A Journey through Categories

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Young Mathematicians Symposium of the Greater Region

September 2018
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A **category** $C$ is a pair $(O, H)$ where $O$ is a set of objects and $H$ a set of arrows (or morphisms) between objects such that

- if $f : o \rightarrow o'$ and $g : o' \rightarrow o''$ then exists $h = g \circ f : o \rightarrow o''$. 
A category $C$ is a pair $(O, H)$ where $O$ is a set of objects and $H$ a set of arrows (or morphisms) between objects such that

- if $f : o \rightarrow o'$ and $g : o' \rightarrow o''$ then exists $h = g \circ f : o \rightarrow o''$.
- exists $1_o : o \rightarrow o$ such that $f \circ 1_o = f$ and $1_o \circ f = f$ (the identity arrow)
A category $C$ is a pair $(O, H)$ where $O$ is a set of objects and $H$ a set of arrows (or morphisms) between objects such that

- if $f : o \to o'$ and $g : o' \to o''$ then exists $h = g \circ f : o \to o''$.
- exists $1_o : o \to o$ such that $f \circ 1_o = f$ and $1_o \circ f = f$ (the identity arrow)
- the composition is associative.
Examples of categories:

- **Set** = sets with functions as arrows,
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- **Top’** = topological spaces with functions,
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Examples of categories:

- **Set** = sets with functions as arrows,
- **Top** = topological spaces with continuous functions,
- **Top’** = topological spaces with functions,
- **Group** = groups with homomorphisms,
- **Cat** = Categories with functors.
A functor $F$ between two categories $\mathbf{C}$ and $\mathbf{D}$ is a map such that

- $\forall o \in \mathbf{C}, F(o) \in \mathbf{D}$
A **functor** $F$ between two categories $\mathbf{C}$ and $\mathbf{D}$ is a map such that

- $\forall o \in \mathbf{C}, F(o) \in \mathbf{D}$
- if $f : o \rightarrow o'$ is an arrow in $\mathbf{C}$, then

Two categories $\mathbf{C}$ and $\mathbf{D}$ are (dually) equivalent if there are (contravariant) covariant functors $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{D} \rightarrow \mathbf{C}$ such that for every $o \in \mathbf{C}$ and every $p \in \mathbf{D}$, $o \sim = G(F(o))$ and $p \sim = F(G(p))$. 
A **functor** $F$ between two categories $C$ and $D$ is a map such that

- $\forall o \in C$, $F(o) \in D$

- if $f : o \rightarrow o'$ is an arrow in $C$, then
  - $F(f) : F(o) \rightarrow F(o')$ is an arrow in $D$ (covariant functor)
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- $\forall o \in \mathbf{C}, F(o) \in \mathbf{D}$
- if $f : o \rightarrow o'$ is an arrow in $\mathbf{C}$, then
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  - $F(f) : F(o') \rightarrow F(o)$ is an arrow in $\mathbf{D}$ (contravariant functor).
A functor $F$ between two categories $C$ and $D$ is a map such that

- $\forall o \in C, F(o) \in D$
- if $f : o \longrightarrow o'$ is an arrow in $C$, then
  - $F(f) : F(o) \longrightarrow F(o')$ is an arrow in $D$ (covariant functor)
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Two categories $C$ and $D$ are (dually) equivalent if there are (contravariant) covariant functors $F : C \longrightarrow D$ and $G : D \longrightarrow C$ such that for every $o \in C$ and for every $p \in D$

$$o \cong G(F(o)) \text{ and } p \cong F(G(p)).$$
A lattice is an ordered set \((L, \leq)\) such that every two elements \(a, b \in L\) have a unique supremum \(a \lor b\) and an unique infimum \(a \land b\).
A **Boolean algebra** is a lattice $B$ with the following conditions

1. $B$ is distributive: meaning that $a \land (b \lor c) = (a \land b) \lor (a \land c)$,
2. $B$ has a top element 1 and a bottom element 0,
3. For every $a \in B$, there exists an unique $\neg a \in B$ such that $a \land \neg a = 0$ and $a \lor \neg a = 1$. 

![Diagram of Boolean algebra](image)
In the beginning

M.H. Stone
In the beginning

\[ \text{BA} \text{Alg} = \text{Boolean algebras with Boolean morphisms} \]

\[ \text{Stone} = \text{zero-dimensional compact Hausdorff spaces with continuous functions} \]
In the beginning

**Theorem**

$\text{BAlg}$ is dually equivalent to $\text{Stone}$

$\text{BAlg} \cong \text{Stone}$
Theorem
\( \text{BAlg} \) is dually equivalent to \( \text{Stone} \)

\[ \begin{align*}
X \in \text{Stone} & \quad \rightarrow \quad B_X = \text{Clp}(X) \in \text{BAlg} \\
B \in \text{Bal} & \quad \rightarrow \quad X_B = \text{Ult}(B) \in \text{Stone}
\end{align*} \]
First extension

**K Haus** = Compact Hausdorff spaces with continuous functions (We dropped the zero-dimensional property).

\[
\overset{?}{\longrightarrow} \overset{\uparrow}{\longrightarrow} \overset{\uparrow}{\longrightarrow} \quad \text{BAlg} \quad \text{─} \quad \text{Stone}
\]
Several answers

\[ \text{KRFrm} \quad \text{BAlg} \quad \text{KHAus} \quad \text{Stone} \quad \text{DeV} \]

(KRFrm (By Isbell) = Compact regular frames with frames homomorphism
DeV (By DeVries) = DeVries algebras with DeVries morphisms)
Several answers

\[ \text{KR Frm (By Isbell)} = \text{Compact regular frames with frames homomorphism} \]
\[ \text{Dev (By de Vries)} = \text{de Vries algebras with de Vries morphisms} \]
**Theorem**

*KHaus* is dually equivalent to *KR Frm*

\[ X \in KHaus \implies L_X = \Omega(X) \in KR Frm \]

\[ L \in KR Frm \implies X_L = pt(L) \in KHaus \]
Definition

A de Vries algebra is a pair \((B, \prec)\) where \(B\) is a complete Boolean algebra and \(\prec\) is a binary relation on \(B\) satisfying

\begin{align*}
&dV1 \quad 1 \prec 1, \\
&dV2 \quad a \prec b \Rightarrow a \leq b, \\
&dV3 \quad a \leq b \prec c \leq d \Rightarrow a \prec d, \\
&dV4 \quad a \prec b, c \Rightarrow a \prec b \land c, \\
&dV5 \quad a \prec b \Rightarrow \neg b \prec \neg a, \\
&dV6 \quad a \prec b \Rightarrow \exists c : a \prec c \prec b, \\
&dV7 \quad a \neq 0 \Rightarrow \exists b \neq 0 : b \prec a.
\end{align*}
Theorem

*KHaus* is dually equivalent to *DeV*

\[ X \in KHaus \rightarrow B_X = RO(X) \in DeV \]

where \( U \in RO(X) \) if and only if \( U = U^{-\circ} \).
Theorem

**KHaus** is dually equivalent to **Dev**

\[ X \in KHaus \longrightarrow B_X = \mathcal{RO}(X) \in DeV \]

where \( U \in \mathcal{RO}(X) \) if and only if \( U = U^{-\circ} \).

On \( \mathcal{RO}(X) \) we have the following operations

1. \( 1 = X \) and \( 0 = \emptyset \)
2. \( U \lor O = (U \cup O)^{-\circ} \)
3. \( U \land O = U \cap O \),
4. \( \neg U = U^{c\circ} \)
5. \( U \prec O \iff U^{-} \subseteq O \)
Definition

If \((B, \prec)\) is a de Vries algebra, a \textbf{round filter} \(F\) of \((B, \prec)\) is a lattice filter such that

\[
a \in F \Rightarrow \exists b \in F : b \prec a.
\]

An \textbf{end} is a round filter maximal among the set of round filters. We denote \(\mathcal{E}(B)\) the set of ends of \((B, \prec)\).

\[
B \in \text{DeV} \longrightarrow X_B = \mathcal{E}(B) \in \text{KHaus}.
\]
Review of the situation

It is possible to determine a compact Hausdorff space $X$ thanks to

1. its associated frame of open sets $\Omega(X)$.
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2. its associated de Vries algebra of regular open sets $R\mathcal{O}(X)$.
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It is possible to determine a compact Hausdorff space $X$ thanks to

1. its associated frame of open sets $\Omega(X)$.
2. its associated de Vries algebra of regular open sets $\mathcal{RO}(X)$.

Can we find something else which determine a compact Hausdorff space?
Uniformly complete bounded Archimedean $\ell$-algebras

$\text{ubal}(\text{By Bezhanishvili}) = \text{Uniformly complete bounded Archimedean } \ell\text{-algebra with morphisms with the right properties.}$
Uniformly complete bounded Archimedean $\ell$-algebras

Definition
An $\ell$-algebra is an algebra $(U, \cdot, +, \land, \lor, 0, 1, r \cdot)$ such that:

1. $(U, \land, \lor)$ is a lattice,
2. $(U, \cdot, +, 0, 1)$ is a ring,
3. $(U, +, 0, r \cdot)$ is a linear space on $\mathbb{R}$,
4. $a \leq b \Rightarrow a + c \leq b + c$,
5. $0 \leq a, b \Rightarrow 0 \leq a \cdot b$,
6. $U \ni a \geq 0, \mathbb{R} \ni r \geq 0 \Rightarrow r \cdot a \geq 0$. 
Uniformly complete bounded Archimedean $\ell$-algebras

**Theorem**

*K Haus is dually equivalent to ubal*

\[
X \in \text{K Haus} \quad \rightarrow \quad U_X = C(X, \mathbb{R}) \in \text{ubal}
\]

\[
U \in \text{KRFrm} \quad \rightarrow \quad X_U = \text{Max}_\ell(U) \in \text{K Haus}
\]
Theorem

*KHaus* is dually equivalent to *ubal*

\[
\begin{align*}
X \in KHaus & \longrightarrow U_X = C(X, \mathbb{R}) \in ubal \\
U \in KRfrm & \longrightarrow X_U = \text{Max}_\ell(U) \in KHaus
\end{align*}
\]

Definition

Let \( U \in ubal \), then \( I \subseteq U \) is an \( \ell \)-ideal if

1. \( I \) is a ring ideal,
2. \( I \) is convex,
3. \( I \) is closed for \( \vee \).
Second extension

StKFrṃ → KRFrm → KHaus → Stone → ubal → PrFrm

StKSp
Second extension
Definition
A topological space $X$ is **stably compact** if it is compact, locally compact, sober and such that saturated compacts are stable by finite intersection.
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A function $f : X \rightarrow Y$ between topological spaces is **proper** if the inverse image of a compact saturated set is a compact saturated set.
Second extension

Definition
A topological space $X$ is **stably compact** if it is compact, locally compact, sober and such that saturated compacts are stable by finite intersection.

A function $f : X \rightarrow Y$ between topological spaces is **proper** if the inverse image of a compact saturated set is a compact saturated set.

$\text{StKSp} = \text{Stably compact spaces with proper continuous functions.}$
Let's shift the problem

Theorem
(Folklore) \( \text{StKSp} \) is equivalent to \( \text{KPSp} \)
Compact po-spaces

Definition
A compact po-space is a pair \((X, \leq)\) where \(X\) is a compact space and where \(\leq\) is an order relation closed in \(X \times X\).
Compact po-spaces

**Definition**
A **compact po-space** is a pair \((X, \leq)\) where \(X\) is a compact space and where \(\leq\) is an order relation closed in \(X \times X\).

\(KPSp =\) compact po-spaces with continuous increasing functions.
Why the modification?

**Theorem**

*If* $X$ *is a compact po-space, then* $X$ *is homeomorphic to* $\text{Con}(I(X, \mathbb{R}^+))$, *where*

- $I(X, \mathbb{R}^+)$ *is the set of the continuous increasing functions from* $X$ *to* $\mathbb{R}^+$,
- $\text{Con}(I(X, \mathbb{R}^+))$ *is the set of maximal congruences on* $X$. 


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**Theorem**

If $X$ is a compact po-space, then $X$ is homeomorphic to $\text{Con}(I(X, \mathbb{R}^+))$, where

- $I(X, \mathbb{R}^+)$ is the set of the continuous increasing functions from $X$ to $\mathbb{R}^+$,
- $\text{Con}(I(X, \mathbb{R}^+))$ is the set of maximal congruences on $X$.

The duality should be

$$X \in \text{KPSp} \quad \rightarrow \quad I(X, \mathbb{R}^+) \in \mathbb{R}^+,$$

$$A \in \mathbb{R}^+ \quad \rightarrow \quad \text{Con}(A) \in \text{KPSp}.$$
\[ X \in \text{KPSp} \rightarrow I(X, \mathbb{R}^+) \in \mathcal{E} \]

\[ A \in \mathcal{E} \rightarrow \text{Con}(A) \in \text{KPSp} \]

**To do list:**

1. Find a category axiomatizing \( I(X, \mathbb{R}^+) \) \( \in \mathcal{E} \)
2. Generalize the category \( \text{ubal} \)
3. Establish a duality between this category and \( \text{KPSp} \)
4. Prove that this duality extends the one between \( \text{ubal} \) and \( \text{KHaus} \)
\[ X \in \text{KPSp} \rightarrow l(X, \mathbb{R}^+) \in ? \]
\[ A \in ? \rightarrow \text{Con}(A) \in \text{KPSp} \]

**To do list:**

1. find a category axiomatizing \( l(X, \mathbb{R}^+) \)
\[ X \in \text{KPSp} \longrightarrow I(X, \mathbb{R}^+) \in ? \]
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1. find a category axiomatizing \( I(X, \mathbb{R}^+) \)
2. generalize the category \text{ubal}
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\[ X \in KPSp \longrightarrow I(X, \mathbb{R}^+) \in ? \]
\[ A \in ? \longrightarrow \text{Con}(A) \in KPSp \]

**To do list**:

1. Find a category axiomatizing \( I(X, \mathbb{R}^+) \)
2. Generalize the category \textbf{ubal}
3. Establish a duality between this category and \textbf{KPSp}
4. Prove that this duality extends the one between \textbf{ubal} and \textbf{KHaus}
1. Axiomatizing

**Definition**

An \( \ell\text{-semi-ring} \) is an algebra \((A, +, \cdot, 0, 1, \land, \lor)\) such that:

1. \((A, \land, \lor)\) is a lattice,
2. \((A, +, 0)\) and \((A, \cdot, 1)\) are commutative monoids,
3. \((A, +, \cdot)\) is distributive,
4. \(a \leq b \iff a + c \leq b + c\),
5. \(a \geq 0\),
6. \(a \leq b \Rightarrow a \cdot c \leq b \cdot c\).
1. Axiomatizing

Definition

1. An $\ell$-semi-ring $A$ is **bounded** if

\[
a \in A \Rightarrow \exists n \in \mathbb{N} : a \leq 1 + \ldots + 1
\]

n times
1. Axiomatizing

Definition

1. An $\ell$-semi-ring $A$ is **bounded** if

\[ a \in A \Rightarrow \exists n \in \mathbb{N} : a \leq 1 + \ldots + 1 := n \cdot 1. \]
1. Axiomatizing

Definition

1. An $\ell$-semi-ring $A$ is **bounded** if

   $a \in A \Rightarrow \exists n \in \mathbb{N} : a \leq 1 + \ldots + 1 := n \cdot 1.$

2. An $\ell$-semi-ring is **Archimedean** if

   $((\forall n \in \mathbb{N})n \cdot a + b \leq n \cdot c + d) \Rightarrow a \leq c.$
1. Axiomatizing

Definition

1. An \(\ell\)-semi-ring \(A\) is **bounded** if

\[ a \in A \Rightarrow \exists n \in \mathbb{N} : a \leq 1 + \ldots + 1 := n \cdot 1. \]

2. An \(\ell\)-semi-ring is **Archimedean** if

\[ ((\forall n \in \mathbb{N}) n \cdot a + b \leq n \cdot c + d) \Rightarrow a \leq c. \]

3. An \(\ell\)-**semi-algebra** is an \(\ell\)-semi-ring which is also an \(\mathbb{R}^+\)-algebra.
1. Axiomatizing

**Theorem**

If $X \in KPSp$ then $I(X, \mathbb{R}^+)$ is a bounded Archimedean $\ell$-semi-algebra.
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**Theorem**

If $X \in KPSp$ then $I(X, \mathbb{R}^+)$ is a bounded Archimedean $\ell$-semi-algebra.

**To do list:**

1. find a category axiomatizing $I(X, \mathbb{R}^+)$
2. generalize the category ubal
3. establish a duality between this category and KPSp
4. Prove that this duality extends the one between ubal and KHaus
1. Axiomatizing

**Theorem**

If $X \in KPSp$ then $I(X, \mathbb{R}^+)$ is a bounded Archimedean $\ell$-semi-algebra.

**To do list:**

1. find a category axiomatizing $I(X, \mathbb{R}^+) : I(X, \mathbb{R}^+) \in \text{sbal}$
2. generalize the category \text{ubal}
3. establish a duality between this category and $KPSp$
4. Prove that this duality extends the one between $\text{ubal}$ and $KHaus$
2. Generalizing

We have to determine the functors between \texttt{ubal} and \texttt{sbal}
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$\triangleright$ $(\textbf{ubal} \rightarrow \textbf{sbal})$

\[ U \in \textbf{ubal} \rightarrow U^+ := \{ a \in U : a \geq 0 \} \in \textbf{sbal}. \]
2. Generalizing

We have to determine the functors between \( \text{ubal} \) and \( \text{sbal} \)

\begin{itemize}
  \item \((\text{ubal} \rightarrow \text{sbal})\)
    \[ U \in \text{ubal} \rightarrow U^+ := \{a \in U : a \geq 0\} \in \text{sbal}. \]
  
  \item \((\text{sbal} \rightarrow \text{ubal})\)
    \[ A \in \text{sbal} \rightarrow A^u := (A \times A/ \sim) \in \text{bal} \]
    where
    \[ (a, b) \sim (c, d) \iff a + d = b + c. \]
\end{itemize}
2. Generalizing

**Theorem**
*If* $U \in \text{ubal}$, *then*

$$U \cong \left( U^+ \times U^+ \right) / \sim .$$
2. Generalizing

**Theorem**
If $U \in \text{ubal}$, then

$$U \cong (U^+ \times U^+)/\sim.$$ 

**To do list**:
1. find a category axiomatizing $I(X, \mathbb{R}^+)$
2. generalize the category $\text{ubal}$
3. establish a duality between this category and $\text{KPSp}$
4. Prove that this duality extends the one between $\text{ubal}$ and $\text{KHaus}$
3. Dualizing

We already know that if $X \in \text{KPSp}$, then

$$X \cong \text{Con}(I(X, \mathbb{R}^+)).$$
3. Dualizing

We already know that if \( X \in \text{KPSp} \), then

\[ X \cong \text{Con}(I(X, \mathbb{R}^+)). \]

On the other side, if \( A \in \text{sbal} \), we do not have

\[ A \cong I(\text{Con}(A), \mathbb{R}^+). \]
3. Dualizing

Theorem
If $A \in \textbf{sbal}$ then $A \cong I(\text{Con}(A), \mathbb{R}^+)$ if and only if

1. $A$ is complete for the uniform norm and
2. $A$ has the difference with constants property, i.e.

$$ (\forall r \in \mathbb{R})(\forall a \in A)(r \cdot 1 \leq a \Rightarrow \exists b \in A : a = b + r \cdot 1). $$
3. Dualizing

Theorem
If \( A \in \textbf{sbal} \) then \( A \cong I(\text{Con}(A), \mathbb{R}^+) \) if and only if

1. \( A \) is complete for the uniform norm and
2. \( A \) has the difference with constants property, i.e.

\[
(\forall r \in \mathbb{R})(\forall a \in A)(r \cdot 1 \leq a \Rightarrow \exists b \in A : a = b + r \cdot 1).
\]

Definition
\textbf{usbal} = uniformly complete bounded Archimedean \( \ell \)-semi-algebra with the difference with constants property with the right morphisms.
3. Dualizing

**Remark**: The category \texttt{usbal} still works for the first (Axiomatizing) and second (Generalizing) point!
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**Remark**: The category \texttt{usbal} still works for the first (Axiomatizing) and second (Generalizing) point!

**To do list**: 

1. find a category axiomatizing \( l(X, \mathbb{R}^+) \)
2. generalize the category \texttt{ubal}
3. establish a duality between this category and \texttt{KPSp}
4. Prove that this duality extends the one between \texttt{ubal} and \texttt{KHaus}
4. Extending

$\text{KHaus} \rightarrow \text{KPSp}$

$\text{ubal} \rightarrow \text{usbal}$
4. Extending

\[ \text{K} \text{Haus} \rightarrow \text{KPSp} \]
\[ \quad \text{ubal} \rightarrow \text{usbal} \]

1. If \( X \in \text{KHaus} \), then

\[ C(X, \mathbb{R})^+ = I(X, \mathbb{R}^+) \]
4. Extending

\[ \text{KHAus} \rightarrow \text{KPSp} \]
\[ \text{ubal} \rightarrow \text{usbal} \]

1. If \( X \in \text{KHAus} \), then

\[ C(X, \mathbb{R})^+ = I(X, \mathbb{R}^+) \]

2. If \( U \in \text{ubal} \), then

\[ \text{Max}_\ell(U) \cong \text{Con}((U^+ \times U^+)/ \sim). \]
Completed square

StKFr

KRFrm

DeV

PrFrm

StKSp

KHaus

BAIg

Stone

ubal

usbal

PrFrm

usbal
Further problems

1. We know how to go from \texttt{usbal} to \texttt{KPSp} and from \texttt{KPSp} to \texttt{StKSp}. Is there a way to bypass this and go directly from \texttt{usbal} to \texttt{StKSp}? (We can ask the existence of the other compositions as well)
Further problems

1. We know how to go from $\text{usbal}$ to $\text{KPSp}$ and from $\text{KPSp}$ to $\text{StKSp}$. Is there a way to bypass this and go directly from $\text{usbal}$ to $\text{StKSp}$? (We can ask the existence of the other compositions as well)

2. $X \in \text{KHaus}$ can be determined by the set $C(X, \mathbb{R}) \in \text{usbal}$, which is mainly a $\mathbb{R}$-linear space, but also by $C(X, \mathbb{C}) \in C^*-\text{alg}$, which is a $\mathbb{C}$-linear space. Do we have a complex counterpart of $I(X, \mathbb{R}^+)$?

\[
\begin{align*}
\text{KHaus} & \rightarrow \text{KPSp} \\
\text{ubal} & \rightarrow \text{usbal}
\end{align*}
\]
Further problems

3. What if we change the way we extended the original problem? Consider

\( \text{DLat} = \text{distributive lattices with lattices morphisms} \)
(We dropped the complemented property)

\[ \text{Priest}(\text{Priestley}) = \text{Priestley spaces with continuous increasing functions.} \]
Further problems

4. If we consider \((B, \prec) \in \text{DeV}\), then the Boolean component \(B \in \text{BAlg}\) has a dual \(X \in \text{Stone}\) and the relation \(\prec\) can be associated to a binary relation \(R\) on \(X\).
Further problems

4. If we consider $(B, \prec) \in \text{DeV}$, then the Boolean component $B \in \text{BAlg}$ has a dual $X \in \text{Stone}$ and the relation $\prec$ can be associated to a binary relation $R$ on $X$. Then, there is a duality between $\text{DeV}$ and the category representing $(X, R)$. 
Further problems

4. If we consider $(B, \prec) \in \text{DeV}$, then the Boolean component $B \in \text{BAalg}$ has a dual $X \in \text{Stone}$ and the relation $\prec$ can be associated to a binary relation $R$ on $X$. Then, there is a duality between $\text{DeV}$ and the category representing $(X, R)$. Do we have the same behaviour if we consider $(L, \prec)$ where $L$ is a distributive lattice and $\prec$ is an adequate binary relation on $L$?
Further problems

4. If we consider \((B, \preceq) \in \text{DeV}\), then the Boolean component \(B \in \text{BAlg}\) has a dual \(X \in \text{Stone}\) and the relation \(\preceq\) can be associated to a binary relation \(R\) on \(X\).

Then, there is a duality between \(\text{DeV}\) and the category representing \((X, R)\).

Do we have the same behaviour if we consider \((L, \preceq)\) where \(L\) is a distributive lattice and \(\preceq\) is an adequate binary relation on \(L\) ?

5. \(\text{DeV}\) can be considered as a extension of modal algebras. Do we have a logic associated to this category ?
Red panda