 2^{-l(r)}$, we have

$$\begin{aligned} 2^{l(r)\frac{d}{p}}\|\Delta_h^{n+1}f\|_{L^p(B_h(x_0,2^{-l(r)}))} \le & \underbrace{\sum_{j=-1}^{k(r)+j_0} 2^{l(r)\frac{d}{p}}\|\Delta_h^{n+1}f_j\|_{L^p(B_h(x_0,2^{-l(r)}))}}_{(1)} \\ & + \underbrace{\sum_{j=k(r)+j_0+1}^{2L(r)} 2^{l(r)\frac{d}{p}}\|\Delta_h^{n+1}f_j\|_{L^p(B_h(x_0,2^{-l(r)}))}}_{(2)} \\ & + \underbrace{\sum_{j=2L(r)+1}^{+\infty} 2^{l(r)\frac{d}{p}}\|\Delta_h^{n+1}f_j\|_{L^p(B_h(x_0,2^{-l(r)}))}}_{(3)}. \end{aligned}$$

(1) By inequality (4.10), we know that for all $j\in\{-1,\dots,k(r)+j_0\}$ and $x\in B_h(x_0,2^{-l(r)})$,

$$|\Delta^{n+1}f_j(x)|\le C_n 2^{-l(r)(n+1)}2^{j(n+1)}\sigma_j^{-1}$$

and it follows that

$$\begin{aligned} (1) &\le C_{B,d}^{\frac{1}{p}} C_n 2^{-l(r)(n+1)}\sum_{j=-1}^{k(r)+j_0} 2^{j(n+1)}\sigma_j^{-1} \\ &\le C_{B,d}^{\frac{1}{p}} C_n C_\sigma 2^{-l(r)(n+1)}2^{(k(r)+j_0)(n+1)}\sigma_{k(r)+j_0}^{-1} \\ &\le C_{B,d}^{\frac{1}{p}} C_n C_\sigma 2^{j_0(n+1)}d_1^{j_0}2^{-l(n+1)}\sigma_{k(r)}^{-1} \\ &\le C_{B,d}^{\frac{1}{p}} C_n C_\sigma 2^{j_0(n+1)}d_1^{j_0}\frac{1}{C_\varepsilon}(2^{n+1-\overline{s}(\sigma)-\varepsilon})^{-l}\sigma_{l(r)}^{-1}. \end{aligned}$$

Now, using the definition of l (4.11), we find that

$$(1)\le \frac{C}{3}\sigma_{l(r)}^{-1}.$$

(2) We know that $B(x_0,2^{-l(r)})\subset\lambda_{l(r)-m'}(x_0)\subset\lambda_{l(r)-l}(x_0)=\lambda_{k(r)}(x_0)$, but if $j>k(r)+j_0$,

$$\operatorname{supp}\psi^{(i)}(2^j\cdot -k)\subset B(\frac{k}{2^j},2^{j_0-j})\subseteq B(\frac{k}{2^j},2^{-k(r)})$$

and if $\lambda = \lambda_{j,k}^{(i)} \not\subseteq 3\lambda_{k(r)}(x_0)$, $B(\frac{k}{2^j}, 2^{-k(r)}) \cap \lambda_{k(r)}(x_0) = \emptyset$ and so $\psi^{(i)}(2^j x - k) = 0$ for all $x \in B(x_0, 2^{-l(r)})$. Therefore, if $p \neq \infty$, for all $x \in B(x_0, 2^{-l(r)})$, we have

$$|f_j(x)|^p \leq C_*^p \sum_{\substack{\lambda \in \Lambda_j \\ \lambda \subseteq 3\lambda_{k(r)}(x_0)}} |c_\lambda|^p |\psi_\lambda(x)|^p.$$

Now, using inequality (4.12), we get

$$\|f_j\|^p_{L^p(B(x_0,2^{-l(r)}))} \leq (C_p^*)^p C_*^p 3^d \left(\frac{\sigma_{k(r)}^{-1}}{|\log_2(2^{-k(r)\frac{d}{p}} \sigma_{k(r)}^{-1})|} \right)^p 2^{-dk(r)} \max_{1 \leq i < 2^d} \|\psi^{(i)}\|_p.$$

Finally, from the choice of C_p^*, if r is large enough such that $L(r) \leq \frac{|\log_2(2^{-l(r)\frac{d}{p}} \sigma_{l(r)}^{-1})|}{s}$, we have

$$(2) \leq \frac{C}{3} \sigma_{l(r)}^{-1}.$$

If $p = \infty$, for all $x \in B(x_0, 2^{-l(r)})$, we have

$$|f_j(x)| \leq \sum |c_\lambda| |\psi_\lambda(x)|$$

where the sum is taken over all $\lambda = \lambda_{j,k}^{(i)} \in \Lambda_j$ such that $\lambda \subset 3\lambda_{k(r)}(x_0)$ and $|\frac{k}{2^j} - x_0| \leq 2^{j_0 - j}$. Therefore,

$$|f_j(x)| \leq C_d C_\infty^* \frac{\sigma_{k(r)}^{-1}}{|\log_2(2^{-k(r)\frac{d}{p}} \sigma_{k(r)}^{-1})|} \max_{1 \leq i < 2^d} \|\psi^{(i)}\|_\infty$$

and, similarly,

$$(2) \leq \frac{C}{3} \sigma_{l(r)}^{-1}.$$

(3) If r is sufficiently large, we have

$$C_\xi 2^{-l(r)(\underline{s}(\sigma) + \frac{d}{p} - \frac{\xi}{2})} < 1$$

and

$$\begin{aligned} L(r) &\geq \frac{1}{s} |\log_2(C_\xi 2^{-l(r)(\underline{s}(\sigma) + \frac{d}{p} - \frac{\xi}{2})})| \\ &= \frac{l(r)}{s} |-(\underline{s}(\sigma) + \frac{d}{p} - \frac{\xi}{2}) + \frac{\log_2(C_\xi)}{l_r}|. \end{aligned}$$

But, as $\underline{s}(\sigma)+\frac{d}{p}-\frac{\xi}{2}>0$, if r is large enough, we can assume that

$$-(\underline{s}(\sigma)+\frac{d}{p}-\frac{\xi}{2})+\frac{\log_2(C_\xi)}{l(r)}<0 \text{ and } \frac{\log_2(C_\xi)}{l(r)}\leq\frac{\xi}{2}.$$

Therefore,

$$\begin{aligned} L(r) &\geq \frac{l(r)}{s}(\underline{s}(\sigma)+\frac{d}{p}-\frac{\xi}{2}-\frac{\log_2(C_\xi)}{l(r)}) \\ &\geq \frac{l(r)}{s}(\underline{s}(\sigma)+\frac{d}{p}-\xi) \\ &\geq \frac{l(r)}{2}. \end{aligned}$$

Thus, for all $j>2L(r)$, $j\geq l(r)$ and, as $f\in\dot{X}^s_{p,q}(x_0)$, we show, in the same way that in the proof of Theorem 3.3.6, the existence of a constant $C'>0$ such that for all such j,

$$\|f_j\|_{L^p(B(x_0,2^{-l(r)}))}\leq C'2^{-sj}$$

and

$$\begin{aligned} (3) &\leq C'\sum_{j=2L(r)+1}^{+\infty}2^{l(r)\frac{d}{p}}2^{n+1}2^{-sj} \\ &\leq 2^{l(r)\frac{d}{p}}2^{-L(r)s}2^{-L(r)s}C''. \end{aligned}$$

But, if r is large enough, we have $2^{-L(r)s}C''\leq\frac{C}{3}$ and finally

$$(3)\leq\frac{C}{3}\sigma^{-1}_{l(r)}.$$

It follows from these three points that there exists $(l(r))_r$ such that $l(r)\to+\infty$ and, for all r,

$$2^{l(r)\frac{d}{p}}\|\Delta_h^{n+1}f\|_{L^p(B_h(x_0,2^{-l(r)}))}\leq C\sigma^{-1}_{l(r)},$$

which implies that $f\in T_p^{\sigma,w}(x_0)$, as the constant C is arbitrary, hence a contradiction.

The proof for the case $\overline{s}(\sigma)<0$ is similar. □

4.2 Prevalence of the logarithmic correction

In this section, we aim at showing that, from the prevalence point of view, for almost every function in a precise functional space, the logarithmic correction induced by Theorem 3.3.6 is necessary. On this purpose, let us first consider the following lemma which gives a way to define the probe we will use afterwards.

Lemma 4.2.1. *Let $\boldsymbol{\sigma} = (\sigma_j)_j$ be an admissible sequence, $x_0 \in \mathbb{R}^d$ and $(E, \mathcal{T})$ be a complete metrisable space of functions defined on $\mathbb{R}^d$ such that*

$$S = \{g \in E : g \in T_p^{\boldsymbol{\sigma}}(x_0)\}$$

is a Borel set of E. If there exists $f \in E$ such that for all $M \in \mathbb{N}$ there exists $j \in \mathbb{N}$ for which

$$\sigma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))} \geq M,$$

then S is Haar-null in E.

Proof. Let us fix $f' \in E$ and $N \in \mathbb{N}$ and consider the set

$$S_N = \{g \in E : \sigma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma) \rfloor + 1} g\|_{L^p(B_h(x_0, 2^{-j}))} \leq N \ \forall j \in \mathbb{N}\}.$$

Assume that there exist $a, b \in \mathbb{R}$ such that $f' + af \in E$ and $f' + bf \in E$. If $M \in \mathbb{N}$, there exists $j \in \mathbb{N}$ for which

$$\sigma_j 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))} \geq M.$$

It follows that

$$\begin{aligned}
|a - b| &= \frac{|a-b| 2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta^{\lfloor \overline{s}(\sigma) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))}}{2^{jd/p} \sup_{|h| \leq 2^{-j}} \|\Delta^{\lfloor \overline{s}(\sigma) \rfloor + 1} f\|_{L^p(B_h(x_0, 2^{-j}))}} \\
&\leq \frac{2^{jd/p}}{M \sigma_j^{-1}} \big(\sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma) \rfloor + 1} f' + af\|_{L^p(B_h(x_0, 2^{-j}))} + \sup_{|h| \leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma) \rfloor + 1} f' + bf\|_{L^p(B_h(x_0, 2^{-j}))} \big) \\
&\leq \frac{2N}{M}
\end{aligned}$$

and so, as M is arbitrary, $a = b$. It follows that the set

$$\{a \in \mathbb{R} : f' + af \in S_N\}$$

contains at most one point. Therefore, the set

$$\{a \in \mathbb{R} : f' + af \in S\} = \bigcup_{N \in \mathbb{N}} \{a \in \mathbb{R} : f' + af \in S_N\}$$

is countable and thus of Lebesgue-measure zero. The conclusion follows. □

Now, let us fix an admissible sequence $\boldsymbol{\sigma} = (\sigma_j)_j$ such that $2^{-j\frac{d}{p}} \sigma_j^{-1}$ tends to 0 as j tends to ∞ and $\underline{\sigma}_1 > 2^{-\frac{d}{p}}$. We define, for all $k \in \mathbb{N}$, the admissible sequence $\boldsymbol{\sigma}^{(\boldsymbol{k})} = (j^{1-\frac{1}{k}} \sigma_j)_j$. As inequalities (1.2) ensure that the sequence $(|\log_2(2^{-jd/p} \sigma_j)|/j)_j$ is bounded, we define the spaces of "under-log" corrected functions in the following way.

Definition 4.2.2. If $x_0 \in \mathbb{R}^d$, a *function* $f \in L^\infty_{loc}$ *belongs to* $T^{\sigma,p}_{/_s \log}(x_0)$ if there exists $k \in \mathbb{N}$ such that $f \in T_p^{\sigma^{(k)}}(x_0)$.

The idea is that a function belongs to $T^{\sigma,p}_{/_s \log}(x_0)$ if its pointwise behaviour at x_0 admits a correction from σ which is asymptotically weaker than the absolute value of the logarithm of $2^{-\cdot d/p}\sigma$.

Let us first consider the case where $p = \infty$ and exhibit a function which satisfies the condition of Theorem 3.3.6 (with $p = q = \infty$) but which does not belong to $T^{\sigma,\infty}_{/_s \log}(0)$. This example is based on one of [67] but some substantive modifications are made to correct some points and adapt it to our context. For the sake of simpleness, we take $d = 1$.

Consider ψ a wavelet of regularity $r > \lfloor \bar{s}(\sigma) \rfloor + 1$ such that $\operatorname{supp}(\psi) \subseteq [-1,1]$ and $\psi(0) = C \neq 0$. Define the sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ by $\varepsilon_m = 2^{-2^{m-1}}$ for all $m \geq 1$. For such a m, define the function

$$f_m = \sigma^{-1}_{2^{m-1}} \sum_{2^m \leq j < 2^{m+1}} \psi(2^j(\cdot - \varepsilon_m)).$$

As $\operatorname{supp}(\psi(2^j(\cdot - \varepsilon_m))) \subseteq [\frac{-1}{2^j} + \varepsilon_m, \varepsilon_m + \frac{1}{2^j}]$, we have

$$\operatorname{supp}(f_m) \subseteq [-\varepsilon_{m+1} + \varepsilon_m, \varepsilon_m + \varepsilon_{m+1}]$$

so $f_m(0) = 0$ and let us remark that those supports are disjoint as soon as $m \geq 2$. It follows that $f_m(\varepsilon_k) = \delta_{m,k} C 2^m \sigma^{-1}_{2^{m-1}}$ for all $m,k \geq 2$. Let us choose $M(\sigma) \geq 2$ big enough such that

$$\varepsilon_m + \varepsilon_{m+1} < l\,\varepsilon_m < -\varepsilon_m + \varepsilon_{m-1} \quad \forall\, m \geq M(\sigma), l \in \{2, \dots, \lfloor \bar{s}(\sigma) \rfloor + 1\}.$$

It follows that, if $m \geq M(\sigma)$, $\Delta^{\lfloor \alpha \rfloor + 1}_{\varepsilon_k} f_m(0) = \delta_{m,k} C 2^m \varepsilon_m^\alpha$. Let us finally consider the function f defined by

$$f = \sum_{m \geq M(\sigma)} f_m,$$

with convergence in L^∞. Its wavelet coefficients are given by

$$c_{j,k} := \begin{cases} \sigma^{-1}_{2^{m-1}} & \text{if } j \geq M(\sigma) \text{ and } k = \varepsilon_m 2^j \\ 0 & \text{otherwise.} \end{cases}$$

At scale $j \in [2^m, 2^{m+1})$ there is only one non-vanishing wavelet coefficient whose value is $\sigma^{-1}_{2^{m-1}}$ and, using (1.2) with $\varepsilon > 0$ small enough such that $\underline{s}(\sigma) - \varepsilon > 0$, we find

$$|c_{j,k}| \leq 2^{-2^{m-1}(\underline{s}(\sigma)-\varepsilon)} \leq 2^{-\frac{(\underline{s}(\sigma)-\varepsilon)}{4} j}.$$

This guarantees, from characterization (3.5), that $f \in B_{\infty,\infty}^{\frac{(\underline{s}(\sigma)-\varepsilon)}{4}}$, and the minimal regularity assumption of Theorem 3.3.6 is satisfied.

For all $j \in \mathbb{N}$, a dyadic cube $[\varepsilon_m, \varepsilon_m + 2^{-j'})$ of scale $j' \in [2^m, 2^{m+1})$ is taken into account in the value of $d_j^\infty(0)$ if the distance between ε_m and the origin is less than $2^{-(j-1)}$ or, in other words, if $j \leq 2^{m-1}+1$. As $\underline{\sigma}_1 > 1$, the sequence $\boldsymbol{\sigma}$ is increasing and, using its admissibility, we can conclude than

$$(\sigma_j d_j^\infty(0))_j \in \ell^\infty.$$

But, from what precedes, we have, for all $m \geq M(\boldsymbol{\sigma})$,

$$|\Delta_{\varepsilon_m}^{\lfloor \overline{s}(\sigma)\rfloor+1} f(0)| = C 2^m \sigma_{2^{m-1}}^{-1} \geq C' |\log(\sigma_{2^{m-1}})| \sigma_{2^{m-1}}^{-1},$$

which shows that f can not belong to $T_{/_s \log}^{\sigma,\infty}(0)$. Of course, by a translation, this construction holds for any $x_0 \in \mathbb{R}^d$.

Using this last function and Lemma 4.2.1, one can establish a first prevalence result concerning the logarithm correction. If $0 < \varepsilon < \frac{\underline{s}(\sigma)}{4}$, $x_0 \in \mathbb{R}^d$, we set

$$E_\infty^\varepsilon(x_0) = \{f \in B_{\infty,\infty}^\varepsilon(\mathbb{R}^d) : (\sigma_j d_j^\infty(x_0))_j \in \ell^\infty\}.$$

Equipped with the norm

$$\|\cdot\|_{E_\infty^\varepsilon(x_0)} : E_\infty^\varepsilon(x_0) \to [0,+\infty) : f \mapsto \|f\|_{B_{\infty,\infty}^\varepsilon} + \|(\sigma_j d_j^\infty(x_0))_j\|_{\ell^\infty},$$

$E_\infty^\varepsilon(x_0)$ is a complete normed space.

Theorem 4.2.3. *If $x_0 \in \mathbb{R}^d$, for all $0 < \varepsilon < \frac{\underline{s}(\sigma)}{4}$, almost every function of $E_\infty^\varepsilon(x_0)$ belongs to $T_{\infty,\log}^{\boldsymbol{\sigma}}(x_0) \setminus T_{/_s \log}^{\sigma,\infty}(x_0)$.*

Proof. We already know that every function of $E_\infty^\varepsilon(x_0)$ belongs to $T_{\infty,\log}^{\boldsymbol{\sigma}}(x_0)$. For all $k \in \mathbb{N}$, let us check that the set

$$B_k = \{g \in E_\infty^\varepsilon(x_0) : f \in T_\infty^{\boldsymbol{\sigma}^{(k)}}\}$$

is Borel. For all $N \in \mathbb{N}$, we define

$$B_{N,k} = \{g \in E_\infty^\varepsilon(x_0) : \sigma_j^{(k)} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma)\rfloor+1} g\|_{L^\infty(B_h(x_0,2^{-j}))} \leq N \ \forall j \in \mathbb{N}\},$$

$B_{N,k}$ is closed as if $(g_m)_{m\in\mathbb{N}}$ is a sequence of functions of $B_{N,k}$ that converges to g in $E_\infty^\varepsilon(x_0)$, then $\|g - g_m\|_{B_h^\varepsilon \infty,\infty} \to 0$ and for all $m, j \in \mathbb{N}$, we have, from (2.2),

$$\begin{aligned}\sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma)\rfloor+1} g\|_{L^\infty(B_h(x_0,2^{-j}))} &\leq \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \overline{s}(\sigma)\rfloor+1} g_m\|_{L^\infty(B_h(x_0,2^{-j}))} + C\|g - g_m\|_{B_{\infty,\infty}^\varepsilon} \\ &\leq N(\sigma_j^{(k)})^{-1} + C\|g - g_m\|_{B_{\infty,\infty}^\varepsilon}.\end{aligned}$$

Taking the limit for $m \to \infty$, we conclude that $g \in B_{N,k}$. It follows that

$$B_k = \bigcup_{N \in \mathbb{N}} B_{N,k}$$

is a Borel set. The function f built above belongs to $E^{\varepsilon}_{\infty}(x_0)$ but, for all $M \in \mathbb{N}$, there exists $j \in \mathbb{N}$ for which

$$\sigma_j^{(k)} \sup_{|h| \leq 2^{-j}} \|\Delta^{\lfloor \overline{s}(\sigma) \rfloor + 1} f\|_{L^\infty(B_h(x_0, 2^{-j}))} \geq M$$

and we conclude from Proposition 4.2.1 that B_k is Haar-null. As we have

$$\{g \in E^{\varepsilon}_{\infty}(x_0) : g \in T^{\sigma,\infty}_{/_s \log}(x_0)\} = \bigcup_{k \in \mathbb{N}} B_k,$$

we conclude that almost every function of $E^{\varepsilon}_{\infty}(x_0)$ belongs to $T^{\sigma}_{\infty,\log}(x_0) \setminus T^{\sigma,\infty}_{/_s \log}(x_0)$. □

Let us now focus on the case $p = 1$. In this setting, we recall that the required property for the admissible sequence $\boldsymbol{\sigma}$ is that $2^{-jd}\sigma_j^{-1}$ tends to 0 as j tends to ∞ and $\underline{\sigma}_1 > 2^{-d}$. Again, we work with $d = 1$. If we have the additional assumption on the wavelet[1] that $\int_0^1 \psi(x)\,dx \neq 0$, and if we redefine the sequence $(f_m)_m$ by

$$f_m = \sigma^{-1}_{2^{m-1}} \varepsilon_m \sum_{2^m \leq j < 2^{m+1}} 2^j \psi(2^j(\cdot - \varepsilon_m)),$$

the functions f, set as previously, but with convergence in $Ł^1$ this time, can be used to define a probe as in the last theorem. Indeed, now, the only non-vanishing coefficient of scale j, when $j \in [2^m, 2^{m+1})$, is now $\sigma^{-1}_{2^{m-1}} \varepsilon^1_m 2^j$. First of all, if $\varepsilon > 0$ is now chosen such that $\underline{s}(\boldsymbol{\sigma}) - \varepsilon > -1$,

$$2^{-j} \sigma^{-1}_{2^{m-1}} \varepsilon_m 2^j \leq 2^{-2^{m-1}(\underline{s}(\sigma)+1-\varepsilon)} \leq 2^{-j\frac{\underline{s}(\sigma)+1/p-\varepsilon}{4}}$$

and $f \in B_{1,\infty}^{\frac{\underline{s}(\sigma)+1/p-\varepsilon}{4}}$. Secondly, if $[\varepsilon_m, \varepsilon_m + 2^{-j'})$ is a dyadic cube of scale $j' \in [2^m, 2^{m+1})$ for which $\varepsilon_m \leq 2^{-(j-1)}$, we have

$$2^{-(j'-j)} \sigma^{-1}_{2^{m-1}} \varepsilon_m 2^{j'} \leq C' \sigma_j^{-1},$$

which ensures that $(\sigma_j d^1_j(0))_j \in \ell^\infty$. Finally, let us remark that, increasing $M(\boldsymbol{\sigma})$ if necessary, one can makes sure that, for all $l \in \{1, ..., \lfloor \overline{s}(\boldsymbol{\sigma}) \rfloor + 1\}$ and $x \in [\varepsilon_m - \varepsilon_{m+1}, \varepsilon_m + \varepsilon_{m+1}]$, we have

$$\varepsilon_m + \varepsilon_{m+1} < x + l\varepsilon_m < \varepsilon_{m-1} - \varepsilon_m$$

[1] This assumption is satisfied for Daubechies wavelets for instance.

and so $f(x+l\varepsilon_m) = \delta_{l,0} f_m(x)$. It follows that

$$\begin{aligned}(3\varepsilon_m)^{-1}\|\Delta_{\varepsilon_m}^{\lfloor \underline{s}(\sigma)\rfloor+1} f\|_{L^1(B_{\varepsilon_m}(0,3\varepsilon_m))} &\geq C_1 \sigma_{2^{m-1}}^{-1} \int_{\varepsilon_m}^{\varepsilon_m+\varepsilon_{m+1}} |\sum_{2^m\leq j<2^{m+1}} 2^j \psi(2^j(x-\varepsilon_m))|\,dx \\ &\geq C_1 \sigma_{2^{m-1}}^{-1} \left| \sum_{2^m\leq j<2^{m+1}} 2^j \int_0^{\varepsilon_{m+1}} \psi(2^j(x-\varepsilon_m))\,dx \right| \\ &= C_1 \sigma_{2^{m-1}}^{-1} \left| \sum_{2^m\leq j<2^{m+1}} \int_0^1 \psi(x)\,dx \right| \\ &= C_2 \sigma_{2^{m-1}}^{-1} 2^m.\end{aligned}$$

If we define, for all $0<\varepsilon<\frac{\underline{s}(\sigma)+d}{4}$ and $x_0 \in \mathbb{R}^d$, the space

$$E_1^\varepsilon(x_0) = \{f \in B_{1,\infty}^\varepsilon(\mathbb{R}^d) : (\sigma_j d_j^1(x_0))_j \in \ell^\infty\},$$

equipped with the obviously modified E_1^ε norm, one can show, in the same way that Theorem 4.2.3, the following result.

Theorem 4.2.4. *If $x_0 \in \mathbb{R}^d$, for all $0<\varepsilon<\frac{\underline{s}(\sigma)+d}{4}$, almost every function of $E_1^\varepsilon(x_0)$ belongs to $T_{1,\log}^\sigma(x_0) \setminus T_{/_s \log}^{\sigma,1}(x_0)$.*

Now, for $1<p<\infty$, from what precedes, a judicious choice to obtain the desired probe seems to take

$$f_m = \sigma_{2^{m-1}}^{-1} \varepsilon_m^{\frac{1}{p}} \sum_{2^m\leq j<2^{m+1}} 2^{\frac{j}{p}} \psi(2^j(\cdot-\varepsilon_m)).$$

Once again, it is easy to check that the obtained function f checks the two first desired properties

$$(\sigma_j d_j^p(0))_j \in \ell^\infty \quad \text{and} \quad f \in B_{p,\infty}^{\frac{\underline{s}(\sigma)+\frac{1}{p}}{4}}.$$

But, unfortunately, for all m, if we compute the L^p norm of f_m, it is proportional (see once again the wavelet characterization of L^p spaces in [102]) to

$$\left(\int_{\mathbb{R}}\left(\sum_{2^m\leq j<2^{m+1}} (\sigma_{2^{m-1}}^{-1}\varepsilon_m^{\frac{1}{p}} 2^{\frac{1}{p}})^2 \chi_{[\varepsilon_m,\varepsilon_m+2^{-j})}\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} = \sigma_{2^{m-1}}^{-1}\left(\sum_{2^m\leq j<2^{m+1}} 2^{-j}\left(\sum_{k=2^m}^{j} 2^{\frac{2k}{p}}\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

and this last term is itself proportional to $\sigma_{2^{m-1}}^{-1} 2^{m/p}$, which is not sufficient to establish a theorem comparable to Theorems 4.2.3 and 4.2.4 for $1<p<\infty$. As the belonging $(\sigma_j d_j^p(0))_j \in \ell^\infty$ is optimal, one can not add a multiplicative term of order $2^{m/q}$ without altering it. We also thought about increasing the number of terms in the sum that defines f_m up to 2^{mp} but it is also impossible without destroying the belonging to an uniform Besov space.

The function f exhibited here guarantees the necessity of a correction of order $(|\log_2(2^{-jd/p}\sigma_j)|)^{1/p}$ for almost every function in[2] E_p^ε, with $0<\varepsilon<\frac{\underline{s}(\sigma)+\frac{d}{p}}{4}$, but cannot be used to prove the following conjecture.

Conjecture 4.2.5. *If $x_0\in\mathbb{R}^d$, for all $1\le p\le\infty$, there exists $\varepsilon_{\sup}^{(p)}>0$ such that, for all $0<\varepsilon<\varepsilon_{\sup}^{(p)}$, almost every function of $E_p^\varepsilon(x_0)$ belongs to $T^\sigma_{p,\log}(x_0)\setminus T^{\sigma,p}_{/_s\log}(x_0)$.*

4.3 About the importance of pointwise spaces of generalized smoothness

In this section, we start by giving, for any admissible sequence σ and $p\in[1,\infty]$, an example of function that belongs to $\overline{T}_p^\sigma(x_0)$. This example leads to discussions concerning the contribution of pointwise spaces of generalized smoothness.

Example 4.3.1. Let us fix $p\in[1,\infty]$ and an admissible sequence σ such that $\underline{s}(\sigma)>-\frac{1}{p}$. Let us also consider a wavelet ψ with compact support included in $[-1,1]$. We define the function f_σ by

$$f_\sigma=\sum_{k\ge2}\sigma_k^{-1}2^{\frac{k}{p}}\psi(2^{2k}(\cdot-2^{-k})),\tag{4.13}$$

with convergence in L^p. For all $k\ge2$, $\psi(2^{2k}(\cdot-2^{-k}))$ is supported in

$$[2^{-k}(1-2^{-k}),2^{-k}(1+2^{-k})]$$

and, in particular, for all $k,k'\ge2$, with $k\neq k'$,

$$\operatorname{supp}(\psi(2^{2k}(\cdot-2^{-k})))\cap\operatorname{supp}(\psi(2^{2k'}(\cdot-2^{-k'})))=\emptyset.$$

Therefore, for all $j\ge2$, we have, with usual modifications if $p=\infty$,

$$\begin{aligned}2^{j/p}\|f_\sigma\|_{L^p(B(0,2^{-j}))}&=2^{j/p}\left(\int_{2^{-j}(1-2^{-2j})}^{2^{-j}}|f_\sigma(x)|^p\,dx+\sum_{k>j}\int_{2^{-k}(1-2^{-k})}^{2^{-k}(1+2^{-k})}|f_\sigma(x)|^p\,dx\right)^{1/p}\\&=2^{j/p}\left(\sigma_j^{-p}2^{-j}\int_{-1}^{0}|\psi(x)|^p\,dx+\sum_{k>j}\sigma_k^{-p}2^{-k}\int_{-1}^{1}|\psi(x)|^p\,dx\right)^{1/p}.\end{aligned}$$

[2] For $1<p<\infty$, E_p^ε is defined in the obvious way, following the definitions of E_∞^ε and E_∞^ε.

Finally, using Lemma 1.2.2, it is clear that we can find constants $C_1, C_2 > 0$ such that, for all $j \geq 2$,

$$C_1 \leq 2^{j/p} \sigma_j \|f_\sigma\|_{L^p(B(0,2^{-j}))} \leq C_2. \tag{4.14}$$

The belonging to $T_p^\sigma(0)$ is immediately guaranteed by (4.14) together with Proposition 3.1.2 (with $P_{j,0} = 0$, for all j). To show that $f \in I_p^\sigma(0)$, it suffices to note that, for all $n \in \mathbb{N}$, one can find an interval $I \subset [0, 2^{-j}]$ such that, for all $x \in [2^{-k}(1-2^{-k}), 2^{-k}(1+2^{-k})]$, $\Delta_h^n f_\sigma = f_\sigma$. If $\overline{s}(\sigma) > 0$, we can also remark that the sequence $(\sigma_j^{-1})_j$ is decreasing, thus, for all $j \geq 2$, $d_j^p(0) \geq \sigma_j^{-1}$ and we can conclude the desired membership by Proposition 4.1.5.

For different values of p and different admissible sequences $\boldsymbol{\sigma}$, Figure 4.1 give a representation of f_σ.

Note that, if we wish to obtain a function in $T_{p,q}^\sigma(0)$, with $q \neq \infty$, it suffices to consider a sequence $(\varepsilon_k)_k \in \ell^q$ and, for all $k \geq 2$, to multiply the k^{th} term in the sum (4.13) by ε_k (the conclusion follows again by Lemma 1.2.2).

Of course, up to a translation, these affirmations hold for arbitrary x_0.

This example is of particular interest to discuss the utility of these new spaces in the precise characterization of the regularity for functions.

Firstly, it shows that for any sequence $\boldsymbol{\sigma}$, there exists functions for which the belonging to $T_p^\sigma(x_0)$ is optimal, as reflected by the membership $f_\sigma \in \overline{T}_p^\sigma(x_0)$.

That being said, let us consider two distinct slowly varying functions Ψ and Φ, $u > -\frac{d}{p}$ and the associated admissible sequences $\boldsymbol{\sigma_{u,\Psi}} = (2^{ju}\Psi(2^j))_j$ and $\boldsymbol{\sigma_{u,\Phi}} = (2^{ju}\Phi(2^j))_j$, see Corollary 1.2.12. If we assume that $\Psi(x) \to 0$ and $\Phi(x) \to 0$ as $x \to \infty$, then for all $\varepsilon > 0$, $f_{\sigma_{u,\Psi}}$ and $f_{\sigma_{u,\Phi}}$ belong to $T_p^{u-\varepsilon}(x_0)$ but not to $T_p^u(x_0)$. The usual spaces of Calderón and Zygmund fail to precisely characterize the regularity at x_0 of $f_{\sigma_{u,\Psi}}$ and $f_{\sigma_{u,\Phi}}$ while the generalized versions are more accurate since $f_{\sigma_{u,\Psi}} \in T_p^{\sigma_{u,\Psi}}(x_0)$ and $f_{\sigma_{u,\Phi}} \in T_p^{\sigma_{u,\Phi}}(x_0)$. Therefore, the notion of regularity underlying the usual spaces may be too coarse in some situations. For example these spaces do not allow to capture the logarithmic correction in the regularity of the Brownian motion [82, 83]. As a consequence, the usual spaces also fail to distinguish $f_{\sigma_{u,\Psi}}$ and $f_{\sigma_{u,\Phi}}$ while, as soon as $\Psi(x) \in o(\Phi(x))$ as $x \to \infty$, $f_{\sigma_{u,\Psi}} \in T_p^{\sigma_{u,\Psi}}(x_0) \setminus T_p^{\sigma_{u,\Phi}}(x_0)$. More generally, if $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ are two admissible sequences such that $\sigma_j \in o(\gamma_j)$ as $j \to +\infty$, $f_\sigma \in T_p^\sigma(x_0) \setminus T_p^\gamma(x_0)$.

This situation occurs in practice when considering for instance the Brownian motion on $(\Omega, \mathcal{B}, \mathbb{P})$. If we take the admissible sequence $\boldsymbol{\sigma} = (2^{\frac{1}{2}j}|\log j|^{-\frac{1}{2}})_j$, from the Khintchine Law of iterated logarithm [82, 60], we know that almost surely for all $\omega \in \Omega$ and for almost every $x_0 \in \mathbb{R}$, $B_\cdot(\omega) \in T_\infty^\sigma(x_0)$ while $B_\cdot(\omega) \notin T_\infty^{\frac{1}{2}}(x_0)$. Being able to make a distinction between a Brownian motion and another process not displaying such logarithmic corrections is an important issue in practice (see [83] and the next section below).

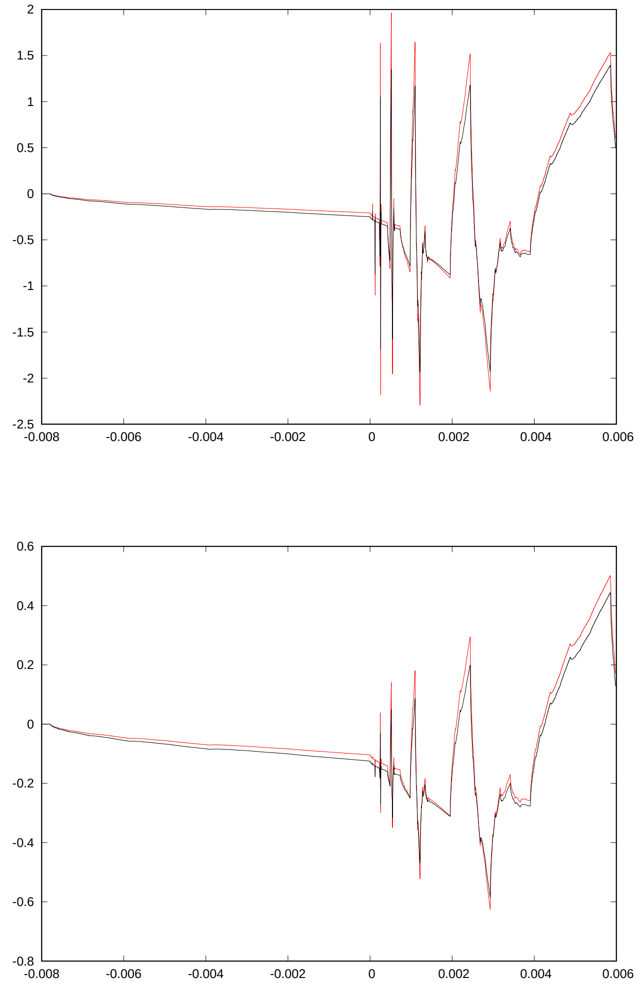


Figure 4.1: Representations of some functions defined in Example 4.3.1 with $p = 2$ (upper panel) and $p = \infty$ (lower panel) and using dyadic sequence (black) and dyadic sequence with a logarithmic correction (red). The wavelet considered is the Daubechies wavelet of order 2.

All these remarks lead to results of prevalence, using again Lemma 4.2.1.

Theorem 4.3.2. *Given $p \in [1,\infty]$, if $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ are admissible sequences such that $\sigma_j \in o(\gamma_j)$ as $j \to +\infty$ and $\overline{s}(\boldsymbol{\sigma}) \leq \overline{s}(\boldsymbol{\gamma})$ then, from the prevalence point of view, almost every function in $T_p^{\sigma}(x_0)$ does not belong to $T_p^{\gamma}(x_0)$.*

Proof. The assumptions made on the admissible sequences insure the inclusion of $T_p^{\gamma}(x_0)$ in $T_p^{\sigma}(x_0)$, see [90]. From the previous remarks on f_σ, we know that for all

$M \in \mathbb{N}$ there exists $j \in \mathbb{N}$ for which

$$\gamma_j 2^{jd/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \bar{s}(\gamma)\rfloor+1} f_\sigma\|_{L^p(B_h(x_0,2^{-j}))} \geq M.$$

Therefore, from Lemma 4.2.1, it suffices to show that $T_p^\gamma(x_0)$ is a Borel set in $(T_p^\sigma(x_0), \|\cdot\|_{T_p^\sigma(x_0)})$. We proceed in the same way that for the proof of Theorem 4.2.3. For all $N \in \mathbb{N}$, we set

$$B_N = \{f \in T_p^\gamma(x_0) : \|f\|_{T_p^\gamma(x_0)} \leq N\}$$

and show that B_N is closed in $(T_p^\sigma(x_0), \|\cdot\|_{T_p^\sigma(x_0)})$: if $(f_k)_k$ is a sequence of functions of B_N that converges to f then, for all $j,k \in \mathbb{N}$,

$$\begin{aligned}
&\|f\|_{L^p(B(0,1))} + \gamma_j 2^{j/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \bar{s}(\gamma)\rfloor+1} f\|_{L^p(B_h(x_0,2^{-j}))}\\
\leq &\|f-f_k\|_{L^p(B(0,1))} + \gamma_j 2^{j/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \bar{s}(\gamma)\rfloor+1} (f-f_k)\|_{L^p(B_h(x_0,2^{-j}))}\\
&+ \|f_k\|_{L^p(B(0,1))} + \gamma_j 2^{j/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \bar{s}(\gamma)\rfloor+1} f_k\|_{L^p(B_h(x_0,2^{-j}))}.
\end{aligned}$$

Of course,

$$\|f_k\|_{L^p(B(0,1))} + \gamma_j 2^{j/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \bar{s}(\gamma)\rfloor+1} f_k\|_{L^p(B_h(x_0,2^{-j}))} \leq N$$

and, using fundamental properties of finite differences,

$$\gamma_j 2^{j/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \bar{s}(\gamma)\rfloor+1} (f-f_k)\|_{L^p(B_h(x_0,2^{-j}))} \leq C\frac{\gamma_j}{\sigma_j}\sigma_j 2^{j/p} \sup_{|h|\leq 2^{-j}} \|\Delta_h^{\lfloor \bar{s}(\sigma)\rfloor+1} (f-f_k)\|_{L^p(B_h(x_0,2^{-j}))}.$$

Taking the limit $k \to +\infty$, we have $\|f\|_{T_p^\gamma(x_0)} \leq N$ and thus B_N is closed. As,

$$T_p^\gamma(x_0) = \bigcup_{N\in\mathbb{N}} B_N,$$

the conclusion follows. □

In particular, while working with a decreasing family of admissible sequences, the assumptions of Theorem 4.3.2 are often met (see [90]) and we can state the following corollary without too much restriction.

Corollary 4.3.3. *Given $p \in [1,\infty]$, if $(\boldsymbol{\sigma}^{(\boldsymbol{h})})_{h>-\frac{1}{p}}$ is a decreasing family of admissible sequences such that $h < h'$ implies $\sigma_j^{(h)} \in o(\sigma_j^{(h')})$ as $j \to +\infty$ and $\bar{s}(\boldsymbol{\sigma}^{(\boldsymbol{h})}) \leq \bar{s}(\boldsymbol{\sigma}^{(\boldsymbol{h'})})$ then, from the prevalence point of view, almost every function in $T_p^{\sigma^{(h)}}(x_0)$ is of exponent h.*

4.4 From theory to practice: open perspectives

This thesis only focuses on the theoretical background needed to establish new methods in multifractal analysis based on admissible sequences and generalized Hölder exponents. Nevertheless, the example presented in the previous section could be used to make first numerical experimentations of our formalism. In a hope to generate interest from programmers and researchers in signal analysis, one can mention results obtained by Thomas Kleyntssens in his own thesis [83]. There, he implemented a numerical method based on a generalization of the S^ν spaces, originally defined by Jaffard [69], adapted to compute generalized Hölder exponents such as the ones we considered before.

Definition 4.4.1. Let ν be a right-continuous increasing function for which there exists $h_{\min} \in \mathbb{R}$ such that $\nu(h) = -\infty$ if $h < h_{\min}$ and $\nu(h) \in [0, d]$ if $h \geq h_{\min}$. Let $(\boldsymbol{\sigma}^{(\boldsymbol{h})})_h$ be a decreasing family of admissible sequences such that $h < h'$ implies that $\sigma_j^{(h)} \in o(\sigma_j^{(h')})$ as $j \to +\infty$.

The set $S^{\nu,\sigma^{(\cdot)}}$ is the set of all complex sequences $c = (c_\lambda)_{\lambda\in\Lambda}$ such that for any $h \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$, there exists $J > 0$ for which for any $j \geq J$, we have

$$\#\{\lambda \in \Lambda_j : \sigma_j^{(h)} |c_\lambda| \geq C\} \leq 2^{(\nu(h)+\varepsilon)j}.$$

Of course, in our context, $c = (c_\lambda)_{\lambda\in\Lambda}$ are the (periodized) wavelet coefficients of a function. One can therefore define the corresponding *generalized wavelet profile* as the function

$$\nu_{c,\sigma^{(\cdot)}} : h \mapsto \lim_{\varepsilon\to 0^+} \limsup_{j\to+\infty} \frac{\log \#\{\lambda \in \Lambda_j : \sigma_j^{(h+\varepsilon)} |c_\lambda| \geq 1\}}{\log 2^j}$$

and show that $c = (c_\lambda)_{\lambda\in\Lambda} \in S^{\nu,\sigma^{(\cdot)}}$ if and only if $\nu_{c\sigma^{(\cdot)}} \leq \nu$. Note that, if we consider, for all $h \in \mathbb{R}$, $\boldsymbol{\sigma}^{(\boldsymbol{h})} = (2^{jh})_j$, we have $S^{\nu,\sigma^{(\cdot)}} = S^\nu$.

Let us highlight the fact that these spaces are strongly connected to the ones considered here. Indeed let us define by $b^{\sigma}_{p,q}$ the set of sequences $(c_\lambda)_{\lambda\in\Lambda}$ that satisfy condition (3.5). We can show [83] that, if we also assume that $\overline{s}(\boldsymbol{\sigma}^{(\boldsymbol{h})}) \to +\infty$ as $h \to +\infty$ then, if for all $p > 0$, $\boldsymbol{\gamma}^{(\boldsymbol{p})}$ is an admissible sequence, we have

$$S^{\nu,\sigma^{(\cdot)}} \subseteq \bigcap_{p>0} \bigcap_{\varepsilon>0} b^{\gamma^{(p)} 2^{\cdot\varepsilon/p}}_{p,\infty}$$

if and only if for any $p, \varepsilon > 0$ and for any $h \geq h_{\min}$, there exists $C > 0$ such that, for all $j \in \mathbb{N}$,

$$2^{j\varepsilon/p} \leq C 2^{jd/p} 2^{-j\nu(h)/p} \gamma_j^{(p)}(\sigma_j^{(h)}).$$

In [83], Thomas Kleyntssens implements an algorithm based on S^ν and $S^{\nu,\sigma^{(\cdot)}}$ spaces to estimate the (standard) Hölderian regularity and explores the numerical contribution of admissible sequences by distinguishing three fundamentals cases:

1. Detection of the Hölderian behaviour for functions with prescribed Hölder exponent defined by their wavelet decomposition.

2. Detection of the Khintchine Law.

3. Detection of the Hölderian behaviour for processes defined in the Schauder basis.

In each case, the methodology is similar: performing repeatedly the algorithm using the families

$$(2^{\alpha j})_j, (2^{\alpha j}|\log|\log 2^j||^{\frac{-1}{2}})_j, (2^{\alpha j}|\log|\log 2^j||^{-1})_j \text{ and } (2^{\alpha j}|\log 2^j|^{\frac{-1}{2}})_j \tag{4.15}$$

of admissible sequences on randomly generated realization of signals with prescribed Hölder exponents and precise regularity known among the four families. Then, the errors between the Hölder exponents estimated by the algorithm and the real exponent are represented by boxplots.

In case 1, signals defined by their wavelet decomposition, of Hölder exponent H and having logarithm corrections $((w(2^j))^{-1})_j$, with

$$w(\cdot) \in \{1, \sqrt{|\log|\log(\cdot)||}, |\log|\log(\cdot)||, \sqrt{|\log(\cdot)|}\}$$

as precise regularity are considered. The results are quite relevant: the algorithm providing the smallest error and the fewest dispersion is the one associated with the good correction in the profile, it is presented in Figure 4.2.

A similar procedure could be used in our setting, using a randomized version of the function described in Example 4.3.1 to check if the algorithm is able to detect the precise regularity given by the used admissible sequence.

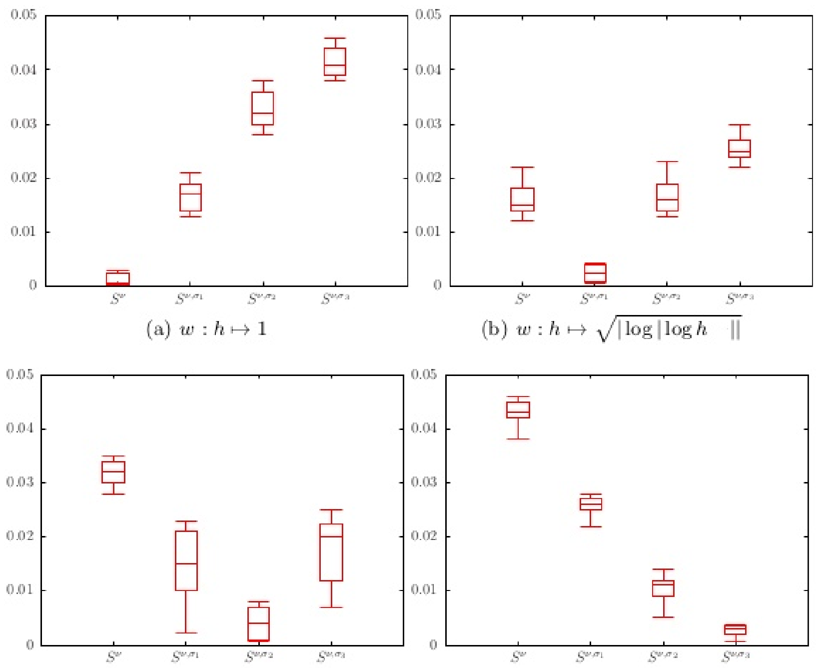
In case 2, Brownian motion is considered: almost surely it is of Hölder exponent 1/2 and, from the Khintchine law of iterated logarithm, we know that it satisfies a $((|\log|\log 2^j||)^{\frac{-1}{2}})_j$ correction. It is compared with a randomized Weierstraß function

$$W(x) = \sum_{j=0} a^j \cos((b^j x + U_j)\pi),$$

where $(U_j)_j$ is an arbitrary sequence of independent random variables with respect to the uniform probability measure on $[0,1]$ and $0 < a < 1 < b$ are chosen such that $-\log(a)/\log(b) = \frac{1}{2}$. In this setting, almost surely, W is of Hölder exponent 1/2 and satisfies a correction of order 1. Again, for each function, the estimations of the Hölder exponent with the smallest error and the fewest dispersion are the ones obtained with the appropriated correction in the profile, as it can be seen in Figure 4.3.

Case 3 is similar to the first two but considers functions represented in Schauder basis and the relevance of the profiles using admissible sequences is again showed.

Thomas Kleyntssens' work highlighted the relevance of dealing with admissible sequences in signal analysis. He provided a method to numerically detect the



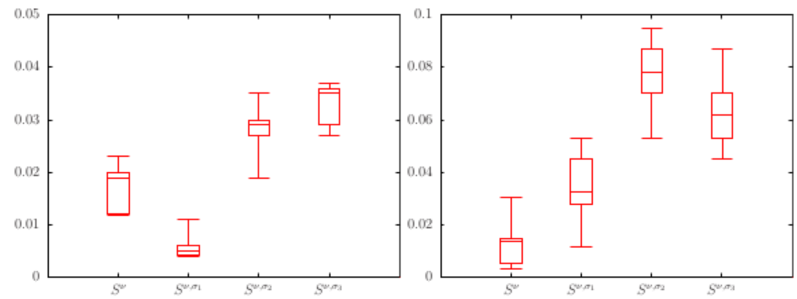
(a) $w : h \mapsto 1$

(b) $w : h \mapsto \sqrt{|\log|\log h||}$

(c) $w : h \mapsto |\log|\log h||$

(d) $w : h \mapsto \sqrt{|\log h|}$

Figure 4.2: Boxplots of the errors of measurement between the prescribed exponent and the one estimated. The function w corresponds to the known correction present in the signal and boxplots correspond to the profile using, from left to right, the admissible sequences $(2^{\alpha j})_j$, $(2^{\alpha j}|\log|\log 2^j||^{\frac{-1}{2}})_j$, $(2^{\alpha j}|\log|\log 2^j||^{-1})_j$ and $(2^{\alpha j}|\log 2^j|^{\frac{-1}{2}})_j$.



(a) Brownian motion

(b) Uniform Weierstraß function of parameters $(a, \exp(-\log(a)/0.5))$ with $a \in \{0.65, 0.7, \ldots, 0.85\}$

Figure 4.3: Boxplots of the errors of measurement between the prescribed exponent and the one estimated. On the left panel, the Hölderian regularity of the Brownian motion is estimated while, on the right panel, a randomized Weierstraß function is considered. Boxplots correspond to the profile using, from left to right, the admissible sequences $(2^{\alpha j})_j$, $(2^{\alpha j}|\log|\log 2^j||^{\frac{-1}{2}})_j$, $(2^{\alpha j}|\log|\log 2^j||^{-1})_j$ and $(2^{\alpha j}|\log 2^j|^{\frac{-1}{2}})_j$.

Khintchine Law, which could be of big interested in stock exchange, for example, as we could detect if a financial market follows a Brownian motion before applying the Black-Scholes model to it ([13, 58]).

Our work here aimed to pave the way for new methods in signal analysis using admissible sequences. Indeed, we establish a new multifractal formalism which relied on them and proved its validity from a prevalence point of view. This gives new opportunities for researchers using the Wavelet Leaders Method as we provided a general framework for it.

Please note also that the L^ν spaces have been introduced recently [4]. They consist in taking advantage of the wavelet leaders by replacing the wavelet coefficients that appear in the profile by themselves. A common generalization of the L^ν and $S^{\nu,\sigma^{(\cdot)}}$ spaces is to consider the p-wavelet leaders profile

$$\nu_{c,\sigma^{(\cdot)}} \,:\, h \mapsto \lim_{\varepsilon\to 0^+} \limsup_{j\to+\infty} \frac{\log \#\{\lambda \in \Lambda_j \,:\, \sigma_j^{(h+\varepsilon)} d_\lambda^p \geq 1\}}{\log 2^j}.$$

We already checked that the corresponding functional spaces satisfy fundamental properties similar to the ones presented in [4, 5], in particular they are independent of the chosen wavelet basis in the Schwartz class, the proofs are straightforward adaptations. Again, an implementation of an algorithm based on theses spaces and profiles could be of great interest to work with the generalized spaces and exponents we introduced in this thesis.

5 The spaces T^p_ϕ and t^p_ϕ

The T^p_u spaces were introduced in essence by Calderón and Zygmund [26]: given a point x_0 of the d-dimensional Euclidean space $\mathbb{R}^d$, $p \in [1,\infty]$ and a number $u \geq -d/p$, $T^p_u(x_0)$ denotes the class of functions f in $L^p(\mathbb{R}^d)$ for which there exists a polynomial P of degree strictly less than u with the property that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C r^u, \tag{5.1}$$

for a constant C (which does not depend on r). If $f \in T^p_u(x_0)$ also satisfies

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} = o(r^u) \qquad \text{as } r \to 0^+,$$

where P is a polynomial of degree less than or equal to u (where we have used the usual Bachmann-Landau notations), then f is said to belong to $t^p_u(x_0)$.

The general idea consists here in replacing the power function $r \mapsto r^u$ appearing in (5.1) with $r \mapsto \phi(r)$ $(r > 0)$, where ϕ is a Boyd function, to obtain generalized spaces T^p_ϕ and t^p_ϕ respectively; typically, such a function ϕ could be $r \mapsto r^u|\ln r|$ for the detection of the logarithmic corrections (such an idea is exploited in [39, 108] in the case of Bessel potential spaces) or more generally $r \mapsto r^u\psi(r)$, where ψ is any slowly varying function. Such a choice is natural and observed in many financial models that are derived from the Brownian motion (e.g. the geometric Brownian motion used in the Black and Scholes model [64], the Hull and White one-factor model [17], etc.).

Proposition 1.2.11 allows us to affirm that we already have investigated those spaces from a multifractal point of view. Now, we wish to explore their properties as regularity spaces and show that they are still related to some notion of smoothness. In this context, we prefer to work with Boyd functions instead of admissible sequences, as it is more convenient to work with a continuum of values, see Remark 5.1.2 below. To achieve this goal, we follow the ideas of Calderón and Zygmund and show that most of the properties established in [26] still hold for the generalized versions T^p_ϕ and t^p_ϕ; we thus introduce in this thesis some generalizations of the results obtained in [26].

In this chapter, we only focus on the standard properties of the generalized spaces T^p_ϕ and t^p_ϕ and establish some basic results concerning them (about completeness, density, embeddings,...). Next, we give a generalization of Whitney extension theorem. Connections with operators and elliptic partial differential equations will be discussed in the next chapter.

Results of the next two chapters were published in [99].

Contents

5.1 Definitions and first properties

Definition 5.1.1. Let $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$. A function $f \in L^p(\mathbb{R}^d)$ belongs to the space $T^p_\phi(x_0)$ if there exist a polynomial P of degree strictly less than $\underline{b}(\phi)$ and a constant $C > 0$ such that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C\phi(r) \qquad \forall r > 0. \tag{5.2}$$

Moreover, if we also have

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \in o(\phi(r)) \qquad \text{as } r \to 0^+, \tag{5.3}$$

we say that f belongs to $t^p_\phi(x_0)$.

Remark 5.1.2. In the previous definition, the condition $\underline{b}(\phi) > -d/p$ is here to ensure that the spaces T^p_ϕ are not degenerated: if $r^{-d/p} < C\phi(r)$ is satisfied in a neighbourhood of the origin, then any function belongs to $T^p_\phi(x_0)$; this inequality is never satisfied if $-d/p < \underline{b}(\phi)$. This condition could be relaxed in Definition 5.1.1, but the interest of such an extended definition is not obvious.

Remark 5.1.3. The definitions of $T^p_\phi(x_0)$ and $T^\sigma_{p,\infty}(x_0)$ slightly differ :

- The use of a Boyd function instead of an admissible sequence is made in order to work with a continuum of values. Proposition 1.2.11 and Theorem 3.2.3 connect the two spaces. Note that the $T^\sigma_{p,\infty}$ regularity, in a multifractal analysis point of view, only focuses on small radii around x_0 and only requires germs of functions. It is then equivalent, and more convenient, in this context, to work with sequences. At the opposite, the $T^p_\phi(x_0)$ regularity scans all values of r, in a more functional analysis approach, and considering the whole function is then necessary. Up to multiplication by cut-off functions, if needed, $T^\sigma_{p,\infty}(x_0)$ can be studied through $T^p_\phi(x_0)$ and $t^p_\phi(x_0)$ spaces, using the Boyd function defined in Proposition 1.2.11.

- In Definition 5.1.1, we ask the polynomial P to be of degree **strictly less** than $\underline{b}(\phi)$, while in Proposition 3.1.2 it is stated to be **less than or equal** to $\bar{s}(\sigma)$. This assures the uniqueness of the polynomial, see Proposition 5.1.5 below, in order to use its coefficients in the $T^p_\phi(x_0)$ norm. Nevertheless, most of the more interesting results exposed in this chapter hold if the Boyd indices are non-integers. In this setting, the spaces $T^p_\phi(x_0)$ and $T^\sigma_{p,\infty}(x_0)$ are identical (up to multiplication by cut-off functions), thanks to Theorem 3.2.3.

Remark 5.1.4. Let us highlight the fact that $t^p_\phi(x_0)$ is a "true subspace" of $T^p_\phi(x_0)$; indeed, under the assumptions of the previous definition, if $f \in L^p(\mathbb{R}^d)$ is such that there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ for which

$$\phi(r)^{-1} r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \to 0 \qquad \text{as } r \to 0^+,$$

then there exists $R > 0$ such that

$$r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \leq \phi(r),$$

for all $r \leq R$. Moreover, for $r \geq R$, we have

$$r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \leq r^{-d/p} \|f\|_{L^p(\mathbb{R}^d)} + C_R(1 + r^n)$$

and an application of Proposition 1.2.6 shows that the right-hand side can be bounded from above by $\phi(r)$, which means that $f \in T^p_\phi(x_0)$.

Let us study the basic properties of the spaces T^p_ϕ.

Proposition 5.1.5. *If $f \in T^p_\phi(x_0)$, then the polynomial P in (5.2) is unique.*

Proof. Of course, if $\underline{b}(\phi) \leq 0$, the polynomial appearing in (5.2) must be 0. Now, if $\underline{b}(\phi) > 0$, let us suppose that there exist two polynomials P and P' of degree strictly less than $\underline{b}(\phi)$ and $C, C' > 0$ such that, for all $r > 0$,

$$r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \leq C\phi(r)$$

and

$$r^{-d/p} \|f - P'\|_{L^p(B(x_0,r))} \leq C'\phi(r).$$

Now, if we define $Q := P - P'$, Q is a polynomial of degree $n < \underline{b}(\phi)$. So, if $\varepsilon > 0$ is such that $n < \underline{b}(\phi) - \varepsilon$, then we have from Proposition 1.2.6 that there exists $C'' > 0$ such that

$$r^{-d/p} \|Q\|_{L^p(B(x_0,r))} \leq C'' r^{\underline{b}(\phi)-\varepsilon}.$$

But, if Q is a non-zero polynomial, than the left-hand side must decrease at most like r^n, which contradicts this last inequality. □

Remark 5.1.6. If $\phi \in \mathcal{B}$ and if the function f belongs to $T^p_\phi(x_0)$ for some $p \in [1,\infty]$, then, in particular, f belongs to $L^1_{\text{loc}}(\mathbb{R}^d)$. Suppose that $\underline{b}(\phi) > 0$ (otherwise, the polynomial P in (5.2) is identically zero) and let us assume that x_0 is a Lebesgue point of f. If P is the polynomial of degree strictly less than $\underline{b}(\phi)$ such that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C\phi(r) \qquad \forall\, r > 0,$$

then we also have

$$r^{-d}\|f - P\|_{L^1(B(x_0,r))} \leq C_d r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C'\phi(r)$$

for all $r > 0$. From the previous relations, we have

$$\begin{aligned}
|f(x_0) - P(x_0)| &\leq C_d\, r^{-d}\|f(x_0) - P(x_0)\|_{L^1(B(x_0,r))}\\
&\leq r^{-d}\|f(x_0) - f\|_{L^1(B(x_0,r))} + r^{-d}\|f - P\|_{L^1(B(x_0,r))}\\
&\quad + r^{-d}\|P - P(x_0)\|_{L^1(B(x_0,r))}\\
&\leq r^{-d}\|f(x_0) - f\|_{L^1(B(x_0,r))} + C'\phi(r)\\
&\quad + C_d \sum_{1\leq|\alpha|<\underline{b}(\phi)} \Big|\frac{D^\alpha P(x_0)}{\alpha!}\Big| r^{|\alpha|}.
\end{aligned}$$

But, as $\underline{b}(\phi) > 0$, Proposition 1.2.6 implies that $\phi(r)$ converges to 0 as r tends to 0^+. As a consequence, as x_0 is supposed to be a Lebesgue-point of f, the last upper bound in the previous inequality tends to 0 as r tends to 0^+, which implies $f(x_0) = P(x_0)$.

Let $f \in T^p_\phi(x_0)$ and

$$P := \sum_{|\alpha|<\underline{b}(\phi)} \frac{D^\alpha P(x_0)}{\alpha!}(x - x_0)^\alpha$$

be the polynomial that appears in (5.2). Let us set

$$|f|_{T^p_\phi(x_0)} := \sup_{r>0} \phi(r)^{-1} r^{-d/p}\|f - P\|_{L^p(B(x_0,r))}$$

and

$$\|f\|_{T^p_\phi(x_0)} := \|f\|_{L^p(\mathbb{R}^d)} + \sum_{|\alpha|<\underline{b}(\phi)} \frac{|D^\alpha P(x_0)|}{\alpha!} + |f|_{T^p_\phi(x_0)}.$$

Proposition 5.1.7. *Let $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$. The space $(T^p_\phi(x_0), \|\cdot\|_{T^p_\phi(x_0)})$ is a Banach space.*

Proof. It is straightforward to show that $\|\cdot\|_{T^p_\phi(x_0)}$ is a norm on $T^p_\phi(x_0)$.

Let us now consider a Cauchy sequence $(f_j)_{j\in\mathbb{N}}$ of $(T^p_\phi(x_0),\|\cdot\|_{T^p_\phi(x_0)})$. For $j\in\mathbb{N}$, let us denote by P_j the polynomial of degree strictly less than $\underline{b}(\phi)$ such that, for all $r>0$,

$$r^{-d/p}\|f_j-P_j\|_{L^p(B(x_0,r))}\le |f_j|_{T^p_\phi(x_0)}\phi(r).$$

Let $f\in L^p(\mathbb{R}^d)$ and $c_\alpha\in\mathbb{C}$ (for $|\alpha|<\underline{b}(\phi)$) be such that $f_j\to f$ in $L^p(\mathbb{R}^d)$ and $D^\alpha P_j(x_0)/\alpha!\to c_\alpha$ in $\mathbb{C}$ for all $|\alpha|<\underline{b}(\phi)$. Let us then define the polynomial P by

$$P:=\sum_{|\alpha|<\underline{b}(\phi)} c_\alpha(x-x_0)^\alpha.$$

For all $q\in\mathbb{N}$, we have

$$\begin{aligned}&\phi(r)^{-1}r^{-d/p}\|(f-f_q)-(P-P_q)\|_{L^p(B(x_0,r))}\\&\quad=\phi(r)^{-1}r^{-d/p}\lim_{s\to\infty}\|(f_s-f_q)-(P_s-P_q)\|_{L^p(B(x_0,r))}\\&\quad\le\limsup_{s\to\infty}\|f_q-f_s\|_{T^p_\phi(x_0)}<\infty.\end{aligned}$$

Taking the supremum over $r>0$ gives us

$$|f-f_q|_{T^p_\phi(x_0)}\le\limsup_{s\to\infty}\|f_q-f_s\|_{T^p_\phi(x_0)}<\infty$$

and passing to the limit for $q\to+\infty$ allows us to get

$$\lim_{q\to+\infty}|f-f_q|_{T^p_\phi(x_0)}=0,$$

which is enough to conclude, as the finiteness of $|f|_{T^p_\phi(x_0)}$ follows from triangular inequality. □

Proposition 5.1.8. *Let $x_0\in\mathbb{R}^d$, $p\in[1,\infty]$ and $\phi\in\mathcal{B}$ be such that $\underline{b}(\phi)>-d/p$; $t^p_\phi(x_0)$ is a closed subspace of $T^p_\phi(x_0)$.*

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of functions in $t^p_\phi(x_0)$ for which there exists $f\in T^p_\phi(x_0)$ such that $f_j\to f$ in $T^p_\phi(x_0)$ and let us show that $f\in t^p_\phi(x_0)$. Let P and P_j ($j\in\mathbb{N}$) be polynomials of degree strictly less than $\underline{b}(\phi)$ such that

$$r^{-d/p}\|f_j-P_j\|_{L^p(B(x_0,r))}\le|f_j|_{T^p_\phi(x_0)}\phi(r)\qquad\forall j\in\mathbb{N}$$

and

$$r^{-d/p}\|f-P\|_{L^p(B(x_0,r))}\le|f|_{T^p_\phi(x_0)}\phi(r).$$

If we set $R:=f-P$ and $R_j:=f_j-P_j$, we know that

$$\sup_{r>0}\phi(r)^{-1}r^{-d/p}\|R_j-R\|_{L^p(B(x_0,r))}\le\|f_j-f\|_{T^p_\phi(x_0)}\to 0\qquad\text{as } j\to\infty$$

and

$$\phi(r)^{-1}r^{-d/p}\|R_j\|_{L^p(B(x_0,r))} \to 0 \qquad \text{as } r \to 0^+.$$

Given $\varepsilon > 0$, let $J \in \mathbb{N}$ be such that $j \geq J$ implies

$$\sup_{r>0} \phi(r)^{-1}r^{-d/p}\|R_j - R\|_{L^p(B(x_0,r))} < \frac{\varepsilon}{2}.$$

There also exists ρ_J such that, for all $r \in (0, \rho_J]$,

$$\phi(r)^{-1}r^{-d/p}\|R_J\|_{L^p(B(x_0,r))} < \frac{\varepsilon}{2}.$$

As a consequence, we have, for such r,

$$\phi(r)^{-1}r^{-d/p}\|R\|_{L^p(B(x_0,r))} < \varepsilon,$$

which proves that $f \in t^p_\phi(x_0)$. □

There is an obvious link, given by the following remark, between the classical spaces C^k of k-times continuously differentiable functions and the spaces $t^p_\phi(x_0)$.

Remark 5.1.9. Let $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$. First, if $\overline{b}(\phi) < 0$ and $f \in C^0(V)$, where V is an open neighbourhood of x_0, then $f \in t^p_\phi(x_0)$. Indeed, if $R > 0$ is such that $B(x_0,R) \subseteq V$ then there exists $C > 0$ such that $|f| \leq C$ on $B(x_0,R)$ and, for $r \in (0,R]$, we have

$$r^{-d/p}\|f\|_{L^p(B(x_0,r))} \leq C.$$

It follows from Proposition 1.2.6 that

$$r^{-d/p}\|f\|_{L^p(B(x_0,r))} \in o(\phi(r)) \qquad \text{as } r \to 0^+.$$

Also, if there exists $n \in \mathbb{N}$ such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$ and $f \in C^{n+1}(V)$, then again $f \in t^p_\phi(x_0)$. Let P be the Taylor expansion of order n of f at x_0. There exists $C > 0$ such that $|f - P| \leq C(\cdot - x_0)^{n+1}$ on $B(x_0,R)$. Therefore

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq Cr^{n+1},$$

for $r \in (0,R]$ and the conclusion comes again from Proposition 1.2.6.

5.2 A density result

Let φ be a non-negative, real-valued function in $\mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \varphi(x)\,dx = 1 \qquad \text{and} \qquad \operatorname{supp}(\varphi) \subset \overline{B(0,1)}.$$

Let f be a function that belongs to $L^p(\mathbb{R}^d)$ for some $p \in [1,\infty)$ and, given $\lambda > 0$, define f_λ by

$$f_\lambda := \lambda^d \varphi(\lambda \cdot) * f. \tag{5.4}$$

It is well known that $f_\lambda \in L^p(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and $\|f_\lambda - f\|_{L^p(\mathbb{R}^d)} \to 0$ as $\lambda \to \infty$. Let us show that if $f \in t^p_\phi(x_0)$, under some basic assumptions on ϕ, then the convergence also holds in $T^p_\phi(x_0)$.

Proposition 5.2.1. *Let $x_0 \in \mathbb{R}^d$, $p \in [1,\infty)$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$ and either $\underline{b}(\phi) \leq 0$ or there exists $n \in \mathbb{N}$ such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$. If a function f belongs to $t^p_\phi(x_0)$, then $\|f_\lambda - f\|_{T^p_\phi(x_0)} \to 0$ as $\lambda \to \infty$.*

Proof. Without loss of generality, we can suppose that $x_0 = 0$. Let us first consider the case where there exists $n \in \mathbb{N}$ such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$. Given $\lambda > 0$, define $R_\lambda := f_\lambda - P_\lambda$ where P_λ is the Taylor expansion of order n of f_λ at 0. Let $R := f - P$, where P is a polynomial of degree n, be such that

$$\phi(r)^{-1} r^{-d/p} \|R\|_{L^p(B(0,r))} \to 0 \qquad \text{as } r \to 0^+.$$

For $r > 0$, we have

$$r^{-d}\|R\|_{L^1(B(0,r))} \leq C_d r^{-d/p}\|R\|_{L^p(B(0,r))} \leq \varepsilon(r)\phi(r),$$

where $\varepsilon(r) \to 0$ as $r \to 0^+$. We can make the assumption that $\varepsilon(r)$ is decreasing to 0 as $r \to 0^+$.

Let us remark that, for $|\alpha| \leq n$, we have $D^\alpha P_\lambda(0) \to D^\alpha P(0)$ as $\lambda \to \infty$. Indeed, for $\lambda > 0$,

$$\begin{aligned} D^\alpha P_\lambda(0) &= D^\alpha f_\lambda(0) \\ &= \int_{\mathbb{R}^d} \lambda^d \varphi(-\lambda y) D^\alpha P(y)\, dy + \int_{\mathbb{R}^d} (-1)^{|\alpha|} \lambda^{d+|\alpha|} D^\alpha \varphi(-\lambda y) R(y)\, dy. \end{aligned}$$

The first term of the right-hand side tends to $D^\alpha P(0)$ as λ tends to infinity and for the second term, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^d} (-1)^{|\alpha|} \lambda^{d+|\alpha|} D^\alpha \varphi(-\lambda y) R(y)\, dy\right| &\leq C_\varphi \lambda^{d+|\alpha|} \int_{B(0,\frac{1}{\lambda})} |R(y)|\, dy \\ &\leq \varepsilon(\frac{1}{\lambda}) \lambda^{|\alpha|} \phi(\frac{1}{\lambda}), \end{aligned}$$

which proves, since $|\alpha| < \underline{b}(\phi)$, that $\int_{\mathbb{R}^d} (-1)^{|\alpha|} \lambda^{d+|\alpha|} D^\alpha \varphi(-\lambda y) R(y)\, dy$ tends to 0 as $\lambda \to \infty$.

Given $r > 0$ and $\lambda > 0$, let us now estimate the quantity $\|R_\lambda\|_{L^p(B(x_0,r))}$. For all $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
R_\lambda(x) &= f_\lambda(x) - P_\lambda(x) \\
&= \int_{\mathbb{R}^d} (\lambda^d \varphi(\lambda(x-y)) - \sum_{|\alpha|\le n} \lambda^{d+|\alpha|} \frac{D^\alpha \varphi(-\lambda y)}{\alpha!} x^\alpha)(P(y) + R(y))\, dy
\end{aligned}$$

and as

$$\int_{\mathbb{R}^d} \lambda^d \varphi(\lambda(x-y)) P(y) - \sum_{|\alpha|\le n} \lambda^{d+|\alpha|} \frac{D^\alpha \varphi(-\lambda y)}{\alpha!} P(y) x^\alpha \, dy$$

is equal to $\lambda^d \varphi(\lambda\cdot) * P$ (which is a polynomial of degree n) minus its Taylor expansion of order n at 0, this last integral is equal to 0. Therefore,

$$R_\lambda(x) = \int_{\mathbb{R}^d} (\lambda^d \varphi(\lambda(x-y)) - \sum_{|\alpha|\le n} \lambda^{d+|\alpha|} \frac{D^\alpha \varphi(-\lambda y)}{\alpha!} x^\alpha) R(y)\, dy.$$

It follows, by Young's inequality, that

$$\begin{aligned}
\|R_\lambda\|_{L^p(B(0,r))} &\le C_\varphi \|R\|_{L^p(B(0,2r))} \\
&\quad + \sum_{\alpha \le n} \lambda^{d+|\alpha|} \|R D^\alpha \varphi(-\lambda\cdot)\|_{L^1(B(0,1/\lambda))} \|\cdot^\alpha\|_{L^p(B(0,r))} \\
&\le C'_\varphi (r^{d/p} \varepsilon(2r)\phi(r) + \sum_{\alpha\le n} \varepsilon(\frac{1}{\lambda}) \lambda^{|\alpha|} \phi(\frac{1}{\lambda}) r^{\frac{d}{p}+|\alpha|}),
\end{aligned}$$

for all $r \ge 1/\lambda$. But, as $\phi(1/\lambda) \le \phi(r)\overline{\phi}(\frac{1}{r\lambda})$ and $\frac{1}{r\lambda} \le 1$, we have, thanks to Remark 1.2.9,

$$\overline{\phi}(\frac{1}{r\lambda})(r\lambda)^{|\alpha|} \le C_\delta (r\lambda)^{-(\underline{b}(\phi)-\delta-|\alpha|)} \le C_\delta,$$

where $\delta > 0$ has been chosen such that $\underline{b}(\phi) - \delta - n \ge 0$. Consequently, given $r, \lambda > 0$ such that $r \ge 1/\lambda$, we have

$$\|R_\lambda\|_{L^p(B(0,r))} \le C r^{d/p} \varepsilon(2r)\phi(r). \tag{5.5}$$

On the other hand, if $r < 1/\lambda$, Taylor's formula provides the following relation:

$$|\lambda^d \varphi(\lambda(x-y)) - \sum_{|\alpha|\le n} \lambda^{d+|\alpha|} \frac{D^\alpha \varphi(-\lambda y)}{\alpha!} x^\alpha| \le C_\varphi (\lambda|x|)^{n+1} \lambda^d,$$

which implies

$$\begin{aligned}
|R_\lambda(x)| &\le C_\varphi (\lambda|x|)^{n+1} \lambda^d \int_{B(0,\frac{2}{\lambda})} |R(y)|\, dy \\
&\le C_{\varphi,d} (\lambda|x|)^{n+1} \varepsilon(\frac{2}{\lambda}) \phi(\frac{2}{\lambda}),
\end{aligned}$$

for all $x \in B(0,r)$. Therefore, we have

$$\|R_\lambda\|_{L^p(B(0,r))} \leq Cr^{d/p}(\lambda r)^{n+1}\varepsilon(\frac{2}{\lambda})\phi(\frac{1}{\lambda}).$$

Now, using the second part of Remark 1.2.9, we can write

$$\begin{aligned}(\lambda r)^{n+1}\phi(\frac{1}{\lambda}) &\leq \phi(r)(\lambda r)^{n+1}\overline{\phi}(\frac{1}{r\lambda})\\ &\leq C_{\delta'}\phi(r)(r\lambda)^{(n+1-\overline{b}(\phi)-\delta')}\\ &\leq C_{\delta'}\phi(r),\end{aligned}$$

where $\delta' > 0$ has been chosen such that $n+1-\overline{b}(\phi)-\delta' \geq 0$. As a consequence, given $R, \lambda > 0$ such that $r < 1/\lambda$, we have

$$\|R_\lambda\|_{L^p(B(0,r))} \leq Cr^{d/p}\varepsilon(\frac{2}{\lambda})\phi(r). \tag{5.6}$$

From relations (5.5) and (5.6), we have

$$\phi(r)^{-1}r^{-d/p}\|R_\lambda\|_{L^p(B(0,r))} \leq C(\varepsilon(2r)+\varepsilon(\frac{2}{\lambda})),$$

for all $r, \lambda > 0$, which naturally implies

$$\phi(r)^{-1}r^{-d/p}\|R - R_\lambda\|_{L^p(B(0,r))} \leq C(\varepsilon(2r)+\varepsilon(\frac{2}{\lambda})). \tag{5.7}$$

Let us now remark that, if we fix $\rho > 0$ and choose $\eta > 0$ such that

$$\underline{b}(\phi) - \eta > n,$$

then, from Proposition 1.2.6, we have

$$\begin{aligned}&\phi(r)^{-1}r^{-d/p}\|R - R_\lambda\|_{L^p(B(0,r))}\\ &\leq \phi(r)^{-1}r^{-d/p}\|f - f_\lambda\|_{L^p(B(0,r))}\\ &\quad + C_d\sum_{|\alpha|\leq n}\frac{|D^\alpha P(x_0) - D^\alpha P_\lambda(x_0)|}{\alpha!}\phi(r)^{-1}r^{|\alpha|}\\ &\leq C_\rho r^{(-\underline{b}(\phi)+\eta-\frac{d}{p})}\|f - f_\lambda\|_{L^p(\mathbb{R}^d)}\\ &\quad + C_{d,\rho}\sum_{|\alpha|\leq n}\frac{|D^\alpha P(x_0) - D^\alpha P_\lambda(x_0)|}{\alpha!}r^{(-\underline{b}(\phi)+\eta+|\alpha|)}\\ &\leq C_\rho \rho^{(-\underline{b}(\phi)+\eta-\frac{d}{p})}\|f - f_\lambda\|_{L^p(\mathbb{R}^d)}\\ &\quad + C_{d,\rho}\sum_{|\alpha|\leq n}\frac{|D^\alpha P(x_0) - D^\alpha P_\lambda(x_0)|}{\alpha!}\rho^{(-\underline{b}(\phi)+\eta+|\alpha|)},\end{aligned}$$

for all $r > \rho$. As we know that $\|f - f_\lambda\|_{L^p(\mathbb{R}^d)} \to 0$ and $D^\alpha P_\lambda(0) \to D^\alpha P(0)$ as $\lambda \to \infty$, for all $|\alpha| \leq n$, we get that

$$\sup_{r\geq\rho} \phi(r)^{-1} r^{-d/p} \|R - R_\lambda\|_{L^p(B(0,r))} \to 0 \qquad \text{as } \lambda \to \infty. \tag{5.8}$$

Gathering (5.7) and(5.8) leads to

$$\sup_{r>0} \phi(r)^{-1} r^{-d/p} \|R - R_\lambda\|_{L^p(B(0,r))} \to 0 \qquad \text{as } \lambda \to \infty, \tag{5.9}$$

since otherwise there exists $\xi > 0$ such that for all $\Lambda > 0$ there exists $\lambda > \Lambda$ for which

$$\sup_{r>0} \phi(r)^{-1} r^{-d/p} \|R - R_\lambda\|_{L^p(B(0,r))} \geq \xi,$$

which makes us able to build a sequence $(\lambda_j)_{j\in\mathbb{N}}$ that converges to ∞ and satisfying

$$\sup_{r>0} \phi(r)^{-1} r^{-d/p} \|R - R_{\lambda_j}\|_{L^p(B(0,r))} \geq \xi,$$

for all j. In particular, given $j \in \mathbb{N}$, there exists $r_j > 0$ such that

$$\phi(r_j)^{-1} r_j^{-d/p} \|R - R_{\lambda_j}\|_{L^p(B(0,r_j))} \geq \frac{\xi}{2}. \tag{5.10}$$

As $\lambda_j \to \infty$, there exists $J_1 \in \mathbb{N}$ such that for all $j \geq J_1$, $\varepsilon(2/\lambda_j) < \xi/(4C)$, where $C > 0$ is the constant appearing in (5.7). Moreover, there also exists $\rho > 0$ such that, for any $r \in (0, \rho]$, $\varepsilon(2r) < \frac{\xi}{4C}$. From (5.8), we know that there exists $J_2 \in \mathbb{N}$ such that, for all $j \geq J_2$,

$$\sup_{r>\rho} \phi(r)^{-1} r^{-d/p} \|R - R_{\lambda_j}\|_{L^p(B(0,r))} < \frac{\xi}{2}. \tag{5.11}$$

Therefore, if $j \geq \max\{J_1, J_2\}$, (5.11) implies $r_j \leq \rho$ and, by (5.7) and (5.10), we finally get a contradiction.

If we now assume that $\overline{b}(\phi) \leq 0$, then $R = f$ and $R_\lambda = f_\lambda$. Therefore, by Young's inequality, we have

$$\|R_\lambda\|_{L^p(B(0,r))} \leq C_\varphi \|R\|_{L^p(B(0,2r))} \leq C\varepsilon(2r)\phi(r).$$

If $r \leq 1/\lambda$, let us recall that $\varepsilon(2r) \leq \varepsilon(2/\lambda)$. As a consequence, relations (5.5), (5.6) and so (5.7) still hold and we can conclude in the same way, using the fact that

$$\phi(r)^{-1} r^{-d/p} \|R - R_\lambda\|_{L^p(B(0,r))} = \phi(r)^{-1} r^{-d/p} \|f - f_\lambda\|_{L^p(B(0,r))}$$

and $\underline{b}(\phi) > \frac{-d}{p}$. □

The last proposition admits the following useful corollary.

Corollary 5.2.2. *Under the assumptions of the preceding proposition, the space $\mathcal{D}(\mathbb{R}^d)$ is a dense subspace of $t^p_\phi(x_0)$.*

Proof. Let us consider $f \in t^p_\phi(x_0)$ and the sequence of functions $(f_j)_{j\in\mathbb{N}}$ defined by

$$f_j := f\chi_{\overline{B(0,2^j)}} \qquad (j \in \mathbb{N}).$$

By Lebesgue's dominated convergence theorem, it is clear that $f_j \to f$ in $L^p(\mathbb{R}^d)$; we will show that f_j belongs to $t^p_\phi(x_0)$ ($j \in \mathbb{N}$) and that the convergence also holds in $T^p_\phi(x_0)$.

Let P be the polynomial of degree strictly less than $\underline{b}(\phi)$ such that

$$\phi(r)^{-1}r^{-d/p}\|f-P\|_{L^p(B(x_0,r))} \to 0 \qquad \text{as } r \to 0^+.$$

First, as $f_j = f$ on $B(x_0,1)$, we have

$$\phi(r)^{-1}r^{-d/p}\|f_j-P\|_{L^p(B(x_0,r))} \to 0 \qquad \text{as } r \to 0^+,$$

for any $j \in \mathbb{N}$. Therefore, given $j \in \mathbb{N}$, $f_j \in t^p_\phi(x_0)$ and

$$\|f-f_j\|_{T^p_\phi(x_0)} = \|f-f_j\|_{L^p(\mathbb{R}^d)} + \sup_{r>0}\phi(r)^{-1}r^{-d/p}\|f_j-f\|_{L^p(B(x_0,r))}.$$

On the one hand, if $r \in (0,2^j]$ then

$$\phi(r)^{-1}r^{-d/p}\|f_j-f\|_{L^p(B(x_0,r))} = 0$$

and, on the other hand, if $r > 2^j$, by Proposition 1.2.6,

$$\begin{aligned}\phi(r)^{-1}r^{-d/p}\|f_j-f\|_{L^p(B(x_0,r))} &\le Cr^{-(\underline{b}(\phi)-\varepsilon+\frac{d}{p})}\|f_j-f\|_{L^p(\mathbb{R}^d)}\\ &\le C2^{-j(\underline{b}(\phi)-\varepsilon+\frac{d}{p})}\|f_j-f\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

where $\varepsilon > 0$ is such that $\underline{b}(\phi)-\varepsilon+\frac{d}{p} \ge 0$ and $C > 0$ is such that $r^{(\underline{b}(\phi)-\varepsilon)} \le C\phi(r)$ for all $r \ge 1$. Therefore, we have

$$\|f-f_j\|_{T^p_\phi(x_0)} \le \|f-f_j\|_{L^p(\mathbb{R}^d)} + C2^{-j(\underline{b}(\phi)-\varepsilon+\frac{d}{p})}\|f_j-f\|_{L^p(\mathbb{R}^d)} \to 0,$$

as $j \to \infty$, which provides the convergence in $T^p_\phi(x_0)$.

The conclusion then follows from Proposition 5.2.1. $\square$

5.3 Some embeddings

Notation 5.3.1. Given $\phi,\psi \in \mathcal{B}$, we will write $\phi \preccurlyeq \psi$ to mean that there exist $R, C > 0$ such that, for all $r \in (0,R)$, we have $\phi(r) \le C\psi(r)$.

Of course, by continuity, one has $\phi \preccurlyeq \psi$ if and only if, for all $R > 0$, there exists $C > 0$ such that $\phi(r) \leq C\psi(r)$ for all $r \in (0,R)$.

Proposition 5.3.2. *Let $\phi, \psi \in \mathcal{B}$; if $\overline{b}(\psi) < \underline{b}(\phi)$ then $\phi \preccurlyeq \psi$. Conversely, if $\phi \preccurlyeq \psi$, then $\underline{b}(\psi) \leq \overline{b}(\phi)$.*

Proof. Let us first assume that $\overline{b}(\psi) < \underline{b}(\phi)$ and let $\varepsilon > 0$ be such that

$$\overline{b}(\psi) + \varepsilon < \underline{b}(\phi) - \varepsilon.$$

By Proposition 1.2.6, given $R > 0$, there exists $C > 0$ such that for all $r \in (0,R)$,

$$\phi(r) \leq C r^{\underline{b}(\phi)-\varepsilon} \leq C' r^{\overline{b}(\psi)+\varepsilon} \leq C'' \psi(r),$$

which means $\phi \preccurlyeq \psi$.

If we now assume $\phi \preccurlyeq \psi$ then, in particular, there exists $C > 0$ such that for all $r \in (0,1)$,

$$\overline{\phi}(1/r)^{-1} \leq C\overline{\psi}(r).$$

Therefore, for such r, we have

$$\frac{\log(\overline{\phi}(1/r))}{\log(1/r)} \geq \frac{\log(C)}{\log(r)} + \frac{\log(\overline{\psi}(r)}{\log(r)}$$

and taking the limit as $r \to 0^+$ gives $\overline{b}(\phi) \geq \underline{b}(\psi)$. □

Proposition 5.3.3. *Let $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and $\phi, \psi \in \mathcal{B}$ be such that either $\overline{b}(\psi) < 0$ or there exists $n \in \mathbb{N}$ for which $n < \underline{b}(\psi) \leq \overline{b}(\psi) < n+1$; if $\phi \preccurlyeq \psi$, then $T^p_\phi(x_0) \hookrightarrow T^p_\psi(x_0)$. Moreover, if $\phi(r) \in o(\psi(r))$ as $r \to 0^+$, then $T^p_\phi(x_0) \hookrightarrow t^p_\psi(x_0)$.*

Proof. Let $f \in T^p_\phi(x_0)$; there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ such that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq |f|_{T^p_\phi(x_0)}\phi(r) \qquad \forall\, r > 0.$$

Let $Q = 0$, $k = l = 0$ if $\overline{b}(\psi) < 0$ and

$$Q = \sum_{|\alpha| \leq n} \frac{D^\alpha P(x_0)}{\alpha!}(\cdot - x_0)^\alpha,$$

$k = n+1$, $l = n$ if $n \in \mathbb{N}$ satisfies $n < \underline{b}(\psi) \leq \overline{b}(\psi) < n+1$. For any $r \leq 1$, we obviously have, by Proposition 1.2.6,

$$\begin{aligned} r^{-d/p}\|f - Q\|_{L^p(B(x_0,r))} &\leq r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} + r^{-d/p}\|P - Q\|_{L^p(B(x_0,r))} \\ &\leq |f|_{T^p_\phi(x_0)}\phi(r) + C_d\|f\|_{T^p_\phi(x_0)} r^k \\ &\leq C_{\phi,\psi}\|f\|_{T^p_\phi(x_0)}\psi(r), \end{aligned}$$

while for $r > 1$,

$$\begin{aligned} r^{-d/p}\|f-Q\|_{L^p(B(x_0,r))} &\le r^{-d/p}\|f\|_{L^p(B(x_0,r))} + r^{-d/p}\|Q\|_{L^p(B(x_0,r))} \\ &\le r^{-d/p}\|f\|_{L^p(\mathbb{R}^d)} + C_{d,p}\|f\|_{T^p_\phi} r^l \\ &\le C_\phi \|f\|_{T^p_\phi(x_0)}\psi(r), \end{aligned}$$

which leads to the first part of the proposition.

The second part comes from the inequality

$$r^{-d/p}\|f-Q\|_{L^p(B(x_0,r))} \le |f|_{T^p_\phi(x_0)}\phi(r) + C_d\|f\|_{T^p_\phi(x_0)} r^k,$$

valid for all $0 < r \le 1$ and the relations $\phi(r) \in o(\psi(r))$ and $r^k \in o(\phi(r))$. □

Proposition 5.3.4. *Let $x_0 \in \mathbb{R}^d$, $p_1, p_2 \in [1,\infty]$, p_3 be such that*

$$0 \le \frac{1}{p_3} := \frac{1}{p_1} + \frac{1}{p_2} \le 1$$

and $\phi \in \mathcal{B}$ be such that there exists $n \in \mathbb{N}$ for which $n < \underline{b}(\phi) \le \overline{b}(\phi) < n+1$. Given $f_1 \in T^{p_1}_\phi(x_0)$ and $f_2 \in T^{p_2}_\phi(x_0)$, we have $f_1 f_2 \in T^{p_3}_\phi(x_0)$, with

$$\|f_1 f_2\|_{T^{p_3}_\phi(x_0)} \le C_{d,p_1,p_2,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)}.$$

Moreover, if $f_1 \in t^{p_1}_\phi(x_0)$ and $f_2 \in t^{p_2}_\phi(x_0)$, then $f_1 f_2 \in t^{p_3}_\phi(x_0)$.

Proof. We know that, given $k \in \{1,2\}$, there exists a polynomial P_k of degree less or equal to n such that $R_k := f_k - P_k$ satisfies

$$r^{-d/p_k}\|R_k\|_{L^{p_k}(B(x_0,r))} \le |f_k|_{T^{p_k}_\phi(x_0)}\phi(r). \tag{5.12}$$

Therefore, if we denote by P the sum of the terms of degree less than or equal to n in P_1P_2, we have

$$f_1 f_2 = P_1P_2 + R_1P_2 + R_2 f_1 = P + P_1P_2 - P + R_1P_2 + R_2 f_1.$$

Let $R := P_1P_2 - P + R_1P_2 + R_2 f_1$; clearly,

$$\sum_{|\alpha|\le n}\frac{|D^\alpha P(x_0)|}{\alpha!} \le \|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)}.$$

Let us first consider $r \le 1$; by Proposition 1.2.6, since

$$|P_1P_2(x) - P(x)| \le (x-x_0)^{n+1}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)},$$

for $x \in B(x_0, r)$, we have

$$\begin{aligned} r^{-d/p_3}\|P_1P_2 - P\|_{L^{p_3}(B(x_0,r))} &\leq C_{d,p_3}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\phi^{p_2}(x_0)}r^{n+1} \\ &\leq C_{d,p_1,p_2,\phi}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\phi^{p_2}(x_0)}\phi(r). \end{aligned}$$

Also, for all $x \in B(x_0, r)$, since $|P_k(x)| \leq \|f_k\|_{T_\phi^{p_k}(x_0)}$ ($k \in \{1, 2\}$),

$$\begin{aligned} r^{-d/p_3}\|R_1P_2\|_{L^{p_3}(B(x_0,r))} &\leq r^{-d/p_2}\|P_2\|_{L^{p_2}(B(x_0,r))}r^{-d/p_1}\|R_1\|_{L^{p_1}(B(x_0,r))} \\ &\leq C_{d,p_2}\|f_2\|_{T_\phi^{p_2}(x_0)}|f_1|_{T_\phi^{p_1}(x_0)}\phi(r). \end{aligned}$$

Using again Proposition 1.2.6, we get

$$\begin{aligned} r^{-d/p_1}\|f_1\|_{L^{p_1}(B(x_0,r))} &\leq r^{-d/p_1}\|f_1 - P_1\|_{L^{p_1}(B(x_0,r))} + r^{-d/p_1}\|P_1\|_{L^{p_1}(B(x_0,r))} \\ &\leq |f_1|_{T_\phi^{p_1}(x_0)}\phi(r) + C_{d,p_1}\|f_1\|_{T_\phi^{p_1}(x_0)}r^n \\ &\leq C_{d,p_1,\phi}\|f_1\|_{T_\phi^{p_1}(x_0)} \end{aligned}$$

and thus

$$\begin{aligned} r^{-d/p_3}\|f_1R_2\|_{L^{p_3}(B(x_0,r))} &\leq r^{-d/p_1}\|f_1\|_{L^{p_1}(B(x_0,r))}r^{-d/p_2}\|R_2\|_{L^{p_2}(B(x_0,r))} \\ &\leq C_{d,p_1,\phi}\|f_1\|_{T_\phi^{p_1}(x_0)}|f_2|_{T_\phi^{p_2}(x_0)}\phi(r). \end{aligned}$$

As a consequence, we can write, for $r < 1$,

$$r^{-d/p_3}\|R\|_{L^{p_3}(B(x_0,r))} \leq C_{d,p_1,p_2,\phi}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\phi^{p_2}(x_0)}\phi(r). \tag{5.13}$$

If we now consider $r > 1$, as $|R| \leq |f_1||f_2| + |P|$, we get

$$\begin{aligned} &r^{-d/p_3}\|R\|_{L^{p_3}(B(x_0,r))} \\ &\leq r^{-d/p_3}\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)} + C_{d,p}r^n\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\phi^{p_2}(x_0)}, \end{aligned}$$

so that inequality (5.13) still holds in this case, by Proposition 1.2.6.

Finally, if $f_1 \in t_\phi^{p_1}(x_0)$ and $f_2 \in t_\phi^{p_2}(x_0)$, we can write

$$r^{-d/p_k}\|R_k\|_{L^{p_k}(B(x_0,r))} \leq \varepsilon_k(r)\phi(r),$$

with $\varepsilon_k(r) > 0$ for $r > 0$ and $\varepsilon_k(r) \to 0$ as $r \to 0^+$ ($k \in \{1, 2\}$). By replacing $|f_k|_{T_\phi^{p_k}(x_0)}$ with $\varepsilon_k(r)$ in the preceding relations, one gets

$$\phi(r)^{-1}r^{-d/p_3}\|R\|_{L^{p_3}(B(x_0,r))} \to 0^+,$$

as $r \to 0^+$, which is sufficient to conclude. □

Corollary 5.3.5. *Let $x_0 \in \mathbb{R}^d$, $p_1, p_2 \in [1,\infty]$, p_3 be such that*

$$0 \leq \frac{1}{p_3} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1$$

and ϕ, ψ be two functions of $\mathcal{B}$ satisfying $\underline{b}(\phi) > 0$, $\underline{b}(\psi) \geq -\frac{d}{p_2}$, $\phi \preccurlyeq \psi$ and either $\underline{b}(\psi) \leq 0$ or $n < \underline{b}(\psi) \leq \overline{b}(\psi) < n+1$ for some $n \in \mathbb{N}$. If $f_1 \in T_\phi^{p_1}(x_0)$ and $f_2 \in T_\psi^{p_2}(x_0)$, then $f_1 f_2 \in T_\psi^{p_3}(x_0)$, with

$$\|f_1 f_2\|_{T_\psi^{p_3}(x_0)} \leq C_{d,p_1,p_2,\phi,\psi} \|f_1\|_{T_\phi^{p_1}(x_0)} \|f_2\|_{T_\psi^{p_2}(x_0)}.$$

Moreover, if $f_1 \in t_\phi^{p_1}(x_0)$ and $f_2 \in t_\psi^{p_2}(x_0)$, then $f_1 f_2 \in t_\psi^{p_3}(x_0)$.

Proof. If $\underline{b}(\psi) \leq 0$, the embedding is obvious since $T_\phi^p(x_0) \hookrightarrow t_0^p(x_0)$ and so, for $r > 0$,

$$\begin{aligned} r^{-d/p_3} \|f_1 f_2\|_{L^{p_3}(B(x_0,r))} &\leq r^{-d/p_1} \|f_1\|_{L^{p_1}(B(x_0,r))} r^{-d/p_2} \|f_2\|_{L^{p_2}(B(x_0,r))} \\ &\leq C_{p_1,\phi,0} \|f_1\|_{T_\phi^{p_1}(x_0)} |f_2|_{T_\psi^{p_2}(x_0)} \psi(r). \end{aligned}$$

Otherwise, we have $\underline{b}(\psi) > 0$ and $f_1 \in T_\psi^{p_1}(x_0)$, with

$$\|f_1\|_{T_\psi^{p_1}(x_0)} \leq C_{\phi,\psi} \|f_1\|_{T_\phi^{p_1}(x_0)}.$$

Using the previous proposition, we get $f_1 f_2 \in T_\psi^{p_3}(x_0)$ and

$$\begin{aligned} \|f_1 f_2\|_{T_\psi^{p_3}(x_0)} &\leq C_{d,p,\psi} \|f_1\|_{T_\psi^{p_1}(x_0)} \|f_2\|_{T_\psi^{p_2}(x_0)} \\ &\leq C_{d,p,\phi,\psi} \|f_1\|_{T_\phi^{p_1}(x_0)} \|f_2\|_{T_\psi^{p_2}(x_0)}, \end{aligned}$$

which allows us to conclude. The second part can be obtained using the usual arguments. □

Proposition 5.3.6. *Let $p_1, p_2 \in [1,\infty]$, p_3 be such that $0 \leq \frac{1}{p_3} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\phi, \varphi \in \mathcal{B}$ be such that $-\frac{d}{p_2} \leq \underline{b}(\varphi)$, $0 < \underline{b}(\phi)$. Let also $f_1 \in T_\phi^{p_1}(x_0)$, $f_2 \in T_\varphi^{p_2}(x_0)$, where x_0 is a Lebesgue-point of f_1; finally let $\psi \in \mathcal{B}$ be such that $\underline{b}(\psi) > -\frac{d}{p_2}$, $\phi \preccurlyeq \psi$ and*

- $\overline{b}(\psi) - \underline{b}(\varphi) < \underline{b}(\phi)$ *if $\underline{b}(\phi) \leq 1$,*
- $\overline{b}(\psi) - \underline{b}(\varphi) < 1$ *if $\underline{b}(\phi) > 1$ and either $\overline{b}(\psi) < 1$ or there exists $n \in \mathbb{N}$ for which $n < \underline{b}(\psi) \leq \overline{b}(\psi) < n+1$.*

There exists a polynomial P of degree strictly less than $\underline{b}(\psi)$ such that, for all $r > 0$,

$$\begin{aligned} &r^{-d/p_3} \|(f_1 - f_1(x_0)) f_2 - P\|_{L^{p_3}(B(x_0,r))} \\ &\leq C_{p_1,p_2,\phi,\varphi,\psi} \|f_1\|_{T_\phi^{p_1}(x_0)} \|f_2\|_{T_\varphi^{p_2}(x_0)} \psi(r). \end{aligned}$$

Consequently, if $f_2 \in L^{p_3}(\mathbb{R}^d)$, then $(f_1 - f_1(x_0)) f_2$ belongs to $T_\psi^{p_3}(x_0)$, with

$$\begin{aligned} &\|(f_1 - f_1(x_0)) f_2\|_{T_\psi^{p_3}(x_0)} \\ &\leq C_{p_1,p_2,\phi,\varphi,\psi} \|f_1\|_{T_\varphi^{p_1}(x_0)} (\|f_2\|_{T_\varphi^{p_2}(x_0)} + \|f_2\|_{L^{p_3}(\mathbb{R}^d)}). \end{aligned}$$

Proof. We keep here the same notations as in the proof of Proposition 5.3.4 and set $g_1 := f_1 - f_1(x_0)$. Let us first consider the case $\underline{b}(\phi) \le 1$; P_1 must be a constant and, by Remark 5.1.6, we have $P_1 = f_1(x_0)$, which allows us to write

$$r^{-d/p_1}\|g_1\|_{L^{p_1}(B(x_0,r))} \le |f_1|_{T^{p_1}_\phi(x_0)}\phi(r). \tag{5.14}$$

Let us consider each case separately. If $\underline{b}(\varphi) \le 0$, then

$$r^{-d/p_2}\|f_2\|_{L^{p_2}(B(x_0,r))} \le |f_2|_{T^{p_2}_\varphi(x_0)}\varphi(r).$$

Therefore, if $\psi \in \mathcal{B}$ is such that $\overline{b}(\psi) < \underline{b}(\phi) + \underline{b}(\varphi)$, then, by choosing $\varepsilon > 0$ such that $\overline{b}(\psi) + \varepsilon < \underline{b}(\phi) + \underline{b}(\varphi) - 2\varepsilon$, we get, by Proposition 1.2.6,

$$\begin{aligned}
r^{-\frac{d}{p_3}}\|g_1 f_2\|_{L^{p_3}(B(x_0,r))} &\le r^{-\frac{d}{p_1}}\|g_1\|_{L^{p_1}(B(x_0,r)}r^{-\frac{d}{p_2}}\|f_2\|_{L^{p_2}(B(x_0,r))}\\
&\le |f_1|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\varphi(x_0)}\phi(r)\varphi(r)\\
&\le C|f_1|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\varphi(x_0)}r^{\underline{b}(\phi)+\underline{b}(\varphi)-2\varepsilon}\\
&\le C'\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\varphi(x_0)}\psi(r),
\end{aligned}$$

for $0 < r \le 1$, where $C, C' > 0$ only depend on ϕ, φ and ψ. If $r > 1$, as $-d/p_2 < \underline{b}(\psi)$, we can use Proposition 1.2.6 to get

$$\begin{aligned}
&r^{-d/p_3}\|g_1 f_2\|_{L^{p_3}(B(x_0,r))}\\
&\le r^{-d/p_3}\|f_1 f_2\|_{L^{p_3}(B(x_0,r))} + r^{-d/p_3}|f_1(x_0)|\|f_2\|_{L^{p_3}(B(x_0,r))}\\
&\le r^{-d/p_3}\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)} + C_{p_2,p_3}r^{-d/p_2}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{L^{p_2}(B(x_0,r))}\\
&\le C_{p_1,p_2,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\varphi(x_0)}\psi(r).
\end{aligned}$$

If $\underline{b}(\varphi) > 0$, let us consider $\psi \in \mathcal{B}$ such that $\underline{b}(\psi) > -\frac{d}{p_2}$, $\overline{b}(\psi) < \underline{b}(\phi) + \underline{b}(\varphi)$ and $\phi \preccurlyeq \psi$. For $0 < r \le 1$, Proposition 1.2.6 allows us to write

$$\begin{aligned}
&r^{-d/p_3}\|g_1 f_2\|_{L^{p_3}(B(x_0,r))}\\
&\le r^{-d/p_3}\|g_1 P_2\|_{L^{p_3}(B(x_0,r))} + r^{-d/p_3}\|g_1 R_2\|_{L^{p_3}(B(x_0,r))}\\
&\le C_{d,p_2}|f_1|_{T^{p_1}_\phi(x_0)}\phi(r)\Big(\sum_{|\alpha|<\underline{b}(\varphi)}\frac{|D^\alpha P_2(x_0)|}{\alpha!}\Big) + |f_1|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\varphi(x_0)}\phi(r)\varphi(r)\\
&\le C_{p_2,\phi,\varphi,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p}_\varphi(x_0)}\psi(r).
\end{aligned}$$

Again, the previous inequality holds for $r > 1$ as well.

Let us now investigate the case $\underline{b}(\phi) > 1$. For $0 < r \le 1$ we have, as we know that $P_1(x_0) = f_1(x_0)$,

$$\begin{aligned}
r^{-d/p_1}\|g_1\|_{L^{p_1}(B(x_0,r))} &\le |f_1|_{T^{p_1}_\phi(x_0)}\phi(r) + C_{d,p_1}\Big(\sum_{1\le|\alpha|<\underline{b}(\phi)}\frac{|D^\alpha P_1(x_0)|}{\alpha!}\Big)r\\
&\le C_{p_1,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}r.
\end{aligned}$$

Obviously, this inequality still holds for $r > 1$. If $\underline{b}(\varphi) \leq 0$, then for all $\psi \in \mathcal{B}$ such that $\underline{b}(\psi) > -\frac{d}{p_2}$ and $\overline{b}(\psi) < \underline{b}(\varphi) + 1$, we have, by Proposition 1.2.6,

$$\begin{aligned} r^{-\frac{d}{p_3}} \|g_1 f_2\|_{L^{p_3}(B(x_0,r))} &\leq C_{p_1,\phi} \|f_1\|_{T^{p_1}_\phi(x_0)} |f_2|_{T^{p_2}_\varphi(x_0)} \varphi(r) r \\ &\leq C_{p_1,\phi,\varphi,\psi} \|f_1\|_{T^{p_1}_\phi(x_0)} \|f_2\|_{T^{p_2}_\varphi(x_0)} \psi(r), \end{aligned}$$

for $0 \leq r < 1$. As $\underline{b}(\psi) > -\frac{d}{p_3}$, this inequality is also satisfied for $r > 1$. If $\underline{b}(\varphi) > 0$, let us consider $\psi \in \mathcal{B}$ such that $\underline{b}(\psi) > -\frac{d}{p_2}$, $\overline{b}(\psi) < \underline{b}(\varphi) + 1$ and $\phi \preccurlyeq \psi$. On the one hand, if $\overline{b}(\psi) < 1$, Proposition 1.2.6 implies

$$\begin{aligned} r^{-\frac{d}{p_3}} \|g_1 f_2\|_{L^{p_3}(B(x_0,r))} &\leq r^{-\frac{d}{p_3}} \|g_1 P_2\|_{L^{p_3}(B(x_0,r))} + r^{-\frac{d}{p_3}} \|g_1 R_2\|_{L^{p_3}(B(x_0,r))} \\ &\leq C_{p_1,\phi} \|f_1\|_{T^{p_1}_\phi(x_0)} \|f_2\|_{T^{p_2}_\varphi(x_0)} r \\ &\quad + C_{p_1,\phi} \|f_1\|_{T^{p_1}_\phi(x_0)} |f_2|_{T^{p_2}_\varphi(x_0)} \varphi(r) r \\ &\leq C_{\phi,\varphi,\psi} \|f_1\|_{T^{p_1}_\phi(x_0)} \|f_2\|_{T^{p_2}_\varphi(x_0)} \psi(r), \end{aligned}$$

for $0 < r \leq 1$; again one easily checks that this inequality also holds for $r > 1$. On the other hand, if $n \in \mathbb{N}$ is such that $n < \underline{b}(\psi) \leq \overline{b}(\psi) < n + 1$, let us define P as the sum of terms of degree less than or equal to n in $(P_1 - f_1(x_0))P_2$; we have

$$\begin{aligned} g_1 f_2 &= (P_1 - f_1(x_0))P_2 + R_1 P_2 + R_2 g_1 \\ &= P + (P_1 - f_1(x_0))P_2 - P + R_1 P_2 + R_2 g_1. \end{aligned}$$

By setting $R := (P_1 - f_1(x_0))P_2 - P + R_1 P_2 + R_2 g_1$, Proposition 1.2.6 gives

$$\begin{aligned} r^{-d/p_3} \|R\|_{L^{p_3}(B(x_0,r))} &\leq r^{-d/p_3} \|g_1 f_2\|_{L^{p_3}(B(x_0,r))} + r^{-d/p_3} \|P\|_{L^{p_3}(B(x_0,r))} \\ &\leq C_{p_3,\psi} \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)} \psi(r) \\ &\quad + C_{p_3,p_2,\psi} \|f_1\|_{T^{p_1}_\phi(x_0)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)} \psi(r) \\ &\quad + C_{d,p_3} \|f_1\|_{T^{p_1}_\phi(x_0)} \|f_2\|_{T^{p_2}_\varphi(x_0)} r^n \\ &\leq C_{\psi,p_1,p_2} \|f_1\|_{T^{p_1}_\phi(x_0)} \|f_2\|_{T^{p_2}_\varphi(x_0)} \psi(r), \end{aligned}$$

for $r > 1$, while for $0 < r < 1$, we have

$$\begin{aligned} r^{-d/p_3} \|R_1 P_2\|_{L^{p_3}(B(x_0,r))} &\leq C_{d,p_2} |f_1|_{T^{p_1}_\phi(x_0)} \|f_2\|_{T^{p_2}_\varphi(x_0)} \phi(r) \\ &\leq C_{p_2,\phi,\psi} |f_1|_{T^{p_1}_\phi(x_0)} \|f_2\|_{T^{p_2}_\varphi(x_0)} \psi(r), \end{aligned}$$

$$\begin{aligned} r^{-d/p_3} \|R_2 g_1\|_{L^{p_3}(B(x_0,r))} &\leq C_{p_1,\phi} \|f_1\|_{T^{p_1}_\phi(x_0)} |f_2|_{T^{p_2}_\varphi(x_0)} \varphi(r) r \\ &\leq C_{p_1 \phi,\varphi,\psi} \|f_1\|_{T^{p_1}_\phi(x_0)} |f_2|_{T^{p_2}_\varphi(x_0)} \psi(r) \end{aligned}$$

and

$$r^{-d/p_3}\|(P_1 - f_1(x_0))P_2 - P\|_{L^{p_3}(B(x_0,r))} \leq C_{d,p_3}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)} r^{n+1}$$
$$\leq C_{p_1,p_2,\psi}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}\psi(r).$$

This proves that there exists a constant $C_{p_1,p_2\phi,\varphi,\psi} > 0$ such that, for all $r > 0$,

$$r^{-d/p_3}\|g_1 f_2 - P\|_{L^{p_3}(B(x_0,r))} \leq C_{p_1,p_2,\phi,\varphi,\psi}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}\psi(r).$$

If $f_2 \in L^{p_3}(\mathbb{R}^d)$, then

$$\|g_1 f_2\|_{L^{p_3}(\mathbb{R}^d)} \leq \|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)} + |f_1(x_0)|\|f_2\|_{L^{p_3}(\mathbb{R}^d)},$$

hence the conclusion. □

Proposition 5.3.7. *Let $x_0 \in \mathbb{R}^d$, $p_1, p_2 \in [1,\infty]$ be such that $p_1 \leq p_2$ and ϕ be a function of $\mathcal{B}$ such that $-d/p_2 < \underline{b}(\phi)$. If f belongs to $T_\phi^{p_2}(x_0) \cap L^{p_1}(\mathbb{R}^d)$, then $f \in T_\phi^{p_1}(\mathbb{R}^d)$, with*

$$\|f\|_{T_\phi^{p_1}(\mathbb{R}^d)} \leq \|f\|_{T_\phi^{p_2}(x_0)} + \|f\|_{L^{p_1}(\mathbb{R}^d)}.$$

Moreover, in this case, $f \in t_\phi^{p_2}(x_0)$ implies $f \in t_\phi^{p_1}(x_0)$.

Proof. Let P be the polynomial of degree strictly less than $\underline{b}(\phi)$ such that, for $r > 0$,

$$r^{-d/p_2}\|f - P\|_{L^{p_2}(B(x_0,r))} \leq |f|_{T_\phi^{p_2}(x_0)}\phi(r).$$

For such r, we have

$$r^{-d/p_1}\|f - P\|_{L^{p_1}(B(x_0,r))} \leq r^{-d/p_1} C_{d,p_1,p_2} r^{\frac{d}{p_1}-\frac{d}{p_2}}\|f - P\|_{L^{p_2}(B(x_0,r))}$$
$$\leq C_{d,p_1,p_2}|f|_{T_\phi^{p_2}(x_0)}\phi(r),$$

which is sufficient to conclude, as $f \in L^{p_1}(\mathbb{R}^d)$.

The second part can be obtained using the same arguments as usual. □

5.4 A generalization of Whitney extension theorem

In this section, we show that some uniform conditions on a closed set E involving spaces T_ϕ^p and t_ϕ^p imply the belonging to spaces $B_\phi(E)$ and $b_\phi(E)$ respectively, that will be defined in this section. Then, we show that a function which has such properties can be extended in an open neighbourhood of E into a function which satisfies generalized Hölderian condition type (see [87]).

In the sequel we will heavily need the following lemma. Its proof can be found in [130] for example.

Lemma 5.4.1. *Given $n \in \mathbb{N}$, there exists a function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ whose support is included in $\overline{B(0,1)}$ such that, for any polynomial P of degree less than or equal to n and any $\varepsilon > 0$, we have*

$$\varphi_\varepsilon * P = P.$$

We now introduce the spaces $B_\phi(E)$ and $b_\phi(E)$ of functions that admit a formal Taylor expansion on a set $E \subset \mathbb{R}^d$ for which the behaviour can be characterized by a Lipschitz-type condition given by a function $\phi \in \mathcal{B}$.

Definition 5.4.2. Let E be a subset of $\mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$; a bounded function f on E belongs to the space $B_\phi(E)$ if there exist $C, M > 0$ such that, for all $x_0 \in E$, there exists a polynomial P_{x_0} of degree strictly less than $\underline{b}(\phi)$,

$$P_{x_0} := \sum_{|\alpha|<\underline{b}(\phi)} \frac{f_\alpha(x_0)}{\alpha!}(\cdot - x_0)^\alpha,$$

such that $f_0(x_0) = f(x_0)$, $|f_\alpha(x_0)| \leq M$ for all $|\alpha| < \underline{b}(\phi)$ and meeting the condition

$$|D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| \leq C\phi(|x - x_0|)|x - x_0|^{-|\alpha|},$$

for all $x \in E$ satisfying $x \neq x_0$ and all $|\alpha| < \underline{b}(\phi)$.

Definition 5.4.3. Let E be a subset of $\mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$; a function f defined on E belongs to the space $b_\phi(E)$ if, for any $x_0 \in E$, there exists a polynomial P_{x_0} of degree strictly less than $\underline{b}(\phi)$,

$$P_{x_0} := \sum_{|\alpha|<\underline{b}(\phi)} \frac{f_\alpha(x_0)}{\alpha!}(\cdot - x_0)^\alpha,$$

for which $f_0(x_0) = f(x_0)$ and

$$\lim_{\substack{x \to x_0 \\ x \in E}} \phi(|x - x_0|)^{-1}|x - x_0|^{|\alpha|}|D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| = 0$$

uniformly in $x_0 \in E$.

Definitions 5.4.2 and 5.4.3 generalize the Taylor chain condition (see Definition 1.7.2) by the mean of T^p_ϕ and t^p_ϕ spaces. Indeed, in Definition 5.4.2, the power function which appears in (1.13) is replaced by a Boyd function while, in Definition 5.4.3 the bound is replaced by an asymptotic behaviour. Our aim here is to show that these adaptations lead to a generalization of Whitney extension theorem.

Proposition 5.4.4. *Let E be a closed subset of $\mathbb{R}^d$ and ϕ be a function of $\mathcal{B}$ satisfying $\underline{b}(\phi) > 0$;*

1. *if there exists $M > 0$ such that $f \in T^p_\phi(x_0)$ with $\|f\|_{T^p_\phi(x_0)} \leq M$ for all $x_0 \in E$, then $f \in B_\phi(E)$ (in the sense that f is equal almost everywhere to a function that belongs to $B_\phi(E)$),*

2. *if $f \in t^p_\phi(x_0)$ for all $x_0 \in E$, with (5.3) holding uniformly in $x_0 \in E$, then $f \in b_\phi(E)$.*

Proof. Let us prove the first point. We know that for any $x_0 \in E$, there exists a polynomial P_{x_0} of degree strictly less than $\underline{b}(\phi)$ such that $R_{x_0} := f - P_{x_0}$ satisfies

$$r^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,r))} \leq M\phi(r), \tag{5.15}$$

for $r > 0$, with $|D^\alpha P_{x_0}(x_0)|/\alpha! \leq M$ for all $|\alpha| < \underline{b}(\phi)$. Moreover, in the light of Remark 5.1.6, one can modify f on a negligible set in order to have $f(x_0) = P_{x_0}(x_0)$ for all $x_0 \in E$. In particular $|f(x_0)| \leq M$ for all $x_0 \in E$ and f is bounded on E.

Let us take a function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such as in Lemma 5.4.1, let x, x_0 be two distinct points of E and set $\varepsilon := |x - x_0|$. Let us define, for $|\alpha| < \underline{b}(\phi)$,

$$I_\alpha := D^\alpha(\varphi_\varepsilon * f)(x).$$

On the one hand, we have

$$\begin{aligned} I_\alpha &= D^\alpha(\varphi_\varepsilon * (P_{x_0} + R_{x_0}))(x) \\ &= (\varphi_\varepsilon * D^\alpha P_{x_0})(x) + (D^\alpha \varphi_\varepsilon * R_{x_0})(x) \\ &= D^\alpha P_{x_0}(x) + (D^\alpha \varphi_\varepsilon * R_{x_0})(x), \end{aligned}$$

and, on the other hand,

$$I_\alpha = D^\alpha P_x(x) + (D^\alpha \varphi_\varepsilon * R_x)(x).$$

Thus we get, for $|\alpha| < \underline{b}(\phi)$,

$$\begin{aligned} D^\alpha P_x(x) &= D^\alpha P_{x_0}(x) + (D^\alpha \varphi_\varepsilon * (R_{x_0} - R_x))(x) \\ &= D^\alpha P_{x_0}(x) + \int_{B(x,\varepsilon)} \varepsilon^{-d+|\alpha|} D^\alpha \varphi(\frac{x-y}{\varepsilon})(R_{x_0}(y) - R_x(y))\,dy. \end{aligned}$$

Setting $C_\varphi := \sup_{|\alpha|<\underline{b}(\phi)} \|D^\alpha \varphi\|_\infty$, we finally get, for $|\alpha| < \underline{b}(\phi)$,

$$\begin{aligned} |D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| &\leq C_\varphi \varepsilon^{-|\alpha|}(\varepsilon^{-d}\|R_{x_0}\|_{L^1(B(x,\varepsilon))} + \varepsilon^{-d}\|R_x\|_{L^1(B(x,\varepsilon))}) \\ &\leq C_\varphi C_d \varepsilon^{-|\alpha|}((2\varepsilon)^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,2\varepsilon))} \\ &\quad + \varepsilon^{-d/p}\|R_x\|_{L^p(B(x,\varepsilon))}) \\ &\leq C\phi(|x - x_0|)|x - x_0|^{-\alpha}, \end{aligned}$$

where the constant $C > 0$ only depends on C_φ, M, d and ϕ.

For the second part of the proposition, let us consider

$$r^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,r))} \in o(\phi(r)) \qquad \text{as } r \to 0^+$$

uniformly in $x_0 \in E$, instead of (5.15). Since the inequality

$$\begin{aligned}|D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| \le C_\varphi C_d \varepsilon^{-|\alpha|}((2\varepsilon)^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,2\varepsilon))} \\ + \varepsilon^{-d/p}\|R_x\|_{L^p(B(x,\varepsilon))})\end{aligned}$$

holds for all $x, x_0 \in E$, we can conclude that, given $C > 0$, there exists $\eta > 0$ such that if $0 < |x - x_0| < \eta$ $(x, x_0 \in E)$ then we have

$$|D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| \le C\phi(|x - x_0|)|x - x_0|^{-\alpha},$$

which means that f belongs to $b_\phi(E)$. □

The theorem concluding this section relies on the following lemma, which establishes the existence of a smooth function on a neighbourhood of a closed subset E which is comparable to the distance to E (see e.g. [130, 26]).

Lemma 5.4.5. *Let $E \subset \mathbb{R}^d$ be a closed set and $U = \{x \in \mathbb{R}^d : d(x,E) < 1\}$; there exist $\delta \in C^\infty(U \setminus E)$ and $C > 0$ such that*

$$C^{-1} d(x,E) \le \delta(x) \le C d(x,E) \qquad \forall\, x \in U \setminus E$$

and

$$|D^\alpha \delta(x)| \le C(\alpha) d(x,E)^{1-|\alpha|} \qquad \forall\, x \in U \setminus E,\, |\alpha| \ge 0.$$

In the sequel, we will also need the following combinatorial lemma, which can be easily proved by induction on $l \in \mathbb{N}$.

Lemma 5.4.6. *Let $l \in \mathbb{N}$;*

- *if $l = 0 \mod 4$, then*

$$-\frac{1}{2}\binom{l}{l/2} = \sum_{j=0}^{\frac{l}{2}-1} (-1)^j \binom{l}{j} = \sum_{j=\frac{l}{2}+1}^{l} (-1)^j \binom{l}{j},$$

- *if $l = 1 \mod 4$, then*

$$\binom{l-1}{\frac{l-1}{2}} = \sum_{j=0}^{\frac{l-1}{2}} (-1)^j \binom{l}{j} = -\sum_{j=\frac{l-1}{2}+1}^{l} (-1)^j \binom{l}{j},$$

- *if $l = 2 \mod 4$, then*

$$\frac{1}{2}\binom{l}{l/2} = \sum_{j=0}^{\frac{l}{2}-1}(-1)^j\binom{l}{j} = \sum_{j=\frac{l}{2}+1}^{l}(-1)^j\binom{l}{j},$$

- *if $l = 3 \mod 4$, then*

$$-\binom{l-1}{\frac{l-1}{2}} = \sum_{j=0}^{\frac{l-1}{2}}(-1)^j\binom{l}{j} = -\sum_{j=\frac{l-1}{2}+1}^{l}(-1)^j\binom{l}{j}.$$

Theorem 5.4.7. *Let $E \subset \mathbb{R}^d$ be a closed set, $U = \{x \in \mathbb{R}^d : d(x,E) < 1\}$, $n \in \mathbb{N}$ and $\phi \in \mathcal{B}$ be such that $n < \underline{b}(\phi)$. If $f \in T^p_\phi(x_0)$ satisfies $\|f\|_{T^p_\phi(x_0)} \leq M$ for some $M > 0$ and all $x_0 \in E$, then there exists $F \in C^n(U)$ such that $F = f$ almost everywhere on E.*

Moreover, if $m \in \mathbb{N}$ is such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < m$, then there exists $C > 0$ such that for any $x \in U$ and any $h \in \mathbb{R}^d \setminus \{0\}$ for which $[x, x+(m-n)h] \subset U$, we have

$$|\Delta_h^{m-n} D^\alpha F(x)| \leq C\phi(|h|)|h|^{-n}, \tag{5.16}$$

for any $|\alpha| = n$.

Proof. Let us consider the functions φ and δ from Lemmata 5.4.1 and 5.4.5 respectively. We know that we can modify f on a set of measure zero so that $f \in B_\phi(E)$. Let us define the function F on U by

$$F(x) := \begin{cases} f(x) & \text{if } x \in E \\ \delta(x)^{-d}\int_{\mathbb{R}^d}\varphi((x-y)\delta(x)^{-1})f(y)\,dy & \text{otherwise.} \end{cases}$$

One obviously has $F \in C^\infty(U \setminus E)$. Let $\overline{x} \in U \setminus E$ and $x_0 \in E$ be such that $|\overline{x} - x_0| = d(\overline{x}, E)$. As $x_0 \in E$, there exists a polynomial P_{x_0} of degree less than or equal to n such that $R_{x_0} := f - P_{x_0}$ satisfies

$$r^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,r))} \leq M\phi(r),$$

for all $r > 0$. For any $x \in U \setminus E$, by setting

$$\Phi_\alpha(x,\cdot) = D_x^\alpha(\delta(x)^{-d}\varphi((x-\cdot)\delta(x)^{-1})),$$

we have, by Lemma 5.4.1,

$$D^\alpha F(x) = D^\alpha P_{x_0}(x) + \int_{\mathbb{R}^d}\Phi_\alpha(x,y)R_{x_0}(y)\,dy.$$

One can easily check (by induction) that $\Phi_\alpha(x,\cdot)$ is of the form

$$\delta(x)^{-d-k}D^\alpha\varphi((x-\cdot)\delta^{-1}(x))(x-\cdot)^\gamma P(x),$$

where $P(x)$ is a product of derivatives of the function δ evaluated at x with t factors and whose sum of orders is equal to w and where $k+w-t-|\gamma| = |\alpha|$. Thanks to the property of the function δ, we have $|P(x)| \le Cd(x,E)^{t-w}$, $\delta(x)^{-d-k} \le C^* d(x,E)^{-d-k}$ and

$$|D^\alpha \varphi((x-\cdot)\delta^{-1}(x))(x-y)^\gamma| \le C_{\gamma,\alpha} d(x,E)^{|\gamma|},$$

as $D^\alpha \varphi((x-\cdot)\delta^{-1}(x))(x-\cdot)^\gamma$ does not vanish if $|x-\cdot| \le \delta(x)$. We thus have the following estimate:

$$|\int_{\mathbb{R}^d} \Phi_\alpha(x,y) R_{x_0}(y)\,dy| \le C_1 d(x,E)^{-d-|\alpha|} \int_{B(x,\delta(x))} |R_{x_0}(y)|\,dy,$$

for all $\alpha \in \mathbb{N}_0^d$ and $x \in U \setminus E$. As there exists $C' > 0$ such that $\delta(x) \le C' d(x,E)$ for all $x \in U \setminus E$, we can write

$$\begin{aligned}
|D^\alpha F(\overline{x}) - D^\alpha P_{x_0}(\overline{x})| &\le C_1 d(\overline{x},E)^{-d-|\alpha|} \int_{B(\overline{x},C'd(\overline{x},E))} |R_{x_0}(y)|\,dy \\
&\le C_1 d(\overline{x},E)^{-|\alpha|} d(\overline{x},E)^{-d} \int_{B(x_0,(C'+1)d(\overline{x},E))} |R_{x_0}(y)|\,dy \\
&\le C_2 M \phi(d(\overline{x},E)) d(\overline{x},E)^{-|\alpha|} \\
&= C_2 M \phi(|\overline{x}-x_0|)(|\overline{x}-x_0|)^{-|\alpha|},
\end{aligned}$$

where $C_2 > 0$ is a constant which only depends on φ, ϕ, C_1, C' and d. Moreover, as $f \in B_\phi(E)$, we know that $P_{x_0}(x_0) = f(x_0)$ and for all $x_1 \in E$ such that $x_1 \neq x_0$, $D^\alpha P_{x_0}(x_0) = D^\alpha P_{x_1}(x_0) + R_\alpha(x_0,x_1)$, where R_α satisfies

$$|R_\alpha(x_0,x_1)| \le C\phi(|x_0-x_1|)(|x_0-x_1|)^{-|\alpha|}, \tag{5.17}$$

for all $|\alpha| \le n$. Therefore, thanks to Taylor's formula, we have, for $|\alpha| \le n$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
D^\alpha P_{x_0}(x) &= \sum_{|\beta|\le n-|\alpha|} \frac{1}{\beta!} D^{\alpha+\beta} P_{x_0}(x_0)(x-x_0)^\beta \\
&= \sum_{|\beta|\le n-|\alpha|} \frac{1}{\beta!} (D^{\alpha+\beta} P_{x_1}(x_0) + R_{\alpha+\beta}(x_0,x_1))(x-x_0)^\beta \\
&= \sum_{|\beta|\le n-|\alpha|} \frac{1}{\beta!} \Big(\sum_{|\gamma|\le n-(|\alpha|+|\beta|)} \frac{1}{\gamma!} D^{\alpha+\beta+\gamma} P_{x_1}(x_1)(x_0-x_1)^\gamma \\
&\quad + R_{\alpha+\beta}(x_0,x_1)\Big)(x-x_0)^\beta
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{|\beta|\le n-|\alpha|}\frac{1}{\beta!}\sum_{|\gamma|\le n-(|\alpha|+|\beta|)}\frac{1}{\gamma!}D^{\alpha+\beta+\gamma}P_{x_1}(x_1)(x_0-x_1)^\gamma(x-x_0)^\beta\\
&=\sum_{|\gamma|\le n-|\alpha|}\frac{1}{\gamma!}\sum_{|\beta|\le n-(|\alpha|+|\gamma|)}\frac{1}{\beta!}D^{\alpha+\beta+\gamma}P_{x_1}(x_1)(x-x_0)^\beta(x_0-x_1)^\gamma\\
&=\sum_{|\gamma|\le n-|\alpha|}\frac{1}{\gamma!}D^{\alpha+\gamma}P_{x_1}(x-x_0+x_1)(x_0-x_1)^\gamma\\
&=D^\alpha P_{x_1}(x).
\end{aligned}$$

Finally, we have

$$D^\alpha P_{x_0}(x)=D^\alpha P_{x_1}(x)+\sum_{|\beta|\le n-|\alpha|}\frac{1}{\beta!}R_{\alpha+\beta}(x_0,x_1)(x-x_0)^\beta,$$

for all $x_0,x_1\in E$ and $x\in\mathbb{R}^d$. In particular, for $|\alpha|\le n$,

$$|D^\alpha P_{x_0}(\overline{x})-D^\alpha P_{x_1}(\overline{x})|\le C\sum_{|\beta|\le n-|\alpha|}\phi(|x_0-x_1|)|x_0-x_1|^{-|\alpha|-|\beta|}|\overline{x}-x_0|^{|\beta|}$$

and as $|\overline{x}-x_0|\le|\overline{x}-x_1|$, we have $|x_0-x_1|\le 2|\overline{x}-x_1|$. Therefore,

$$\begin{aligned}
&\phi(|x_0-x_1|)|x_0-x_1|^{-|\alpha|-|\beta|}\\
&\le\phi(|\overline{x}-x_1|)|\overline{x}-x_1|^{-|\alpha|-|\beta|}\overline{\phi}(\frac{|x_0-x_1|}{|\overline{x}-x_1|})(\frac{|x_0-x_1|}{|\overline{x}-x_1|})^{-|\alpha|-|\beta|}
\end{aligned}$$

and, as $|\alpha|+|\beta|\le n<\underline{b}(\phi)$, Remark 1.2.9 implies that

$$\overline{\phi}(\frac{|x_0-x_1|}{|\overline{x}-x_1|})(\frac{|x_0-x_1|}{|\overline{x}-x_1|})^{-|\alpha|-|\beta|}$$

is bounded (by a constant which only depends on ϕ). We thus have

$$|D^\alpha P_{x_0}(\overline{x})-D^\alpha P_{x_1}(\overline{x})|\le C\phi(|\overline{x}-x_1|)|\overline{x}-x_1|^{-|\alpha|},$$

for $|\alpha|\le n$. This inequality and the upper bound obtained for $D^\alpha F(\overline{x})-D^\alpha P_{x_0}(\overline{x})$ give the following relation, valid for all $x_1\in E$:

$$\begin{aligned}
&|D^\alpha F(\overline{x})-D^\alpha P_{x_1}(\overline{x})|\\
&\le C(\phi(|\overline{x}-x_0|)|\overline{x}-x_0|^{-|\alpha|}+\phi(|\overline{x}-x_1|)|\overline{x}-x_0|^{-|\alpha|})
\end{aligned}$$

and as $|\overline{x}-x_0|\le|\overline{x}-x_1|$, we get, as before,

$$|D^\alpha F(\overline{x})-D^\alpha P_{x_1}(\overline{x})|\le C\phi(|\overline{x}-x_1|)|\overline{x}-x_1|^{-|\alpha|}. \tag{5.18}$$

Let F_α be the function defined on U by

$$F_\alpha(x) := \begin{cases} D^\alpha P_x(x) & \text{if } x \in E \\ D^\alpha F(x) & \text{otherwise.} \end{cases}$$

We have proved that, for $|\alpha| \le n$, $F_\alpha \in C^\infty(U \setminus E)$ and for $x \in E$ and $h \neq 0$ such that $x + h \in U$, we have

$$F_\alpha(x+h) = \sum_{|\beta| \le n-|\alpha|} D^{\alpha+\beta} P_x(x) h^\beta + R_\alpha(x, x+h), \tag{5.19}$$

where

$$|R_\alpha(x, x+h)| \le C\phi(|h|)|h|^{-|\alpha|},$$

with a uniform constant. More precisely, if h is such that $x+h \in E$, the previous inequality is satisfied because f belongs to $B_\phi(E)$; otherwise $x + h \in U \setminus E$ and the inequality follows from (5.18). This is sufficient to show that $F \in C^n(U)$ and $D^\alpha F = F_\alpha$ on U for all $|\alpha| \le n$. Indeed, (5.19) implies that F_α is continuous on E and so on U. Given $n \ge 1$, let us fix $x \in E$; if $h \in \mathbb{R} \setminus \{0\}$ is sufficiently small, for $j \in \{1, \ldots, d\}$, we have

$$F(x + he_j) - F(x) = \sum_{|\beta|=1}^{n} D^\beta P_x(x)(he_j)^\beta + R_0(x, x+h),$$

which allows us to write

$$\begin{aligned} \Big|\frac{F(x+he_j) - F(x)}{h} - F_{e_j}(x)\Big| &\le \sum_{|\beta|=2}^{n} |D^\beta P_x(x)||h|^{|\beta|-1} + \frac{|R_0(x,x+h)|}{|h|} \\ &\le \sum_{|\beta|=2}^{n} |D^\beta P_x(x)||h|^{|\beta|-1} + C\frac{\phi(|h|)}{|h|} \end{aligned}$$

and, as the right-hand side tends to 0 as h tends to 0, we can conclude, since $1 \le n < \underline{b}(\phi)$, that F is differentiable at x and $D_j F(x) = F_{e_j}(x)$. If we now assume that F is $(n-1)$-times continuously differentiable at x, with $D^\alpha F(x) = F_\alpha(x)$ for every $|\alpha| \le n-1$, we have, for $|\alpha| = n-1$, $h \in \mathbb{R} \setminus \{0\}$ sufficiently small and $j \in \{1, \ldots, d\}$,

$$\begin{aligned} &\Big|\frac{F_\alpha(x+he_j) - F_\alpha(x)}{h} - F_{\alpha+e_j}(x)\Big| \\ &\le \sum_{|\beta|=1} |D^{\alpha+\beta} P_x(x)|\, |h|^{|\beta|-1} + \frac{|R_\alpha(x,x+h)|}{|h|} \\ &\le \sum_{|\beta|=1} |D^{\alpha+\beta} P_x(x)|\, |h|^{|\beta|-1} + C\frac{\phi(|h|)}{|h|^n} \end{aligned}$$

and we can conclude, in the same way, that F_α is differentiable at x, with $D_j F_\alpha(x) = F_{\alpha+e_j}(x)$.

Let us now prove that if $n < \underline{b}(\phi) \le \overline{b}(\phi) < m$, then there exists $C > 0$ such that, for all $x \in U$ and $h \in \mathbb{R}^d$ such that $[x, x + mh] \subset U$, we have

$$|\Delta_h^{m-n} D^\alpha F(x)| \le C\phi(|h|)|h|^{-n},$$

for all $|\alpha| = n$. So far, we know from (5.17) and (5.18) that the following inequality holds for all $|\alpha| = n$, $x \in U$ and y in E satisfying $x \neq y$:

$$|F_\alpha(x) - F_\alpha(y)| \le C\phi(|x-y|)|x-y|^{-n}.$$

If $x \in U$ and $h \in \mathbb{R}^d \setminus \{0\}$ are such that there exists $k \in \{0, \dots, m-n\}$ for which $x + kh \in E$, we can use Lemma 5.4.6 to obtain, setting $l = m - n$,

$$\begin{aligned}
|\Delta_h^l D^\alpha F(x)| &= |\sum_{j=0}^{l} (-1)^j \binom{l}{j} D^\alpha F(x + jh)| \\
&= |\sum_{j=0}^{l} (-1)^j \binom{l}{j} (D^\alpha F(x + jh) - D^\alpha F(x + kh))| \\
&\le \sum_{j=0}^{l} \binom{l}{j} C\phi(|(j-k)h|)|(j-k)h|^{-n} \\
&\le C'\phi(|h|)|h|^{-n}.
\end{aligned}$$

Let us now consider the case for which we have, for all $k \in \{0, \dots, l\}$, $x + kh \in U \setminus E$; let us first suppose that $d(x, E) \le (l+1)|h|$ and take $x_0 \in E$ such that $|x_0 - x| = d(x, E)$. Of course $|x_0 - x| \le (l+1)|h|$ and, for all $j \in \{0, \dots, l\}$, we have $|x_0 - (x + jh)| \le (2l+1)|h|$. As before, we have

$$\begin{aligned}
|\Delta_h^l D^\alpha F(x)| &\le \sum_{j=0}^{l} \binom{l}{j} |D^\alpha F(x + jh) - D^\alpha F(x_0)| \\
&\le C \sum_{j=0}^{l} \binom{l}{j} \phi(|x + jh - x_0|)|x + jh - x_0|^{-n}
\end{aligned}$$

and, for all $j \in \{0, \dots, l\}$,

$$\begin{aligned}
&\phi(|x + jh - x_0|)|x + jh - x_0|^{-n} \\
&\le \phi(|h|)|h|^{-n} \overline{\phi}\Big(\frac{|x + jh - x_0|}{|h|}\Big)\Big(\frac{|x + jh - x_0|}{|h|}\Big)^{-n}.
\end{aligned}$$

That being said, we have $\frac{|x+jh-x_0|}{h} \le 2l + 1$ and so, by Remark 1.2.9,

$$\overline{\phi}\Big(\frac{|x + jh - x_0|}{|h|}\Big)\Big(\frac{|x + jh - x_0|}{|h|}\Big)^{-n} \le C,$$

where the constant C only depends on ϕ and l. Therefore, we can write

$$|\Delta_h^l D^\alpha F(x)| \leq C'\phi(|h|)|h|^{-n}.$$

It remains to consider the case where $x + kh \in U \setminus E$ for all $k \in \{0,\dots,l\}$ and $(l+1)|h| < d(x,E)$. As before, let x_0 stand for a point in E such that $|x_0 - x| = d(x,E)$. We already know that, for any $y \in U \setminus E$,

$$D^\alpha F(y) = D^\alpha P_{x_0}(y) + \int_{\mathbb{R}^d} \Phi_\alpha(y,\xi) R_{x_0}(\xi)\, d\xi.$$

The function $y \mapsto \int_{\mathbb{R}^d} \Phi_\alpha(y,\xi) R_{x_0}(\xi)\, d\xi$ belongs to $C^\infty(U \setminus E)$ and, for all $\beta \in \mathbb{N}_0^d$,

$$D^\beta \int_{\mathbb{R}^d} \Phi_\alpha(y,\xi) R_{x_0}(\xi)\, d\xi = \int_{\mathbb{R}^d} \Phi_{\alpha+\beta}(y,\xi) R_{x_0}(\xi)\, d\xi.$$

As the segment $[x, x+lh]$ is included in $U \setminus E$, we know, by Taylor's formula, that there exist points x_β with $|\beta| = l$ on the segment $[x, x+lh]$ such that

$$\begin{aligned}
\Delta_h^l D^\alpha F(x) &= \Delta_h^l \int_{\mathbb{R}^d} \Phi_\alpha(x,\xi) R_{x_0}(\xi)\, d\xi \\
&= \sum_{|\beta|=l} h^\beta \int_{\mathbb{R}^d} \Phi_{\alpha+\beta}(x_\beta,\xi) R_{x_0}(\xi)\, d\xi \\
&= \sum_{|\beta|=l} h^\beta \int_{B(x_\beta, Cd(x_\beta,E))} \Phi_{\alpha+\beta}(x_\beta,\xi) R_{x_0}(\xi)\, d\xi,
\end{aligned}$$

where C is a constant such that $\delta(y) \leq Cd(y,E)$ for all $y \in U \setminus E$. Moreover, for such y, we have already obtained that

$$|\Phi_{\alpha+\beta}(y)| \leq C' d(y,E)^{-d-(|\alpha|+|\beta|)} = C' d(y,E)^{-d-m}.$$

If $|\beta| = l$, as $x_\beta \in [x, x+lh]$, we have

$$d(x_\beta, E) \geq d(x,E) - |x - x_\beta| \geq (l+1)|h| - l|h| = |h|$$

and so, if $\xi \in B(x_\beta, Cd(x_\beta,E))$,

$$\begin{aligned}
|\xi - x_0| &\leq |\xi - x_\beta| + |x_\beta - x| + |x - x_0| \\
&\leq Cd(x_\beta,E) + l|h| + d(x,E) \\
&\leq Cd(x_\beta,E) + ld(x_\beta,E) + d(x_\beta,E) + l|h| \\
&\leq C'' d(x_\beta,E).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\int_{B(x_\beta,Cd(x_\beta,E))} \Phi_{\alpha+\beta}(x_\beta,\xi)R_{x_0}(\xi)\,d\xi| \\
&\leq C'd(x_\beta,E)^{-d-m}\int_{B(x_0,C''d(x_\beta,E))}|R_{x_0}(\xi)|\,d\xi \\
&\leq C'Md(x_\beta,E)^{-m}\phi(d(x_\beta,F)) \\
&\leq C'M\phi(|h|)|h|^{-m}\overline{\phi}(\frac{d(x_\beta,E)}{|h|})(\frac{d(x_\beta,E)}{|h|})^{-m}.
\end{aligned}$$

Now, as $d(x_\beta,E)/|h| \geq 1$, and $\overline{b}(\phi) < m$, we know that

$$\overline{\phi}(\frac{d(x_\beta,E)}{|h|})(\frac{d(x_\beta,E)}{|h|})^{-m}$$

is bounded by a constant which only depends on ϕ and m. We can thus write

$$|\Delta^l_h D^\alpha F(x)| \leq C'\phi(|h|)|h|^{-n},$$

which is what we need to conclude the proof. □

Theorem 5.4.8. *Let $E \subset \mathbb{R}^d$ be a closed set, $U = \{x \in \mathbb{R}^d : d(x,E) < 1\}$, $n \in \mathbb{N}$ and ϕ be a function of $\mathcal{B}$ such that $n < \underline{b}(\phi)$. If $f \in t^p_\phi(x_0)$ for all $x_0 \in E$, with (5.3) holding uniformly in $x_0 \in E$, then there exists $F \in C^n(U)$ such that $F = f$ almost everywhere on E.*

Moreover, if $m \in \mathbb{N}$ is such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < m$, then, for all $|\alpha| = n$, $x \in E$, and $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $0 < |h| \leq \eta$ for which $[x, x+(m-n)h] \subset E$,

$$|\Delta^{m-n}_h D^\alpha F(x)| \leq \varepsilon\phi(|h|)|h|^{-n}.$$

Proof. The proof is essentially the same as the previous one, using this time the fact that $f \in b_\phi(E)$ and

$$r^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,r))} \in o(\phi(r)) \qquad \text{as } r \to 0^+,$$

uniformly in $x_0 \in E$. □

Remark 5.4.9. Let us highlight the fact that inequality (5.16) characterizes the belonging to a generalized Hölder space, originally stated in [87], and corresponds to the condition in Corollary 2.3.3 for $p = q = \infty$, $\boldsymbol{\gamma} = (2^j)_j$ and $\boldsymbol{\sigma} = (\phi^{-1}(2^j))_j$.

6 T_ϕ^p regularity, operators and elliptic partial differential equations

In their seminal paper [26], Calderón and Zygmund use the T_u^p and t_u^p spaces to obtain pointwise estimates for solutions of elliptic partial differential equations $\mathcal{E}f = g$. Such equations are remarkable because the coefficients in the differential operator $\mathcal{E}$, which are functions, satisfy some kind of invertibility condition, see Definition 6.4.1 below. This condition is expressed in term of so-called symbols which link elliptic partial differential equations to standard operators such as the Bessel transform, Laplace operator and convolution singular integrals.

The main theorem in [26] can be stated as follows: if all the coefficients of $\mathcal{E}$ are of class $T_u^\infty(x_0)$, if all components f_j and g_k are of class L^p and $g_k \in T_v^p$ with $p \in (1,\infty)$, $-d/p \leq v \leq u$, $v \notin \mathbb{Z}$, then there exists a constant C for which

$$\|D^\alpha f_j\|_{T^q_{v+m-|\alpha|}(x_0)} \leq C(\sum_k \|g_k\|_{T_v^p(x_0)} + \sum_j \|f_j\|_{W_m^p}), \tag{6.1}$$

for all j, $|\alpha| \leq m$, where q is a number satisfying

- $p \leq q \leq \infty$ if $1/p < (m - |\alpha|)/d$,
- $p \leq q < \infty$ if $1/p = (m - |\alpha|)/d$,
- $1/p \leq 1/q \leq 1/p - (m - |\alpha|)/d$ otherwise.

Moreover, if g belongs to $t_v^p(x_0)$, then $D^\alpha f$ belongs to $t^q_{v+m-|\alpha|}(x_0)$. Another theorem states that if $\mathcal{E}$ is elliptic almost everywhere on a set of positive measure whose points x_0 satisfy $\mu(x_0) > c$ for some constant $c > 0$, if the coefficients of $\mathcal{E}$ are in $T_u^\infty(x_0)$ and $g \in T_v^p(x_0)$ for almost every x_0 and if $f \in L_m^p$, then $D^\alpha f$ belongs to $t^q_{v+m-|\alpha|}(x_0)$ for almost every x_0. Let us remark that there is a common misunderstanding when stating the hypothesis of this main theorem: the coefficients of $\mathcal{E}$ have to belong to $T_u^\infty(x_0)$ (see page 172 of [26], where T_u is defined as T_u^∞); the case where these coefficients belong to $T_u^p(x_0)$ with $p < \infty$ is not considered in [26].

In this chapter, we wish to extend this result by considering both T_ϕ^p functions and conditions based on L^p norm, with $p < \infty$, for the coefficients of $\mathcal{E}$. To achieve this goal, we first need to investigate the action of the Bessel operator, derivatives and singular integrals operators on a T_ϕ^p function. Let us highlight the fact that the main source of difficulty is the introduction of L^p conditions for the coefficients of $\mathcal{E}$. As usual, most of the properties of the standard spaces are preserved in the generalized version.

Contents

6.1 The Bessel operator

In this section we look at the action of the Bessel operator of order s,

$$\mathcal{J}^s f := \mathcal{F}^{-1}\Big((1+|\cdot|^2)^{-s/2}\mathcal{F}f\Big) \qquad (s\in\mathbb{R},\ f\in\mathcal{S}'),$$

onto spaces $T^p_\phi(x_0)$ and $t^p_\phi(x_0)$. If ϕ is a function of $\mathcal{B}$ and s belongs to $\mathbb{R}$, then ϕ_s will denote the function

$$\phi_s : (0,+\infty)\to(0,+\infty) \quad x\mapsto \phi(x)x^s.$$

It is obvious that ϕ_s is again a function of $\mathcal{B}$ such that $\underline{b}(\phi_s)=\underline{b}(\phi)+s$ and $\overline{b}(\phi_s)=\overline{b}(\phi)+s$.

Let us recall that if $0<s<d+1$, then we have $\mathcal{J}^s f = u_s * f$, where u_s is the function defined for $x\neq 0$ by

$$u_s(x) = \frac{1}{(2\pi)^{\frac{d-1}{2}}2^{s/2}\Gamma(s/2)\Gamma(\frac{d-s+1}{2})}e^{|x|}\int_0^{+\infty} e^{-|x|t}(t+t^2/2)^{\frac{d-s-1}{2}}\,dt.$$

The following inequality holds for all $0<s<d$ and $\alpha\in\mathbb{N}_0^d$:

$$D^\alpha u_s(x)\leq C_{s,\alpha}e^{-|x|}(1+|x|^{-d+s-|\alpha|}). \tag{6.2}$$

For the sake of simplicity, let us introduce the notion of admissible value for a real number.

Definition 6.1.1. Given $\phi\in\mathcal{B}$, a value $s>0$ is said to be *admissible (for ϕ)* if one of the following two conditions is satisfied:

- $\overline{b}(\phi)+s<0$,
- there exists $n\in\mathbb{N}$ such that $n<\underline{b}(\phi)+s\leq\overline{b}(\phi)+s<n+1$.

Theorem 6.1.2. *Let $x_0\in\mathbb{R}^d$, $p\in(1,\infty]$, $\phi\in\mathcal{B}$ be such that $\underline{b}(\phi)>-d/p$ and $s>0$ be an admissible value for ϕ. The operator $\mathcal{J}^s$ maps continuously $T^p_\phi(x_0)$ into $T^q_{\phi_s}(x_0)$, where*

- $1/p \geq 1/q \geq \frac{1}{p} - \frac{s}{d}$ *if* $p < d/s$,
- $p \leq q \leq \infty$ *if* $d/s < p \leq \infty$,
- $p \leq q < \infty$ *if* $d/s = p$.

Proof. Let f be a function of $T^p_\phi(x_0)$; we know that there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ such that $R := f - P$ satisfies

$$r^{-d/p}\|R\|_{L^p(B(x_0,r))} \leq |f|_{T^p_\phi(x_0)}\phi(r), \tag{6.3}$$

for all $r > 0$. Without loss of generality, we can assume that $x_0 = 0$. We first want to estimate the following two quantities, for all $r > 0$ and $u \in \mathbb{R}$:

$$\int_{B(0,r)} |R(x)||x|^{-u}\,dx \qquad \text{and} \qquad \int_{\mathbb{R}^d\setminus B(0,r)} |R(x)|\,|x|^{-u}\,dx.$$

For this purpose, we use the same idea that in Lemma 3.3.7 and set

$$\varphi(r) := \int_{B(0,r)} |R(x)|\,dx;$$

from inequality (6.3), we have

$$\varphi(r) \leq C_d|f|_{T^p_\phi(0)} r^d\phi(r). \tag{6.4}$$

Moreover, using the spherical coordinates in $\mathbb{R}^d$, we can write

$$\varphi(r) = \int_0^r \psi(\rho)\,d\rho, \tag{6.5}$$

where

$$\psi(\rho) := \rho^{d-1}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} |R(x(\rho,\theta_1,\cdots,\theta_{d-1}))|\,d\Omega_d,$$

and we recall that $d\Omega_d$ stands for $\sin^{d-2}(\theta_1)\cdots\sin(\theta_{d-2})d\theta_1\cdots d\theta_{d-1}$. Therefore, we have, for $\varepsilon > 0$,

$$\begin{aligned}\varphi(r)r^{-u} - \varphi(\varepsilon)\varepsilon^{-u} &= \int_\varepsilon^r \rho^{-u}\psi(\rho)\,d\rho - \int_\varepsilon^r u\rho^{-(u+1)}\varphi(\rho)\,d\rho \\ &= \int_{B(0,r)\setminus B(0,\varepsilon)} |R(x)||x|^{-u}\,dx - \int_\varepsilon^r u\rho^{-(u+1)}\varphi(\rho)\,d\rho.\end{aligned}$$

Consequently,

$$\int_{B(0,r)\setminus B(0,\varepsilon)} |R(x)||x|^{-u}\,dx \leq \varphi(r)r^{-u} + u\int_0^r \rho^{-(u+1)}\varphi(\rho)\,d\rho.$$

If $\underline{b}(\phi)+d-u>0$, then

$$\begin{aligned}\int_0^r \rho^{-(u+1)}\varphi(\rho)\,d\rho &\le C_d|f|_{T^p_\phi(0)}\int_0^r \rho^{d-u-1}\phi(\rho)\,d\rho\\ &\le C_d|f|_{T^p_\phi(0)}\phi(r)\int_0^r \rho^{d-u-1}\overline{\phi}(\frac{\rho}{r})\,d\rho\\ &= C_d|f|_{T^p_\phi(0)}\phi(r)r^{d-u}\int_0^1 \frac{\overline{\phi}(\xi)\xi^{d-u}}{\xi}\,d\xi\\ &\le C_u|f|_{T^p_\phi(0)}\phi(r)r^{d-u},\end{aligned}$$

thanks to Proposition 1.2.10. Hence, for all $r>0$ and $u\in\mathbb{R}$ such that $\underline{b}(\phi)+d-u>0$,

$$\int_{B(0,r)}|R(x)||x|^{-u}\,dx\le C_{d,u}|f|_{T^p_\phi(x_0)}\phi(r)r^{d-u}. \tag{6.6}$$

If we now assume that $\overline{b}(\phi)+d-u<0$, then, for all $N>0$,

$$\int_{B(0,N)\setminus B(0,r)}|R(x)||x|^{-u}\,dx=\varphi(N)N^{-u}-\varphi(r)r^{-u}+u\int_r^N \rho^{-u-1}\varphi(\rho)\,d\rho$$

and, since $\varphi(N)N^{-u}$ tends to 0 as $N\to\infty$, we get, thanks to (6.4) and Proposition 1.2.6,

$$\int_{\mathbb{R}^d\setminus B(0,r)}|R(x)||x|^{-u}\,dx\le C_u|f|_{T^p_\phi(x_0)}\phi(r)r^{d-u}, \tag{6.7}$$

using the same technique as before.

Let us first assume that $0<s<d$; we have

$$\mathcal{J}^s f=u_s*P+u_s*R,$$

where u_s*P is a polynomial of degree strictly less than $\underline{b}(\phi)$ whose sum of coefficients is bounded by the sum of the coefficients of P. We thus need to estimate u_s*R. Let us fix $r>0$ and $x\in\mathbb{R}^d$ such that $2|x|<r$; if there exists $n\in\mathbb{N}$ for which $n<\underline{b}(\phi)+s\le\overline{b}(\phi)+s<n+1$, by Taylor's formula, we find

$$\begin{aligned}(u_s*R)(x)=&\int_{B(0,r)}u_s(x-y)R(y)\,dy+\sum_{|\alpha|\le n}\frac{x^\alpha}{\alpha!}\int_{\mathbb{R}^d}D^\alpha u_s(-y)R(y)\,dy\\ &-\sum_{|\alpha|\le n}\frac{x^\alpha}{\alpha!}\int_{B(0,r)}D^\alpha u_s(-y)R(y)\,dy\\ &+\sum_{|\alpha|=n+1}\int_{\mathbb{R}^d\setminus B(0,r)}D^\alpha u_s(\Theta(x)x-y)R(y)\,dy,\end{aligned}$$

for a $\Theta(x) \in (0,1)$. Using inequalities (6.2) and then (6.6), we get, for all $|\alpha| \le n$,

$$\begin{aligned}
|\int_{\mathbb{R}^d} D^\alpha u_s(-y)R(y)\,dy| &\le C\Big(\int_{B(0,1)} |y|^{-d+s-|\alpha|}|R(y)|\,dy \\
&\quad + \int_{\mathbb{R}^d\setminus B(0,1)} e^{-|y|}|f(y)|\,dy + \int_{\mathbb{R}^d\setminus B(0,1)} e^{-|y|}|P(y)|\,dy\Big) \\
&\le C_{\alpha,s}|f|_{T^p_\phi(0)} + C_p\|f\|_{L^p(\mathbb{R}^d)} \\
&\quad + C\sum_{|\beta|<\underline{b}(\phi)} \frac{|D^\beta P(0)|}{\beta!}\int_{\mathbb{R}^d\setminus B(0,1)} e^{-|y|}|y|^\beta\,dy \\
&\le C_{\alpha,s,p,d}\|f\|_{T^p_\phi(0)},
\end{aligned}$$

so that

$$\sum_{|\alpha|\le n} \frac{x^\alpha}{\alpha!}\int_{\mathbb{R}^d} D^\alpha u_s(-y)R(y)\,dy$$

is a polynomial of degree n whose coefficients are bounded by $\|f\|_{T^p_\phi(0)}$. For all $|\alpha| \le n$, we also have, thanks to (6.6),

$$\begin{aligned}
|\int_{B(0,r)} D^\alpha u_s(-y)R(y)\,dy| &\le C\int_{B(0,r)} |y|^{-d+s-|\alpha|}|R(y)|\,dy \\
&\le C_\alpha |f|_{T^p_\phi(x_0)}\phi(r)r^{s-|\alpha|}.
\end{aligned}$$

Now, if $|\alpha| = n+1$ and if $|y| \ge r$, then $|\Theta(x)x - y| \ge |y|/2$ and, assuming that $s < d$,

$$|D^\alpha u_s(\Theta(x)x-y)| \le C|\Theta(x)x-y|^{-d+s-|\alpha|} \le C'|y|^{-d+s-|\alpha|}.$$

From (6.7), we get

$$|\int_{\mathbb{R}^d\setminus B(0,r)} D^\alpha u_s(\Theta(x)x-y)R(y)\,dy| \le C_\alpha|f|_{T^p_\phi(x_0)}\phi(r)r^{s-|\alpha|}.$$

If we also assume that $\frac{1}{p} - \frac{s}{d} < 0$ and if p' is the conjugate exponent of p, then, from $-(d-s)p' < d$, since

$$|u_s| \le C|\cdot|^{-d+s},$$

we can affirm that $u_s \in L^{p'}(\mathbb{R}^d)$ and, for all $r > 0$,

$$\|u_s\|_{L^{p'}(B(0,2r))} \le Cr^s r^{-d/p}.$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
|\int_{B(0,r)} u_s(x-y)R(y)\,dy| &\le \|u_s\|_{L^{p'}(B(0,2r))}\|R\|_{L^p(B(0,r))} \\
&\le Cr^s r^{-d/p}\|R\|_{L^p(B(0,r))} \\
&\le C|f|_{T^p_\phi(0)}r^s\phi(r).
\end{aligned}$$

This shows that

$$P' = u_s * P - \sum_{|\alpha|\le n} \frac{.^\alpha}{\alpha!}\int_{\mathbb{R}^d} D^\alpha u_s(-y)R(y)\,dy$$

is a polynomial of degree n such that

$$\|\mathcal{J}^s f - P'\|_{L^\infty(B(0,2r))} \le C_{s,\phi,p,d}|f|_{T^p_\phi(0)}\phi_s(2r), \tag{6.8}$$

which means that $\mathcal{J}^s f$ belongs to $\in T^\infty_{\phi_s}(0)$. Moreover, by Young's inequality,

$$\|\mathcal{J}^s f\|_{L^\infty(\mathbb{R}^d)} \le \|u_s\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)}. \tag{6.9}$$

From this relation, inequality (6.8) and the fact that the sum of the coefficients of P' is bounded by $\|f\|_{T^p_\phi(0)}$, we get

$$\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}.$$

If we now assume that $\frac{1}{p} - \frac{s}{d} > 0$, then

$$\begin{aligned} |\int_{B(0,r)} u_s(x-y)R(y)\,dy| &\le C\int_{\mathbb{R}^d} \frac{|R\chi_{B(0,r)}|}{|x-y|^{d-s}}\,dy \\ &= CI_s(|R\chi_{B(0,r)}|), \end{aligned}$$

for $r > 0$, where I_s is the Riesz potential of order s. As a consequence, if q is such that $1/q = \frac{1}{p} - \frac{s}{d}$, we have, by the Hardy-Littlewood-Sobolev lemma (see e.g. [118]),

$$\begin{aligned} \|I_s(R\chi_{B(0,r)})\|_{L^q(\mathbb{R}^d)} &\le C\|R\|_{L^p(B(0,r))} \\ &\le C|f|_{T^p_\phi(0)} r^{d/p}\phi(r) \\ &= C|f|_{T^p_\phi(0)} r^{\frac{d}{q}} r^s\phi(r). \end{aligned}$$

This implies

$$r^{-\frac{d}{q}}\|\mathcal{J}^s f - P'\|_{L^q(B(0,2r))} \le C_{s,\phi,p,d}|f|_{T^p_\phi(0)}\phi_s(2r),$$

for $r > 0$, which means that $\mathcal{J}^s f$ belongs to $T^q_{\phi_s}(0)$. One more use of the Hardy-Littlewood-Sobolev lemma gives

$$\|\mathcal{J}^s f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}$$

and we obtain, using the same arguments as before,

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}. \tag{6.10}$$

If $\overline{b}(\phi)+s<0$, let us decompose $(u_s * R)(x)$ as follows:

$$(u_s * R)(x) = \int_{B(0,r)} u_s(x-y)R(y)\,dy + \int_{\mathbb{R}^d\setminus B(0,r)} u_s(x-y)R(y)\,dy.$$

We can use inequality (6.7) again to estimate the second term in this equality; more precisely, we have

$$\Big|\int_{\mathbb{R}^d\setminus B(0,r)} u_s(x-y)R(y)\,dy\Big| \leq C_s|f|_{T^p_\phi(x_0)}\phi(r)r^s.$$

That being done, we can use the same reasoning to show that (6.9) and (6.10) still hold in this case.

Let us extend inequalities (6.9) and (6.10) to all admissible values of $s>0$. If $s=d$, let $0<\varepsilon<d$ be such that the quantity $v:=s-\varepsilon$ satisfies $0<v<d$ and $n<\underline{b}(\phi)+v\leq\overline{b}(\phi)+v<n+1$. Suppose first that $\frac{1}{p}-\frac{s}{d}>0$; we have $\frac{1}{p}-\frac{v}{d}>0$, which implies $\mathcal{J}^v f\in T^r_{\phi_v}(0)$, with $1/r=\frac{1}{p}-\frac{v}{d}$ and $\|\mathcal{J}^v f\|_{T^r_{\phi_v}(0)}\leq C\|f\|_{T^p_\phi(0)}$. From $\mathcal{J}^s f=\mathcal{J}^\varepsilon\mathcal{J}^v f$ and

$$\frac{1}{r}-\frac{\varepsilon}{d}=\frac{1}{p}-\frac{s}{d}>0,$$

we know that $\mathcal{J}^s f\in T^q_{\phi_s}(0)$ with $1/q:=\frac{1}{r}-\frac{\varepsilon}{d}=\frac{1}{p}-\frac{s}{d}$ and

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)}\leq C\|\mathcal{J}^v f\|_{T^r_{\phi_v}(0)}\leq C\|f\|_{T^p_\phi(0)}.$$

Now, let us suppose that $\frac{1}{p}-\frac{s}{d}<0$; by choosing ε such that $\frac{1}{p}-\frac{v}{d}<0$, we get $\mathcal{J}^v f\in T^\infty_{\phi_v}(0)$, with $\|\mathcal{J}^v f\|_{T^\infty_{\phi_v}(0)}\leq C\|f\|_{T^p_\phi(0)}$ and we obtain $\mathcal{J}^s f\in T^\infty_{\phi_s}(0)$, with $\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)}\leq C\|f\|_{T^p_\phi(0)}$.

Let us consider the case $s=kd+v$ with $k\in\mathbb{N}$ and $0<v\leq d$; let us first remark that if $n\in\mathbb{N}$ satisfies

$$n<\underline{b}(\phi)+s<\overline{b}(\phi)+s<n+1,$$

then $d\leq n$ implies

$$0\leq n-d<\underline{b}(\phi)+s-d\leq\overline{b}(\phi)+s-d<n-d+1$$

and $s-d$ is still an admissible value. Otherwise, $n<d$ and so $n+1\leq d$, which means that we have $\overline{b}(\phi)+s-d<0$ and therefore that $s-d$ is also an admissible value. Suppose first that $\frac{1}{p}-\frac{s}{d}>0$; let us prove by induction that $\mathcal{J}^s f$ belongs to $T^q_{\phi_s}(0)$ with $1/q:=\frac{1}{p}-\frac{s}{d}$ and $\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)}\leq C\|f\|_{T^p_\phi(0)}$. The case $k=0$ being already known, let us show that if the assertion is true for $k-1$, then it is also true for k ($k\geq 1$). Since $s-d$ is an admissible value, $\mathcal{J}^{s-d}f\in T^r_{\phi_{s-d}}(0)$ with $1/r=\frac{1}{p}-\frac{s-d}{d}$ and

$$\|\mathcal{J}^{s-d}f\|_{T^r_{\phi_{s-d}}(0)}\leq C\|f\|_{T^p_\phi(0)}.$$

As

$$\frac{1}{r} - \frac{d}{d} = \frac{1}{p} - \frac{s}{d} > 0,$$

we have $\mathcal{J}^s f \in T^q_{\phi_s}(0)$ with $1/q := \frac{1}{p} - \frac{s}{d}$ and $\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}$. Now, let us suppose that $\frac{1}{p} - \frac{s}{d} < 0$; let us prove by induction that $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}.$$

It remains to show that if the assertion is true for $k-1$, then it is also true for k ($k \geq 1$). If $\frac{1}{p} - \frac{s-d}{d} < 0$, then $\mathcal{J}^{s-d} f \in T^\infty_{\phi_{s-d}}(0)$ and $\|\mathcal{J}^{s-d} f\|_{T^\infty_{\phi_{s-d}}(0)} \leq C\|f\|_{T^p_\phi(0)}$. From what we have obtained before for the case $s = d$, we get $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}.$$

Otherwise, if $\frac{1}{p} - \frac{s-d}{d} > 0$, from the previous point, $\mathcal{J}^{s-d} f \in T^r_{\phi_{s-d}}(0)$ with $1/r := \frac{1}{p} - \frac{s-d}{d}$, $\|\mathcal{J}^{s-d} f\|_{T^r_{\phi_{s-d}}(0)} \leq C\|f\|_{T^p_\phi(0)}$ and

$$\frac{1}{r} - \frac{d}{d} = \frac{1}{p} - \frac{s}{d} < 0.$$

The case $s = d$ leads to $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$ and $\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}$. Finally, if $\frac{1}{p} - \frac{s-d}{d} = 0$, let $0 < \varepsilon < d$ be such that $s - d + \varepsilon$ is still an admissible value. As $\frac{1}{p} - \frac{s-d+\varepsilon}{d} < 0$, we have $\mathcal{J}^{s-d+\varepsilon} f \in T^\infty_{\phi_{s-d+\varepsilon}}(0)$ and $\|\mathcal{J}^{s-d+\varepsilon} f\|_{T^\infty_{\phi_{s-d+\varepsilon}}(0)} \leq C\|f\|_{T^p_\phi(0)}$. We can thus write

$$\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}.$$

Let us now remark that if $f \in T^p_\phi(x_0)$ and $\mathcal{J}^s f \in T^q_{\phi_s}(x_0)$ with $q > p$, then we can define $R_s := \mathcal{J}^s f - P_s$ where P_s is a polynomial of degree strictly less than $\underline{b}(\phi) + s$ such that

$$r^{-d/q}\|R_s\|_{L^q(B(x_0,r))} \leq |\mathcal{J}^s f|_{T^q_{\phi_s}(x_0)}\phi_s(r).$$

If $p \leq p' \leq q$ and $q' \geq 1$ is such that $\frac{1}{q} + \frac{1}{q'} = 1/p'$, for $r > 0$, we have

$$\begin{aligned} r^{-d/p'}\|R_s\|_{L^{p'}(B(x_0,r))} &\leq C_d r^{-d/p'} r^{d/q'}\|R_s\|_{L^q(B(x_0,r))} \\ &\leq C_d |\mathcal{J}^s f|_{T^q_{\phi_s}(x_0)}\phi_s(r), \end{aligned}$$

which means that $\mathcal{J}^s f$ belongs to $T^{p'}_{\phi_s}(x_0)$ (using the estimation made by the same polynomial as the one that gives the belonging to $T^q_{\phi_s}(x_0)$). Moreover, if $0 \leq \theta \leq 1$ is such that $1/p' = \frac{\theta}{q} + \frac{1-\theta}{p}$, we know that

$$\begin{aligned}\|\mathcal{J}^s f\|_{L^{p'}(\mathbb{R}^d)} &\leq \|\mathcal{J}^s f\|^{\theta}_{L^q(\mathbb{R}^d)} \|\mathcal{J}^s f\|^{1-\theta}_{L^p(\mathbb{R}^d)} \\ &\leq C\|\mathcal{J}^s f\|^{\theta}_{L^q(\mathbb{R}^d)} \|f\|^{1-\theta}_{L^p(\mathbb{R}^d)} \\ &\leq \|\mathcal{J}^s f\|_{L^q(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

We are finally able to prove the three points of the theorem. If $p < d/s$, let us set $1/p^* := \frac{1}{p} - \frac{s}{d}$; $p^* \geq 1$ and from the first part of the proof, $\mathcal{J}^s f$ belongs to $T^{p^*}_{\phi_s}(0)$ and $\|\mathcal{J}^s f\|_{T^{p^*}_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}$. Now, from the second part, for q satisfying $1/p \geq 1/q \geq 1/p^*$, $\mathcal{J}^s f$ belongs to $T^{p^*}_{\phi_s}(0)$ and

$$\begin{aligned}\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} &\leq C(\|\mathcal{J}^s f\|_{T^{p^*}_{\phi_s}(0)} + \|f\|_{L^p(\mathbb{R}^d)}) \\ &\leq c\|f\|_{T^p_\phi(0)}.\end{aligned}$$

Let us consider the case $p > d/s$. The first part of the proof implies that $\mathcal{J}^s f$ belongs to $T^\infty_{\phi_s}(0)$ and $\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}$. Using the second part of the proof, for $p \leq q \leq \infty$, we get that $\mathcal{J}^s f$ belongs to $T^q_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}.$$

For the case $p = d/s$, let $0 < \varepsilon < s$ be such that

$$\frac{1}{p} - \frac{\varepsilon}{d} > \frac{1}{p} - \frac{s}{d} = 0,$$

ε being chosen sufficiently close to s so that it is an admissible value; the first point of the proof gives that $\mathcal{J}^\varepsilon f$ belongs to $T^q_{\phi_\varepsilon}(0)$ and $\|\mathcal{J}^\varepsilon f\|_{T^q_{\phi_\varepsilon}(0)} \leq C\|f\|_{T^p_\phi(0)}$ for q such that $1/p \geq 1/q > \frac{1}{p} - \frac{\varepsilon}{d}$. Now,

$$\frac{1}{q} - \frac{s-\varepsilon}{d} > \frac{1}{p} - \frac{\varepsilon}{d} - \frac{s-\varepsilon}{d} = 0$$

and, from the first part of the proof, $\mathcal{J}^s f$ belongs to $T^q_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}.$$

We can conclude by letting ε tends to s^-. $\square$

This theorem admits the following corollary, regarding $t^p_\phi(x_0)$ spaces.

Corollary 6.1.3. *Let $x_0 \in \mathbb{R}^d$, $p \in (1,\infty)$, $\phi \in \mathcal{B}$ be such that either $\underline{b}(\phi) > -d/p$ and $\underline{b}(\phi) \leq 0$ or there exists $n \in \mathbb{N}$ such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$. Let us consider an admissible value $s > 0$ for ϕ; if the function f belongs to $t^p_\phi(x_0)$, then $\mathcal{J}^s f \in t^q_{\phi_s}(x_0)$, where*

- $1/p \geq 1/q \geq \frac{1}{p} - \frac{s}{d}$, *if* $p < d/s$,
- $p \leq q \leq \infty$, *if* $d/s < p < \infty$,
- $p \leq q < \infty$, *if* $d/s = p$.

Proof. By Corollary 5.2.2, we know that there exists a sequence of functions $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R}^d) \cap t^p_\phi(x_0)$ such that $f_j \to f$ in $T^p_\phi(x_0)$. For such a function, $\mathcal{J}^s f_j \in C^\infty(\mathbb{R}^d)$ and Remark 5.1.9 implies that $\mathcal{J}^s f_j$ belongs to $t^r_{\phi_s}(x_0)$, for all $r \in [1,\infty]$. But, for all values of q that we consider, the preceding theorem implies

$$\|\mathcal{J}^s(f_j - f)\|_{T^q_{\phi_s}(x_0)} \leq C\|f_j - f\|_{T^p_\phi(x_0)}.$$

Therefore, $\mathcal{J}^s f_j$ converges to $\mathcal{J}^s f$ in $T^q_{\phi_s}(x_0)$. From Proposition 5.1.8, we know that $t^q_{\phi_s}(x_0)$ is a closed subspace of $T^q_{\phi_s}(x_0)$, which gives us the conclusion. □

6.2 Derivatives

In this section, we investigate the estimates that can be made for a function whose derivatives are known to belong to $T^p_\phi(x_0)$ (or $t^p_\phi(x_0)$). For such a task, we will need the following classical lemma of Sobolev spaces theory (see e.g. [130]).

Lemma 6.2.1. *Let* $1 \leq p < d$ *and* q *be defined by* $1/q := \frac{1}{p} - \frac{1}{d}$. *There exists* $C_{p,d} > 0$ *such that, for all* $f \in \mathcal{D}(\mathbb{R}^d)$,

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C_{p,d} \sum_{j=1}^{d} \|D_j f\|_{L^p(\mathbb{R}^d)}.$$

Let us remind that, if $\phi \in \mathcal{B}$, then ϕ_1 is the Boyd function defined by

$$\phi_1(x) = x\phi(x) \quad \forall\, x > 0.$$

Theorem 6.2.2. *Let* $x_0 \in \mathbb{R}^d$, $p \in [1,\infty)$, $\phi \in \mathcal{B}$ *be such that* $\underline{b}(\phi) > -d/p$ *and either* $\overline{b}(\phi) < -1$ *or there exists* $n \in \mathbb{N} \cup \{-1\}$ *for which* $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$. *If* f *is such that* $D_j f \in T^p_\phi(x_0)$ *for all* $j \in \{1,\ldots,d\}$ *and*

1. *if* $1 \leq p < d$ *and* $f \in L^q(\mathbb{R}^d)$ *with* $1/q := \frac{1}{p} - \frac{1}{d}$, *then* $f \in T^q_{\phi_1}(x_0)$ *and*

$$\|f\|_{T^q_{\phi_1}(x_0)} \leq C_{p,\phi} \sum_{j=1}^{d} \|D_j f\|_{T^p_\phi(x_0)}, \tag{6.11}$$

2. *if* $f \in L^q(\mathbb{R}^d)$ *where* $q \in [1,\infty)$ *is such that* $1/p \geq 1/q > \frac{1}{p} - \frac{1}{d}$, *then* $f \in T^q_{\phi_1}(x_0)$ *and*

$$\|f\|_{T^q_{\phi_1}(x_0)} \leq C_{p,\phi} \sum_{j=1}^{d} \|D_j f\|_{T^p_\phi(x_0)} + \|f\|_{L^q(\mathbb{R}^d)}. \tag{6.12}$$

Moreover, if $D_j f$ belongs to $t^p_\phi(x_0)$ for all $j \in \{1,\dots,d\}$, then f also belongs to $t^q_{\phi_1}(x_0)$, with q satisfying one of the two preceding points.

Proof. Let us first suppose that f belongs to $\mathcal{D}(\mathbb{R}^d)$; for $j \in \{1,\dots,d\}$, let us set

$$k_j : \mathbb{R}^d \setminus \{0\} \to \mathbb{R} \quad x \mapsto \frac{1}{\omega_d}\frac{x_j}{|x|^d},$$

where ω_d is the area of the hyper-sphere in $\mathbb{R}^d$. It is easy to check that for $x \neq 0$, we have $\sum_{j=1}^d D_j k_j(x) = 0$.

Let us fix $x \in \mathbb{R}^d$, set, given $r > 0$, $\Omega_r := \{y \in \mathbb{R}^d : |x-y| \geq r\}$ and denote by $\partial\Omega_r := \{y \in \mathbb{R}^d : |x-y| = r\}$ the boundary of this set. Using Green's first identity (see Theorem 1.7.4), we get

$$\sum_{j=1}^d \int_{\Omega_r} D_j f(y) k_j(x-y)\, dy = \frac{1}{\omega_d}\int_{\partial\Omega_r} \frac{f(y)}{|x-y|^{d-1}}\, d\sigma.$$

Lebesgue's dominated convergence theorem implies that the right-hand side tends to $f(x)$ as r tends to 0^+, while the left-hand side tends to

$$\sum_{j=1}^d \int_{\mathbb{R}^d} D_j f(y) k_j(x-y)\, dy.$$

Therefore, we have the following representation for f:

$$f = \sum_{j=1}^d \int_{\mathbb{R}^d} D_j f(y)\, k_j(\cdot - y)\, dy. \tag{6.13}$$

Let us prove the second point in the case $q = p$. Let us first deal with the case $\overline{b}(\phi) < -1$; for $r > 0$ and $x \in \mathbb{R}^d$ such that $|x - x_0| \leq r$, we can write

$$f(x) = \sum_{j=1}^d (f_{1,j}(x) + f_{2,j}(x)),$$

where we have set

$$f_{1,j}(x) := \int_{B(x_0,2r)} D_j f(y) k_j(x-y)\, dy$$

and

$$f_{2,j}(x) := \int_{\mathbb{R}^d \setminus B(x_0,2r)} D_j f(y) k_j(x-y)\, dy.$$

By Young's inequality, we have

$$\begin{aligned} r^{-d/p}\|f_{1,j}\|_{L^p(B(x_0,r))} &\le r^{-d/p}\|D_j f\|_{L^p(B(x_0,2r))}\|k_j\|_{L^1(B(x_0,3r))} \\ &\le C\overline{\phi}(2)|D_j f|_{T^p_\phi(x_0)}\phi(r)r. \end{aligned} \tag{6.14}$$

To estimate $r^{-d/p}\|f_{2,j}\|_{L^p(B(x_0,r))}$, let us define the function F_j for $r>0$ by

$$F_j(r) := \int_{B(x_0,r)} |D_j f(y)|\,dy = \int_0^r \psi_j(\rho)\,d\rho,$$

where we have set, using spherical coordinates in $\mathbb{R}^d$ centered at x_0,

$$\psi_j(\rho) := \rho^{d-1}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}|D_j f(y(\rho,\theta_1,\cdots,\theta_{d-1}))|\,d\Omega_d.$$

We know that, for $r>0$,

$$r^{-d}F_j(r) \le C_d|D_j f|_{T^p_\phi(x_0)}\phi(r) \tag{6.15}$$

and, for all $R>0$, we have

$$\begin{aligned} &F_j(R)R^{1-d} - F_j(2r)(2r)^{1-d} \\ &= \int_{2r}^R \psi_j(\rho)\rho^{1-d}\,d\rho + \int_{2r}^R F_j(\rho)(1-d)\rho^{-d}\,d\rho. \end{aligned} \tag{6.16}$$

Thanks to inequality (6.15) and Proposition 1.2.6, as $\overline{b}(\phi) < -1$, $F_j(R)R^{1-d}$ tends to 0 as R tends to $+\infty$. Therefore,

$$\begin{aligned} \int_{2r}^{+\infty}\psi_j(\rho)\rho^{1-d}\,d\rho &\le (d-1)\int_{2r}^{+\infty}F_j(\rho)\rho^{-d}\,d\rho \\ &\le C_d(d-1)|D_j f|_{T^p_\phi(x_0)}\int_{2r}^{+\infty}\phi(\rho)\,d\rho \\ &\le C_d(d-1)|D_j f|_{T^p_\phi(x_0)}\overline{\phi}(2)\phi(r)\int_{2r}^{+\infty}\overline{\phi}(\frac{\rho}{2r})\,d\rho \\ &= C_d(d-1)|D_j f|_{T^p_\phi(x_0)}\overline{\phi}(2)2\phi(r)r\int_1^{+\infty}\overline{\phi}(t)\,dt. \end{aligned}$$

By Proposition 1.2.10, this last integral is bounded and thus

$$\int_{2r}^{+\infty}\psi_j(\rho)\rho^{1-d}\,d\rho \le C_d C_{\phi,1}|D_j f|_{T^p_\phi(x_0)}\phi_1(r),$$

where

$$C_{\phi,1} := \overline{\phi}(2)\int_1^{+\infty}\overline{\phi}(t)\,dt. \tag{6.17}$$

As

$$
\begin{aligned}
|f_{j,2}(x)| &\leq \int_{\mathbb{R}^d \setminus B(x_0,2r)} \frac{|D_j f(y)|}{|x-y|^{d-1}}\, dy \\
&\leq \int_{\mathbb{R}^d \setminus B(x_0,2r)} \frac{|D_j f(y)|}{(\frac{1}{2}|x_0-y|)^{d-1}}\, dy \\
&= C_d \int_{2r}^{+\infty} \rho^{1-d} \psi_j(\rho)\, d\rho,
\end{aligned}
$$

we finally obtain

$$
r^{-d/p} \|f_{j,2}\|_{L^p(B(x_0,r))} \leq C_d C_{\phi,1} |D_j f|_{T^p_\phi(x_0)} \phi_1(r).
$$

Inequality (6.12) follows from this estimate and (6.14). Now, let us suppose that $-1 < \underline{b}(\phi) \leq \overline{b}(\phi) < 0$ and fix $r > 0$. For any $x \in B(x_0, r)$, we have

$$
f(x) - f(x_0) = \sum_{j=1}^{d} (f_{j,1} + f_{j,2} - f_{j,3})(x),
$$

where we have set,

$$
f_{j,1}(x) := \int_{B(x_0,2r)} D_j f(y) k_j(x-y)\, dy,
$$

$$
f_{j,2}(x) := \int_{\mathbb{R}^d \setminus B(x_0,2r)} D_j f(y)(k_j(x-y) - k_j(x_0-y))\, dy
$$

and

$$
f_{j,3}(x) := \int_{B(x_0,2r)} D_j f(y) k_j(x_0-y)\, dy.
$$

Once again, we have

$$
r^{-d/p} \|f_{1,j}\|_{L^p(B(x_0,r))} \leq C\overline{\phi}(2) |D_j f|_{T^p_\phi(x_0)} \phi_1(r).
$$

Moreover, if $x \in B(x_0, r)$ and $|x_0 - y| \geq 2r$, then, for all $|h| \leq |x - x_0|$, $|x_0 - y + h| \geq |x_0 - y|/2$ and so, by the mean value theorem and the fact that $|D^\alpha k_j(z)| \leq C/|z|^d$ for all $z \neq 0$ and $|\alpha| = 1$,

$$
|k_j(x-y) - k_j(x_0-y)| \leq Cr|x_0-y|^{-d}.
$$

Therefore,

$$
|f_{j,2}(x)| \leq Cr \int_{2r}^{+\infty} \psi_j(\rho) \rho^{-d}\, d\rho
$$

and by the same reasoning as before, using this time $\overline{b}(\phi) < 0$, we get

$$r^{-d/p}\|f_{j,2}\|_{L^p(B(x_0,r))} \leq C_d C_{\phi,2}|D_j f|_{T^p_\phi(x_0)}\phi_1(r),$$

where

$$C_{\phi,2} := \overline{\phi}(2)\int_1^{+\infty}\frac{\overline{\phi}(t)}{t}\,dt. \tag{6.18}$$

For the last term, we have

$$|f_{j,3}(x)| \leq r\int_0^{2r}\psi_j(\rho)\rho^{1-d}\,d\rho$$

and using an equality similar to (6.16), we have

$$\int_0^{2r}\psi_j(\rho)\rho^{1-d}\,d\rho \leq F_j(2r)(2r)^{1-d} + d\int_0^{2r}F_j(\rho)\rho^{-d}\,d\rho.$$

As $-1 < \underline{b}(\phi)$, we have

$$r^{-d/p}\|f_{j,3}\|_{L^p(B(x_0,r))} \leq C_d C_{\phi,3}|D_j f|_{T^p_\phi(x_0)}\phi_1(r),$$

where

$$C_{\phi,3} := \overline{\phi}(2)(1+\int_0^1\overline{\phi}(t)\,dt). \tag{6.19}$$

Again, inequality (6.12) follows from the estimate made of $r^{-d/p}\|f_{j,k}\|_{L^p(B(x_0,r))}$, for all $r > 0$ and $k \in \{1,2,3\}$. Finally, if there exists $n \in \mathbb{N}$ such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$, let P be the Taylor expansion of f at x_0 of order $n+1$, set $\widetilde{f} := f - P$ and, for $j \in \{1,\dots,d\}$, $\widetilde{f_j} := D_j\widetilde{f}$. For $r > 0$, we have

$$\begin{aligned}
&\int_{B(x_0,r)}|\widetilde{f}(y)|^p\,dy \\
&= \int_0^r\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}|\widetilde{f}(x_0 + y_{(\rho,\theta_1,\dots,\theta_{d-1})})|^p\rho^{d-1}\,d\Omega_d\,d\rho,
\end{aligned}$$

where $y_{(\rho,\theta_1,\dots,\theta_{d-1})}$ is the point defined by

$$[y_{(\rho,\theta_1,\dots,\theta_{d-1})}]_j := \rho\prod_{k<j}\sin(\theta_k)\cos(\theta_j)\forall j \in \{0,\dots,d-1\}$$

and

$$[y_{(\rho,\theta_1,\dots,\theta_{d-1})}]_d := \rho\prod_{k<d}\sin(\theta_k).$$

Let us set

$$g_j(\theta_1,\ldots,\theta_{d-1}) := \prod_{k<j} \sin(\theta_k)\cos(\theta_j)$$

and

$$g_d(\theta_1,\ldots,\theta_{d-1}) := \prod_{k<d} \sin(\theta_k).$$

Using Taylor's formula, we have, as $\widetilde{f}(x_0) = 0$,

$$\widetilde{f}(x_0 + y_{(\rho,\theta_1,\ldots,\theta_{d-1})}) = \sum_{j=1}^{d} \int_0^\rho \widetilde{f_j}(x_0 + y_{(t,\theta_1,\ldots,\theta_{d-1})}) g_j(\theta_1,\ldots,\theta_{d-1})\, dt.$$

Therefore, as $|g_j| \leq 1$, Hölder's inequality leads to

$$\begin{aligned}
&\int_{B(x_0,r)} |\widetilde{f}(y)|^p\, dy \\
&\leq C_{d,p} \int_0^r \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \rho^{d-1} \int_0^\rho \sum_{j=1}^{d} |\widetilde{f_j}(x_0 + y_{(t,\theta_1,\ldots,\theta_{d-1})})|^p\, dt \rho^{p-1}\, d\Omega_d d\rho \\
&\leq C_{d,p} r^{d+p-2} \sum_{j=1}^{d} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \int_0^r \int_t^r |\widetilde{f_j}(x_0 + y_{(t,\theta_1,\ldots,\theta_{d-1})})|^p\, d\rho dt d\Omega_d \\
&\leq C_{d,p} r^{d+p-1} \sum_{j=1}^{d} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \int_0^r |\widetilde{f_j}(x_0 + y_{(t,\theta_1,\ldots,\theta_{d-1})})|^p\, dt d\Omega_d \\
&= C_{d,p} r^{d+p-1} \sum_{j=1}^{d} \int_{B(x_0,r)} \frac{|\widetilde{f_j}(y)|^p}{|y - x_0|^{d-1}}\, dy.
\end{aligned}$$

Moreover, using a similar technique as before, we have, for $j \in \{1,\ldots,d\}$,

$$\int_{B(x_0,r)} \frac{|\widetilde{f_j}(y)|^p}{|y - x_0|^{d-1}}\, dy \leq |D_j f|^p_{T^p_\phi(x_0)} \phi(r)^p r (1 + \int_0^1 \overline{\phi}(t)^p\, dt),$$

which allows us to conclude, as $\underline{b}(\phi) > 0$, that

$$r^{-d/p} \|\widetilde{f}\|_{L^p(B(x_0,r))} \leq C_{d,p} C_{\phi,4} \sum_{j=1}^{d} |D_j f|_{T^p_\phi(x_0)} \phi_1(r), \tag{6.20}$$

where

$$C_{\phi,4} := (1 + \int_0^1 \overline{\phi}(t)^p\, dt)^{1/p}. \tag{6.21}$$

In order to estimate $\|f\|_{T^p_{\phi_1}(x_0)}$, we need information about $\sum_{|\alpha|\leq n+1}|D^\alpha P(x_0)|/\alpha!$. We have

$$\sum_{0<|\alpha|\leq n+1}\frac{|D^\alpha P(x_0)|}{\alpha!}\leq C\sum_{j=1}^{d}\sum_{0<|\beta|\leq n}\frac{D^\beta P_j(x_0)}{\beta!}, \tag{6.22}$$

where, given $j\in\{1,\ldots,d\}$, P_j is the Taylor expansion of D_jf at x_0 of order n. It remains to work on $P(x_0)=f(x_0)$. For this purpose, let us choose $\varphi\in\mathcal{D}(\mathbb{R}^d)$ such that $\varphi=1$ on $\overline{B(0,1)}$ and $\operatorname{supp}(\varphi)\subseteq\overline{B(0,2)}$. Using representation (6.13), we obtain

$$\begin{aligned}f(x_0)=f(x_0)\varphi(x_0-x_0)=\sum_{j=1}^{d}\Big(&\int_{\mathbb{R}^d}k_j(x_0-y)D_jf(y)\varphi(y-x_0)\,dy\\&+\int_{\mathbb{R}^d}k_j(x_0-y)f(y)D_j\varphi(y-x_0)\,dy\Big).\end{aligned}$$

For the first term of the right-hand side, we have

$$\Big|\int_{\mathbb{R}^d}k_j(x_0-y)D_jf(y)\varphi(y-x_0)\,dy\Big|\leq C_\varphi\int_{B(x_0,2)}\frac{|D_jf(y)|}{|x_0-y|^{d-1}}\,dy.$$

For $r>0$, we have

$$r^{-d/p}\|D_jf-P_j\|_{L^p(B(x_0,r))}\leq|D_jf|_{T^p_\phi(x_0)}\phi(r)$$

and so

$$r^{-d/p}\|D_jf\|_{L^p(B(x_0,r))}\leq|D_jf|_{T^p_\phi(x_0)}\phi(r)+C_d\sum_{|\beta|\leq n}\frac{|D^\beta P_j(x_0)|}{\beta!}r^{|\beta|}.$$

As

$$\sum_{|\beta|\leq n}\frac{|D^\beta P_j(x_0)|}{\beta!}\leq\|D_jf\|_{T^p_\phi(x_0)},$$

we can write, using the same technique as before,

$$\int_{B(x_0,2)}\frac{|D_jf(y)|}{|x_0-y|^{d-1}}\,dy\leq C_dC_{\phi,5}\|D_jf\|_{T^p_\phi(x_0)},$$

where

$$C_{\phi,5}:=\phi(2)+2^n+2\phi(2)\int_0^1\overline{\phi}(t)\,dt. \tag{6.23}$$

For the second term, we have

$$\begin{aligned}
&|\int_{\mathbb{R}^d} k_j(x_0-y)f(y)D_j\varphi(y-x_0)\,dy| \\
&\le \int_{B(x_0,2)\setminus B(x_0,1)} |k_j(x_0-y)||f(y)||D_j\varphi(y-x_0)|\,dy \\
&\le C_\varphi \int_{B(x_0,2)\setminus B(x_0,1)} |f(y)|\,dy \\
&\le C_{\varphi,d}\|f\|_{L^p(\mathbb{R}^d)},
\end{aligned}$$

which gives

$$|f(x_0)| \le C_{\varphi,d}(C_{\phi,5}\sum_{j=1}^d \|D_j f\|_{T_\phi^p(x_0)} + \|f\|_{L^p(\mathbb{R}^d)}).$$

This relation, equations (6.20) and (6.22) lead to inequality (6.12). That being done, we have thus obtained the second part of the theorem in the case $p=q$.

Let us now prove the first point of the theorem, still considering a function f from $\mathcal{D}(\mathbb{R}^d)$. As previously, let us denote by φ a function in $\mathcal{D}(\mathbb{R}^d)$ such that $\varphi=1$ on $\overline{B(0,1)}$ and $\mathrm{supp}(\varphi)\subseteq\overline{B(0,2)}$. If there exists $n\in\mathbb{N}\cup\{-1\}$ such that $n<\underline{b}(\phi)\le\overline{b}(\phi)<n+1$, let P be the Taylor expansion of f at x_0 of order $n+1$, otherwise we set $P=0$. Finally, define $\widetilde{f} := f-P$ and, for $j\in\{1,\dots,d\}$, $\widetilde{f_j} := D_j\widetilde{f}$. If $1/q := \frac{1}{p}-\frac{1}{d}$, thanks to Lemma 6.2.1, we have, for all $r>0$,

$$\begin{aligned}
r^{-d/q}\|\widetilde{f}\|_{L^q(B(x_0,r))} &\le r^{-d/q}\|\widetilde{f}\varphi(\frac{\cdot - x_0}{r})\|_{L^p(\mathbb{R}^d)} \\
&\le C_{p,d}r^{-d/q}\sum_{j=1}^d(\|\widetilde{f_j}\varphi(\frac{\cdot - x_0}{r})\|_{L^p(\mathbb{R}^d)} \\
&\quad + r^{-1}\|\widetilde{f}D_j\varphi(\frac{\cdot - x_0}{r})\|_{L^p(\mathbb{R}^d)}) \\
&= C_\varphi C_{p,d}\sum_{j=1}^d(rr^{-d/p}\|\widetilde{f_j}\|_{L^p(B(x_0,2r))} \\
&\quad + r^{-d/p}\|\widetilde{f}\|_{L^p(B(x_0,2r))}).
\end{aligned}$$

Moreover, by hypothesis,

$$rr^{-d/p}\|\widetilde{f_j}\|_{L^p(B(x_0,2r))} \le 2^{d/p}\overline{\phi}(2)|D_jf|_{T_\phi^p(x_0)}\phi_1(r) \tag{6.24}$$

and, using what we have proved so far,

$$r^{-d/p}\|\widetilde{f}\|_{L^p(B(x_0,2r))} \le C_{d,p}C_\phi\sum_{j=1}^d|D_jf|_{T_\phi^p(x_0)}\phi_1(r). \tag{6.25}$$

As before,

$$\sum_{|\alpha|\le n+1} \frac{|D^\alpha P(x_0)|}{\alpha!} \le C_{\varphi,d}(C_{\phi,5}\sum_{j=1}^d \|D_j f\|_{T^p_\phi(x_0)} + \|f\|_{L^q(\mathbb{R}^d)}). \tag{6.26}$$

That being done, another use of Lemma 6.2.1 gives

$$\|f\|_{L^q(\mathbb{R}^d)} \le C_{p,d}\sum_{j=1}^d \|D_j f\|_{L^p(\mathbb{R}^d)} \tag{6.27}$$

and inequality (6.11) is proved, thanks to relations (6.24) to (6.27).

Now, let us come back to the second point of the theorem and investigate the case where $q \ge 1$ is such that $1/p \ge 1/q > \frac{1}{p} - \frac{1}{d}$; we still consider a function $f \in \mathcal{D}(\mathbb{R}^d)$. Again, we use equality (6.13); as $1/p \ge 1/q > \frac{1}{p} - \frac{1}{d}$, there exists $p' \in [1,\infty)$ such that $1/q = \frac{1}{p} + \frac{1}{p'} - 1$ and, by Young's inequality,

$$\begin{aligned}&\|\int_{\mathbb{R}^d} k_j(\cdot - y)\widetilde{f_j}(y)\varphi(\frac{y-x_0}{r})\,dy\|_{L^q(B(x_0,r))}\\ &\le C_\varphi \|k_j\|_{L^{p'}(B(x_0,3r))}\|\widetilde{f_j}(y)\|_{L^p(B(x_0,2r))}\end{aligned}$$

and

$$\|k_j\|_{L^{p'}(B(x_0,3r))} \le C_{d,p}((3r)^{(d-1)(1-p')+1})^{1/p'} = C_{d,p}(3r)^{\frac{d}{q}-\frac{d}{p}+1},$$

which gives us

$$\begin{aligned}&r^{-d/q}\|\int_{\mathbb{R}^d} k_j(\cdot - y)\widetilde{f_j}(y)\varphi(\frac{y-x_0}{r})\,dy\|_{L^q(B(x_0,r))}\\ &\le C_{\varphi,d,p}\overline{\phi}(2)|D_j f|_{T^p_\phi(x_0)}\phi_1(r).\end{aligned}$$

Similarly, using the first part of the proof, we obtain

$$\begin{aligned}&\|\int_{\mathbb{R}^d} k_j(\cdot - y)\widetilde{f}(y)r^{-1}D_j\varphi(\frac{y-x_0}{r})\,dy\|_{L^q(B(x_0,r))}\\ &\le C_{\varphi,d,p}r^{-d/p}r^{\frac{d}{q}}\|\widetilde{f}(y)\|_{L^p(B(x_0,2r))}\\ &\le C_{\varphi,d,p}C_\phi\overline{\phi}(2)r^{\frac{d}{q}}\sum_{j=1}^d |D_j f|_{T^p_\phi(x_0)}\phi_1(r).\end{aligned}$$

This upper bound and equation (6.26) lead to inequality (6.12).

Now that the theorem has been obtained for the functions belonging to $\mathcal{D}(\mathbb{R}^d)$, let us consider a compactly-supported function f such that $D_j f \in t^p_\phi(x_0)$, for all $j \in \{1,\ldots,d\}$. Given $\lambda > 0$, let f_λ be the function defined by (5.4) and, for $j \in \{1,\ldots,d\}$, define $f_{\lambda,j} := D_j f_\lambda$. By Proposition 5.2.1, we know that $f_{\lambda,j}$ converges to $D_j f$ in $T^p_\phi(x_0)$ ($j \in \{1,\dots,d\}$). Inequalities (6.11) and (6.12) imply that $(f_\lambda)_{\lambda>0}$ is a Cauchy sequence in $T^q_{\phi_1}(x_0)$ (with appropriate q) and thus, by Proposition 5.1.7, $(f_\lambda)_{\lambda>0}$ converges in $T^q_{\phi_1}(x_0)$. As f_λ converges to f in $L^q(\mathbb{R}^d)$, we conclude that f_λ converges to f in $T^q_{\phi_1}(x_0)$. Moreover, by passing to the limit, we can affirm that inequalities (6.11) and (6.12) still hold for f. Now, as f_λ belongs to $\mathcal{D}(\mathbb{R}^d)$ and $t^q_{\phi_1}(x_0)$ for all $\lambda > 0$, by Proposition 5.1.8, f also belongs to $t^q_{\phi_1}(x_0)$.

Let us now consider a general function f such that, for all $j \in \{1,\dots,d\}$, $D_j f$ belongs to $t^p_\phi(x_0)$ and let us again take $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\varphi = 1$ on $\overline{B(0,1)}$ and $\operatorname{supp}(\varphi) \subseteq \overline{B(x_0,2)}$. Given $\varepsilon > 0$, we define

$$f_\varepsilon := f\varphi(\varepsilon(\cdot - x_0)).$$

By assumption, we know that, for all $j \in \{1,\dots,d\}$, there exists a polynomial P_j of degree strictly less than $\underline{b}(\phi)$ such that

$$\phi(r)^{-1} r^{-d/p} \|D_j f - P_j\|_{L^p(B(x_0,r))} \to 0 \qquad \text{as } r \to 0^+.$$

Moreover, since we assume that $f \in L^q(\mathbb{R}^d)$ for some $q \geq p$, f belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$D_j f_\varepsilon = D_j f \varphi(\varepsilon(\cdot - x_0)) + \varepsilon f D_j\varphi(\varepsilon(\cdot - x_0))$$

belongs to $L^p(\mathbb{R}^d)$ for all $\varepsilon > 0$. Of course, we have

$$\begin{aligned}
&\phi(r)^{-1} r^{-d/p} \|D_j f_\varepsilon - P_j\|_{L^p(B(x_0,r))} \\
&\leq \phi(r)^{-1} r^{-d/p} \|D_j f \varphi(\varepsilon(\cdot - x_0)) - P_j\|_{L^p(B(x_0,r))} \\
&\quad + \phi(r)^{-1} r^{-d/p} \|\varepsilon f D_j\varphi(\varepsilon(\cdot - x_0))\|_{L^p(B(x_0,r))}.
\end{aligned}$$

Now, for r sufficiently small, we have $\varphi(\varepsilon(\cdot - x_0)) = 1$ and $D_j\varphi(\varepsilon(\cdot - x_0)) = 0$ on $B(x_0,r)$ and, for such r,

$$\phi(r)^{-1} r^{-d/p} \|D_j f_\varepsilon - P_j\|_{L^p(B(x_0,r))} \leq \phi(r)^{-1} r^{-d/p} \|D_j f - P_j\|_{L^p(B(x_0,r))},$$

which shows that $D_j f_\varepsilon$ belongs to $t^p_\phi(x_0)$. As f_ε is compactly-supported, the previous case reveals that f_ε belongs to $t^q_{\phi_1}(x_0)$ (for appropriate q). Let us prove that $D_j f_\varepsilon$ tends to $D_j f$ in $T^p_\phi(x_0)$, as ε tends to 0^+. We have

$$\begin{aligned}
&\|D_j f_\varepsilon - D_j f\|_{T^p_\phi(x_0)} \\
&= \sup_{r>0} \phi(r)^{-1} r^{-d/p} \|D_j f_\varepsilon - D_j f\|_{L^p(B(x_0,r))} + \|D_j f_\varepsilon - D_j f\|_{L^p(\mathbb{R}^d)}
\end{aligned}$$

and

$$D_j f_\varepsilon - D_j f = D_j f(\varphi(\varepsilon(\cdot - x_0)) - 1) + \varepsilon f D_j\varphi(\varepsilon(\cdot - x_0)). \tag{6.28}$$

A simple application of Lebesgue's dominated convergence theorem shows that the L^p norm of the first term of the right-hand side of (6.28) tends to 0 as ε tends to 0^+, while

$$\begin{aligned}\|\varepsilon f D_j\varphi(\varepsilon(\cdot - x_0))\|_{L^p(\mathbb{R}^d)} &\leq C_\varphi \varepsilon \|f\|_{L^p(\overline{B(x_0,2/\varepsilon)}\setminus B(x_0,1/\varepsilon))}\\ &\leq C_{\varphi,p,q,d}\,\varepsilon^{1-\frac{d}{p}+\frac{d}{q}}\|f\|_{L^q(\mathbb{R}^d\setminus B(x_0,1/\varepsilon))}.\end{aligned}$$

Since $1-\frac{d}{p}+\frac{d}{q} \geq 0$ by hypothesis and $\|f\|_{L^q(\mathbb{R}^d\setminus B(x_0,1/\varepsilon))}$ tends to 0 as ε tends to 0^+, so does $\|D_j f_\varepsilon - D_j f\|_{L^p(\mathbb{R}^d)}$. Moreover, for $0<\varepsilon<1$, if $0<r<1/\varepsilon$, then $D_j f_\varepsilon - D_j f$ vanishes on $B(x_0,r)$. If $r>1/\varepsilon$, then $r>1$ and if $\delta>0$ satisfies $\underline{b}(\phi)-\delta+\frac{d}{p}>0$, then by Proposition 1.2.6,

$$\phi(r)^{-1}r^{-d/p} \leq C_{\delta,\phi} r^{-(\underline{b}(\phi)-\delta+\frac{d}{p})} \leq C_{\delta,\phi}\varepsilon^{(\underline{b}(\phi)-\delta+\frac{d}{p})},$$

which finally leads to

$$\begin{aligned}&\sup_{r>0}\phi(r)^{-1}r^{-d/p}\|D_j f_\varepsilon - D_j f\|_{L^p(B(x_0,r))}\\ &\leq C_{\delta,\phi}\varepsilon^{(\underline{b}(\phi)-\delta+\frac{d}{p})}\|D_j f_\varepsilon - D_j f\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

so that $D_j f_\varepsilon$ tends to $D_j f$ in $T^p_\phi(x_0)$ as $\varepsilon \to 0^+$. Using again the completeness of the space $T^q_{\phi_1}(x_0)$ and the closeness of $t^q_{\phi_1}(x_0)$, we conclude, by inequalities (6.11) and (6.12), that f_ε tends to f in $T^q_{\phi_1}(x_0)$ and $f \in t^q_{\phi_1}(x_0)$. By passing to the limit in (6.11) and (6.12), we obtain that those inequalities still hold for f.

It remains to consider the case of a function f such that, for $j\in\{1,\ldots,d\}$, $D_j f$ belongs to $T^p_\phi(x_0)$. Let $\varepsilon>0$ be such that

$$-d/p < \underline{b}(\phi)-\varepsilon \leq \overline{b}(\phi)-\varepsilon < -1,$$

if $\overline{b}(\phi) < -1$ and

$$n < \underline{b}(\phi)-\varepsilon \leq \overline{b}(\phi)-\varepsilon < n+1$$

if $n \in \mathbb{N}\cup\{-1\}$ satisfies $n<\underline{b}(\phi)\leq\overline{b}(\phi)<n+1$. For such a number, $D_j f$ belongs to $t^p_{\phi_{-\varepsilon}}(x_0)$ for $j\in\{1,\ldots,d\}$ and it follows from the previous case that $D_j f$ belongs to $t^q_{\phi_{1-\varepsilon}}(x_0)$. Moreover, if $1\leq p<d$ and f belongs to $L^q(\mathbb{R}^d)$ with $1/q := \frac{1}{p}-\frac{1}{d}$, then f belongs to $T^q_{\phi_{1-\varepsilon}}(x_0)$ and

$$\|f\|_{T^q_{\phi_{1-\varepsilon}}(x_0)} \leq C_{p,\phi_{-\varepsilon}}\sum_{j=1}^d \|D_j f\|_{T^p_{\phi_{-\varepsilon}}(x_0)}. \tag{6.29}$$

Otherwise, if f belongs to $L^q(\mathbb{R}^d)$ with $q\in[1,\infty)$ satisfying $1/p \geq 1/q > \frac{1}{p}-\frac{1}{d}$, then f belongs to $T^q_{\phi_{1-\varepsilon}}(x_0)$ and

$$\|f\|_{T^q_{\phi_{1-\varepsilon}}(x_0)} \leq C_{p,\phi-\varepsilon}\sum_{j=1}^d \|D_j f\|_{T^p_{\phi-\varepsilon}(x_0)} + \|f\|_{L^q(\mathbb{R}^d)}. \tag{6.30}$$

Let us analyse the constants defined in (6.17), (6.18), (6.19), (6.21) and (6.23). For a chosen $\varepsilon > 0$, we have for example

$$\begin{aligned} C_{\phi_{-\varepsilon},1} &= \overline{\phi_{-\varepsilon}}(2) \int_1^{+\infty} \overline{\phi_{-\varepsilon}}(t)\,dt \\ &= \overline{\phi}(2)2^{-\varepsilon} \int_1^{+\infty} \overline{\phi}(t)t^{-\varepsilon}\,dt \\ &\leq C_\phi \end{aligned}$$

and a similar reasoning applied to (6.18), (6.19), (6.21) and (6.23) shows that we can find a constant $C > 0$ such that, for ε small enough, the constant $C_{p,\phi_{-\varepsilon}}$ appearing in (6.29) and (6.30) is bounded by $CC_{p,\phi}$. Moreover, since

$$\|D_j f\|_{T^p_{\phi_{-\varepsilon}}(x_0)} \leq \|D_j f\|_{T^p_\phi(x_0)},$$

we can conclude by taking the limit as ε tends to 0^+. □

6.3 Singular integral operators

Let us now study the action of convolution singular integral operators on the space $T^p_\phi(x_0)$. This class of operators was particularly studied by Calderón and Zygmund in [24, 25], where the authors proved the following crucial theorem.

Theorem 6.3.1. *Let us set, for $\varepsilon > 0$,*

$$\mathcal{K}_\varepsilon f = \int_{\mathbb{R}^d \setminus B(\cdot,\varepsilon)} k(\cdot - y) f(y)\,dy,$$

where

- *k is homogeneous[1] of degree $-d$,*
- *k has mean value zero on the sphere $\Sigma = \{x \in \mathbb{R}^d : |x| = 1\}$,*
- *$k \in L^q(\Sigma)$ for a $1 < q < \infty$,*
- *$f \in L^p(\mathbb{R}^d)$ with $1 < p < \infty$.*

Then, there exists $\mathcal{K}f \in L^p(\mathbb{R}^d)$ such that $\mathcal{K}_\varepsilon f$ tends to $\mathcal{K}f$ in $L^p(\mathbb{R}^d)$, and pointwise almost everywhere as $\varepsilon \to 0^+$. Moreover, if we set

$$\mathcal{K}^* f = \sup_{\varepsilon > 0} |\mathcal{K}_\varepsilon f|,$$

then $\mathcal{K}^* f$ belongs to $L^p(\mathbb{R}^d)$ and

$$\|\mathcal{K}^* f\|_{L^p(\mathbb{R}^d)} \leq C_{p,q} \|k\|_{L^q(\Sigma)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{6.31}$$

[1] It means that $k(\lambda x) = \lambda^{-d} k(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$.

Remark 6.3.2. In the theorem originally stated by Calderón and Zygmund, the integrability assumption made on k is the following: $k + k(-\cdot) \in L\log L(\Sigma)$. This condition is a little less restrictive, since for a finite measure space $(X, \mathscr{A}, \mu)$, we have (see [6] for example)

$$L^q(X, \mathscr{A}, \mu) \hookrightarrow L\log L(X, \mathscr{A}, \mu),$$

for all $1 < q < \infty$. In the sequel, we will need to consider $k \in L^q(\Sigma)$, with $1 < q < \infty$, in order to take advantage of inequality (6.31).

We will use the following notation:

Notation 6.3.3. Given $\phi \in \mathcal{B}$, we set

$$\lceil \overline{b}(\phi) \rceil_{\mathbb{N}} := \inf\{k \in \mathbb{N} : \overline{b}(\phi) < k\}.$$

Proposition 6.3.4. *Let $\mathcal{K}$ be the convolution singular integral operator defined by*

$$\mathcal{K}f = p.v. \int k(\cdot - y) f(y)\, dy,$$

where the kernel $k \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is homogeneous of degree $-d$. We also assume that k has mean value zero on the sphere Σ.

Let $p \in (1, \infty)$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that $-d/p < \underline{b}(\phi)$ and either $\overline{b}(\phi) < 0$ or there exists $n \in \mathbb{N}$ for which

$$n < \underline{b}(\phi) \leq \overline{b}(\phi) < n + 1. \tag{6.32}$$

If a function f belongs to $T^p_\phi(x_0)$, then $\mathcal{K}f \in T^p_\phi(x_0)$ and

$$\|\mathcal{K}f\|_{T^p_\phi(x_0)} \leq C_{\phi,p} M \|f\|_{T^p_\phi(x_0)}, \tag{6.33}$$

where we have set

$$M = \sup_{\substack{|x|=1 \\ 0 \leq |\alpha| \leq \lceil \overline{b}(\phi) \rceil_{\mathbb{N}}}} |D^\alpha k(x)|.$$

Moreover, if $f \in t^p_\phi(x_0)$, then we also have $\mathcal{K}f \in t^p_\phi(x_0)$.

Proof. We can assume, without loss of generality, that $x_0 = 0$. If $f \in T^p_\phi(0)$ then there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ such that, for all $r > 0$,

$$r^{-d/p} \|f - P\|_{L^p(B(x_0, r))} \leq |f|_{T^p_\phi(0)} \phi(r).$$

Let φ be a function in $\mathcal{D}(\mathbb{R}^d)$ such that $\varphi = 1$ on $\overline{B(0,1)}$ and $\operatorname{supp}(\varphi) \subseteq \overline{B(0,2)}$; we set

$$f_1 := \varphi P \qquad \text{and} \qquad f_2 := f - f_1.$$

If $\overline{b}(\phi) < 0$, then $f_1 = 0$ and obviously $f_1 \in T^p_\phi(0)$ with $\|f_1\|_{T^p_\phi(0)} \leq \|f\|_{T^p_\phi(0)}$. Otherwise, (6.32) holds and if $r \leq 1$, $r^{-d/p}\|f_1 - P\|_{L^p(B(x_0,r))} = 0$. If $r > 1$, then, by Proposition 1.2.6,

$$\begin{aligned} r^{-d/p}\|f_1 - P\|_{L^p(B(x_0,r))} &\leq r^{-d/p} C_{\varphi,p}\|P\|_{L^p(B(x_0,r))} \\ &\leq C_{\varphi,d,p} \sum_{|\alpha|\leq n} \frac{|D^\alpha P(0)|}{\alpha!} r^{|\alpha|} \\ &\leq C_{\varphi,d,p} C_\phi \|f\|_{T^p_\phi(0)} \phi(r), \end{aligned}$$

which means that $f_1 \in T^p_\phi(0)$, with

$$\|f_1\|_{T^p_\phi(0)} \leq C_{\varphi,d,p} C_\phi \|f\|_{T^p_\phi(0)}.$$

As a consequence, we have

$$\|f_2\|_{T^p_\phi(0)} \leq (1 + C_{\varphi,d,p} C_\phi)\|f\|_{T^p_\phi(0)}.$$

Let us now consider $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\operatorname{supp}(\psi) \subseteq \overline{B(0,2)}$ and set, for $\varepsilon > 0$ and $x \in \mathbb{R}^d$,

$$I_\varepsilon(x) = \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} k(y)\psi(x-y)\,dy = \int_{B(0,2+|x|)\setminus B(0,\varepsilon)} k(y)\psi(x-y)\,dy.$$

We have, using the notation used in the proof of Theorem 6.1.2, as k is homogeneous of degree $-d$,

$$\begin{aligned} &\int_{B(0,2+|x|)\setminus B(0,\varepsilon)} k(y)\psi(x)\,dy \\ &= \psi(x) \int_\varepsilon^{2+|x|} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi k(y_{(\rho,\theta_1,\ldots,\theta_{d-1})})\rho^{d-1}\,d\Omega_d\,d\rho \\ &= \psi(x) \int_\varepsilon^{2+|x|} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi k(y_{(1,\theta_1,\ldots,\theta_{d-1})})\rho^{-1}\,d\Omega_d\,d\rho \\ &= \psi(x)(\ln(2+|x|) - \ln(\varepsilon)) \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi k(y_{(1,\theta_1,\ldots,\theta_{d-1})})\,d\Omega_d \\ &= 0, \end{aligned}$$

as k has mean value zero on Σ. Therefore, for $\varepsilon > 0$ and $x \in \mathbb{R}^d$,

$$I_\varepsilon(x) = \int_{B(0,2+|x|)\setminus B(0,\varepsilon)} k(y)(\psi(x-y) - \psi(x))\,dy.$$

We will use this last equality to show that the sequence $(I_\varepsilon)_{\varepsilon>0}$ converges uniformly as ε tends to 0^+. Indeed, for all $x \in \mathbb{R}^d$, if $0 < \varepsilon < \varepsilon'$, we have, since for all $y \neq 0$, $|k(y)| \leq M|y|^{-d}$ by the homogeneity of k,

$$\begin{aligned} |I_{\varepsilon'}(x) - I_\varepsilon(x)| &\leq M \int_{B(0,\varepsilon')\setminus B(0,\varepsilon)} |y|^{-d}|y| \sup_{|\alpha|=1} \|D^\alpha \psi\|_\infty\,dy \\ &= C_{\psi,d} M(\varepsilon' - \varepsilon), \end{aligned}$$

which shows that $(I_\varepsilon)_{\varepsilon>0}$ is uniformly Cauchy. It follows that $\mathcal{K}\psi$ is well defined and I_ε uniformly converges to $\mathcal{K}(\psi)$ as ε tends to 0^+. Moreover, for $0<\varepsilon<1$, we have

$$\begin{aligned}|I_\varepsilon(x)| &\leq |I_1(x)-I_\varepsilon(x)|+|I_1(x)|\\ &\leq C_{\psi,d}M(1-\varepsilon)+M\int_{\mathbb{R}^d\setminus B(0,1)}|y|^{-d}|\psi(x-y)|\,dy\\ &\leq C_{\psi,d}M(1-\varepsilon)+M\int_{\mathbb{R}^d}|\psi(y)|\,dy\\ &\leq C'_{\psi,d}M,\end{aligned}$$

so that $\|\mathcal{K}(\psi)\|_{\mathbb{R}^d}\leq C'_{\psi,d}M$. Using the same reasoning, we can show that, for $\varepsilon>0$ and $\alpha\in\mathbb{N}_0^d$,

$$D^\alpha I_\varepsilon=\int_{\mathbb{R}^d\setminus B(0,\varepsilon)}k(y)D^\alpha\psi(\cdot-y)\,dy,$$

$D^\alpha I_\varepsilon$ uniformly converges to $D^\alpha\mathcal{K}(\psi)$ and $\|\mathcal{K}(D^\alpha\psi)\|_{\mathbb{R}^d}\leq C_{\psi,d,\alpha}M$. As a consequence, $\mathcal{K}(\psi)$ belongs to $C^\infty(\mathbb{R}^d)$ with $D^\alpha\mathcal{K}(\psi)=\mathcal{K}(D^\alpha\psi)$. Moreover, if $|x|\geq 3$, then, for $\varepsilon>0$,

$$\begin{aligned}|I_\varepsilon(x)| &\leq M\int_{\{(x,y):|x-y|>\varepsilon,|y|<2\}}|x-y|^{-d}|\psi(y)|\,dy\\ &\leq M3^d|x|^{-d}\int_{\mathbb{R}^d}|\psi(y)|\,dy\\ &= C_\psi M3^d|x|^{-d}\end{aligned}$$

and so, by Lebesgue's dominated convergence theorem, $\mathcal{K}(\psi)\in L^p(\mathbb{R}^d)$, with $\|\mathcal{K}(\psi)\|_{L^p(\mathbb{R}^d)}\leq C_{\psi,d,p}M$. Gathering all these relations, we can claim, using Remark 5.1.9, that $\mathcal{K}(\psi)$ belongs to $T^p_\phi(0)$ and there exists $C_{\psi,d,p}>0$ such that $\|\mathcal{K}(\psi)\|_{T^p_\phi(x_0)}\leq C_{\psi,d,p}M$.

Now, let us apply this result to the function $x\mapsto x^\alpha\varphi(x)$ in order to obtain a constant $C_{\varphi,\alpha,d,p}$ such that $\|\mathcal{K}(\cdot^\alpha\varphi)\|_{T^p_\phi(0)}\leq C_{\varphi,\alpha,d,p}M$, which gives

$$\|\mathcal{K}(f_1)\|_{T^p_\phi(0)}\leq\sum_{|\alpha|\leq n}\frac{|D^\alpha P(x_0)|}{\alpha!}\|\mathcal{K}(\cdot^\alpha\varphi)\|_{T^p_\phi(0)}\leq C_{\varphi,d,p}M\|f\|_{T^p_\phi(0)}. \tag{6.34}$$

For $\|\mathcal{K}(f_2)\|_{T^p_\phi(0)}$, we use Hölder's inequality to get, for $r>0$,

$$r^{-d}\int_{B(0,r)}|f_2(y)|\,dy\leq C'_{\varphi,d,p}\|f\|_{T^p_\phi(0)}\phi(r)$$

and, as for (6.6) and (6.7), we can write

$$\int_{B(0,r)}|f_2(y)||y|^{-s}\,dy\leq C_{\varphi,d,p,s}\|f\|_{T^p_\phi(0)}\phi(r)r^{d-s}, \tag{6.35}$$

if $\underline{b}(\phi) + d - s > 0$, and

$$\int_{\mathbb{R}^d \setminus B(0,r)} |f_2(y)||y|^{-s}\, dy \leq C_{\varphi,d,p,s} \|f\|_{T^p_\phi(0)} \phi(r) r^{d-s} \tag{6.36}$$

if $\overline{b}(\phi) + d - s < 0$.

That being done, let us consider the case where condition (6.32) holds and fix $r > 0$; for $x \in B(0, r/2)$, we have, using Taylor's formula,

$$\begin{aligned}
\mathcal{K} f_2(x) = &\lim_{\varepsilon \to 0^+} \int_{\{(x,y):|x-y|>\varepsilon,|y|\leq r\}} k(x-y) f_2(y)\, dy \\
&+ \lim_{\varepsilon \to 0^+} \int_{\{(x,y):|x-y|>\varepsilon,|y|>r\}} k(x-y) f_2(y)\, dy \\
= &\lim_{\varepsilon \to 0^+} \int_{\{(x,y):|x-y|>\varepsilon,|y|\leq r\}} k(x-y) f_2(y)\, dy \\
&+ \int_{\mathbb{R}^d \setminus B(0,r)} k(x-y) f_2(y)\, dy \\
= &\lim_{\varepsilon \to 0^+} \int_{\{(x,y):|x-y|>\varepsilon,|y|\leq r\}} k(x-y) f_2(y)\, dy \\
&+ \sum_{|\alpha| \leq n} \frac{x^\alpha}{\alpha!} \left(\int_{\mathbb{R}^d} D^\alpha k(-y) f_2(y)\, dy - \int_{B(0,r)} D^\alpha k(-y) f_2(y)\, dy \right) \\
&+ \sum_{|\alpha| \leq n+1} \frac{x^\alpha}{\alpha!} \int_{\mathbb{R}^d \setminus B(0,r)} D^\alpha k(\Theta(x) x - y) f_2(y)\, dy,
\end{aligned}$$

for a $\Theta(x)$ belonging to $(0,1)$.

Thanks to the homogeneity of k, we have, for $|\alpha| \leq n+1$ and $y \neq 0$, $|D^\alpha k(-y)| \leq M|y|^{-d-|\alpha|}$. Using inequality (6.35) and Hölder's inequality, we get, if $q \in (1, \infty)$ is the conjugate exponent of p,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} D^\alpha k(-y) f_2(y)\, dy \right| \\
&\leq \int_{B(0,1)} |f_2(y)||y|^{-d-|\alpha|}\, dy + \int_{\mathbb{R}^d \setminus B(0,1)} |f_2(y)||y|^{-d-|\alpha|}\, dy \\
&\leq C \|f\|_{T^p_\phi(0)} \phi(1) + \|f_2\|_{L^p(\mathbb{R}^d)} \| |\cdot|^{-d-|\alpha|} \|_{L^q(\mathbb{R}^d \setminus B(0,1))} \\
&\leq C' \|f\|_{T^p_\phi(0)} + C'' \|f\|_{T^p_\phi(0)},
\end{aligned}$$

for $|\alpha| \leq n$. As a consequence,

$$P' := \sum_{|\alpha| \leq n} \frac{\cdot^\alpha}{\alpha!} \int_{\mathbb{R}^d} D^\alpha k(-y) f_2(y)\, dy$$

is a polynomial whose sum of the coefficients is bounded by $C_{\phi,p}\|f\|_{T_\phi^p(0)}$. Similarly, we have, for $|\alpha| \leq n$,

$$|\int_{B(0,r)} D^\alpha k(-y) f_2(y)\,dy| \leq C_{\alpha,d}\phi(r) r^{-\alpha}.$$

Given $x \in B(0,r/2)$ and $|y| \geq r$, we have $|\Theta(x)x - y| \geq |y|/2$ and so, by inequality (6.36),

$$\begin{aligned} |\int_{\mathbb{R}^d \setminus B(0,r)} D^\alpha k(\Theta(x)x - y) f_2(y)\,dy| &\leq M 2^{d+|\alpha|} \int_{\mathbb{R}^d \setminus B(0,r)} |f_2(y)||y|^{-d-|\alpha|}\,dy \\ &\leq M C_{\alpha,d}\phi(r) r^{-\alpha}, \end{aligned}$$

for $|\alpha| = n+1$. Finally, using Theorem 6.3.1, we have

$$\begin{aligned} &\| \lim_{\varepsilon\to 0^+} \int_{\{(\cdot,y):|\cdot - y|>\varepsilon, |y|\leq r\}} k(\cdot - y) f_2(y)\,dy\|_{L^p(\mathbb{R}^d)} \\ &\leq C_p M \|f_2\|_{L^p(B(x_0,r))} \\ &\leq (1 + C_{\varphi,d,p} C_\phi) M \|f\|_{T_\phi^p(0)} \phi(r) r^{d/p} \end{aligned}$$

and we can conclude that there exists a constant $C_{\phi,p,d} > 0$ such that, for $r > 0$,

$$r^{-d/p} \|\mathcal{K} f_2 - P'\|_{L^p(B(0,r))} \leq C_{\phi,p,d} M \phi(r).$$

If we now assume $\overline{b}(\phi) < 0$, then, for $r > 0$ and $x \in B(0,r/2)$, we have

$$\begin{aligned} \mathcal{K} f_2(x) = &\lim_{\varepsilon\to 0^+} \int_{\{(x,y):|x-y|>\varepsilon, |y|\leq r\}} k(x-y) f_2(y)\,dy \\ &+ \int_{\mathbb{R}^d \setminus B(0,r)} k(x-y) f_2(y)\,dy. \end{aligned}$$

We can deal with the first term of the right-hand side just as we did before, while for the second we use the estimation

$$\begin{aligned} |\int_{\mathbb{R}^d \setminus B(0,r)} k(x-y) f_2(y)\,dy| &\leq M \int_{\mathbb{R}^d \setminus B(0,r)} |y|^{-d} |f_2(y)|\,dy \\ &\leq C_d M \phi(r), \end{aligned}$$

which follows from (6.36). This leads to the following relation, holding for $r > 0$,

$$r^{-d/p} \|\mathcal{K} f_2\|_{L^p(B(0,r))} \leq C_{\phi,p,d} M \phi(r).$$

One more use of Theorem 6.3.1 ensures

$$\|\mathcal{K} f_2\|_{L^p(\mathbb{R}^d)} \leq C_p M \|f_2\|_{L^p(\mathbb{R}^d)},$$

which allows us to conclude, with (6.34), that the desired inequality (6.33) holds.

If we moreover assume that f belongs to $t^p_\phi(0)$, then we know that there exists a sequence of functions $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R}^d)$ such that f_j converges to f in $T^p_\phi(0)$ as j tends to infinity. By a reasoning similar to the one we made for the function ψ at the beginning of the proof, we can conclude that, for all $j \in \mathbb{N}$, $\mathcal{K}f_j$ belongs to $C^\infty(\mathbb{R}^d)$ and so to $t^p_\phi(0)$ as well, by Remark 5.1.9. In addition, it follows from inequality (6.33) that $\mathcal{K}f_j$ converges to $\mathcal{K}f$ in $T^p_\phi(0)$ as j tends to infinity and, as $t^p_\phi(0)$ is a closed subspace, we get that $\mathcal{K}f \in t^p_\phi(0)$. □

Corollary 6.3.5. *Let us denote by $\mathcal{Y}_{l,m}$ the convolution singular integral operator defined as*

$$\mathcal{Y}_{l,m} f := p.v. \int k_{l,m}(\cdot - y) f(y)\, dy,$$

whose kernel is

$$k_{l,m} := Y_{l,m}\left(\frac{\cdot}{|\cdot|}\right)|\,.\,|^{-d},$$

where $(Y_{l,m})_{l,m}$ forms a complete system of orthogonal spherical harmonics (for more details on spherical harmonics , see e.g. [104, 106]), m being the degree of the harmonic. Under the assumption of Proposition 6.3.4, there exist constants $C_p, C_{\phi,p} > 0$ such that

$$\|\mathcal{Y}_{l,m} f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}, \qquad \|\mathcal{Y}^*_{l,m} f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)} \tag{6.37}$$

and

$$\|\mathcal{Y}_{l,m} f\|_{T^p_\phi(x_0)} \leq C_{\phi,p}\, m^{\frac{d-2}{2} + \lceil \overline{b}(\phi) \rceil_{\mathbb{N}}} \|f\|_{T^p_\phi(x_0)}. \tag{6.38}$$

Proof. Inequalities (6.37) come from (6.31) and the fact that $\|k_{l,m}\|_{L^2(\Sigma)}$ is equal to 1. Inequality (6.38) is obtained from (6.33), using the fact that, for $\alpha \in \mathbb{N}_0^d$, we have $|D^\alpha Y_{l,m}| \leq C_\alpha m^{(\frac{d-2}{2} + |\alpha|)}$ on Σ (see [25]). □

A fundamental example of convolution singular integral operators is given by the Riesz transform $(\mathcal{R}_j)_{1\leq j\leq d}$, defined for $j \in \{1, \ldots, d\}$ by

$$\mathcal{R}_j f(x) := p.v. \frac{-i\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \int \frac{(x_j - y_j)}{|x-y|^{d+1}} f(y)\, dy.$$

Let us fix $1 < p < \infty$ and $k \geq 1$; it is known that the following facts hold (see e.g. [25, 26]):

- if $f \in W^p_k(\mathbb{R}^d)$, we have $\mathcal{R}_j f \in W^p_k(\mathbb{R}^d)$ and $\mathcal{R}_j$ is a continuous operator on $W^p_k(\mathbb{R}^d)$,
- for $l \in \{1, \ldots, d\}$ and $f \in W^p_k(\mathbb{R}^d)$, we have $D_l(\mathcal{R}_j f) = \mathcal{R}_j(D_l f)$ and $\mathcal{R}_j(D_l f) = \mathcal{R}_l(D_j f)$,

- if $f \in L^p(\mathbb{R}^d)$, we have $\sum_{j=1}^d \mathcal{R}_j^2 f = f$.

The operator

$$\Lambda := i \sum_{j=1}^{d} \mathcal{R}_j D_j$$

continuously maps $W_k^p(\mathbb{R}^d)$ into $W_{k-1}^p(\mathbb{R}^d)$ and, if $k \geq 2$,

$$\Lambda^2 f = -\Delta f,$$

for all $f \in W_k^p(\mathbb{R}^d)$. We also have the identity $D_j f = -i\mathcal{R}_j \Lambda f$ for all $f \in W_k^p(\mathbb{R}^d)$. It can also be shown that for all $m \in \mathbb{N}$ such that $2m+1 \geq d$, there exist $a_1, \ldots a_m < 0$ and a positive integrable function h_m with derivatives continuous and bounded up to order $2m+1-d$ such that

$$\Lambda \mathcal{J} f = f + \sum_{j=1}^{m} a_j \mathcal{J}^{2j} f - h_m * f,$$

for all $f \in L^p(\mathbb{R}^d)$ (see [26]).

Proposition 6.3.6. *Let $p \in (1,\infty)$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that either $\overline{b}(\phi) < -1$ or there exists $n \in \mathbb{N} \cup \{-1\}$ for which $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$. The operator $D_j \mathcal{J}$ continuously maps $T_\phi^p(x_0)$ into itself.*

Proof. Let f be a function of $T_\phi^p(x_0)$; from what precedes, we have

$$D_j \mathcal{J} f = -i\mathcal{R}_j \Lambda \mathcal{J} f = -i\mathcal{R}_j (f + \sum_{j=1}^{m} a_j \mathcal{J}^{2j} f - h_m * f),$$

where m has been chosen sufficiently large so that h_m belongs to $C^{\lceil \overline{b}(\phi) \rceil_{\mathbb{N}}}(\mathbb{R}^d)$. Using Remark 5.1.9, we thus have $h_m * f \in t_\phi^p(x_0)$. Moreover, by Theorem 6.1.2 and Proposition 5.3.3, we know that $\mathcal{J}$ continuously maps $T_\phi^p(x_0)$ into itself. The conclusion is obtained by applying Proposition 6.3.4 to $\mathcal{R}_j$. □

The decomposition of functions into spherical harmonics will lead us to singular integral operators whose kernel depends on several variables.

Definition 6.3.7. Let $q \in [1,\infty]$, $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $x_0 \in \mathbb{R}^d$. Let $\mathcal{K}$ be the singular integral operator of the form

$$f \mapsto a(\cdot) f(\cdot) + p.v. \int k(\cdot, \cdot - y) f(y)\, dy,$$

where

- a is a bounded measurable function,
- for all $x \in \mathbb{R}^d$, $k(x,\cdot)$ is homogeneous of degree $-d$, has mean value zero on Σ and belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$.

The symbol of $\mathcal{K}$ is the function

$$\sigma(\mathcal{K}) : (x,z) \mapsto a(x) + \widehat{k}(x,z),$$

where, given $x \in \mathbb{R}^d$, $\widehat{k}(x,\cdot)$ is the Fourier transform of $k(x,\cdot)$ (understood in the distribution sense). We know that for all $x \in \mathbb{R}^d$, $\widehat{k}(x,\cdot)$ belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and is homogeneous of degree 0 (see e.g. [55]). We say that $\mathcal{K}$ is in the class $T^q_\phi(x_0)$ if, for all $|\alpha| \leq 2d + \lceil \overline{b}(\phi) \rceil_{\mathbb{N}}$ and $z \neq 0$, the function

$$x \mapsto D^\alpha_z \sigma(\mathcal{K})(x,z)$$

is in $T^q_\phi(x_0) \cap L^\infty(\mathbb{R}^d)$, uniformly on Σ. We then define

$$\begin{aligned} \|\mathcal{K}\|_{T^q_\phi(x_0)} = \max\{ & \sup_{\substack{|z|=1 \\ 0\leq|\alpha|\leq 2d+\lceil \overline{b}(\phi)\rceil_{\mathbb{N}}}} \|D^\alpha_z \sigma(\mathcal{K})(\cdot,z)\|_{T^q_\phi(x_0)}, \\ & \sup_{\substack{|z|=1 \\ 0\leq|\alpha|\leq 2d+\lceil \overline{b}(\phi)\rceil_{\mathbb{N}}}} \|D^\alpha_z \sigma(\mathcal{K})(\cdot,z)\|_{L^\infty(\mathbb{R}^d)}\}. \end{aligned}$$

If moreover, for all $|\alpha| \leq 2d + \lceil \overline{b}(\phi) \rceil_{\mathbb{N}}$ and $z \neq 0$, the function $x \mapsto D^\alpha_z k(x,z)$ belongs to $t^q_\phi(x_0)$ uniformly on Σ, then we say that $\mathcal{K}$ is in the class $t^q_\phi(x_0)$.

Remark 6.3.8. Given $x \in \mathbb{R}^d$, $\sigma(\mathcal{K})(x,\cdot)$ is an homogeneous function of degree zero; it is proved in [104, 25] that the following decompositions hold: for $(x,z) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$, we have

$$k(x,z) = \sum_{l,m} a_{l,m}(x) Y_{l,m}(\frac{z}{|z|})|z|^{-d}$$

and

$$\sigma(\mathcal{K})(x,z) = a(x) + \sum_{l,m} a_{l,m}(x) \gamma_m Y_{l,m}(\frac{z}{|z|}),$$

where $\gamma_m := \frac{i^m \pi^{\frac{d}{2}} \Gamma(\frac{m}{2})}{\Gamma(\frac{m+d}{2})}$ and

$$\begin{aligned} a_{l,m}(x) &:= (-1)^v (m(m+d-2))^{-v} \int_\Sigma Y_{l,m} L^v k(x,\cdot)\, d\sigma \\ &= (-1)^v (m(m+d-2))^{-v} \gamma_m^{-1} \int_\Sigma Y_{l,m} L^v \sigma(\mathcal{K})(x,\cdot)\, d\sigma, \end{aligned}$$

with $LF(z) = |z|^2 \Delta F(z)$ and $v \in \mathbb{N}$.

Theorem 6.3.9. *Let $q \in [1,\infty]$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$. Let $\mathcal{K}$ be a singular integral operator of class $T^q_\phi(x_0)$; we have*

1. *$a_{l,m} \in T^q_\phi(x_0) \cap L^\infty(\mathbb{R}^d)$ and*
$$\max\{\|a_{l,m}\|_{T^q_\phi(x_0)}, \|a_{l,m}\|_{L^\infty(\mathbb{R}^d)}\} \le C_\phi m^{\frac{d}{2}-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)},$$
2. *if $p \in (1,\infty)$ is such that $0 \le 1/p^* := \frac{1}{q}+\frac{1}{p} \le 1$ and if $f \in L^p(\mathbb{R}^d)$, then, for almost every $x \in \mathbb{R}^d$, $\mathcal{K}f(x)$ and $\mathcal{Y}_{l,m}f(x)$ exist and the series*
$$a(x)f(x) + \sum_{l,m} a_{l,m}(x)\mathcal{Y}_{l,m}f(x)$$
converges absolutely to $\mathcal{K}f(x)$,
3. *$\mathcal{K}$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^{p^*}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$: there exists a constant $C_{p,q} > 0$ such that, for all $f \in L^p(\mathbb{R}^d)$,*
$$\max\{\|\mathcal{K}f\|_{L^{p^*}(\mathbb{R}^d)}, \|\mathcal{K}f\|_{L^p(\mathbb{R}^d)}\} \le C_{p,q}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|f\|_{L^p(\mathbb{R}^d)},$$
4. *let $\psi \in \mathcal{B}$ be such that $\underline{b}(\psi) \ge -d/p$, $\phi \preccurlyeq \psi$ and either $\underline{b}(\psi) \le 0$ or $n < \underline{b}(\psi) \le \overline{b}(\psi) < n+1$ for some $n \in \mathbb{N}$; $\mathcal{K}$ is a bounded operator from $T^p_\psi(x_0)$ to $T^{p^*}_\psi(x_0)$: there exists a constant $C_{p,q,\phi,\psi} > 0$ such that, for all $f \in T^p_\psi(x_0)$,*
$$\|\mathcal{K}f\|_{T^{p^*}_\psi(x_0)} \le C_{p,q,\phi,\psi}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)},$$
5. *if moreover $\mathcal{K}$ is of class $t^q_\phi(x_0)$, then $a_{l,m}$ belongs to $t^q_\phi(x_0)$ and, for all $f \in t^p_\psi(x_0)$, $\mathcal{K}f$ belongs to $t^{p^*}_\psi(x_0)$.*

Proof. We keep the same notations as in Remark 6.3.8 with $v := d + \lceil \frac{\overline{b}(\phi)-1}{2} \rceil_{\mathbb{N}}$.

Let us show the first point. For all $x \in \mathbb{R}^d$ and $z \in \Sigma$, let us write
$$L^v\sigma(\mathcal{K})(x,z) := \sum_{|\alpha|\le 2v} g_\alpha(z) D^\alpha_z \sigma(\mathcal{K})(x,z),$$
where g_α is a product of powers of z_j ($j \in \{1,\dots,d\}$). From the definition of the class of operators in $T^q_\phi(x_0)$, for $z \in \Sigma$, we have
$$\|L^v\sigma(\mathcal{K})(\cdot,z)\|_{L^q(\mathbb{R}^d)} \le C_v\|\mathcal{K}\|_{T^q_\phi(x_0)}.$$
Let us also recall that $\|Y_{l,m}\|_{L^2(\Sigma)}$ is equal to 1. If $q \ge 2$, then, if we denote by q' the conjugate exponent of q, $q' \le 2$, we have by Hölder's inequality (with usual modification if

$q = \infty$),

$$\begin{aligned}
&\|a_{l,m}\|_{L^q(\mathbb{R}^d)} \\
&= (m(m+d-2))^{-v}\gamma_m^{-1}\Big(\int_{\mathbb{R}^d}\Big|\int_{\Sigma} Y_{l,m}(z)L^v\sigma(\mathcal{K})(x,z)\,d\sigma(z)\Big|^q\,dx\Big)^{1/q} \\
&\le C_d m^{\frac{d}{2}-2v}\Big(\int_{\mathbb{R}^d}\|Y_{l,m}\|^q_{L^{q'}(\Sigma)}\|L^v\sigma(\mathcal{K})(x,\cdot)\|^q_{L^q(\Sigma)}\,dx\Big)^{1/q} \\
&\le C_d m^{\frac{d}{2}-2v}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{\frac{1}{q'}-\frac{1}{2}}\|Y_{l,m}\|_{L^2(\Sigma)}\Big(\int_{\Sigma}\int_{\mathbb{R}^d}|L^v\sigma(\mathcal{K})(x,z)|^q\,dx\,d\sigma(z)\Big)^{1/q} \\
&\le C_{d,v}m^{\frac{d}{2}-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1/2} \\
&= C_{d,v}m^{\frac{d}{2}-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}.
\end{aligned}$$

From this, we get $\|a_{l,m}\|_{L^q(\mathbb{R}^d)} \le Cm^{\frac{d}{2}-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}$ and a similar argument can be applied to obtain the same inequality with $\|a_{l,m}\|_{L^\infty(\mathbb{R}^d)}$. Now, if $q \le 2$, we have

$$\begin{aligned}
&\|a_{l,m}\|_{L^q(\mathbb{R}^d)} \\
&= (m(m+d-2))^{-v}\gamma_m^{-1}\Big(\int_{\mathbb{R}^d}\Big|\int_{\Sigma} Y_{l,m}(z)L^v\sigma(\mathcal{K})(x,z)\,d\sigma(z)\Big|^q\,dx\Big)^{1/q} \\
&\le C_d m^{\frac{d}{2}-2v}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1-\frac{1}{q}}\Big(\int_{\mathbb{R}^d}\int_{\Sigma}|Y_{l,m}(z)|^q|L^v\sigma(\mathcal{K})(x,z)|^q\,d\sigma(z)\,dx\Big)^{1/q} \\
&\le C_{d,v}m^{\frac{d}{2}-2v}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1-\frac{1}{q}}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|Y_{l,m}\|_{L^q(\Sigma)} \\
&\le C_{d,v}m^{\frac{d}{2}-2v}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1/2}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|Y_{l,m}\|_{L^2(\Sigma)} \\
&= C_{d,v}m^{\frac{d}{2}-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}.
\end{aligned}$$

Moreover, we know that, for $|\alpha| \le 2d + \overline{b}(\phi) + 1$ and $z \in \Sigma$, there exists a polynomial

$$P_{\alpha,z} := \sum_{|\beta|\le n} C^{(\beta)}_{z,\alpha}(\cdot - x_0)^\beta$$

of degree n such that

$$\sum_{|\beta|\le n}|C^{(\beta)}_{z,\alpha}| \le \|\mathcal{K}\|_{T^q_\phi(x_0)}$$

and, for $r > 0$,

$$r^{-d/q}\|D^\alpha_z\sigma(\mathcal{K})(\cdot,z) - P_{\alpha,z}\|_{L^q(B(x_0,r)} \le \|\mathcal{K}\|_{T^q_\phi(x_0)}\phi(r).$$

Thus,

$$P = \sum_{|\beta|\le n}(-1)^v(m(m+d-2))^{-v}\gamma_m^{-1}\int_{\Sigma} Y_{l,m}(z)\Big(\sum_{|\alpha|\le 2v} g_\alpha(z)C^{(\beta)}_{z,\alpha}\Big)\,d\sigma(\cdot - x_0)^\beta$$

is a polynomial of degree n for which

$$\sum_{|\beta|\leq n} |(m(m+d-2))^{-v}\gamma_m^{-1}\int_\Sigma Y_{l,m}(z)(\sum_{|\alpha|\leq 2v} g_\alpha(z)C^{(\beta)}_{z,\alpha})\,d\sigma| \\ \leq C_\phi m^{\frac{d}{2}-2v}\|\mathcal{K}\|_{T_\phi(x_0)}$$

and, for $r>0$, we can show, in the same way as before, that

$$r^{-d/q}\|a_{l,n}-P\|_{L^q(B(x_0,r))}\leq C_{d,q}m^{\frac{d}{2}-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}\phi(r).$$

Let us show the second point. It is well known that there exists a constant $C_d>0$ such that, for $m\in\mathbb{N}$, the number of spherical harmonics of degree m is bounded by $C_d m^{d-2}$ (see e.g. [106]). Moreover, if f belongs to $L^p(\mathbb{R}^d)$, from Corollary 6.3.5, we also know that $\|\mathcal{Y}^*_{l,m}f\|_{L^p(\mathbb{R}^d)}\leq C_p\|f\|_{L^p(\mathbb{R}^d)}$. From this, using the first point, we can claim that if $p^*\geq 1$ is such that $1/p^*:=\frac{1}{p}+\frac{1}{q}$, then

$$\sum_{l,m} a_{l,m}\mathcal{Y}^*_{l,m}f$$

converges in $L^{p^*}(\mathbb{R}^d)$. As a consequence, for almost every $x\in\mathbb{R}^d$,

$$\sum_{l,m} a_{l,m}(x)\mathcal{Y}^*_{l,m}f(x)$$

is finite.

Let us fix $\varepsilon>0$ and $x\in\mathbb{R}^d$ such that $|a_{l,m}(x)|\leq C_\phi m^{\frac{d}{2}-2v}$; we have

$$\begin{aligned}&\int_{\mathbb{R}^d\setminus B(x,\varepsilon)} k(x,x-y)f(y)\,dy\\ &=\int_{\mathbb{R}^d\setminus B(x,\varepsilon)}\sum_{l,m} a_{l,m}(x)Y_{l,m}(\frac{x-y}{|x-y|})|x-y|^{-d}f(y)\,dy\\ &=\sum_{l,m} a_{l,m}(x)\int_{\mathbb{R}^d\setminus B(x,\varepsilon)} Y_{l,m}(\frac{x-y}{|x-y|})|x-y|^{-d}f(y)\,dy,\end{aligned}$$

because, $y\mapsto|x-y|^{-d}f(y)$ is integrable (using Hölder's inequality) on $\mathbb{R}^d\setminus B(x,\varepsilon)$ and

$$\begin{aligned}|\sum_{l,m} a_{l,m}(x)Y_{l,m}(\frac{x-\cdot}{|x-\cdot|})| &\leq C_{d,q}\sum_{m\in\mathbb{N}} m^{\frac{d}{2}-2v}m^{d-2}m^{\frac{d-2}{2}}\|\mathcal{K}\|_{T^q_\phi(x_0)}\\ &\leq C_{d,q}\|\mathcal{K}\|_{T^q_\phi(x_0)}.\end{aligned}$$

Now, if x is a point for which $\sum_{l,m}a_{l,m}(x)\mathcal{Y}^*_{l,m}f(x)$ is finite and $\mathcal{Y}_{l,m}f(x)$ exists for all l,m, then, for $\varepsilon>0$,

$$\int_{\mathbb{R}^d\setminus B(x,\varepsilon)} Y_{l,m}(\frac{x-y}{|x-y|})|x-y|^{-d}f(y)\,dy\leq\mathcal{Y}^*_{l,m}f(x),$$

which allows us to take the limit as ε tends to 0^+ to obtain

$$\mathcal{K}f(x) = a(x)f(x) + \sum_{l,m} a_{l,m}(x)\mathcal{Y}_{l,m}f(x).$$

The conclusion follows from the fact that almost every $x \in \mathbb{R}^d$ is such that the quantity $\sum_{l,m} a_{l,m}(x)\mathcal{Y}^*_{l,m}f(x)$ is finite, $|a_{l,m}(x)| \leq C_\phi m^{\frac{d}{2}-2v}$ and $\mathcal{Y}^*_{l,m}f(x)$ exists for all l,m, by countable intersection.

Let us prove the third point. For $f \in L^p(\mathbb{R}^d)$, we have, from the previous point and Corollary 6.3.5,

$$\begin{aligned}
\|\mathcal{K}f\|_{L^{p^*}(\mathbb{R}^d)} &= \|af + \sum_{l,m} a_{l,m}\mathcal{Y}_{l,m}f\|_{L^{p^*}(\mathbb{R}^d)} \\
&\leq \|a\|_{L^q(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} + \sum_{l,m}\|a_{l,m}\|_{L^q(\mathbb{R}^d)}\|\mathcal{Y}_{l,m}f\|_{L^p(\mathbb{R}^d)} \\
&\leq \|a\|_{L^q(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)} \\
&\quad + C_{p,q,d}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|f\|_{L^p(\mathbb{R}^d)}\sum_{m\in\mathbb{N}} m^{\frac{d}{2}-2v}m^{d-2} \\
&\leq C_{p,q}\|\mathcal{K}\|_{T_\phi(x_0)}\|f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

The upper bound for $\|\mathcal{K}f\|_{L^p(\mathbb{R}^d)}$ can be obtained in the same way.

Let us prove the fourth point. Again, from point 2, Proposition 5.3.2, Corollaries 6.3.5 and 5.3.5, for $f \in T^p_\phi(x_0)$, we have

$$\begin{aligned}
\|\mathcal{K}f\|_{T^{p^*}_\psi(x_0)} &\leq C_{p,q,\phi,\psi}(\|a\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)} + \sum_{l,m}\|a_{l,m}\|_{T^q_\phi(x_0)}\|\mathcal{Y}_{l,m}f\|_{T^p_\psi(x_0)}) \\
&\leq C_{p,q,\phi,\psi}(\|a\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)} \\
&\quad + \|f\|_{T^p_\psi(x_0)}\|\mathcal{K}\|_{T^q_\phi(x_0)}\sum_{m\in\mathbb{N}} m^{d-2}m^{\frac{d}{2}-2v}m^{\frac{d-2}{2}+\lceil\overline{b}(\psi)\rceil_{\mathbb{N}}}) \\
&\leq C_{p,q,\phi,\psi}(\|a\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)} \\
&\quad + \|f\|_{T^p_\psi(x_0)}\|\mathcal{K}\|_{T^q_\phi(x_0)}\sum_{m\in\mathbb{N}} m^{d-2}m^{\frac{d}{2}-2v}m^{\frac{d-2}{2}+\lceil\overline{b}(\phi)\rceil_{\mathbb{N}}}) \\
&\leq C_{p,q,\phi,\psi}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)}.
\end{aligned}$$

Let us prove the last point. We keep here the notations from the first point. By definition of the class $t^q_\phi(x_0)$, there exist $\varepsilon > 0$ and $\varepsilon(r)$ converging to 0 as $r \to 0^+$ such that, for $|\alpha| \leq 2d + \overline{b}(\phi) + 1$, $z \in \Sigma$ and $r > 0$ sufficiently small,

$$r^{-d/q}\|D^\alpha\sigma(\mathcal{K})(\cdot,z) - P_{\alpha,z}\|_{L^q(B(x_0,r)} \leq \varepsilon(r)\phi(r).$$

As a consequence, for such r,

$$r^{-d/q}\|a_{l,n} - P\|_{L^q(B(x_0,r))} \leq C\varepsilon(r)m^{\frac{d}{2}-2v}\phi(r)$$

and $a_{l,n}$ belongs to $t^q_\phi(x_0)$. The conclusion comes from the second part of Corollary 5.3.5 and the fact that $t^{p^*}_\psi(x_0)$ is closed. □

Remark 6.3.10. Let us come back to the convolution singular integral operators we considered in Theorem 6.3.4. For such an operator, the kernel k is independent of the variable x and $\|\mathcal{K}\|^*_{T^p_\phi(x_0)}$ is bounded by the derivatives of k on Σ. Following the path taken in the last theorem, we can also bound this norm using now the derivatives of $\sigma(\mathcal{K})$. Indeed, as k does not depend on x, so do $\sigma(\mathcal{K})$ and $a_{l,m}$. Let us consider $p \in (1,\infty)$ and $\phi \in \mathcal{B}$ as in Theorem 6.3.4, define

$$v(\phi) := \begin{cases} d & \text{if } \overline{b}(\phi) < 0 \\ d + \lceil \frac{\overline{b}(\phi)-1}{2} \rceil_{\mathbb{N}} & \text{otherwise} \end{cases}$$

and

$$N := \sup_{\substack{|z|=1 \\ 0\le|\alpha|\le v(\phi)}} |D^\alpha \sigma(\mathcal{K})(z)|.$$

Using an argument similar to the one used in Theorem 6.3.9, we have

$$|a_{l,m}| \le C m^{\frac{d}{2}-2v} N,$$

for all l,m. For all $f \in L^p(\mathbb{R}^d)$,

$$\mathcal{K}f = \sum_{l,m} a_{l,m} \mathcal{Y}_{l,m} f$$

almost everywhere, $\mathcal{K}f \in L^p(\mathbb{R}^d)$ and, if $f \in T^p_\phi(x_0)$, then $\mathcal{K}f \in T^p_\phi(x_0)$ with $\|\mathcal{K}f\|_{T^p_\phi(x_0)} \le C_{p,\phi} N \|f\|_{T^p_\phi(x_0)}$.

6.4 Elliptic partial differential equations

Definition 6.4.1. *An elliptic partial differential equation* at $x_0 \in \mathbb{R}^d$ of order $m \in \mathbb{N}$ is a partial differential equation of the form

$$\mathcal{E}f = \sum_{|\alpha|\le m} a_\alpha D^\alpha f = g,$$

where, for all $|\alpha| \le m$, a_α is a $s \times r$ matrix of functions and

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}, \qquad g = \begin{pmatrix} g_1 \\ \vdots \\ g_s \end{pmatrix}$$

are vector valued functions with $f_j \in W^p_m(\mathbb{R}^d)$ for all $j \in \{1,\ldots,r\}$; D^α stands for the weak derivative and

$$\mu(x_0) := \inf_{|\xi|=1} \det[(\sum_{|\alpha|=m} a^*_\alpha(x_0)\xi^\alpha)(\sum_{|\alpha|=m} a_\alpha(x_0)\xi^\alpha)] > 0$$

is the ellipticity constant of $\mathcal{E}$ at x_0.

In [25], Calderón and Zygmund proved that if $\mathcal{E}$ is elliptic with constant coefficients $(a_\alpha)_{|\alpha|=m}$ all of the same order, then we can write

$$\mathcal{E} = \mathcal{K}\Lambda^m,$$

where $\mathcal{K}$ is a $s \times r$ matrix of convolution singular operators, whose matrix of symbols is, for $z \neq 0$,

$$\sigma(\mathcal{K})(z) = (-i)^m \sum_{|\alpha|=m} a_\alpha z^\alpha |z|^{-m}.$$

They also showed in [26] that, in this case, there exists a $r \times s$ matrix of convolution singular operators whose matrix of symbols is[2]

$$\sigma(\mathcal{H}) = [\sigma(\mathcal{K})^*\sigma(\mathcal{K})]^{-1}\sigma(\mathcal{K})^*$$

and for which $\mathcal{H}\mathcal{K}$ is the identity operator. From Remark 6.3.10, we can estimate the dual norm of $\mathcal{H}$ on the spaces $T^p_\phi(x_0)$, using the ellipticity constant of $\mathcal{E}$ and $(|a_\alpha|)_{|\alpha|=m}$.

Now, if

$$\mathcal{E}f = \sum_{|\alpha|\leq m} a_\alpha D^\alpha f = g$$

is a general elliptic partial differential equation at $x_0 \in \mathbb{R}^d$ of order $m \in \mathbb{N}$, we set

$$\mathcal{E}_{x_0} := \sum_{|\alpha|=m} a_\alpha(x_0).$$

By what precedes, we have $\mathcal{E}_{x_0} = \mathcal{K}\Lambda^m$, where $\mathcal{K}$ is a matrix of convolution singular operators for which $\mathcal{H}\mathcal{K}$ is the identity operator. Then, let us define

$$h := \begin{cases} (1-\Delta)^{m/2} f & \text{if } m \text{ is even} \\ (i+\Lambda)(1-\Delta)^{\frac{m-1}{2}} f & \text{if } m \text{ is odd.} \end{cases}$$

Applying $\mathcal{H}$ on $\mathcal{E}_{x_0} f + (\mathcal{E} - \mathcal{E}_{x_0}) f = g$ gives

$$\Lambda^m f = \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E}) f$$

[2]The ellipticity of the equation allows us to take the inverse matrix of $\sigma(\mathcal{K})^*\sigma(\mathcal{K})$.

and, as $\Lambda^2 = -\Delta$, we obtain, if m is even,

$$\begin{aligned} h &= \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + [(1-\Delta)^{m/2} - (-\Delta)^{m/2}]f \\ &= \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + L_1(D)f, \end{aligned}$$

where $L_1(D)$ is a differential operator of order $m-2$ with constant coefficients. Assuming that m is odd, we get

$$\begin{aligned} h &= \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + [(i+\Lambda)(1-\Delta)^{\frac{m-1}{2}} - \Lambda(-\Delta)^{\frac{m-1}{2}}]f \\ &= \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + L_2(D)f + \Lambda L_3(D)f, \end{aligned}$$

where $L_2(D)$ (resp. $L_3(D)$) is a differential operator of order $m-1$ (resp. $m-3$) with constant coefficients.

In the sequel, we choose as the norm of a vector-valued function the sum of the norm of its components.

Proposition 6.4.2. *Let $p_1 \in (1,\infty)$ and $p_2 \in [1,\infty]$ be such that*

$$0 \le \frac{1}{p_3} := \frac{1}{p_1} + \frac{1}{p_2} \le 1,$$

$x_0 \in \mathbb{R}^d$ and $\phi, \varphi, \psi \in \mathcal{B}$ be such that

- *$0 < \underline{b}(\phi)$ and the coefficients of $\mathcal{E}$ are functions in $T^{p_1}_\phi(x_0)$ for which x_0 is a Lebesgue-point,*
- *$\phi \preccurlyeq \psi$,*
- *$-d/p_2 < \underline{b}(\psi)$ and there exists $n \in \mathbb{Z}$ such that $n < \underline{b}(\psi) \le \overline{b}(\psi) < n+1$ and $g \in T^{p_3}_\psi(x_0)$,*
- *$-d/p_2 < \underline{b}(\varphi)$ and there exists $l \in \mathbb{Z}$ such that $l < \underline{b}(\varphi) \le \overline{b}(\varphi) < l+1$ and $h \in T^{p_2}_\varphi(x_0)$,*
- *$\overline{b}(\psi) - \underline{b}(\varphi) < \min\{\underline{b}(\phi), 1\}$.*

We also assume that there exists $p^* \in [1, p_3]$ such that $f \in W^{p^*}_m(\mathbb{R}^d)$. In this case, h belongs to $T^{p_3}_\psi(x_0)$ with

$$\|h\|_{T^{p_3}_\psi(x_0)} \le \|\mathcal{H}g\|_{T^{p_3}_\psi(x_0)} + C_{p_1,p_2,\varphi,\psi,\phi}((1+MN)\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}),$$

where M is the least upper bound of the norm of the coefficients of $\mathcal{E}$ in $T^{p_2}_\phi(x_0)$ and

$$N = \sup_{\substack{|z|=1 \\ 0 \le |\alpha| \le v(\psi)}} |D^\alpha \sigma(\mathcal{K})(z)|,$$

where $v(\psi)$ is defined as in Remark 6.3.10.

Proof. Let us first consider the case where m is even; we have $f = \mathcal{J}^m h$ and therefore[3], for $|\alpha| \le m$,

$$D^\alpha f = (D\mathcal{J})^\alpha \mathcal{J}^{m-|\alpha|} h.$$

As a consequence, for $|\alpha| < m$, we have $\overline{b}(\psi) < \underline{b}(\varphi) + 1$, $\varphi_{m-|\alpha|} \preccurlyeq \psi$ and, following Proposition 6.3.6 and Theorem 6.1.2,

$$\begin{aligned}
\|D^\alpha f\|_{T^{p_2}_\psi(x_0)} &\le C_{p_2,\psi} \|\mathcal{J}^{m-|\alpha|} h\|_{T^{p_2}_\psi(x_0)} \\
&\le C_{p_2,\varphi,\psi} \|\mathcal{J}^{m-|\alpha|} h\|_{T^{p_2}_{\varphi_{m-|\alpha|}}(x_0)} \\
&\le C_{p_2,\varphi,\psi} \|h\|_{T^{p_2}_\varphi(x_0)}.
\end{aligned}$$

If $|\alpha| = m$, Proposition 6.3.6 gives

$$\begin{aligned}
\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} &= \|(D\mathcal{J})^\alpha h\|_{T^{p_2}_\varphi(x_0)} \\
&\le C_{p_2,\varphi} \|h\|_{T^{p_2}_\varphi(x_0)}.
\end{aligned}$$

Let us consider the operators

$$\mathcal{E}_1 = \sum_{|\alpha|<m} a_\alpha D^\alpha \qquad \text{and} \qquad \mathcal{E}_2 = \sum_{|\alpha|=m} (a_\alpha(x_0) - a_\alpha) D^\alpha;$$

by Corollary 5.3.5, we have

$$\begin{aligned}
\|\mathcal{H}\mathcal{E}_1 f\|_{T^{p_3}_\psi(x_0)} &\le C_{p_3,\psi} N \|\mathcal{E}_1 f\|_{T^{p_3}_\psi(x_0)} \\
&\le C_{p_1,p_2,\phi,\psi} NM \sum_{|\alpha|<m} \|D^\alpha f\|_{T^{p_2}_\psi(x_0)} \\
&\le C_{p_1,p_2,\phi,\varphi,\psi} NM \|h\|_{T^{p_2}_\varphi(x_0)}.
\end{aligned}$$

Let us remark that the assumption $\overline{b}(\psi) - \underline{b}(\varphi) < \min\{\underline{b}(\phi), 1\}$ allows us to use Proposition 5.3.6 to get

$$\begin{aligned}
\|\mathcal{H}\mathcal{E}_2 f\|_{T^{p_3}_\psi(x_0)} &\le N \|\mathcal{E}_2 f\|_{T^{p_3}_\psi(x_0)} \\
&\le C_{p_1,p_2,\phi,\psi} NM \sum_{|\alpha|=m} (\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} + \|D^\alpha f\|_{L^{p_3}(\mathbb{R}^d)}) \\
&\le C_{p_1,p_2,\phi,\varphi,\psi} NM (\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}).
\end{aligned}$$

Finally, by Proposition 5.3.7, we have

$$\begin{aligned}
\|L_1(D) f\|_{T^{p_3}_\psi(x_0)} &\le C \sum_{|\alpha|\le m-2} \|D^\alpha f\|_{T^{p_3}_\psi(x_0)} \\
&\le C_{p_2,p_3} \sum_{|\alpha|\le m-2} \|D^\alpha f\|_{T^{p_2}_\psi(x_0)} + \|D^\alpha f\|_{L^{p_3}(\mathbb{R}^d)} \\
&\le C_{p_2,p_3,\varphi,\psi} (\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}),
\end{aligned}$$

[3] $(D\mathcal{J})^\alpha$ stands for $(D_1\mathcal{J})^{\alpha_1} \dots (D_d\mathcal{J})^{\alpha_d}$

which leads to the conclusion.

Let us now assume that m is odd; in this case, we have

$$\begin{aligned}\mathcal{J}^{m+1}h = (i+\Lambda)\mathcal{J}^2 f &\Rightarrow (-i+\Lambda)\mathcal{J}^{m+1}h = (1-\Delta)\mathcal{J}^2 f\\ &\Rightarrow (-i+\Lambda)\mathcal{J}^{m+1}h = f\end{aligned}$$

and therefore, for $|\alpha| \leq m$,

$$D^\alpha f = (i\sum_{j=1}^{d}\mathcal{R}_j(D_j\mathcal{J}) - i\mathcal{J})(D\mathcal{J})^\alpha \mathcal{J}^{m-|\alpha|}h.$$

Given $|\alpha| < m$ and $j \in \{1,\dots,d\}$, we have, by Proposition 6.3.4, Proposition 6.3.6 and Theorem 6.1.2,

$$\|\mathcal{R}_j(D_j\mathcal{J})(D\mathcal{J})^\alpha \mathcal{J}^{m-|\alpha|}h\|_{T^{p_2}_\psi(x_0)} \leq C_{p_2,\varphi,\psi}\|h\|_{T^{p_2}_\varphi(x_0)}.$$

From Theorem 6.1.2 and Proposition 5.3.3, we know that $\mathcal{J}$ continuously maps $T^{p_2}_\psi(x_0)$ into itself and so we also have

$$\|\mathcal{J}(D_j\mathcal{J})\mathcal{J}^{m-|\alpha|}h\|_{T^{p_2}_\psi(x_0)} \leq C_{p_2,\varphi,\psi}\|h\|_{T^{p_2}_\varphi(x_0)}.$$

As a consequence, the inequality

$$\|D^\alpha f\|_{T^{p_2}_\psi(x_0)} \leq C_{p_2,\varphi,\psi}\|h\|_{T^{p_2}_\varphi(x_0)}$$

still holds for all $|\alpha| < m$. By a similar reasoning, we can get the following inequality:

$$\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} \leq C_{p_2,\varphi}\|h\|_{T^{p_2}_\varphi(x_0)},$$

for $|\alpha| = m$. Therefore, the upper bounds for $\|\mathcal{HE}_1 f\|_{T^{p_3}_\psi(x_0)}$ and $\|\mathcal{HE}_2 f\|_{T^{p_3}_\psi(x_0)}$ are still satisfied. Finally, we also have

$$\|L_2(D)f\|_{T^{p_3}_\psi(x_0)} \leq C_{p_2,p_3,\varphi,\psi}(\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)})$$

and, as $\Lambda = i\sum_{j=1}^d \mathcal{R}_j D_j$, Proposition 6.3.4 implies

$$\begin{aligned}\|\Lambda L_3(D)f\|_{T^{p_3}_\psi(x_0)} &\leq C_{p_3,\psi}\sum_{|\alpha|\leq m-2}\|D^\alpha f\|_{T^{p_3}_\psi(x_0)}\\ &\leq C_{p_2,p_3,\varphi,\psi}(\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}),\end{aligned}$$

which gives the conclusion in this second case. □

Remark 6.4.3. It is still possible to obtain an inequality of this kind if we consider the case $\varphi(r) = r^{-d/p_2}$.

- If $d/p_2 \notin \mathbb{N}$, then Theorem 6.1.2 still holds for φ, since the assumption $\underline{b}(\phi) > -d/p$ is just assumed in order to guarantee the relation $r^{-d/p} \leq C\phi(r)$ for r sufficiently large; it can thus be relaxed in this case. Therefore, Proposition 6.3.6 can also be applied with φ and the inequalities

$$\|D^\alpha f\|_{T^{p_2}_\psi(x_0)} \leq C_{p_2,\varphi,\psi}\|h\|_{T^{p_2}_\varphi(x_0)} \qquad \forall|\alpha| < m$$

and

$$\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} \leq C_{p_2,\varphi}\|h\|_{T^{p_2}_\varphi(x_0)} \qquad \forall|\alpha| = m$$

are still valid in this case. Let us also remark that we have

$$\|h\|_{T^{p_2}_\varphi(x_0)} \leq 2\|h\|_{L^{p_2}(\mathbb{R}^d)} \leq C_{m,p_2}\|f\|_{W^{p_2}_m(\mathbb{R}^d)}.$$

- If $d/p_2 \in \mathbb{N}$ with $p_2 < d$, let us consider $|\alpha| < m$; we have

$$D^\alpha f \in W^{p_2}_1(\mathbb{R}^d) \hookrightarrow L^{p*}(\mathbb{R}^d),$$

with $1/p^* := \frac{1}{p_2} - \frac{1}{d}$, by Sobolev's embedding. Therefore, for $r > 0$,

$$\begin{aligned} r^{-d/p_2}\|D^\alpha f\|_{L^{p_2}(B(x_0,r))} &\leq C_{d,p_2,p^*} r^{-d/p_2} r^{d(\frac{1}{p_2}-\frac{1}{p^*})}\|D^\alpha f\|_{L^{p^*}(B(x_0,r))} \\ &\leq C_{d,p_2,p*}\|D^\alpha f\|_{W^{p^*}_1(\mathbb{R}^d)} r^{-d/p^*} \end{aligned}$$

and $D^\alpha f \in T^{p_2}_{-d/p^*}(x_0)$, with

$$\|D^\alpha f\|_{T^{p_2}_{-d/p^*}(x_0)} \leq C_{d,p_2,p^*}\|f\|_{W^{p_2}_m(\mathbb{R}^d)}.$$

Moreover, as $\overline{b}(\psi) < -\frac{d}{p_2} + 1 = -d/p^*$, we get

$$\|D^\alpha f\|_{T^{p_2}_\psi(x_0)} \leq C_{d,p_2,p^*,\psi}\|f\|_{W^{p_2}_m(\mathbb{R}^d)}.$$

Of course, for $|\alpha| = m$, we have

$$\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} \leq 2\|D^\alpha f\|_{L^{p_2}(\mathbb{R}^d)} \leq 2\|f\|_{W^{p_2}_m(\mathbb{R}^d)}$$

and we can now conclude that

$$\begin{aligned} &\|h\|_{T^{p_3}_\psi(x_0)} \\ &\leq \|\mathcal{H}g\|_{T^{p_3}_\psi(x_0)} + C_{p_1,p_2,\varphi,\psi,\phi}((1+MN)\|f\|_{W^{p_2}_m(\mathbb{R}^d)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}). \end{aligned}$$

- If $d/p_2 \in \mathbb{N}$, let us first prove the following lemma.

Lemma 6.4.4. *If $d > 1$, for $d \le q < \infty$, we have the continuous embedding $W^d_1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$.*

Proof. Let g be a function of $W^d_1(\mathbb{R}^d)$; let us first remark that $g \in L^{\frac{d^2}{d-1}}(\mathbb{R}^d)$. Indeed, $g^d \in L^1(\mathbb{R}^d)$, with

$$\|g^d\|_{L^1(\mathbb{R}^d)} = \|g\|^d_{L^d(\mathbb{R}^d)} \le \|g\|^d_{W^d_1(\mathbb{R}^d)}$$

and, for $|\alpha| = 1$, by Hölder's inequality,

$$\begin{aligned}\|D^\alpha g^d\|_{L^1(\mathbb{R}^d)} &= \|d g^{d-1} D^\alpha g\|_{L^1(\mathbb{R}^d)}\\ &\le d\|g\|^{d-1}_{L^d(\mathbb{R}^d)}\|D^\alpha g\|_{L^d(\mathbb{R}^d)}\\ &\le d\|g\|^d_{W^d_1(\mathbb{R}^d)}.\end{aligned}$$

Therefore, g^d belongs to $W^1_1(\mathbb{R}^d)$ with $\|g^d\|_{W^1_1(\mathbb{R}^d)} \le C\|g\|^d_{W^d_1(\mathbb{R}^d)}$ and, as $d > 1$, Sobolev's embedding gives $W^1_1(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{d-1}}(\mathbb{R}^d)$ and finally $g \in L^{\frac{d^2}{d-1}}(\mathbb{R}^d)$, with

$$\|g\|_{L^{\frac{d^2}{d-1}}(\mathbb{R}^d)} \le C\|g\|_{W^d_1(\mathbb{R}^d)}.$$

Let us prove by induction that any $g \in W^d_1(\mathbb{R}^d)$ belongs to $L^{\frac{(d+k)d}{d-1}}(\mathbb{R}^d)$ with

$$\|g\|_{L^{\frac{d(d+k)}{d-1}}(\mathbb{R}^d)} \le C_k\|g\|_{W^d_1(\mathbb{R}^d)},$$

for all $k \in \mathbb{N}$. Let us suppose that this property holds for some $k \in \mathbb{N}$ and let $(\varphi_j)_{j\in\mathbb{N}}$ be a sequence of functions in $\mathcal{D}(\mathbb{R}^d)$ such that φ_j converges to g in $W^d_1(\mathbb{R}^d)$. In particular, by induction, φ_j converges to g in $L^{(d+k)\frac{d}{d-1}}(\mathbb{R}^d)$. Let us recall that for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have (see e.g. Lemma 8.7. in [121])

$$\begin{aligned}&\Big(\int_{\mathbb{R}^d} |\varphi(x)|^{(d+k+1)\frac{d}{d-1}}\,dx\Big)^{\frac{d-1}{d}}\\ &\le \frac{d+k+1}{2}\Big(\prod_{l=1}^d \|D_l\varphi\|_{L^d(\mathbb{R}^d)}\Big)^{1/d}\Big(\int_{\mathbb{R}^d} |\varphi(x)|^{(d+k)\frac{d}{d-1}}\,dx\Big)^{\frac{d-1}{d}},\end{aligned}$$

which holds if and only if

$$\|\varphi\|^{d+k+1}_{L^{(d+k+1)\frac{d}{d-1}}(\mathbb{R}^d)} \le \frac{d+k+1}{2}\Big(\prod_{l=1}^d \|D_l\varphi\|_{L^d(\mathbb{R}^d)}\Big)^{1/d}\|\varphi\|^{d+k}_{L^{(d+k)\frac{d}{d-1}}(\mathbb{R}^d)}.$$

This proves that $(\varphi_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^{(d+k+1)\frac{d}{d-1}}(\mathbb{R}^d)$. As a consequence, g belongs to $L^{(d+k+1)\frac{d}{d-1}}(\mathbb{R}^d)$, with

$$\|g\|_{L^{(d+k+1)\frac{d}{d-1}}(\mathbb{R}^d)} \le C_k\Big(\frac{d+k+1}{2}\Big)^{\frac{1}{d+k+1}}\|g\|_{W^d_1(\mathbb{R}^d)}.$$

□

Let us finish the ongoing remark. If $|\alpha| < m$, then $D^\alpha f$ belongs to $W_1^d(\mathbb{R}^d)$, so as $-1 < \overline{b}(\psi) < 0$, we can choose $d \leq q < \infty$ such that $\overline{b}(\psi) < -d/q$. By the previous lemma, $D^\alpha f$ belongs to $L^q(\mathbb{R}^d)$ and

$$\|D^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_q \|D^\alpha f\|_{W_1^d(\mathbb{R}^d)}.$$

It follows that, for $r > 0$,

$$\begin{aligned} r^{-1}\|D^\alpha f\|_{L^d(B(x_0,r))} &\leq C_{d,q} r^{-1} r^{d(\frac{1}{d}-\frac{1}{q})} \|D^\alpha f\|_{L^q(B(x_0,r))} \\ &\leq C_{d,q} r^{-d/q} \|D^\alpha f\|_{W_1^d(\mathbb{R}^d)}. \end{aligned}$$

Hence, $D^\alpha f$ belongs to $T^p_{-d/q}(x_0)$, with

$$\|D^\alpha f\|_{T^p_{-d/q}(x_0)} \leq C_{d,q} \|D^\alpha f\|_{W_1^d(\mathbb{R}^d)}.$$

Since $\overline{b}(\psi) < -d/q$, we can write

$$\|D^\alpha f\|_{T^d_\psi(x_0)} \leq C_{\psi,q} \|D^\alpha f\|_{T^d_{-d/q}(x_0)} \leq C_{d,q,\psi} \|D^\alpha f\|_{W_1^d(\mathbb{R}^d)}.$$

The previous reasoning for the case $|\alpha| = m$ is still valid and we get again

$$\begin{aligned} &\|h\|_{T^{p_3}_\psi(x_0)} \\ &\leq \|\mathcal{H}g\|_{T^{p_3}_\psi(x_0)} + C_{p_1,p_2,\varphi,\psi,\phi}((1+MN)\|f\|_{W_m^{p_2}(\mathbb{R}^d)} + \|f\|_{W_m^{p_3}(\mathbb{R}^d)}). \end{aligned}$$

Definition 6.4.5. Let, $p \in (1,\infty)$, $\phi, \varphi \in \mathcal{B}$ be such that $0 < \underline{b}(\phi)$, $-d/p < \underline{b}(\varphi)$ and such that there exists $n \in \mathbb{Z}$ such that $n < \underline{b}(\varphi) \leq \overline{b}(\varphi) < n+1$; let us define k_p as follows:

- if $\underline{b}(\varphi) = \overline{b}(\varphi)$,

$$k_p(\phi,\varphi) := \min\{k \in \mathbb{N} : \frac{1}{k}(\underline{b}(\varphi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\}\},$$

- if $n < \underline{b}(\varphi) < \overline{b}(\varphi) < n+1$,

$$k_p(\phi,\varphi) := k_p(\phi, \cdot^{\underline{b}(\varphi)}) + \min\{k \in \mathbb{N} : \frac{\overline{b}(\varphi) - \underline{b}(\varphi)}{k} < \min\{1, \underline{b}(\phi)\}\}.$$

Theorem 6.4.6. *Let $p \in (1,\infty)$, $q \in (1,\infty]$, $x_0 \in \mathbb{R}^d$ and $\phi, \varphi \in \mathcal{B}$ be such that $-d/p < \underline{b}(\varphi)$, $0 < \underline{b}(\phi)$ and such that there exists $n \in \mathbb{Z}$ such that $n < \underline{b}(\varphi) < \overline{b}(\varphi) < n+1$. Let $\mathcal{E}f = g$ be an elliptic differential equation of order m at x_0 such that the coefficients of $\mathcal{E}$ are functions in $T^q_\phi(x_0)$ for which x_0 is a Lebesgue-point. Let us suppose that*

- *$g \in T^{p_1}_\varphi(x_0)$ with $1/p_1 := \frac{1}{p} + \frac{1}{q}$,*

- $\phi \preccurlyeq \varphi$ *and* $\overline{b}(\varphi) \le \underline{b}(\phi)$ *or* $\overline{b}(\varphi) - \underline{b}(\varphi) \le \min\{1, \underline{b}(\phi)\}$,
- $0 < \dfrac{1}{p'} := \dfrac{k_p(\phi,\varphi)}{q} + \dfrac{1}{p} < 1$,
- $f \in W^p_m(\mathbb{R}^d)$ *and* $p^* := \inf\{s \ge 1 : f \in W^s_m(\mathbb{R}^d)\} \le p'$.

Then there exists a constant $C_{p',\phi,\varphi,m}$ such that, for all $|\alpha| \le m$, $D^\alpha f$ belongs to the space $T^{q'}_{\varphi_{m-|\alpha|}}(x_0)$ and

$$\begin{aligned}\|D^\alpha f\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} \le C_{p',\phi,\varphi}(&N(1+MN)^{k_p(\phi,\varphi)-1}\|g\|_{T^{p_1}_{\varphi}(x_0)} \\ &+ k_p(\phi,\varphi)(1+MN)^{k_p(\phi,\varphi)}(\|f\|_{W^p_m(\mathbb{R}^d)} + \|f\|_{W^{p'}_m(\mathbb{R}^d)})),\end{aligned}$$

for all $q' \ge 1$ such that

- $1/p' \ge 1/q' \ge \frac{1}{p'} - \frac{m-|\alpha|}{d}$ *if* $1/p' > \frac{m-|\alpha|}{d}$,
- $p' \le q' \le \infty$ *if* $1/p' < \frac{m-|\alpha|}{d}$,
- $p' \le q' < \infty$ *if* $1/p' = \frac{m-|\alpha|}{d}$,

where M is the least upper bound of the norm of the coefficients of $\mathcal{E}$ in $T^q_\phi(x_0)$ and

$$N = \sup_{\substack{|z|=1 \\ 0\le|\alpha|\le v(\varphi)}} |D^\alpha \sigma(\mathcal{K})(z)|.$$

Proof. Let us first suppose that $\underline{b}(\varphi) = \overline{b}(\varphi)$ and set $k = k_p(\phi,\varphi)$. Let us choose $0 \le \varepsilon < 1$ such that

- $0 < \frac{(1-\varepsilon)}{k}(\underline{b}(\varphi) + \frac{d}{p}) \le \frac{(1+\varepsilon)}{k}(\underline{b}(\varphi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\}$,
- $-\frac{d}{p} + \frac{j+\varepsilon}{k}(\underline{b}(\varphi) + \frac{d}{p}) \notin \mathbb{Z}$ for all $j \in \{1,\dots,k-1\}$.

We can then define, for $j \in \{0,\dots,k\}$, the function ψ_j by

$$\psi_j(r) := \begin{cases} r^{-d/p} & \text{if } j = 0 \\ r^{-d/p}(\varphi(r) r^{d/p})^{\frac{j+\varepsilon}{k}} & \text{if } 1 \le j < k \\ \varphi & \text{if } j = k. \end{cases}$$

For $0 \le j < k$, we have $\overline{b}(\psi_j) < \underline{b}(\varphi)$ and so $\varphi \preccurlyeq \psi_j$. Moreover, for $1 \le j \le k$,

$$\underline{b}(\psi_j) = \overline{b}(\psi_j) = -\frac{d}{p} + \frac{j+\varepsilon}{k}(\underline{b}(\varphi) + \frac{d}{p}) \notin \mathbb{Z}.$$

We also have

$$\overline{b}(\psi_1) - \underline{b}(\psi_0) = \frac{(1+\varepsilon)}{k}(\underline{b}(\varphi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\}$$

and, for $1 \leq j < k$,

$$\begin{aligned}\overline{b}(\psi_{j+1}) - \underline{b}(\psi_j) &= -\frac{d}{p} + \frac{j+1+\varepsilon}{k}(\overline{b}(\varphi) + \frac{d}{p}) + \frac{d}{p} - \frac{j+\varepsilon}{k}(\underline{b}(\varphi) + \frac{d}{p}) \\ &= \frac{1}{k}(\underline{b}(\varphi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\},\end{aligned}$$

as well as

$$\overline{b}(\psi_k) - \underline{b}(\psi_{k-1}) = \frac{(1-\varepsilon)}{k}(\underline{b}(\varphi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\}.$$

Given $j \in \{0, \dots, k\}$, let us also define $p_j \in (1, \infty)$ by

$$\frac{1}{p_j} := \frac{j}{q} + \frac{1}{p}.$$

Since we have $h \in L^p(\mathbb{R}^d)$, $h \in T^{p_0}_{\psi_0}(x_0)$ and $\phi \preccurlyeq \psi_1$, we can write, using the previous remark,

$$\|h\|_{T^{p_1}_{\psi_1}(x_0)} \leq \|\mathcal{H}g\|_{T^{p_1}_{\psi_1}(x_0)} + C_1(1+MN)(\|f\|_{W^p_m(\mathbb{R}^d)} + \|f\|_{W^{p_1}_m(\mathbb{R}^d)}).$$

Now, since f belongs to $W^{p_1}_m$ and the coefficients of $\mathcal{E}$ are in $L^q(\mathbb{R}^d)$, g belongs to $L^{p_2}(\mathbb{R}^d)$ and, from Proposition 5.3.7, also to $T^{p_2}_{\psi_2}(x_0)$. Furthermore, by Proposition 6.4.2, we have

$$\|h\|_{T^{p_2}_{\psi_2}(x_0)} \leq \|\mathcal{H}g\|_{T^{p_2}_{\psi_2}(x_0)} + C_0(1+MN)(\|h\|_{T^{p_1}_{\psi_1}(x_0)} + \|f\|_{W^{p_2}_m(\mathbb{R}^d)}).$$

By iterating the reasoning, we find, for $1 \leq j \leq k$,

$$\|h\|_{T^{p_j}_{\psi_j}(x_0)} \leq \|\mathcal{H}g\|_{T^{p_j}_{\psi_j}(x_0)} + C_j(1+MN)(\|h\|_{T^{p_{j-1}}_{\psi_{j-1}}(x_0)} + \|f\|_{W^{p_j}_m(\mathbb{R}^d)}).$$

Now, for $1 \leq j \leq k$, we have

$$\begin{aligned}\|\mathcal{H}g\|_{T^{p_j}_{\psi_j}(x_0)} &\leq C_{p_j,\psi_j} N \|g\|_{T^{p_j}_{\psi_j}(x_0)} \\ &\leq C_{p_1,p_j,\psi_j} N \|g\|_{T^{p_1}_{\psi_j}(x_0)} + N\|g\|_{L^{p_1}(\mathbb{R}^d)} \\ &\leq C_{p_1,p',\phi} N \|g\|_{T^{p_1}_{\phi}(x_0)}\end{aligned}$$

and

$$\|f\|_{W^{p_j}_m(\mathbb{R}^d)} \leq \|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m(\mathbb{R}^d)},$$

this allows us to claim the existence of a constant $C_{p,p',\phi,\varphi} > 0$ such that

$$\|h\|_{T^{p'}_\varphi(x_0)} \\ \leq C_{p,p',\phi,\varphi}(N(1+MN)^{k-1}\|g\|_{T^{p_1}_\varphi(x_0)} + k(1+MN)^k(\|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m(\mathbb{R}^d)})).$$

That being done, let us establish the same inequality under the assumption $n < \underline{b}(\varphi) < \overline{b}(\varphi) < n+1$. If $\overline{b}(\varphi) \leq \underline{b}(\phi)$, then we set $k_1 := k_p(\phi, \cdot^{\underline{b}(\varphi)})$ and

$$k_2 := \min\{k \in \mathbb{N} : \frac{\overline{b}(\varphi) - \underline{b}(\varphi)}{k} < \min\{1, \underline{b}(\phi)\}\}.$$

We also define, for $0 \leq j < k_2$,

$$\psi_j(r) := r^{\underline{b}(\varphi) + \frac{j}{k_2}(\overline{b}(\varphi) - \underline{b}(\varphi))},$$

and $\psi_{k_2} := \varphi$. For $0 \leq j < k$, we have

$$\overline{b}(\psi_j) = \underline{b}(\varphi) + \frac{j}{k_2}(\overline{b}(\varphi) - \underline{b}(\varphi)) < \overline{b}(\varphi) \leq \underline{b}(\phi),$$

and so $\phi \preccurlyeq \psi_j$. Also,

$$\overline{b}(\psi_{j+1}) - \underline{b}(\psi_j) = \frac{1}{k_2}(\overline{b}(\varphi) - \underline{b}(\varphi)) < \min\{1, \underline{b}(\phi)\}.$$

From the first part of the proof, we can write, if p_0 is defined by $1/p_0 := \frac{k_1}{q} + \frac{1}{p}$,

$$\|h\|_{T^{p_0}_{\psi_0}(x_0)} \leq C_{p,p_0,\phi,\varphi}(N(1+MN)^{k_1-1}\|g\|_{T^{p_1}_\varphi(x_0)} \\ + (k_1)(1+MN)^{k_1}(\|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m(\mathbb{R}^d)})).$$

We can proceed as we did in the first part to get the desired inequality.

Let us now consider the case where $\overline{b}(\varphi) > \underline{b}(\phi)$ and $\overline{b}(\varphi) - \underline{b}(\varphi) < \min\{1, \underline{b}(\phi)\}$. Let us choose α such that $\max\{-d/p, n\} < \alpha < \underline{b}(\varphi)$ and $\overline{b}(\varphi) - \alpha < \underline{b}(\phi)$; in particular, α is not an integer. From the first part of the proof, we know that there exists a constant $C_{p,p'\phi,\varphi} > 0$ such that

$$\|h\|_{T^{p''}_\alpha(x_0)} \leq C_{p,p',\phi,\varphi}(N(1+MN)^{k-2}\|g\|_{T^q_\varphi(x_0)} \\ + (k-1)(1+MN)^{k-1}(\|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m})),$$

with $1/p'' := \frac{k-1}{q} + \frac{1}{p}$. Now, Proposition 6.4.2 implies

$$\|h\|_{T^{p'}_\varphi(x_0)} \leq C_{p,p',\phi,\varphi}(N\|g\|_{T^{p'}_\varphi(x_0)} + (1+MN)(\|h\|_{T^{p''}_\alpha(x_0)} + \|f\|_{W^{p'}_m(\mathbb{R}^d)})) \\ \leq C_{p,p',\phi,\varphi}(N(1+MN)^{k-1}\|g\|_{T^{p_1}_\varphi(x_0)} \\ + k(1+MN)^k(\|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m})),$$

which gives the desired inequality.

Let us now consider $|\alpha| \leq m$ and $q' \geq 1$ such as in the assumption. If m is even then

$$\begin{aligned}\|D^\alpha f\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} &= \|\mathcal{J}^{m-|\alpha|}(DJ)^\alpha h\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} \\ &\leq C_\varphi \|(DJ)^\alpha h\|_{T^{p'}_\varphi(x_0)} \\ &\leq C_\varphi \|h\|_{T^{p'}_\varphi(x_0)},\end{aligned}$$

by Theorem 6.1.2 and Proposition 6.3.6. If m is odd, we get

$$\begin{aligned}\|D^\alpha f\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} &= \|\mathcal{J}^{m-|\alpha|}(i\sum_{j=1}^{d}\mathcal{R}_j(D_j\mathcal{J}) - i\mathcal{J})(D\mathcal{J})^\alpha h\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} \\ &\leq C_\varphi \|(i\sum_{j=1}^{d}\mathcal{R}_j(D_j\mathcal{J}) - i\mathcal{J})(D\mathcal{J})^\alpha h\|_{T^{p'}_\varphi(x_0)} \\ &\leq C_\varphi \|h\|_{T^{p'}_\varphi(x_0)},\end{aligned}$$

by Theorem 6.1.2, Proposition 6.3.4 and Proposition 6.3.6. From this, the inequality obtained in the first part of the proof allows us to conclude the desired membership and inequality. □

7 Continuously differentiable functions on compact sets

An intermediate between uniform regularity, as we studied in Chapter 2 and pointwise regularity, Chapters 3 to 6, is to consider functions defined on compact sets. There, using the structure of the compact, one can define a notion of differentiability.

In most analysis textbooks differentiability is only treated for functions on open domains and, if needed, e.g., for the divergence theorem, an ad hoc generalization for functions on compact sets is given. We propose instead to define differentiability on arbitrary sets as the usual affine-linear approximability – the price one has to pay is then the definite article: instead of *the* derivative there can be many. We will only consider compact domains in order to have a natural norm on our space. The results are easily extended to σ-compact (and, in particular, closed) sets.

An $\mathbb{R}^n$-valued function f on a compact set $K \subseteq \mathbb{R}^d$ is said to belong $C^1(K,\mathbb{R}^n)$ if there exists a continuous function df on K with values in linear maps from $\mathbb{R}^d$ to $\mathbb{R}^n$ such that, for all $x \in K$,

$$\lim_{\substack{y\to x\\ y\in K}} \frac{f(y)-f(x)-df(x)(y-x)}{|y-x|} = 0, \tag{7.1}$$

where $|\cdot|$ is the Euclidean norm. For $n=1$ we often identify $\mathbb{R}^d$ with its dual and write $\langle\cdot,\cdot\rangle$ for the evaluation which is then the scalar product. Questions about $C^1(K,\mathbb{R}^n)$ easily reduce to the case $C^1(K) = C^1(K,\mathbb{R})$.

Of course, equality (7.1) means that df is a continuous (Fréchet) derivative of f on K. As in the case of open domains, every $f \in C^1(K)$ is continuous and we have the chain rule: for all (continuous) derivatives df of f on K and dg of g on $f(K)$ the map $x \mapsto dg(f(x)) \circ df(x)$ is a (continuous) derivative of $g\circ f$ on K.

In general, a derivative need not be unique. For this reason, a good tool to study $C^1(K)$ is the jet space

$$\mathcal{J}^1(K) = \{(f,df) : df \text{ is a continuous derivative of } f \text{ on } K\}$$

endowed with the norm

$$\|(f,df)\|_{\mathcal{J}^1(K)} = \|f\|_K + \|df\|_K,$$

where $\|\cdot\|_K$ is the uniform norm on K and $|df(x)| = \sup\{|df(x)(v)| : |v| \le 1\}$. For the projection $\pi(f, df) = f$ we have $C^1(K) = \pi(\mathcal{J}^1(K))$, and we equip $C^1(K)$ with the quotient norm, i.e.

$$\|f\|_{C^1(K)} = \|f\|_K + \inf\{\|df\|_K : df \text{ is a continuous derivative of } f \text{ on } K\}.$$

It seems that the space $C^1(K)$ did not get much attention in the literature. This is in sharp contrast to the "restriction space" $C^1(\mathbb{R}^d|K) = \{f|_K : f \in C^1(\mathbb{R}^d)\}$. Obviously, the inclusion $C^1(\mathbb{R}^d|K) \subseteq C^1(K)$ holds but it is well known that, in general, it is strict. Simple examples are domains with inward directed cusps like

$$K = \{(x,y) \in [-1,1]^2 : |y| \ge e^{-1/x} \text{ for } x > 0\}.$$

The function $f(x,y) = e^{-1/(2x)}$ for $x, y > 0$ and $f(x,y) = 0$ elsewhere is in $C^1(K)$ but it is not the restriction of a C^1-function on $\mathbb{R}^2$ because it is not Lipschitz continuous near the origin.

In a famous paper from 1934 [128], Whitney proved Theorem 1.7.3 which, in this context, states that $C^1(\mathbb{R}^d|K) = \pi(\mathscr{E}^1(K))$ where $\mathscr{E}^1(K)$ is the space of jets (f, df) for which the limit (7.1) is uniform in $x \in K$. Moreover, $\mathscr{E}^1(K)$ endowed with the norm

$$\|(f, df)\|_{\mathscr{E}^1(K)} = \|(f, df)\|_{\mathcal{J}^1(K)} + \sup\left\{\frac{|f(y) - f(x)|}{|y - x|} : x, y \in K, y \neq x\right\}$$

is a Banach space. Thus, $C^1(\mathbb{R}^d|K)$ equipped with the quotient norm $\|\cdot\|_{C^1(\mathbb{R}^d|K)}$ inherited from $\|\cdot\|_{\mathscr{E}^1(K)}$ is also a Banach space.

Since their introduction, Whitney jets (also of higher orders) have been widely studied, in particular in the context of extension operators [46, 51, 52, 53]. Generalizations of them have been defined in various contexts such as Baire functions [85], holomorphic functions [18], Sobolev spaces [131, 132], so-called $C^{m,\omega}(\mathbb{R}^d)$ spaces [45] or (generalized) Hölder spaces as we did in Chapter 5.

In this chapter, we prove that $\mathscr{E}^1(K)$ is always a dense subset of $\mathcal{J}^1(K)$. The density of $C^1(\mathbb{R}^d|K)$ in $C^1(K)$ is then an immediate consequence. Together with a characterization of the completeness of $(C^1(K), \|\cdot\|_{C^1(K)})$, it leads to a simple geometric criterion for the equality $C^1(K) = C^1(\mathbb{R}^d|K)$ as Banach spaces. In the one-dimensional case, we also give a characterization of the mere algebraic equality.

If the compact set K is topologically regular, i.e. the closure of its interior, another common way to define differentiability is the space

$$C^1_{\text{int}}(K) = \{f \in C(K) : f|_{\mathring{K}} \in C^1(\mathring{K}) \text{ and } df \text{ extends continuously to } K\},$$

see for instance [49, 130]. For $f \in C^1_{\text{int}}(K)$ we will denote again by df the unique continuous extension to K of the derivative.

In this topologically regular situation, the derivative of a continuously differentiable function on K is uniquely determined by the function, which means that the projection π is injective on $\mathcal{J}^1(K)$ and therefore $C^1(K)$ and $\mathcal{J}^1(K)$ as well as $C^1(\mathbb{R}^d|K)$ and $\mathscr{E}^1(K)$, respectively, can be identified.

Equipped with the norm $\|f\|_K + \|df\|_K$, it is clear that $C^1_{\text{int}}(K)$ is always a Banach space that contains $C^1(K)$. Despite this nice aspect we will see by an example of Sauter [114] that $C^1_{\text{int}}(K)$ has a dramatic drawback: compositions of $C^1_{\text{int}}(K)$-functions need not be differentiable.

We will present some results about equalities between $C^1_{\text{int}}(K)$, $C^1(\mathbb{R}^d|K)$ and $C^1(K)$ which are related the so-called "Whitney conjecture" ([132, 129]).

Results in this chapter were found during a research stay in Trier Universität with Leonhard Frerick and Jochen Wengenroth and were published in [54].

Contents

7.1 Path integrals

A function $f \in C^1(K)$ need not be Lipschitz continuous because segments with endpoints in K, to which one would like to apply the mean value theorem, need not be contained in K. Instead of segments one then has to consider rectifiable paths in K, i.e. continuous functions $\gamma : [a,b] \to K$ such that the length

$$L(\gamma) = \sup\left\{\sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| : a = t_0 < \cdots < t_n = b\right\}$$

is finite. The function $\ell(t) = L(\gamma|_{[a,t]})$ is then continuous: Given $\varepsilon > 0$ and a partition such that the length of the corresponding polygon is bigger than $L(\gamma) - \varepsilon$ every interval $[r,s]$ lying between two consecutive points of the partition satisfies

$$\ell(s) - \ell(r) = L(\gamma_{[r,s]}) \leq |\gamma(s) - \gamma(r)| + \varepsilon.$$

For the minimal length of the subintervals of the partition one then easily gets the required continuity estimate.

Proposition 7.1.1 (Mean value inequality). *Let $f \in C^1(K)$ and $x, y \in K$. If df is a derivative of f on K and if x and y are joined by a rectifiable path $\gamma : [a,b] \to K$, then*

$$|f(y) - f(x)| \leq L(\gamma) \sup\{|df(z)| : z \in \gamma([a,b])\}. \tag{7.2}$$

Proof. We essentially repeat Hörmander's proof [63, Theorem 1.1.1]. For each $c > \sup\{|df(z)| : z \in \gamma([a,b])\}$ the set $T = \{t \in [a,b] : |f(\gamma(t)) - f(x)| \le c\ell(t)\}$ is non-empty and closed because of the continuity of $f \circ \gamma$ and ℓ, hence it has a largest element $t \in [a,b]$. If t is different from b, the differentiability of f at $z = \gamma(t)$ gives a neighbourhood U of z such that

$$|f(z) - f(w)| \le |f(z) - f(w) - df(z)(z-w)| + |df(z)(z-w)| \le c|z-w|$$

for all $w \in U$. By the continuity of γ we find $s > t$ with $\gamma(s) \in U$ so that

$$|f(\gamma(s)) - f(x)| \le |f(\gamma(s)) - f(\gamma(t))| + c\ell(t) \le c|\gamma(s) - \gamma(t)| + c\ell(t) \le c\ell(s),$$

contradicting the maximality of t. □

The mean value inequality does not use the continuity of a derivative and has the usual consequences. For example, if $df = 0$ is a derivative of f and K is *rectifiably pathwise connected* (a certainly self-explaining notion) then f is constant.

Our next aim is to show that a continuous derivative integrates back to the function along rectifiable paths. We first recall the relevant notions. If $F : K \to \mathbb{R}^d$ is continuous and γ is a rectifiable path in K we define the path integral $\int_\gamma F$ as the limit of Riemann-Stieltjes sums

$$\sum_{j=1}^{n} \langle F(\gamma(\tau_j)), \gamma(t_j) - \gamma(t_{j-1})\rangle$$

where $a = t_0 < \ldots < t_n = b$ are partitions with $\max\{t_j - t_{j-1} : 1 \le j \le n\} \to 0$ and $t_{j-1} \le \tau_j \le t_j$. The existence of the limit is seen from an appropriate Cauchy condition (or by using the better known one-dimensional case where rectifiable paths are usually called functions of bounded variation). If γ is even absolutely continuous, i.e. there is a Lebesgue integrable $\dot\gamma : [a,b] \to \mathbb{R}^d$ with $\gamma(\beta) - \gamma(\alpha) = \int_\alpha^\beta \dot\gamma(t)dt$ for all $\alpha \le \beta$, one gets from the uniform continuity of $F \circ \gamma$ the familiar representation

$$\int_\gamma F = \int_a^b \langle F(\gamma(t)), \dot\gamma(t)\rangle dt.$$

If γ is even continuously differentiable and $F = df$ for a function $f \in C^1(K)$, the integrand in the last formula is the derivative of $f \circ \gamma$ (by the chain rule) and the fundamental theorem of calculus gives $\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$. Since continuous differentiability of γ is a not a realistic assumption in our considerations (interesting phenomena typically occur for quite rough compact sets K), we need a more general version.

Theorem 7.1.2 (Fundamental theorem of calculus)**.** *For each $f \in C^1(K)$ with a continuous derivative df and each rectifiable path $\gamma : [a,b] \to K$ we have*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)). \tag{7.3}$$

Proof. Given a partition $a = t_0 < \ldots < t_n = b$ and a fixed $j \in \{1,\ldots,n\}$ we set $z = \gamma(t_j)$ and apply the mean value inequality to the function

$$g(x) = f(x) - f(z) - \langle df(z), x - z\rangle$$

on $\gamma([t_{j-1}, t_j])$. Since $dg(x) = df(x) - df(z)$ is a derivative of g, we obtain

$$\begin{aligned}
&\left|f(\gamma(t_j)) - f(\gamma(t_{j-1})) - \langle df(\gamma(t_j)), \gamma(t_j) - \gamma(t_{j-1})\rangle\right| \\
&= |g(\gamma(t_{j-1})) - g(z)| \\
&\leq L(\gamma|_{[t_{j-1},t_j]}) \sup\{|df(\gamma(t)) - df(\gamma(t_{j-1})| : t \in [t_{j-1}, t_j]\}.
\end{aligned}$$

The uniform continuity of $df \circ \gamma$ yields that this supremum is small whenever the partition is fine enough. The theorem then follows by writing $f(\gamma(b)) - f(\gamma(a))$ as a telescoping sum and inserting these estimates together with the obvious additivity of the length. □

Remark 7.1.3. Below, we will need a slightly more general version of the fundamental theorem: The formula $\int_\gamma df = f \circ \gamma|_a^b$ holds if f and df are continuous on K and $df(x)$ is a derivative of f at x for all but finitely many $x \in \gamma([a,b])$.

Indeed, if only the endpoints $\gamma(a)$ and $\gamma(b)$ are exceptional, this follows from a simple limiting argument, the general case is then obtained by decomposing the integral $\int_\gamma df$ into a sum.

In the proof of Proposition 7.2.5, we will have to find a rectifiable path by using the Arzelá-Ascoli theorem. It is then essential to have a "tame" parametrization which we explain briefly; more details can be found, e.g., in [56]. Given a rectifiable path $\gamma : [a,b] \to \mathbb{R}^d$ with length $L = L(\gamma)$ and length function $\ell(t) = L(\gamma|_{[a,t]})$, the function $\alpha(s) = \inf\{t \in [a,b] : \ell(t) \geq s\}$ is again increasing but not necessarily continuous, it jumps over the intervals where ℓ is constant. Nevertheless, $\tilde{\gamma} = \gamma \circ \alpha : [0,L] \to \mathbb{R}^d$ is a continuous path with $\tilde{\gamma}([0,L]) = \gamma([a,b])$ such that all path integrals along γ and $\tilde{\gamma}$ coincide and such that $L(\tilde{\gamma}|_{[0,t]}) = t$ for all $t \in [0,L]$; in particular, $\tilde{\gamma}$ is Lipschitz with constant 1. This path $\tilde{\gamma}$ is called the parametrization of γ by arclength.

If $\{\gamma_i : i \in I\}$ is a family of curves with equal length, it then follows that $\{\tilde{\gamma}_i : i \in I\}$ is equicontinuous. Moreover, Rademacher's theorem implies that $\tilde{\gamma}$ is almost everywhere differentiable and absolutely continuous.

We have seen that the behaviour of functions $f \in C^1(K)$ concerning compositions and the fundamental theorem together with its consequences is essentially as in the case of open domains. We will now present Sauter's example [114] showing that is not the case for $f \in C^1_{\mathrm{int}}(K)$.

Let C be the ternary Cantor set and U its complement in $(0,1)$. The open set Ω is constructed from $U \times (0,1)$ by removing disjoint closed balls $(B_j)_{j\in\mathbb{N}}$ that accumulate precisely at $C \times [0,1]$ and such that the sum of the diameters is less than 1/4. This implies that there exist horizontal lines in $K = \overline{\Omega}$ that do not intersect any of the balls.

If f is the Cantor function on $[0,1]$, we consider the function F defined on K by $F(x,y) = f(x)$. We have $F \in C^1_{\mathrm{int}}(K)$ because it is continuous and $dF = 0$ on $\Omega = \mathring{K}$, as f is locally constant on U. If now $\gamma : [0,1] \to K$ is the obvious left-to-right arclength parametrization of one of the horizontal lines crossing K, we have

$$\int_\gamma dF = 0$$

while

$$F(\gamma(1)) - F(\gamma(0)) = f(1) - f(0) = 1.$$

This proves $F \notin C^1(K)$. This example shows that the fundamental theorem does not hold for C^1_{int} and also reveals the catastrophe that compositions (namely $F \circ \gamma$) of C^1_{int}-functions need not be C^1_{int}.

7.2 Completeness

We study here the completeness of $(C^1(K), \|\cdot\|_{C^1(K)})$ and $(\mathcal{J}^1(K), \|\cdot\|_{\mathcal{J}^1(K)})$. We show that, if K has infinitely many connected components, then these spaces are not complete. In contrast, if K has finitely many connected components, the completeness of both spaces is characterized by a pointwise geometric condition whose uniform version goes back to Whitney in [129]. It is interesting to note that this characterization is conjectured in [34] in the context of complex differentiability.

First we consider the case of compact sets with infinitely many connected components. This is similar to [14, Theorem 2.3].

Proposition 7.2.1. *If K is a compact set with infinitely many connected components, then $(C^1(K), \|\cdot\|_{C^1(K)})$ is incomplete.*

Proof. We can partition $S_0 = K$ into two non-empty, disjoint sets S_1 and K_1, both closed and open subsets of K, such that S_1 has infinitely many connected components. Iterating this procedure we obtain a sequence $(K_j)_{j\in\mathbb{N}}$ of pairwise disjoint non-empty closed and open subsets of K.

We fix $x_j \in K_j$ and, by compactness and passing to a subsequence, we can assume that x_j converges in K. The limit x_0 cannot belong to any K_j because they are open and pairwise disjoint.

We consider the functions $f_n : K \to \mathbb{R}$ defined by $f_n(x) = |x_j - x_0|$ for $x \in K_j$ with $1 \le j \le n$ and $f_n(x) = 0$, otherwise. These functions are locally constant and hence $f_n \in C^1(K)$. It is easy to check that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C^1(K), \|\cdot\|_{C^1(K)})$.

The only possible limit is the function $f(x) = |x_j - x_0|$ for all $x \in K_j$ and $j \in \mathbb{N}$ and $f(x) = 0$ otherwise. But, for all $j \in \mathbb{N}$, we have

$$\frac{|f(x_j) - f(x_0)|}{|x_j - x_0|} = 1,$$

and since $df_n = 0$ this shows that f cannot be the limit in $C^1(K)$. □

The characterization of the completeness of $C^1(K)$ will rely on the following notion.

Definition 7.2.2. A set $K \subseteq \mathbb{R}^d$ is called *Whitney regular* if there exists $C > 0$ such that any two points $x, y \in K$ can be joined by a rectifiable path in K of length bounded by $C|x - y|$; sometimes this condition is called quasiconvexity, e.g., in the book [19].

We say that K is *pointwise Whitney regular* if, for every $x \in K$, there are a neighbourhood V_x of x and $C_x > 0$ such that any $y \in V_x$ is joined to x by a rectifiable path in K of length bounded by $C_x|x - y|$.

The inward cusp mentioned in the introduction distinguishes these two notions. If K is *geodesically bounded* (i.e. any two points can be joined by a curve of length bounded by a fixed constant) one can take $V_x = K$ in the definition so that the crucial difference is then the non-uniformity of the constants C_x.

Proposition 7.2.3. *If K is a pointwise Whitney regular compact set, then the space $(\mathcal{J}^1(K), \|\cdot\|_{\mathcal{J}^1(K)})$ is complete.*

Proof. For a Cauchy sequence $((f_j, df_j))_{j\in\mathbb{N}}$ in $\mathcal{J}^1(K)$ we get, from the completeness of $C(K)$, uniform limits f and df and we only have to show that df is a derivative of f.

Given $x \in K$ and a path γ from x to y of length $L(\gamma) \le C_x|x - y|$, the formula in the fundamental theorem of calculus immediately extends from f_j and df_j to the limits and thus gives

$$f(y) - f(x) - \langle df(x), y - x\rangle = \int_\gamma (df - df(x)).$$

The continuity of df and the bound on $L(\gamma)$ then easily imply the desired differentiability. □

To obtain the converse of this simple result we first apply the uniform boundedness principle to show that the completeness of $(C^1(K), \|\cdot\|_{C^1(K)})$ is equivalent to some bounds for the difference quotient of a function $f \in C^1(K)$. This is the same as in the case of complex differentiability [62, 14].

Proposition 7.2.4. *The following assertions are equivalent:*

1. *The space* $(\mathcal{J}^1(K), \|\cdot\|_{\mathcal{J}^1(K)})$ *is a Banach space;*

2. *The space* $(C^1(K), \|\cdot\|_{C^1(K)})$ *is a Banach space;*

3. *For every* $x \in K$*, there exists* $C_x > 0$ *such that for all* $f \in C^1(K)$ *and* $y \in K \setminus \{x\}$

$$\frac{|f(y) - f(x)|}{|y - x|} \leq C_x \|f\|_{C^1(K)}. \tag{7.4}$$

Proof. The fact that assertion 1 implies assertion 2 is a standard fact from Banach space theory. Let us show that the second assertion implies the third one. For fixed $x \in K$ and each $y \in K \setminus \{x\}$ we define a linear and continuous functional on $C^1(K)$ by

$$\Phi_y(f) = \frac{f(y) - f(x)}{|y - x|}.$$

For fixed $f \in C^1(K)$, we get a bound for $\sup_{y \in K \setminus \{x\}} |\Phi_y(f)|$ because of the differentiability at x.

The Banach-Steinhaus theorem thus gives

$$C_x = \sup\{|\Phi_y(f)| : \|f\|_{C^1(K)} \leq 1, y \in K \setminus \{x\}\} < \infty.$$

Now we assume that inequality (7.4) holds and show that $(\mathcal{J}^1(K), \|\cdot\|_{\mathcal{J}^1(K)})$ is complete. For a Cauchy sequence $((f_j, df_j))_{j \in \mathbb{N}}$ in $\mathcal{J}^1(K)$ we have uniform limits f and df. In particular, for all $\varepsilon > 0$, $x \in K$, and $p < q$ big enough, we have

$$\|f_p - f_q\|_{C^1(K)} \leq \|(f_p, df_p) - (f_q, df_q)\|_{\mathcal{J}^1(K)} < \frac{\varepsilon}{4C_x} \quad \text{and} \quad \|df_p - df\|_K < \frac{\varepsilon}{4}.$$

Now, there exists $\delta > 0$ such that, for all $y \in B(x, \delta) \setminus \{x\}$,

$$B = \frac{|f_p(y) - f_p(x) - \langle df_p(x), y - x \rangle|}{|y - x|} < \frac{\varepsilon}{4}.$$

Finally, for all such y, if q is large enough,

$$A = \frac{|(f(y) - f_q(y)) - (f(x) - f_q(x))|}{|y - x|} < \frac{\varepsilon}{4}$$

and

$$\begin{aligned}
&\frac{|f(y) - f(x) - \langle df(x), y - x \rangle|}{|x - y|} \\
&\leq A + \frac{|(f_p(y) - f_q(y)) - (f_p(x) - f_q(x))|}{|y - x|} + B + |df_p(x) - df(x)| \\
&< \varepsilon,
\end{aligned}$$

which shows that df is a derivative of f on K. $\square$

Next we show that, for connected sets K, inequality (7.4) implies pointwise regularity. This is a simple adaptation of a result in [63, Theorem 2.3.9]; we repeat the proof for the sake of completeness.

Proposition 7.2.5. *Let K be a compact connected set. If, for any $x \in K$, there exists $C_x > 0$ such that for all $f \in C^1(K)$ and $y \in K \setminus \{x\}$ we have*

$$\frac{|f(y) - f(x)|}{|y - x|} \leq C_x \|f\|_{C^1(K)}, \tag{7.5}$$

then K is pointwise Whitney regular.

Proof. For any $\varepsilon > 0$,

$$K_\varepsilon = \{x \in \mathbb{R}^d : \inf_{y \in K} |x - y| < \varepsilon\}$$

is an open connected neighbourhood of K. Let us fix $x \in K$ and define the function d_ε on $K_{2\varepsilon}$ by

$$d_\varepsilon(y) = \inf\{L(\gamma) : \gamma \text{ is a rectifiable path from } x \text{ to } y \text{ in } K_{2\varepsilon}\}.$$

Then, for fixed $y_0 \in K$, we set $u_\varepsilon(y) = \min\{d_\varepsilon(y), d_\varepsilon(y_0)\}$. If y and y' are close enough in $K_{2\varepsilon}$, we have

$$|u_\varepsilon(y) - u_\varepsilon(y')| \leq |y - y'|, \tag{7.6}$$

as any rectifiable path from x to y can be prolonged by the segment between y and y' to a rectifiable path from x to y'.

If ϕ is a positive smooth function with support in $B(0,\varepsilon)$ and integral 1, the convolution $u_\varepsilon * \phi$, defined in K_ε, is a smooth function for which $|d(u_\varepsilon * \phi)| \leq 1$ on K, because of inequality (7.6). Then, from (7.5), we have

$$|(u_\varepsilon * \phi)(x) - (u_\varepsilon * \phi)(y_0)| \leq C_x(d_\varepsilon(y_0) + 1)|x - y_0|$$

which gives us, passing to the limit $\mathrm{supp}(\phi) \to \{0\}$,

$$d_\varepsilon(y_0) \leq C_x(d_\varepsilon(y_0) + 1)|x - y_0|.$$

For $y_0 \in B(x, \frac{1}{2C_x}) \cap K$, this implies $d_\varepsilon(y_0) \leq 1$ and thus $d_\varepsilon(y_0) \leq 2C_x|x - y_0|$. Hence, there exists a rectifiable path from x to y_0 in $K_{2\varepsilon}$ of length bounded by $2C_x|x - y_0| + \varepsilon$. Using the parametrization by arclength gives an equicontinuous family of paths and the conclusion follows from the Arzelá-Ascoli Theorem 1.7.7. □

Remark 7.2.6. If the constant C_x in the previous proposition is uniform with respect to $x \in K$, then inequality (7.6) is equivalent to the Whitney regularity of K, as stated in Hörmander's book [63].

Collecting all results of this section, we have the following characterization of the completeness of $(C^1(K), \|\cdot\|_{C^1(K)})$.

Theorem 7.2.7. $(C^1(K), \|\cdot\|_{C^1(K)})$ *is complete if and only if K has finitely many components which are pointwise Whitney regular.*

Remark 7.2.8. In this pointwise Whitney regular situation, the jet space $\mathcal{J}^1(K)$ can be described as a space of continuous irrotational vector fields F on K, i.e. vector fields F for which $\int_\gamma F = 0$ for all closed rectifiable paths γ in K. More precisely, if $(f, df) \in \mathcal{J}^1(K)$, the fundamental theorem of calculus implies that df is circulation free and if F is circulation free and continuous we can define, for some fixed $x_0 \in K$, for all $x \in K$

$$f(x) = \int_\gamma F,$$

where γ is a path in K from x_0 to x. This definition makes sense as F is circulation free and F is a continuous derivative of f on K, by a similar argument as in the proof of Proposition 7.2.3.

7.3 Density of restrictions

In this section we will show that the space $C^1(\mathbb{R}^d|K)$ of restrictions of continuously differentiable functions on $\mathbb{R}^d$ to K is always dense in $C^1(K)$. As $\mathcal{D}(\mathbb{R}^d)$, the space of C^∞-functions with compact support, is dense in $C^1(\mathbb{R}^d)$; this is the same as the density of test functions restricted to K in $C^1(K)$ and again, it is advantageous to consider this question on the level of jets, that is, we will show that

$$i : \mathcal{D}(\mathbb{R}^d) \to \mathcal{J}^1(K), \, \varphi \mapsto (\varphi|_K, d\varphi|_K)$$

has dense range.

For general K, the standard approximation procedures like convolution with smooth bump functions do not apply easily, and we will use the Hahn-Banach theorem instead, see Theorem 1.7.8 and the remark below.

A continuous linear functional Φ on $\mathcal{J}^1(K) \subseteq C(K)^{d+1}$ is, by the Hahn-Banach and Riesz representation theorem, given by signed measures $\mu, \mu_1, \cdots, \mu_d$ on K via

$$\Phi(f, df) = \int f \, d\mu + \sum_{j=1}^{d} \int d_j f \, d\mu_j,$$

where $d_j f$ are the components of df. If Φ vanishes on the image of i we have, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int \varphi d\mu + \sum_{j=1}^{d} \int \partial_j \varphi d\mu_j = 0.$$

For the distributional derivatives of the measures this means that

$$\mu = \sum_{j=1}^{d} \partial_j \mu_j = \operatorname{div}(T)$$

where $T = (\mu_1, \dots, \mu_d)$ is a vector field of measures or a *charge*.

Fortunately, such charges were thoroughly investigated by Smirnov in [117]. Roughly speaking, he proved a kind of Choquet representation of charges in terms of very simple ones induced by Lipschitz paths in K. If $\gamma : [a,b] \to K$ is Lipschitz with a.e. derivative $\dot\gamma = (\dot\gamma_1, \dots, \dot\gamma_d)$ and $F = (F_1, \dots, F_d)$ is a continuous vector field, we have, as noted in section 7.2,

$$\int_\gamma F = \int_a^b \langle F(\gamma(t)), \dot\gamma(t)\rangle dt = \sum_{j=1}^{d} \int_a^b F_j(\gamma(t))\dot\gamma_j(t)dt.$$

In order to see this as the action $\langle T, F\rangle = \sum\limits_{j=1}^{d} \int F_j d\mu_j$ of a charge $T = (\mu_1, \dots, \mu_d)$, we denote by μ_j the image (or push-forward) under γ of the measure with density $\dot\gamma_j$ on $[a,b]$ so that $\int F_j(\gamma(t))\dot\gamma_j(t)dt = \int F_j d\mu_j$. For the charge $T_\gamma = (\mu_1, \dots, \mu_d)$ we then have

$$\langle T_\gamma, F\rangle = \int_\gamma F.$$

The fundamental theorem of calculus for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with derivative $d\varphi$ then gives

$$\operatorname{div}(T_\gamma)(\varphi) = -\int_\gamma d\varphi = \varphi(\gamma(a)) - \varphi(\gamma(b)) = (\delta_{\gamma(a)} - \delta_{\gamma(b)})(\varphi),$$

that is

$$\operatorname{div}(T_\gamma) = \delta_{b(\gamma)} - \delta_{e(\gamma)},$$

where $b(\gamma)$ and $e(\gamma)$ denote the beginning and the end of γ (the change of signs comes from the minus sign in the definition of distributional derivatives).

To formulate Smirnov's results we write Γ for the set of all Lipschitz paths in $\mathbb{R}^d$. Moreover, for a charge T we denote by

$$\|T\|(E) = \sup\left\{ \sum_{j\in\mathbb{N}} |T(E_j)| : (E_j)_{j\in\mathbb{N}} \text{ is a partition of } E \right\}$$

the corresponding variation measure.

Given a set $\mathcal{S}$ of charges, endowed with the Borel σ-algebra with respect to the weak topology induced by the evaluation

$$\langle (\mu_1, \dots, \mu_d), (\varphi_1, \dots, \varphi_d)\rangle = \sum_{j=1}^{d} \int \varphi_j \, d\mu_j, \ \varphi_j \in \mathcal{D}(\mathbb{R}^d),$$

a charge T is said to decompose into elements of $\mathcal{S}$ if there is a finite, positive measure ν on $\mathcal{S}$ such that

$$T = \int_{\mathcal{S}} R \, d\nu(R) \text{ and } \|T\| = \int_{\mathcal{S}} \|R\| d\nu(R),$$

where these integrals are meant in the weak sense, i.e. $\langle T, \varphi \rangle = \int_{\mathcal{S}} \langle R, \varphi \rangle d\nu(R)$ for all $\varphi \in (\mathcal{D}(\mathbb{R}^d))^d$. By density and the continuity of charges with respect to the uniform norm, this extends to all $\varphi \in (C_c(\mathbb{R}^d))^d$, where $C_c(\mathbb{R}^d)$ is the space of continuous functions with compact support.

We can now state a consequence of Smirnov's results (theorem C of [117] is somewhat more precise than we need).

Theorem 7.3.1. *Every charge T with compact support such that* $\operatorname{div}(T)$ *is a signed measure can be decomposed into elements of* Γ, *i.e. there is a positive finite measure ν on* Γ *such that*

$$T = \int_{\Gamma} T_\gamma d\nu(\gamma) \quad \textit{and} \quad \|T\| = \int_{\Gamma} \|T_\gamma\| d\nu(\gamma).$$

The decomposition of the corresponding variation measures has the important consequence that the supports of ν-almost all T_γ are contained in the support of T (where the supports are meant as the supports of signed measures which coincide with the supports of the corresponding distributions). After removing a set of ν-measure 0 we can thus assume that all paths involved in the decomposition of T have values in the support of T. Using the definition of the distributional derivative we also obtain a decomposition of the divergences:

$$\operatorname{div}(T) = \int_{\Gamma} \operatorname{div}(T_\gamma) d\nu(\gamma) = \int_{\Gamma} \delta_{b(\gamma)} - \delta_{e(\gamma)} d\nu(\gamma).$$

We are now prepared to state and prove the main result of this section.

Theorem 7.3.2. *For each compact set K, the space $C^1(\mathbb{R}^d|K)$ is dense in $C^1(K)$.*

Proof. We will show that $i : \mathscr{D}(\mathbb{R}^d) \to \mathcal{J}^1(K) : \varphi \mapsto (\varphi|_K, d\varphi|_K)$ has dense range, the conclusion then follows by projecting onto the first components.

Let us consider $\Phi \in (C(K)^{d+1})'$ such that Φ vanishes on the range of i. By the Hahn-Banach theorem it is enough to show that $\Phi|_{\mathcal{J}^1(K)} = 0$.

As explained at the beginning of this section we get signed measures μ and μ_j on K with

$$\Phi((f, f_1, \cdots, f_d)) = \int f \, d\mu + \int f_1 d\mu_1 + \cdots + \int f_d d\mu_d$$

for all $(f, f_1, \cdots, f_d) \in C(K)^{d+1}$, and $T = (\mu_1, \cdots, \mu_d)$ satisfies $\operatorname{div}(T) = \mu$. We can thus apply Theorem 7.3.1 and get a measure ν and $\mathcal{S} \subseteq \Gamma$ such that all paths in $\mathcal{S}$ have values in K and

$$T = \int_{\mathcal{S}} T_\gamma d\nu(\gamma).$$

For $(f, df) = (f, d_1 f, \ldots, d_d f) \in \mathcal{J}^1(K)$ we extend all components to $C_c(\mathbb{R}^d)$ by Tietze extension theorem and obtain from the fundamental theorem of calculus for $C^1(K)$-functions

$$\begin{aligned}
\int d_1 f \, d\mu_1 + \cdots + \int d_d f \, d\mu_d &= \langle T, df \rangle \\
&= \int_{\mathcal{S}} \langle T_\gamma, df \rangle d\nu(\gamma) \\
&= \int_{\mathcal{S}} \delta_{e(\gamma)}(f) - \delta_{b(\gamma)}(f) d\nu(\gamma) \\
&= -\operatorname{div}(T)(f) \\
&= -\int f \, d\mu,
\end{aligned}$$

which means that $\Phi|_{\mathcal{J}^1(K)} = 0$. □

The use of the Hahn-Banach theorem has the disadvantage of not giving any concrete approximations. Let us therefore very briefly mention two situations where approximations can be described explicitly.

A natural idea is to glue the local approximation given by the definition of differentiability together with a partition of unity. If K is Whitney regular, there exists $C > 0$ such that any two points $x, y \in K$ can be joined by a rectifiable path in K of length at most $C|x - y|$. For all $\delta > 0$, let $\mathscr{P}_\delta = (\varphi_j, C_j)_{j \in \mathbb{N}}$be a grid partition of unity of $\mathbb{R}^d$, such as considered in Theorem 1.4.6 of [63], made of cubes $(C_j)_{j \in \mathbb{N}}$ of diameter δ. We know there exist $C^{(1)}, C^{(2)} > 0$, independent of δ, such that

$$\sup_{\substack{x \in \mathbb{R}^d \\ k \in \{1, \ldots, d\} \\ j \in \mathbb{N}}} |\partial_k \varphi_j(x)| \le \frac{C^{(1)}}{\delta}, \tag{7.7}$$

and such that at most $C^{(2)}$ cubes C_j have no empty intersection. We set $C^* = CC^{(1)}C^{(2)}d$.

Let $f \in C^1(K)$, df be a continuous derivative of f on K and $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$\begin{aligned}
|f(y) - f(x)| &< \frac{\varepsilon}{4} \quad \forall x, y \in K \, : \, |x - y| < \lambda, \\
|df_k(y) - df_k(x)| &< \frac{\varepsilon}{4d} \quad \forall x, y \in K \, : \, |x - y| < \lambda, \, k \in \{1, \ldots, d\}, \\
|df(y) - df(x)| &< \frac{\varepsilon}{4C^*} \quad \forall x, y \in K \, : \, |x - y| < C\lambda.
\end{aligned}$$

Then, we take $\delta < \min(\frac{\varepsilon}{4(\|df\|_K+1}), \lambda)$; if $C_j \cap K \neq \emptyset$, we choose x_j in this intersection, otherwise we remove C_j from $\mathscr{P}_\delta$. The function

$$f_{\mathscr{P}_\delta} = \sum_j \varphi_j (f(x_j) + \langle df(x_j), \cdot - x_j \rangle)$$

belongs to $C^\infty(\mathbb{R}^d)$ and, for all $k \in \{1, \dots, d)$, we have

$$\partial_k f_{\mathscr{P}_\delta} = \sum_j \partial_k \varphi_j (f(x_j) + \langle df(x_j), \cdot - x_j \rangle) + \sum_j \varphi_j f_k(x_j).$$

Then, $f_{\mathscr{P}_\delta|K} \in C^1(\mathbb{R}^d|K)$ and it is easy to check, with Theorem 7.1.2, that

$$\|f_{\mathscr{P}_\delta|K} - f\|_{C^1(K)} < \varepsilon.$$

The next family of compact sets we consider is defined as follows in [47].

Definition 7.3.3. We say that $S \subseteq \mathbb{R}^d$ is *radially self-absorbing* if for each $r > 1$, we have

$$S \subseteq \overset{\circ}{(rS)}, \tag{7.8}$$

where $rS := \{rx : x \in S\}$. A set S is *locally radially self-absorbing* if any point $x \in S$ admits a radially self-absorbing neighbourhood in E.

It is easy to see that a compact set is radially self-absorbing if and only if $0 \in \overset{\circ}{K}$ and K is star-shaped from 0 in such a way that for all $x \in K$, the segment $[0, x)$ is included in $\overset{\circ}{K}$. If K is locally radially self-absorbing[1], we can cover it by finitely many radially self-absorbing sets $(S_j)_{1 \leq j \leq J}$, star-shaped from x_j. Then, if $(\varphi_j, V_j)_{1 \leq j \leq J}$ is a partition of unity associated to this covering and if $f \in C^1(K)$, then, for all $n \in \mathbb{N}$, from (7.8),

$$f_n = \sum_{j=1}^{J} \varphi_j f(\frac{1}{n+1}(n \cdot + x_j))$$

is defined and differentiable on an open neighbourhood of K, so $f_{n|K} \in C^1(\mathbb{R}^d|K)$. If $x \in V_j$, we have

$$|\frac{1}{n+1}(n \cdot + x_j) - x| \leq \frac{\operatorname{diam}(V_j)}{n+1}$$

and the convergence of $(f_{n|K})_{n \in \mathbb{N}}$ to f in $C^1(K)$ is then straightforward.

Finally, the following strategy can be applied for compact sets where finitely many points forbid to use one of the two preceding cases. Namely, if $x_1, \cdots, x_J \in K$ are such that, for any δ small enough,

$$K^{(\delta)} = K \setminus \bigcup_{j=1}^{J} B(x_j, \delta)$$

[1] In particular, K is topologically regular.

is a compact set for which an explicit approximation of $\mathcal{J}^1(K^{(\delta)})$ jets by $\mathscr{E}^1(K^{(\delta)})$ jets is known, then we build an explicit approximation for $C^1(K)$ functions by $C^1(\mathbb{R}^d|K)$ functions. Indeed, we know that for all $j \in \{1,\dots,J\}$ and $\delta > 0$, one can find $\varphi_j^{(\delta)}$ smooth function, supported on $B(x_j,\delta)$ such that $0 \le \varphi_j^{(\delta)} \le 1$ and $\varphi_j^{(\delta)} = 1$ on $\overline{B(x_j,\frac{\delta}{3})}$. Moreover, if $\varphi_0^{(\delta)} := 1 - \sum_{j=1}^{J} \varphi_j^{(\delta)}$, $\operatorname{supp}(\varphi_0^{(\delta)}) \subset K^{(\delta')}$ for all $\delta' < \frac{\delta}{3}$. Moreover, one can find $C^{(1)} > 0$ such that inequality (7.7) holds for all $j \in \{0,\dots,J\}$.

Let $f \in C^1(K)$, df be a continuous derivative of f and $\varepsilon > 0$; if $\delta > 0$ is small enough, $\delta < \frac{\varepsilon}{4(\|df\|_K+1)}$, the balls $(B(x_j,\delta))_{1\le j\le J}$ are disjoint and for all $j \in \{1,\dots,J\}$ and $x \in B(x_j,\delta) \cap K$,

$$\begin{aligned}|f(x) - f(x_j) - df(x_j)(x - x_j)| &\le \frac{\varepsilon}{4C_1}|x - x_j|,\\ |f(x) - f(x_j)| &< \frac{\varepsilon}{4},\\ |df_k(x) - df_k(x_j)| &< \frac{\varepsilon}{4d} \quad \forall\, k \in \{1,\dots,d\}.\end{aligned}$$

Then, if $\delta' < \frac{\delta}{3}$, let us take $(g;dg) \in \mathscr{E}^1(K^{(\delta')})$ such that

$$\|(f;df) - (g;dg)\|_{\mathcal{J}^1(K^{(\delta')})} < \frac{\varepsilon}{4d}\frac{\delta}{C_1}.$$

It is easy to check that if we define on K

$$f_\varepsilon = \varphi_0^{(\delta)} g + \sum_{j=1}^{J} \varphi_j^{(\delta)}(f(x_j) + \langle df(x_j), \cdot - x_j\rangle),$$

and

$$df_{\varepsilon,k} = \partial_k \varphi_0^{(\delta)} g + \varphi_0^{(\delta)} g_k + \sum_{=1}^{J} \partial_k \varphi_j^{(\delta)}(f(x_j) + \langle df(x_j), \cdot - x_j\rangle) + \sum_{j=1}^{J} \varphi_j^{(\delta)} f_k(x_j),$$

for all $k \in \{1,\dots,d\}$, then $(f_\varepsilon; df_\varepsilon) \in \mathscr{E}^1(K)$ and

$$\|(f;df) - (f_\varepsilon;df_\varepsilon)\|_{\mathcal{J}^1(K)} < \varepsilon.$$

In particular, $f_\varepsilon \in C^1(\mathbb{R}^d|K)$ and

$$\|f - f_\varepsilon\|_{C^1(K)} < \varepsilon.$$

7.4 Comparison

In this section, we compare the spaces $C^1(\mathbb{R}^d|K)$, $C^1(K)$ and $C^1_{\mathrm{int}}(K)$.

Theorem 7.4.1. $C^1(K) = C^1(\mathbb{R}^d|K)$ *with equivalent norms if and only if K has only finitely many components which are all Whitney regular.*

Proof. Assuming the stated isomorphism of normed spaces we get that $C^1(K)$ is complete and Proposition 7.2.1 implies that K has only finitely many components. Moreover, the equivalence of norms implies $\frac{|f(y)-f(x)|}{|y-x|} \leq C\|f\|_{C^1(K)}$ for some constant so that Remark 7.2.6 implies that each component is Whitney regular.

For the other implication we first note that the global Whitney condition for each of the finitely many components implies, by the mean value inequality, the equivalence of the norms $\|\cdot\|_{C^1(\mathbb{R}^d|K)}$ and $\|\cdot\|_{C^1(K)}$ on $C^1(\mathbb{R}^d|K)$. This is thus a complete and hence closed subspace of $C^1(K)$ and, on the other hand, it is dense by Theorem 7.3.2. □

If we assume, a priori, the completeness of $C^1(K)$, i.e. K has finitely many components which are pointwise Whitney regular, then the algebraic equality $C^1(K) = C^1(\mathbb{R}^d|K)$ already implies the equivalence of norms by the open mapping theorem. However, in the next section we will see that $K = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ satisfies $C^1(K) = C^1(\mathbb{R}|K)$ although $C^1(K)$ is incomplete. This means that the algebraic equality, in general, does not imply the equivalence of norms. Except for the one-dimensional case, we do not know a characterization of the algebraic equality $C^1(K) = C^1(\mathbb{R}^d|K)$. Nevertheless, we would like to remark that this property has very poor stability properties. The example of the inward directed cusp mentioned in the introduction is the union of two convex sets whose intersection is an interval (sadly, the two halfs of a broken heart behave better than the intact heart). More surprising is perhaps the following example showing that the property $C^1(K) = C^1(\mathbb{R}^d|K)$ is not stable with respect to cartesian products.

Example 7.4.2. For $M = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ and $K = M \times [0,1]$ we have $C^1(K) \neq C^1(\mathbb{R}^2|K)$.

Proof. We construct a function $f \in C^1(K)$ which is equal to 0 everywhere except for some tiny bumps on the segments $S_n = \{2^{-n}\} \times [0,1]$. More precisely, we fix $\varphi \in C^\infty(\mathbb{R})$ with support in $[-1,1]$ which is bounded in absolute value by 1, and satisfies $\varphi(0) = 1$. For $(x,y) \in S_n$ we then set $f(x,y) = n^{-3}\varphi(n^2(y - 1/n))$. It is easy to check that f is differentiable on K (the only non-obvious point is $(0,0)$ where the derivative is 0), and that one can choose a continuous derivative (because the second partial derivatives on S_n are bounded by c/n where c is a bound for the derivative of φ). Hence $f \in C^1(K)$ but $f \notin C^1(\mathbb{R}^2|K)$ because f is not Lipschitz continuous as $f(2^{-n}, 1/n) - f(2^{-n+1}, 1/n)) = n^{-3}$ which is much bigger than the distance between the arguments. □

Let us consider now a topologically regular compact set $K \subseteq \mathbb{R}^d$. We can formulate the main theorem of [129] in this context as follows.

Theorem 7.4.3. *Let K be a topologically regular compact set. If* $\mathring{K}$ *is Whitney regular, then* $C^1_{\text{int}}(K) = C^1(\mathbb{R}^d|K)$.

In Example 7.4.5 we proved that the reverse implication doesn't hold. This should be compared with a theorem of [132] about Sobolev regularity: For an open, connected, and finitely connected set $\Omega \subseteq \mathbb{R}^2$ every element of

$$\widetilde{W}^k_\infty(\Omega) = \{f \in C^{k-1}(\Omega) : \partial^\alpha f \in L_\infty(\Omega) \text{ for all } |\alpha| = k\}$$

is the restriction of a function in $W^k_\infty(\mathbb{R}^2)$ if *and only if* Ω is Whitney regular. As a preparation, we establish the following proposition.

Proposition 7.4.4. *Let K be a topologically regular compact set and assume that, for all $x \in \partial K$, there exist $C_x > 0$ and a neighbourhood V_x of x in K such that each $y \in V_x$ can be joined from x by a rectifiable path in $\mathring{K} \cup \{x,y\}$ of length bounded by $C_x|x-y|$. Then $C^1_{\text{int}}(K) = C^1(K)$.*

Proof. Let us take $f \in C^1_{\text{int}}(K)$. In order to prove that $f \in C^1(K)$, we just have to show the differentiability at $x \in \partial K$. For all $y \in V_x$ we get, from Remark 7.1.3,

$$f(y) - f(x) - \langle df(x), y-x\rangle = \int_\gamma (df - df(x)),$$

where γ is as stated in the assumptions. This is enough to get the differentiability at x, as we did previously in Proposition 7.2.3. □

We now construct a topologically regular compact connected set whose interior is not Whitney regular, but where equality $C^1_{\text{int}}(K) = C^1(\mathbb{R}^d|K)$ holds.

Example 7.4.5. Let Ω be the open unit disk in $\mathbb{R}^2$ from which we remove, as in Sauter's example, sufficiently tiny disjoint balls which accumulate precisely at $S = \{0\} \times [-\frac{1}{2}, \frac{1}{2}]$. Then $K = \overline{\Omega}$ is connected, topologically regular and Whitney regular (by the same argument as explained below). In particular, from Theorem 7.4.1, we know that $C^1(\mathbb{R}^2|K) = C^1(K)$.

Of course, $\mathring{K}$ is not Whitney regular, because S is not contained in $\mathring{K}$, but the assumptions of Proposition 7.4.4 are satisfied and hence $C^1(K) = C^1_{\text{int}}(K)$: Indeed, a boundary point x of K is either a boundary point of the unit disc, or of one of the tiny removed discs in which cases the condition is clear, or x is on the segment S. If then y is a point of $\mathring{K}$ not lying on the segment $\{0\} \times [-1,1]$, we consider the line from y to x and, whenever this line intersects one of the removed discs, we replace this intersection by a path through $\mathring{K}$ which is parallel to the boundary of the little disc. The total length increase of this new path is by a factor π. Finally, if $z \in K$ is arbitrary, we can use the preceding argument to connect z by a very short path to some y as considered before that we then connect to x.

To give a partial converse of Whitney's theorem 7.4.3, we state the following consequence of Theorem 7.2.7.

Proposition 7.4.6. *Let K be a topologically regular compact set. If $C^1_{\text{int}}(K) = C^1(K)$ (in particular, if $C^1_{\text{int}}(K) = C^1(\mathbb{R}^d|K)$ holds), then K has only finitely many connected components which are all pointwise Whitney regular.*

Proof. If $C^1_{\text{int}}(K) = C^1(K)$, then $(C^1(K), \|\cdot\|_{C^1(K)})$ is complete and hence Theorem 7.2.7 implies the stated properties of K. □

7.5 The one-dimensional case

In this last section we completely characterize the equality between the three spaces of C^1-functions for compact subsets of $\mathbb{R}$. Of course, all three spaces coincide for topologically regular compact sets with only finitely many components, and otherwise $C^1(K)$ is incomplete by Proposition 7.2.1 and thus different from $C^1_{\text{int}}(K)$. The remaining question of when $C^1(K) = C^1(\mathbb{R}|K)$ will depend on the behaviour of the bounded connected components of $\mathbb{R} \setminus K$ which we call *gaps of K*. These are thus maximal bounded open intervals G in the complement, and we denote their length by $\ell(G)$.

The simple idea is that small gaps are dangerous for the Lipschitz continuity on K which is a necessary condition for C^1-extendability. In fact, we will show that $C^1(K) \neq C^1(\mathbb{R}|K)$ whenever there are $\xi \in K$ and nearby gaps of K of length much smaller than the distance of the gap to ξ. To be precise, we define, for positive ε,

$$\sigma_\varepsilon(\xi) = \sup\left\{ \frac{\sup\{|y-\xi| : y \in G\}}{\ell(G)} : G \subseteq (\xi - \varepsilon, \xi + \varepsilon) \text{ is a gap of } K \right\},$$

with $\sup \emptyset = 0$. Of course, these $[0,\infty]$-valued functions are increasing with respect to ε and thus we can define the *gap-structure function*

$$\sigma(\xi) = \lim_{\varepsilon \to 0} \sigma_\varepsilon(\xi).$$

Theorem 7.5.1. *For a compact set $K \subseteq \mathbb{R}$ we have $C^1(K) = C^1(\mathbb{R}|K)$ if and only if $\sigma(\xi) < \infty$ for all $\xi \in K$.*

Before giving the proof let us discuss some examples. The Cantor set K satisfies $\sigma(\xi) = \infty$ for all $\xi \in K$ so that $C^1(K) \neq C^1(\mathbb{R}|K)$.

Other simple examples are sets of the form $K = \{0\} \cup \{x_n : n \in \mathbb{N}\}$ for decreasing sequences $x_n \to 0$. Then $\sigma(x_n) = 0$ for all $n \in \mathbb{N}$ and only the behaviour of $\sigma(0)$ depends on the sequence. Since the gaps of K are (x_{n+1}, x_n) we get $\sigma(0) = \limsup \frac{x_n}{x_n - x_{n+1}}$. This quantity is finite for fast sequences like $x_n = a^{-n}$ with $a > 1$ but infinite for slower sequences like $x_n = n^{-p}$ for $p > 0$.

This class of examples can be easily modified to topologically regular sets of the form $K = \{0\} \cup \bigcup_{n \in \mathbb{N}} [x_n, x_n + r_n]$. For $r_n = e^{-2n}$ we get $\sigma(0) < \infty$, e.g., for $x_n = e^{-n}$ and $\sigma(0) = \infty$ for $x_n = 1/n$.

We are now going to prove Theorem 7.5.1. We invite the reader to have Figure 7.1 in mind while discovering the proof, to have a correct image of the built and described functions.

Proof. We will use Whitney's characterization that $f \in C^1(\mathbb{R}|K)$ if and only if, for all non-isolated $\xi \in K$,

$$\lim_{x,y\to\xi} \frac{f(x)-f(y)}{x-y} = f'(\xi),$$

see [128]. Let us first assume $\sigma(\xi) = \infty$ for some $\xi \in K$. There is thus a sequence of gaps $G_n = (a_n, b_n) \subseteq (\xi - 1/n, \xi + 1/n)$ with $\sup\{|y-\xi| : y \in G_n\}/|a_n - b_n| > 2n$. Passing to a subsequence, we may assume that all these gaps are on the same side of ξ, say $\xi < a_n < b_n$, so that $b_n - \xi > 2n(b_n - a_n)$.

Moreover, again by passing to a subsequence and using $\sigma_\varepsilon(\xi) = \infty$ for $\varepsilon = (b_n - a_n)/2$, we can reach $b_{n+1} < a_n$ and that the midpoints $y_n = (a_n + b_n)/2$ of the gaps satisfy

$$\frac{y_n - y_{n+1}}{b_n - a_n} \geq n.$$

We now define $f : K \to \mathbb{R}$ by $f(x) = (y_n - \xi)/n$ for $x \in K \cap (y_n, y_{n-1})$ (with $y_0 = \infty$) and $f(x) = 0$ for $x \leq \xi$. Since the jumps of f are outside K it is clear that f is differentiable at all points $x \in K \setminus \{\xi\}$ with $f'(x) = 0$. To show the differentiability at ξ with $f'(\xi) = 0$ we calculate for $x \in K \cap (y_n, y_{n-1})$,

$$\left|\frac{f(x)-f(\xi)}{x-\xi}\right| = \left|\frac{(y_n-\xi)/n}{x-\xi}\right| \leq \left|\frac{(y_n-\xi)/n}{y_n-\xi}\right| \leq \frac{1}{n}.$$

Thus, $f \in C^1(K)$ but $f \notin C^1(\mathbb{R}|K)$ because

$$\frac{f(b_n)-f(a_n)}{b_n-a_n} = \frac{(y_n-\xi)/n-(y_{n+1}-\xi)/(n+1)}{b_n-a_n} \geq \frac{(y_n-y_{n+1})/n}{b_n-a_n} \geq 1.$$

Let us now assume $\sigma(\xi) < \infty$ for all $\xi \in K$. To prove that every $f \in C^1(K)$ belongs to $C^1(\mathbb{R}|K)$, we first show that we can assume $f' = 0$. Indeed, we extend $f' : K \to \mathbb{R}$ to a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ and consider $g(x) = f(x) - \int_0^x \varphi(t)dt$. Then $g \in C^1(K)$ satisfies $g' = 0$ and $g \in C^1(\mathbb{R}|K)$ implies $f \in C^1(\mathbb{R}|K)$.

Let us thus fix $f \in C^1(K)$ with $f' = 0$. We have to show Whitney's condition stated above at any non-isolated point ξ which, for notational convenience, we may assume to be $\xi = 0$. We fix $c > \max\{\sigma(0), 1\}$ and $\varepsilon \in (0,1)$. There is thus $\delta > 0$ such that, because of the differentiability at $\xi = 0$ with $f'(0) = 0$, we have

$$\left|\frac{f(x)-f(0)}{x-0}\right| < \frac{\varepsilon}{2c}, \tag{7.9}$$

for all $x \in K$ with $|x| < \delta$ and, because of $\sigma_\delta(\xi) < c$ for small enough δ,

$$\sup\{|y| : y \in G\} \leq c\ell(G),$$

for all gaps $G \subseteq (-\delta, \delta)$. For $x, y \in K \cap (-\delta, \delta)$ we will show

$$\left| \frac{f(x) - f(y)}{x - y} \right| \le \varepsilon.$$

If x, y are in the same component of K this quotient is 0 because f is locally constant. Moreover, if x, y are on different sides of 0, the quotient is bounded by ε because of (7.9) and $c \ge 1$. It remains to consider the case $0 < x < y$. Then there is a gap G between x and y and, since f is locally constant, we may decrease y so that $y \in \partial K$ without changing $f(y)$ which thus increases the difference quotient we have to estimate. This implies that y is the endpoint of gap $G = (a, y)$ with $a \ge x$, which leads to

$$|y - x| \ge |y - a| = \ell(G) \ge y/c \ge x/c.$$

Therefore,

$$\begin{aligned} \left| \frac{f(x) - f(y)}{x - y} \right| &\le \left| \frac{f(x) - f(0)}{x - y} \right| + \left| \frac{f(y) - f(0)}{x - y} \right| \\ &\le c \left| \frac{f(x) - f(0)}{x - 0} \right| + c \left| \frac{f(y) - f(0)}{y - 0} \right| \le \varepsilon. \end{aligned}$$ □

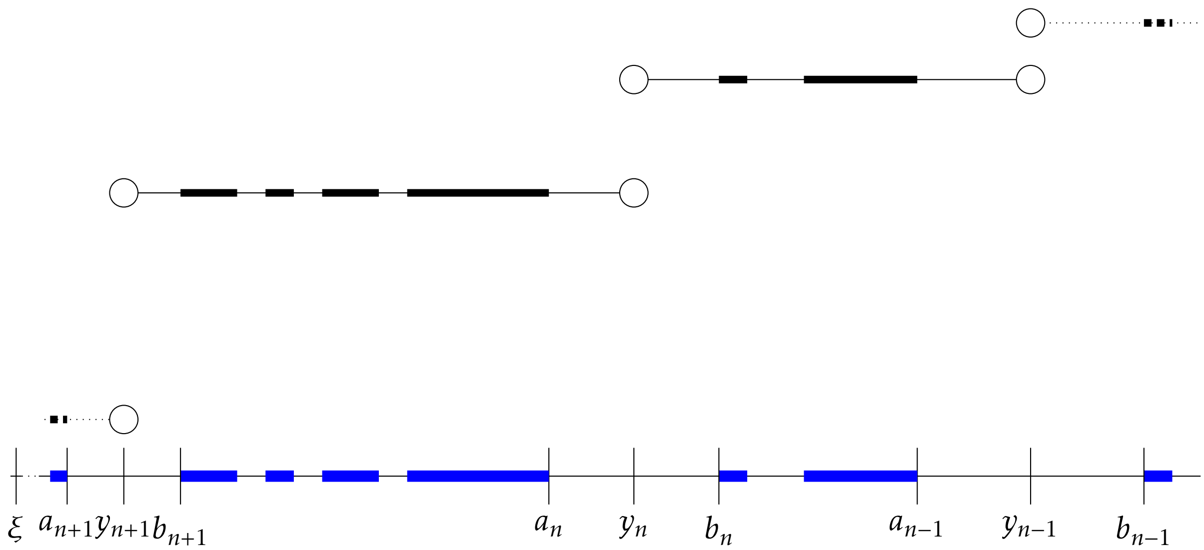


Figure 7.1: Part of the allure of function which causes the inequality $C^1(\mathbb{R}|K) \ne C^1(K)$. The compact set K, partially drawn in blue, is included in $\mathbb{R} \setminus \bigcup_n (a_n, b_n)$. The value of the function on K, in thick black, is determined by jumps occurring outside of K, in $y_n = (a_n + b_n)/2$.

Index

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