Algebra|Coalgebra Seminar

Institute for Logic, Language and Computation

Slanted Canonicity of Analytic Inductive Inequalities

Laurent De Rudder and Alessandra Palmigiano

June 17, 2020

Introduction

In this talk, we will broach an new algebraic environment which:

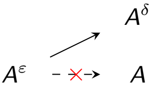
1. extends the theory of **canonical extensions**,
2. extends the theory of **subordination algebras**,
3. solves an open problem related to the **(multi-)modal classical companions** of DLE-logics,
4. allows for a formal-topological characterization of **analytic inductive inequalities.**

Slanted operators

Definition

Let A be a lattice

- a coordinatewise finitely join-preserving n_f-ary map $f : A^\varepsilon \to A^\delta$ is a **c-slanted operator on** A if its range is included in $K(A^\delta)$.
- a coordinatewise finitely meet-preserving n_g-ary map $g : A^\varepsilon \to A^\delta$ is an **o-slanted operator on** A if its range is included in $O(A^\delta)$.



Examples of slanted operators

Examples of slanted operators occur in the literature in connection with:

- **Residuals of σ and π-extensions of standard operators**:
  - $\mathbb{A} = (A, \Box, \Diamond)$ a modal algebra, $\mathbb{A}^\delta = (A^\delta, \Box^\delta, \Diamond^\delta)$ its canonical extension and $\blacklozenge$, $\blacksquare$ the respective adjoints of $\Box^\delta$ and $\Diamond^\delta$, then

    $$\blacklozenge\mid_A : A \to A^\delta \text{ and } \blacksquare\mid_A : A \to A^\delta$$

    are respectively c-slanted and o-slanted.
  - $\mathbb{A} = (A, \mathcal{F}, \mathcal{G})$ a lattice expansion, the residuals of every $f \in \mathcal{F}$ and every $g \in \mathcal{G}$ are o-slanted or c-slanted operators.
- **Quasi-modal algebras and generalised implication lattices**:
  - A quasi-modal algebra is a pair $\mathbb{Q} = (Q, \triangle)$ where Q is a modal algebra and $\triangle$ is a map $Q \to \mathcal{I}(Q)$ such that:
    - $\triangle 1 = A$,
    - $\triangle(a \wedge b) = \triangle a \cap \triangle b$,
  - A generalised implication lattice is a pair $\mathbb{G} = (G, \Rightarrow)$ where G is a bounded distributive lattice and $\Rightarrow$ is a map $G \times G \to \mathcal{I}(G)$ such that, among other properties:
    - $(a \vee b) \Rightarrow c = (a \Rightarrow c) \cap (b \Rightarrow c)$,
    - $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \cap (a \Rightarrow c)$.

Examples of slanted operators

Examples of slanted operators occur in the literature in connection with:

- **Subordination algebras**
  A subordination algebra is a pair $\mathbb{S} = (S, \prec)$ where S is Boolean algebra and $\prec \subseteq S^2$ is such that
  - $\prec(a, -) := \{b \in S \mid a \prec b\}$ is an filter,
  - $\prec(-, a) := \{b \in S \mid b \prec a\}$ is an ideal.

  Then the operators $\lozenge : S \to S^\delta$ and $\blacksquare : S \to S^\delta$ defined as

  $$\lozenge : S \to S^\delta : a \mapsto \bigwedge \prec(a, -) \text{ and } \blacksquare : S \to S^\delta : a \mapsto \bigvee \prec(-, a)$$

  are respectively c-slanted and o-slanted.
- **Gödel-McKinsey-Tarski translation**
  For every Heyting algebra A whose Esakia dual is $(X, \leq)$, then

  $$[\leq] : \mathrm{Clop}(X) \to \mathcal{P}(X)$$

  is an o-slanted operator. This semantic box provides the interpretation for the $\Box$ of the Gödel translation

$\mathcal{L}_{\mathrm{LE}}$ languages

- The language $\mathcal{L}_{\mathrm{LE}}(\mathcal{F}, \mathcal{G})$ is constituted by
  - a denumerable set PROP $= \{p, q, r, \ldots\}$ of propositional variables,
  - the classical lattices connectives $\wedge$ and $\vee$,
  - the classical lattices constants $\top$ and $\bot$,
  - disjoint sets of connectives $\mathcal{F}$ and $\mathcal{G}$. Each connective $h \in \mathcal{F} \cup \mathcal{G}$ has an associated arity n_h and an associated order-type ε_h.
- The formulas of $\mathcal{L}_{\mathrm{LE}}$ are defined recursively as follow

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid h(\overline{\varphi})$$

where $p \in \mathrm{PROP}$ and $h \in \mathcal{F} \cup \mathcal{G}$.

Slanted algebras

Definition

A **slanted $\mathcal{L}_{\mathrm{LE}}$-algebra** is a tuple $\mathbb{A} = (A, \mathcal{F}, \mathcal{G})$ such that:

- A is a bounded lattice;
- every $f \in \mathcal{F}$ is an n_f-ary c-slanted operator.
- every $g \in \mathcal{G}$ is an n_g-ary o-slanted operator.

Remark

Since

$$A \subseteq O(A^{\delta}) \cap K(A^{\delta}),$$

every standard $\mathcal{L}_{\mathrm{LE}}$-algebra is in particular a slanted $\mathcal{L}_{\mathrm{LE}}$-algebra.

Canonical extensions of slanted algebras

Let $f : A^n \to A^\delta$ be a c-slanted operator, then, we should have

$$\begin{array}{ccc} (A^\varepsilon)^\delta & \xrightarrow{f^\sigma} & (A^\delta)^\delta \\ \uparrow & & \uparrow \\ A^\varepsilon & \xrightarrow{f} & A^\delta \end{array} \quad \text{and} \quad \begin{array}{ccc} (A^\varepsilon)^\delta & \xrightarrow{g^\pi} & (A^\delta)^\delta \\ \uparrow & & \uparrow \\ A^\varepsilon & \xrightarrow{g} & A^\delta \end{array}$$

Canonical extensions of slanted algebras

Let $f : A^n \to A^\delta$ be a c-slanted operator, then, we should have

$$\begin{array}{ccc} (A^\varepsilon)^\delta & \xrightarrow{f^\sigma} & (A^\delta)^\delta \\ \uparrow & & \uparrow \\ A^\varepsilon & \xrightarrow{f} & A^\delta \end{array} \quad \text{and} \quad \begin{array}{ccc} (A^\varepsilon)^\delta & \xrightarrow{g^\pi} & (A^\delta)^\delta \\ \uparrow & & \uparrow \\ A^\varepsilon & \xrightarrow{g} & A^\delta \end{array}$$

Instead, we will have (as in Gehrke-Jónsson Math Scand section 2.3)

$$\begin{array}{ccc} (A^\delta)^\varepsilon & \xrightarrow{f^\sigma} & A^\delta \\ \uparrow & \nearrow_{f} & \\ A^\varepsilon & & \end{array} \qquad \begin{array}{ccc} (A^\delta)^\varepsilon & \xrightarrow{g^\pi} & A^\delta \\ \uparrow & \nearrow_{g} & \\ A^\varepsilon & & \end{array}$$

Canonical extensions of slanted algebras

σ-extension for c-slanted	π-extension for o-slanted
$f^\sigma(\overline{k}) = \bigwedge\{f(\overline{a}) \mid \overline{a} \geq^{\varepsilon_f} \overline{k}\}$	$g^\pi(\overline{o}) = \bigvee\{g(\overline{a}) \mid \overline{a} \leq^{\varepsilon_g} \overline{o}\}$
$f^\sigma(\overline{u}) = \bigvee\{f^\sigma(\overline{k}) \mid \overline{k} \leq^{\varepsilon_f} \overline{u}\}$	$g^\pi(\overline{v}) = \bigwedge\{g^\pi(\overline{o}) \mid \overline{o} \geq^{\varepsilon_g} \overline{v}\}$

Lemma

1. *f^σ and g^π are monotone;*
2. *f^σ is coordinatewise completely join-preserving;*
3. *g^π is coordinatewise completely meet-preserving.*

Definition

The **canonical extension** of a slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A} = (A, \mathcal{F}, \mathcal{G})$ is the perfect standard $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}^\delta := (A^\delta, \mathcal{F}^\delta, \mathcal{G}^\delta)$ where

- $\mathcal{F}^\delta := \{f^\sigma \mid f \in \mathcal{F}\}$,
- $\mathcal{G}^\delta := \{g^\pi \mid g \in \mathcal{G}\}$.

Slanted canonicity

Definition

Let $\mathbb{A} = (A, \mathcal{F}, \mathcal{G})$ be a slanted $\mathcal{L}_{\mathrm{LE}}$-algebra, an **(admissible) assignment** into $\mathbb{A}$ is a map

$$V : \mathsf{Prop} \to A.$$

Definition

Let $\varphi \leq \psi$ be a $\mathcal{L}_{\mathrm{LE}}$-inequality and $\mathbb{A}$ be a slanted $\mathcal{L}_{\mathrm{LE}}$-algebra.

1. $(\mathbb{A}, V) \models \varphi \leq \psi$ if $(\mathbb{A}^{\delta}, e \circ V) \models \varphi \leq \psi$ in the standard sense.
2. $\mathbb{A} \models \varphi \leq \psi$ (or $\mathbb{A}^{\delta} \models_{\mathbb{A}} \varphi \leq \psi$) if $(A^{\delta}, e \circ V) \models \varphi \leq \psi$ for any admissible assignment.

Definition

An $\mathcal{L}_{\mathrm{LE}}$-inequality $\varphi \leq \psi$ is **s-canonical** if for every slanted $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}$,

$$\mathbb{A}^{\delta} \models_{\mathbb{A}} \varphi \leq \psi \quad \text{implies} \quad \mathbb{A}^{\delta} \models \varphi \leq \psi.$$

Theorem

Every analytic inductive inequality is s-canonical.

Slanted canonicity projects onto standard canonicity

Let $\mathbb{A} = (A, \mathcal{F}, \mathcal{G})$ be a standard $\mathcal{L}_{\mathrm{LE}}$-algebra.

- The canonical extensions of $\mathbb{A}$ qua slanted $\mathcal{L}_{\mathrm{LE}}$-algebra qua standard $\mathcal{L}_{\mathrm{LE}}$-algebra correspond.
- An inequality $\varphi \leq \psi$ is valid in $\mathbb{A}$ qua slanted algebra if and only if it is valid in $\mathbb{A}$ qua standard algebra.

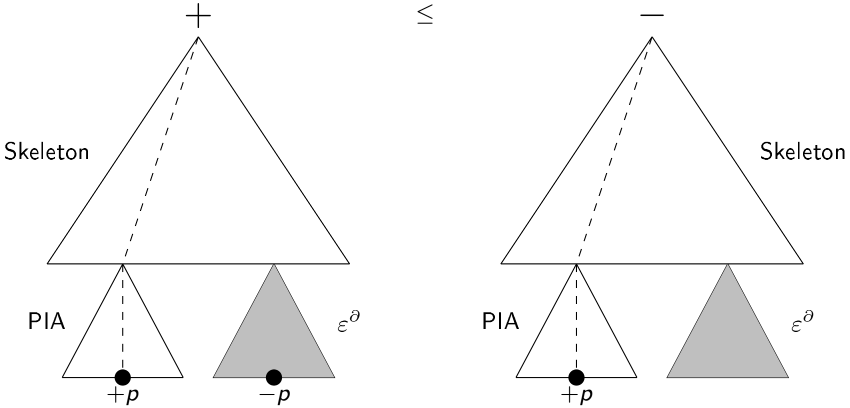
Moreover

- If $\varphi \leq \psi$ is s-canonical, then it is canonical.
- **Examples**:
  - $\Diamond\Box p \leq \Box\Diamond p$ is s-canonical (and hence canonical);
  - $p \leq \Diamond\Box p$ is canonical but not s-canonical.

Inductive inequalities

Every ε-branch must be good, but no restrictions for ε^∂-branches.

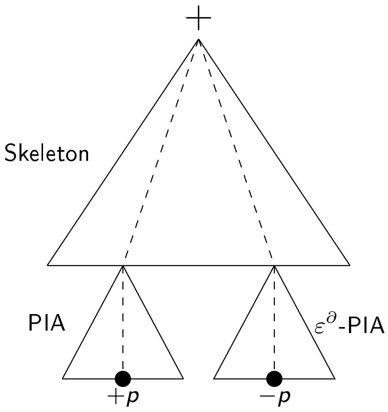
$$p \quad \leq \Diamond\Box p$$



Analytic inductive inequalities

Every branch must be good.

$$\Diamond\Box p \leq \Box\Diamond p$$



Canonicity via correspondence

Proof of standard canonicity

If $\varphi \leq \psi$ is a inductive inequality, then for any standard algebra $\mathbb{A}$, we have

$$\begin{array}{ccc} \mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\ \Updownarrow & & \Updownarrow \\ \mathbb{A}^\delta \models_{\mathbb{A}} \mathrm{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\delta \models \mathrm{ALBA}(\varphi \leq \psi) \end{array}$$

Proof of slanted canonicity

If $\varphi \leq \psi$ is an analytic inductive inequality, then for any slanted algebra $\mathbb{A}$, we have

$$\begin{array}{ccc} \mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\ \Updownarrow & & \Updownarrow \\ \mathbb{A}^\delta \models_{\mathbb{A}} \mathrm{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\delta \models \mathrm{ALBA}(\varphi \leq \psi) \end{array}$$

Main ingredients for topological Ackermann: compactness and intersection lemma.

(Strictly) syntactically closed and open formulas

1. **Syntactically closed** and **syntactically open** $\mathcal{L}_{\mathrm{LE}}^{+}$-formulas: for every $f^* \in \mathcal{F}^*$, $f \in \mathcal{F}$, $g^* \in \mathcal{G}^*$, and $g \in \mathcal{G}$,

$$\mathsf{SC} \ni \varphi ::= p \mid \mathbf{j} \mid \top \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid f^*(\overline{\varphi}, \overline{\psi}) \mid g(\overline{\varphi}, \overline{\psi})$$

$$\mathsf{SO} \ni \psi ::= p \mid \mathbf{m} \mid \top \mid \bot \mid \psi \vee \psi \mid \psi \wedge \psi \mid g^*(\overline{\psi}, \overline{\varphi}) \mid f(\overline{\psi}, \overline{\varphi}).$$

2. **Strictly syntactically closed** and **strictly syntactically open** $\mathcal{L}_{\mathrm{LE}}^{+}$-formulas: for every $f^* \in \mathcal{F}^*$, and $g^* \in \mathcal{G}^*$,

$$\mathsf{SSC} \ni \varphi ::= p \mid \mathbf{j} \mid \top \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid f^*(\overline{\varphi}, \overline{\psi}),$$

$$\mathsf{SSO} \ni \psi ::= p \mid \mathbf{m} \mid \top \mid \bot \mid \psi \vee \psi \mid \psi \wedge \psi \mid g^*(\overline{\psi}, \overline{\varphi}).$$

Applications

- Generalise the Sahlqvist theorem for subordination algebras:
  1. from tense/modal signatures to arbitrary signatures,
  2. from Boolean setting to general lattice one,
  3. from a duality based canonical extension to a constructive one,
- The canonicity via the Gödel-McKinsey-Tarski translation is now accessible to arbitrary distributive lattices

$$\begin{array}{ccc} \mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\ \Updownarrow & & \Updownarrow \\ \mathbb{B} \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) & \Leftrightarrow & \mathbb{B}^\delta \models \tau_\varepsilon(\varphi) \leq \tau_\varepsilon(\psi) \end{array}$$

Conclusions and future work

- New algebraic structures, generalising previous concepts.
- New models for $\mathcal{L}_{\mathrm{LE}}$-logics, suitable to solve translations issues.
- An unexpected (?) link with display calculi.

- A categorical and universal algebraic approach of slanted algebras.
- Arbitrary slanted maps.
- Topology in display calculi?

Mandatory last slide

Thank you for your attention!!

Signed generation tree

Example

Consider the language $\mathcal{L} = (f_1, f_2, g)$ with $\varepsilon_{f_1} = (1, \partial, \partial)$, $\varepsilon_{f_2} = (\partial)$ and $\varepsilon_g = (\partial, 1)$. Then, the positive generation tree of the formula

$$\varphi := f_1(p, q, f_2(p)) \vee g(p \wedge q, r)$$

is given by