GENERALIZED SHAPE OPTIMIZATION USING XFEM AND LEVEL SET METHODS

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OUTLINE

- Introduction
- eXtended Finite Element Method (XFEM)
- Level Set Method
- Problem Formulation
- Sensitivity Analysis
- Applications
  - Implementation
  - Plate with a hole
- Conclusion
INTRODUCTION

- Types of variables
  - (a) Sizing optimization
  - (b) shape optimization
  - (c) topology optimization
  - (d) material selection

- Types of problems
  - Structural
  - Multidisciplinary
INTRODUCTION

- TOPOLOGY OPTIMIZATION (Bendsøe & Kikuchi, 1988)
  - Optimal material distribution
  - Optimal topology without any a priori
  - Fixed mesh
  - Design variables
    - = Local density parameters
    - Many thousands of design variables
  - Simple design problem:
    - Minimum compliance s.t. volume constraint
    - Local constraints are difficult to handle
    - Geometrical constraints (often manufacturing constraints) are difficult to define and to control
  - Preliminary design: interpretation phase necessary to come to a CAD model
  - Great industrial applications
INTRODUCTION

Perimeter constraint
INTRODUCTION

Min max compliance design

Min max stress design
INTRODUCTION

- SHAPE OPTIMIZATION
  - Modification of boundaries of CAD model
  - Fixed topology a priori
  - Design variables
    - = CAD model parameters
    - Small number of design variables
  - Quite complex design problems:
    - Large number of global and local constraints
    - Geometrical constraints easily included
  - Detailed design
  - Mesh management problems
    - Mesh modification / mesh distortion
    - Velocity field
  - Small number of industrial applications
**INTRODUCTION**

- Position of a point after a perturbation of the design variable $d_i$

  \[ X(d_i + \delta d_i) = X(d_i) + V_i \delta d_i \]

  with \( V_i = \frac{\partial X}{\partial d_i} \)

- Derivative of a response in a given point:

  \[
  \frac{DR}{Dd_i} = \frac{\partial R}{\partial d_i} \sum_k \frac{\partial R}{\partial X_k} \frac{\partial X_k}{\partial d_i} = \frac{\partial R}{\partial d_i} + V_i \nabla R
  \]

Conclusion: determine the velocity field at first
**INTRODUCTION**

- Practical calculation of velocity field
  - Transfinite mapping
  - Natural / mechanical approach (Belegundu & Rajan, Zhang & Beckers)
  - Laplacian smoothing
  - Relocation schemes
INTRODUCTION

Without error control

With error control
INTRODUCTION

- EXTENDED FINITE ELEMENT METHOD (XFEM)
  - alternative to remeshing methods

- LEVEL SET METHOD
  - alternative description to parametric description of curves

- XFEM + LEVEL SET METHODS
  - Efficient treatment of problem involving discontinuities and propagations
  - Early applications to crack problems. Moës et al. (1999)
  - Applications to topology optimisation Belytschko et al. (2003), Wang et al. (2003), Allaire et al. (2004)
INTRODUCTION

- THIS WORK
  - XFEM + Level Set methods = alternative method to shape optimisation
  - Intermediate approach between shape and topology optimisation
  - XFEM
    - work on fixed mesh
    - no mesh problems
  - Level Set
    - smooth curve description
    - modification of topology is possible
  - Problem formulation:
    - global and local constraints
    - small number of design variables
EXTENDED FINITE ELEMENT METHOD

- Early motivation: study of propagating crack in mechanical structures → avoid the remeshing procedure

- Principle:
  - Allow the model to handle discontinuities that are non-conforming with the mesh
  - Add internal degree of freedom $a_i$
  - Add special shape functions $H(x)N_i(x)$ (discontinuous)

\[
u = \sum_{i \in I} u_i N_i(x) + \sum_{i \in L} u_i N_i(x) H(x)
\]

\[
Kq = g \iff \begin{bmatrix} K_{uu} & K_{ua} \\ K_{au} & K_{aa} \end{bmatrix} \begin{bmatrix} u \\ a \end{bmatrix} = \begin{bmatrix} f_u \\ f_a \end{bmatrix}
\]
EXTENDED FINITE ELEMENT METHOD

- Representing holes or material – void interfaces
  - Remove empty elements
  - Keep partially filled elements
  - Use XFEM numerical integration

\[ u = \sum_{i \in I} N_i(x) V(x) u_i \]

\[ V(x) = \begin{cases} 1 & \text{if node } \in \text{solid} \\ 0 & \text{if node } \in \text{void} \end{cases} \]
EXTENDED FINITE ELEMENT METHOD

- Quadrangles and triangles XFEM elements
- Numerical integration
  - Division into sub-triangles
  - Integration over sub-triangles
  - Gauss points

Material 1

Material 2

Boundary

void

solid

Boundary
THE LEVEL SET METHOD

- Principle (Sethian, 1999)
  - Introduce a higher dimension
  - Implicit representation
  - Interface = the zero level of a function \( \psi(x, t) = 0 \)

- Possible practical implementation:
  - Approximated on a fixed mesh by the signed distance function to curve \( \Gamma \):
    \[
    \psi(x, t) = \pm \min_{x_{\Gamma} \in \Gamma(t)} \|x - x_{\Gamma}\|
    \]

- Advantages:
  - 2D / 3D
  - Combination of entities: e.g. min / max
THE LEVEL SET METHOD

- Level Set of a square hole
- Combination of two holes
**THE LEVEL SET METHOD**

- **Evolution of interface**

\[
\frac{\partial \psi}{\partial t} + V \| \nabla \psi \| = 0
\]

\[
\psi(x,t) = 0 \quad \text{given}
\]

- \( V \): velocity function of \( \Gamma \) in the outward normal direction to interface
THE LEVEL SET METHOD

- In XFEM framework,
  - Each node has a Level Set dof
  - Interpolation using classical shape functions
    \[ \psi(x,t) = \sum_i \psi_i N_i(x) \]
  - Material assigned to a part of the Level Set (positive or negative)

- Building a library of graphic primitives and features
  - Lines
  - Circles, ellipses, rectangles, triangles
  - NURBS
  - ...
THE LEVEL SET METHOD

Level set defined by a set of points

Level set associated to a NURBS
PROBLEM FORMULATION

- Geometry description and material layout:
  - Using Level Sets
  - Basic Level Set features: circles, ellipses, rectangles, etc.

- Design Problem
  - Find the best shape to minimize a given objective functions while satisfying design constraints

- Design variables:
  - Parameters of Level Sets

- Objective and constraints
  - Mechanical responses: global (compliance) or local (displacement, stress)
  - Geometrical characteristics: volume, distance

- Problem formulation similar to shape optimization but simplified thanks to XFEM and Level Set!
PROBLEM FORMULATION

BECAUSE OF XFEM AND LEVEL SET

- The mesh has not to coincide with the geometry
- Work on a fixed mesh

- Sensitivity analysis: no velocity field and no mesh perturbation required

- Topology can be altered as entities can be merged or separated → generalized shape
- Introduction of new holes requires a topological derivatives

- Topology optimization can be simulated using a design universe of holes and an optimal selection problem (Missoum et al. 2000)
SENSITIVITY ANALYSIS

- Classical approach for sensitivity analysis in industrial codes: *semi analytical* approach

- Discretized equilibrium

\[ K \frac{\partial u}{\partial x} = \left( \frac{\partial f}{\partial x} - \frac{\partial K}{\partial x} u \right) \]

- Derivatives of displacement

- Semi analytical approach

\[
\frac{\partial K}{\partial x} \approx \frac{K(x + \delta x) - K(x)}{\delta x} \quad \frac{\partial f}{\partial x} \approx \frac{f(x + \delta x) - f(x)}{\delta x}
\]
SENSITIVITY ANALYSIS

- Fixed mesh $\rightarrow$ no mesh perturbation
- However finite differences of stiffness matrix have to be made with a **frozen number of dof**
- Critical situations happen when some empty elements become partly filled with solid after perturbating of the level set:

![Reference Level Set](image1)

![Level Set after perturbation](image2)

- Node with dof
- New nodes with dof
SENSITIVITY ANALYSIS

Reference configuration

After level set perturbation
SENsitIVITY ANALYSIS

- Strategies to freeze the number of dof
  - analytical derivatives of stiffness matrix:
    - not general!
  - boundary layer in which all elements are retained
    - rigid modes, larger size of the problem
  - boundary layer with softening material (SIMP law)
    - lost of void / solid approximation
- ignore the new elements that become solid or partly solid
  - small errors, but minor contributions
  - practically, no problem observed
  - efficiency and simplicity
  - validated on benchmarks
**SENSITIVITY ANALYSIS**

- Summary of the semi-analytical approach strategy

\[
\frac{\partial K}{\partial x} \approx \frac{K(x + \delta x) - K(x)}{\delta x}
\]

<table>
<thead>
<tr>
<th>Element initially</th>
<th>→ Solid</th>
<th>→ Cut</th>
<th>→ Void</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid</td>
<td>OK</td>
<td>OK</td>
<td></td>
</tr>
<tr>
<td>Cut</td>
<td>OK</td>
<td>OK</td>
<td></td>
</tr>
<tr>
<td>Void</td>
<td>Ignored</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
IMPLEMENTATION

- Preliminary investigations by coupling a standard XFEM code by Moës with a general open optimisation code (Boss Quattro)

- New implementation in a multiphysic finite element code in C++ (OOFELIE from Open Engineering www.open-engineering.com)

- XFEM library: 2D problems with a library of quadrangles and triangles.

- Available results for optimization:
  - Compliance
  - Displacements
  - Strains, Stresses
  - Energy per element

- Visualization:
  - Level Sets
  - Results
Direct solution of the original optimisation problem which is generally non-linear, implicit in the design variables

Minimise $f(x)$

s. t.: $g_j(x) \leq g_j^{\text{max}}$ $j=1,m$

is replaced by a sequence of optimisation sub-problems

Minimise $F(x)$

s. t.: $G_j(x) \leq G_j^{\text{max}}$ $j=1,m$

by using approximations of the responses and using powerful mathematical programming algorithms (Lagrangian duality methods or Quadratic Programming)
CONLIN OPTIMIZATION SOLVER

- FORTRAN computer programme can be used as a standalone software or an optimizer in open optimization tools
- General solver for structural and multidisciplinary problems: Sizing, shape, and topology problems
- Robust and Efficient
- Large scale problems:
  - 100,000 design variables (topology)
  - 5,000 constraints (shape)
  - 5,000 constraints and 5,000 design variables (topology)
- Implemented in several industrial optimisation tools: BOSS-Quattro, MBB-Lagrange, OptiStruct (Altair)
APPLICATIONS

CLASSICAL PROBLEM OF PLATE WITH A HOLE REVISITED

- Square plate with a hole
- Bidirectional stress field
  \( \sigma_x = 2 \sigma_0 \quad \sigma_y = \sigma_0 \)
- \( E = 1 \text{ N/m}^2, \quad \nu = 0.3 \)

- Minimize compliance
  - st volume constraint
- Design variables: major axis a and orientation \( \theta \)

- Mesh 30 x 30 nodes
**APPLICATIONS**

Min Compliance
s.t. Volume constraint

14 it.
Min Compliance
s.t. Volume constraint

11 it.
APPLICATIONS

- Discretization error of the geometry using approx of level set
  Over-estimating geometric values:

<table>
<thead>
<tr>
<th>Objective function Constraint</th>
<th>Xfem Iteration 1</th>
<th>Xfem Iteration 11</th>
<th>Fem Iteration 1</th>
<th>Fem Iteration 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimise $U$, Surface $&lt; 3.45$</td>
<td>27.9, 3.59</td>
<td>20.2, 3.45</td>
<td>26, 3.50</td>
<td>18.3, 3.45</td>
</tr>
<tr>
<td>Variable $1e^{-4} &lt; \theta &lt; 90$</td>
<td>45</td>
<td>1e-4</td>
<td>45</td>
<td>0</td>
</tr>
<tr>
<td>Variable $1e^{-4} &lt; a &lt; 1$</td>
<td>0.5</td>
<td>1.06</td>
<td>0.5</td>
<td>0.88</td>
</tr>
</tbody>
</table>

- Representing interfaces inside an element:
APPLICATIONS

- Linear interpolation of the Level Set may introduce discontinuity:
  - Parametric study of the surface of the plate
  - Variation of 1%

- Take care of numerical noise
APPLICATIONS
APPLICATIONS

- Topology modification during optimization
  - Two variables: center $x_1$, center $x_2$
  - Min. potential energy under a surface constraint
  - Uniform Biaxial loading: $\sigma_x = \sigma_0$, $\sigma_y = \sigma_0$

12 it.
APPLICATIONS

- Evolution of the objective function

![Graph showing the evolution of the objective function over iterations.]

- Evolution of the Level Set

![Images showing the Level Set evolution with iteration numbers: Iteration 1 and Iteration 12.]

<table>
<thead>
<tr>
<th>Objective function</th>
<th>Minimise $U$</th>
<th>Iteration 1</th>
<th>Iteration 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constraint</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Surface &gt; 7.8</td>
<td>6.9</td>
<td>7.95</td>
<td></td>
</tr>
<tr>
<td>Variable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.5 &lt; x_1$</td>
<td>0.5</td>
<td>-0.0662076</td>
<td></td>
</tr>
<tr>
<td>Variable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.5 &lt; x_2$</td>
<td>-0.5</td>
<td>0.0457915</td>
<td></td>
</tr>
</tbody>
</table>
APPLICATIONS

- Mesh refinement for the Level Set representation of sharp parts
- Accuracy of stresses

Mesh refinement for the Level Set representation of sharp parts

12 it.
Applications

- Design universe of holes (Missoum et al., 2000)
  - Selection and sizing of basic Level Set entities with a GA in classical topology
- Find a result as close as possible to MBB topology solution
  - 14 triangles are « well » placed.
  - Variables : presence of a triangle
- The optimum is reached after 36
CONCLUSION

■ XFEM and Level Set gives ride to a generalized shape optimisation technique

■ Intermediate to shape and topology optimisation
  ■ Work on a fixed mesh
  ■ Topology can be modified:
    ■ Holes can merge and disappear
    ■ New holes cannot be introduced without topological derivatives
  ■ Smooth curves description
  ■ Void-solid description
  ■ Small number of design variables
  ■ Global or local response constraints
  ■ No velocity field and mesh perturbation problems
CONCLUSION

- Contribution of this work
  - New perspectives of XFEM and Level Set
  - Investigation of semi-analytical approach for sensitivity analysis
  - Implementation in a general C++ multiphysics FE code

- Concept just validated

- Perspectives:
  - Sensitivity analysis (to be continued)
  - 3D problems
  - Stress constrained problems
  - Dynamic problems
  - Multiphysic simulation problems with free interfaces
Thank you for your invitation

Thank you for your attention
ACKNOWLEDGEMENTS

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  - ARC MEMS, Action de recherche concertée 03/08-298 'Modeling, Multi-physic Simulation, and Optimization of Coupled Problems - Application to Micro-Electro-Mechanical Systems' funded by the Communauté Française de Belgique
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