A Journey through Categories

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Young Mathematicians Symposium of the Greater Region

September 2018

Preamble - Category theory

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- exists $1_o : o \longrightarrow o$ such that $f \circ 1_o = f$ and $1_o \circ f = f$ (the identity arrow)

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- the composition is associative.

Preamble - Category theory

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Examples of categories :

- **Set** = sets with functions as arrows,
- **Top** = topological spaces with continuous functions,
- **Top'** = topological spaces with functions,
- **Group** = groups with homomorphisms,
- **Cat** = Categories with functors.

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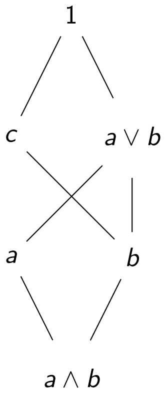
Two categories $\mathbf{C}$ et $\mathbf{D}$ are **(dually) equivalent** if there are (contravariant) covariant functors $F : \mathbf{C} \longrightarrow \mathbf{D}$ and $G : \mathbf{D} \longrightarrow \mathbf{C}$ such that for every $o \in \mathbf{C}$ and for every $p \in \mathbf{D}$

$$o \cong G(F(o)) \text{ and } p \cong F(G(p)).$$

Preamble - Lattice theory

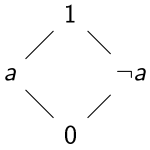
A **lattice** is an ordered set $(L, \leq)$ such that every two elements $a, b \in L$ have an unique supremum $a \vee b$ and an unique infimum $a \wedge b$.



Preamble - Lattice theory

A **Boolean algebra** is a lattice B with the following conditions

1. B is distributive : meaning that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
2. B has a top element 1 and a bottom element 0,
3. For every $a \in B$, there exists an unique $\neg a \in B$ such that $a \wedge \neg a = 0$ and $a \vee \neg a = 1$.



In the beginning

M.H. Stone

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BAlg = Boolean algebras with Boolean morphisms
Stone = zero-dimensional compact Hausdorff spaces with continuous functions

In the beginning

Theorem

BAlg *is dually equivalent to* ***Stone***

BAlg — ***Stone***

In the beginning

Theorem

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$$\boldsymbol{BAlg} \text{ — } \boldsymbol{Stone}$$

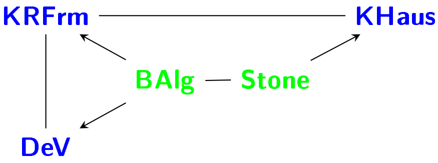
$$X \in \textbf{Stone} \longrightarrow B_X = \mathrm{Clp}(X) \in \textbf{BAlg}$$
$$B \in \textbf{Balg} \longrightarrow X_B = \mathrm{Ult}(B) \in \textbf{Stone}$$

First extension

KHaus = Compact Hausdorff spaces with continuous functions
(We dropped the zero-dimensional property).

$$\begin{array}{ccc} \textbf{?} & \text{- -} & \textbf{KHaus} \\ \uparrow & & \uparrow \\ \textbf{BAlg} & \text{—} & \textbf{Stone} \end{array}$$

Several answers



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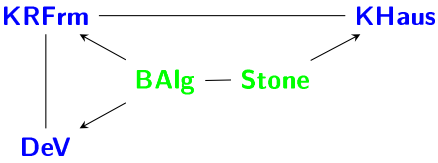

KRFrm (By Isbell) = Compact regular frames with frames homomorphism
DeV (By de Vries) = de Vries algebras with de Vries morphisms

Theorem

KHaus *is dually equivalent to* ***KRFrm***

$$X \in \mathbf{KHaus} \longrightarrow L_X = \Omega(X) \in \mathbf{KRFrm}$$
$$L \in \mathbf{KRFrm} \longrightarrow X_L = \mathrm{pt}(L) \in \mathbf{KHaus}$$

Definition

A **de Vries algebra** is a pair $(B, \prec)$ where B is a complete Boolean algebra and $\prec$ is a binary relation on B satisfying

dV1 $1 \prec 1$,

dV2 $a \prec b \Rightarrow a \leq b$,

dV3 $a \leq b \prec c \leq d \Rightarrow a \prec d$,

dV4 $a \prec b, c \Rightarrow a \prec b \wedge c$,

dV5 $a \prec b \Rightarrow \neg b \prec \neg a$,

dV6 $a \prec b \Rightarrow \exists c : a \prec c \prec b$,

dV7 $a \neq 0 \Rightarrow \exists b \neq 0 : b \prec a$.

Theorem

KHaus *is dually equivalent to* ***DeV***

$$X \in \textbf{KHaus} \longrightarrow B_X = \mathcal{RO}(X) \in \textbf{DeV}$$

where $U \in \mathcal{RO}(X)$ if and only if $U = U^{-\circ}$

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where $U \in \mathcal{RO}(X)$ if and only if $U = U^{-\circ}$.
On $\mathcal{RO}(X)$ we have the following operations

1. $1 = X$ and $0 = \emptyset$
2. $U \vee O = (U \cup O)^{-\circ}$
3. $U \wedge O = U \cap O$,
4. $\neg U = U^{c\circ}$
5. $U \prec O \Leftrightarrow U^{-} \subseteq O$

Definition

If $(B, \prec)$ is a de Vries algebra, a **round filter** F of $(B, \prec)$ is a lattice filter such that

$$a \in F \Rightarrow \exists b \in F : b \prec a.$$

An **end** is a round filter maximal among the set of round filters. We denote $\mathcal{E}(B)$ the set of ends of $(B, \prec)$.

$$B \in \mathbf{DeV} \longrightarrow X_B = \mathcal{E}(B) \in \mathbf{KHaus}.$$

Review of the situation

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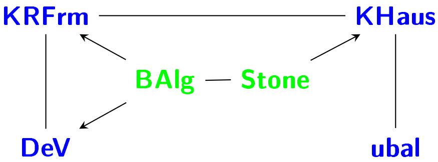
Can we find something else which determine a compact Hausdorff space ?

Uniformly complete bounded Archimedean ℓ-algebras



ubal(By Bezhanishvili) = Uniformly complete bounded Archimedan ℓ-algebra with morphisms with the right properties.

Uniformly complete bounded Archimedean ℓ-algebras

Definition

An ℓ-**algebra** is an algebra $(U, \cdot, +, \wedge, \vee, 0, 1, r\cdot)$ such that :

1. $(U, \wedge, \vee)$ is a lattice,
2. $(U, \cdot, +, 0, 1)$ is a ring,
3. $(U, +, 0, r\cdot)$ is a linear space on $\mathbb{R}$,
4. $a \leq b \Rightarrow a + c \leq b + c$,
5. $0 \leq a, b \Rightarrow 0 \leq a \cdot b$,
6. $U \ni a \geq 0, \mathbb{R} \ni r \geq 0 \Rightarrow r \cdot a \geq 0$.

Uniformly complete bounded Archimedean ℓ-algebras

Theorem

KHaus *is dually equivalent to* ***ubal***

$$X \in \textbf{KHaus} \longrightarrow U_X = C(X, \mathbb{R}) \in \textbf{ubal}$$

$$U \in \textbf{KRFrm} \longrightarrow X_U = \mathrm{Max}_\ell(U) \in \textbf{KHaus}$$

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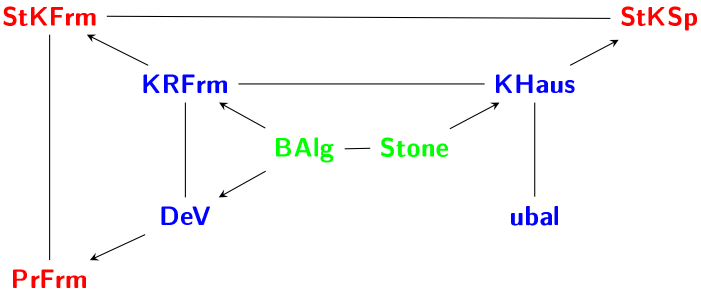
Definition

Let $U \in \mathbf{ubal}$, then $I \subseteq U$ is an ℓ-ideal if

1. I is a ring ideal,
2. I is convex,
3. I is closed for $\vee$.

Second extension



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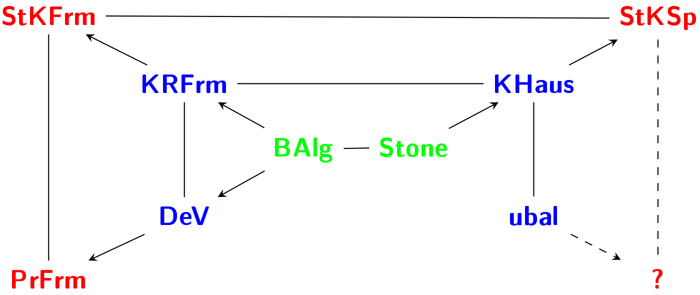

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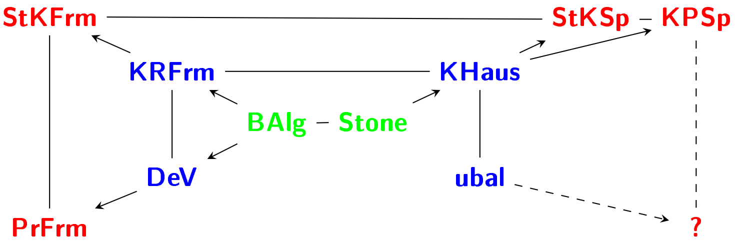
StKSp = Stably compact spaces with proper continuous functions.

Let's shift the problem



Theorem

(Folklore) ***StKSp*** *is equivalent to* ***KPSp***

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KPSp = compact po-spaces with continuous increasing functions.

Why the modification ?

Theorem

If X is a compact po-space, then X is homeomorph to $\mathrm{Con}(I(X, \mathbb{R}^+))$, *where*

- $I(X, \mathbb{R}^+)$ *is the set of the continuous increasing functions from X to* $\mathbb{R}^+$,
- $\mathrm{Con}(I(X, \mathbb{R}^+))$ *is the set of maximal congruences on X.*

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The duality should be

$$X \in \mathbf{KPSp} \longrightarrow I(X,\mathbb{R}^+) \in \mathbf{?}$$
$$A \in \mathbf{?} \longrightarrow \mathrm{Con}(A) \in \mathbf{KPSp}$$

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To do list :

1. find a category axiomatizing $I(X, \mathbb{R}^+)$
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4. Prove that this duality extends the one between **ubal** and **KHaus**

1. Axiomatizing

Definition

An ℓ-**semi-ring** is an algebra $(A, +, \cdot, 0, 1, \wedge, \vee)$ such that :

1. $(A, \wedge, \vee)$ is a lattice,
2. $(A, +, 0)$ and $(A, \cdot, 1)$ are commutative monoids,
3. $(A, +, \cdot)$ is distributive,
4. $a \leq b \Leftrightarrow a + c \leq b + c$,
5. $a \geq 0$,
6. $a \leq b \Rightarrow a \cdot c \leq b \cdot c$.

1. Axiomatizing

Definition

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3. An ℓ-**semi-algebra** is an ℓ-semi-ring which is also an $\mathbb{R}^+$-algebra.

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Theorem

If $X \in$ ***KPSp*** *then* $I(X, \mathbb{R}^+)$ *is a bounded Archimedean* ℓ*-semi-algebra.*

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To do list :

1. ~~find a category axiomatizing $I(X, \mathbb{R}^+)$~~ : $I(X, \mathbb{R}^+) \in$ **sbal**
2. generalize the category **ubal**
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2. Generalizing

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- (**ubal** $\longrightarrow$ **sbal**)

$$U \in \textbf{ubal} \longrightarrow U^+ := \{a \in U : a \geq 0\} \in \textbf{sbal}.$$

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- (**sbal** $\longrightarrow$ **ubal**)

$$A \in \mathbf{sbal} \longrightarrow A^u := (A \times A/ \sim) \in \mathbf{bal}$$

where

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c.$$

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3. Dualizing

We already know that if $X \in \mathbf{KPSp}$, then

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On the other side, if $A \in \mathbf{sbal}$, we do **not** have

$$A \cong I(\mathrm{Con}(A), \mathbb{R}^+).$$

3. Dualizing

Theorem

If $A \in \textbf{\textit{sbal}}$ *then* $A \cong I(\mathrm{Con}(A), \mathbb{R}^+)$ *if and only if*

1. A *is complete for the uniform norm and*
2. A *has the difference with constants property, i.e.*

$$(\forall r \in \mathbb{R})(\forall a \in A)(r \cdot 1 \leq a \Rightarrow \exists b \in A : a = b + r \cdot 1).$$

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Definition

usbal = uniformly complete bounded Archimedean ℓ-semi-algebra with the difference with constants property with the right morphisms.

3. Dualizing

Remark : The category **usbal** still works for the first (Axiomatizing) and second (Generalizing) point !

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4. Extending

$$\begin{array}{ccc} \mathbf{KHaus} & \longrightarrow & \mathbf{KPSp} \\ \Big| & & \Big| \\ \mathbf{ubal} & \longrightarrow & \mathbf{usbal} \end{array}$$

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1. If $X \in \mathbf{KHaus}$, then

$$C(X,\mathbb{R})^+ = I(X,\mathbb{R}^+)$$

4. Extending

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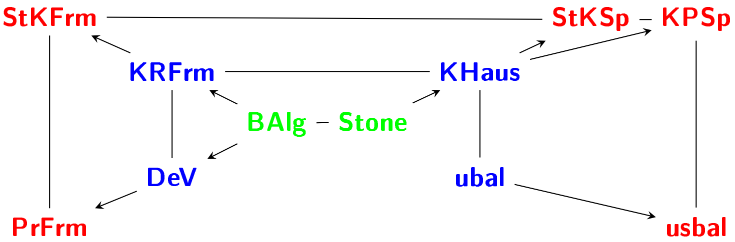
1. If $X \in$ **KHaus**, then

$$C(X, \mathbb{R})^{+} = I(X, \mathbb{R}^{+})$$

2. If $U \in$ **ubal**, then

$$\mathrm{Max}_{\ell}(U) \cong \mathrm{Con}((U^{+} \times U^{+})/\sim).$$

Completed square



Further problems

1. We know how to go from **usbal** to **KPSp** and from **KPSp** to **StKSp**. Is there a way to bypass this and go directly from **usbal** to **StKSp** ? (We can ask the existence of the other compositions as well)

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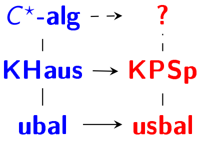
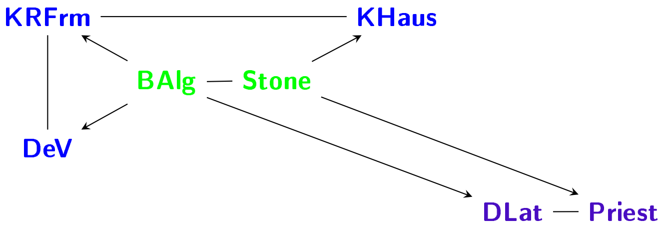
1. We know how to go from **usbal** to **KPSp** and from **KPSp** to **StKSp**. Is there a way to bypass this and go directly from **usbal** to **StKSp** ? (We can ask the existence of the other compositions as well)

2. $X \in$ **KHaus** can be determined by the set $C(X, \mathbb{R}) \in$ **usbal**, which is mainly a $\mathbb{R}$-linear space, but also by $C(X, \mathbb{C}) \in C^\star$-**alg**, which is a $\mathbb{C}$-linear space. Do we have a complex counterpart of $I(X, \mathbb{R}^+)$?



Further problems

3. What if we change the way we extended the original problem ?
   Consider
   DLat = distributive lattices with lattices morphisms
   (We dropped the complemented property)



Priest(Priestley) = Priestley spaces with continuous increasing functions.

Further problems

4. If we consider $(B, \prec) \in \mathbf{DeV}$, then the Boolean component $B \in \mathbf{BAlg}$ has a dual $X \in \mathbf{Stone}$ and the relation $\prec$ can be associated to a binary relation R on X.

Further problems

4. If we consider $(B, \prec) \in \mathbf{DeV}$, then the Boolean component $B \in \mathbf{BAlg}$ has a dual $X \in \mathbf{Stone}$ and the relation $\prec$ can be associated to a binary relation R on X.
   Then, there is a duality between **DeV** and the category representing (X, R).

Further problems

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Further problems

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   Then, there is a duality between $\mathbf{DeV}$ and the category representing (X, R).
   Do we have the same behaviour if we consider $(L, \prec)$ where L is a distributive lattice and $\prec$ is an adequate binary relation on L ?
5. $\mathbf{DeV}$ can be considered as a extension of modal algebras. Do we have a logic associated to this category ?

Red panda