LIÈGE université
Applied Sciences

Structural & Stochastic Dynamics
Urban & Environmental Engineering

Influence of a small flexibility of connections on the elastic structural response of frames

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FREEDOM TO RESEARCH

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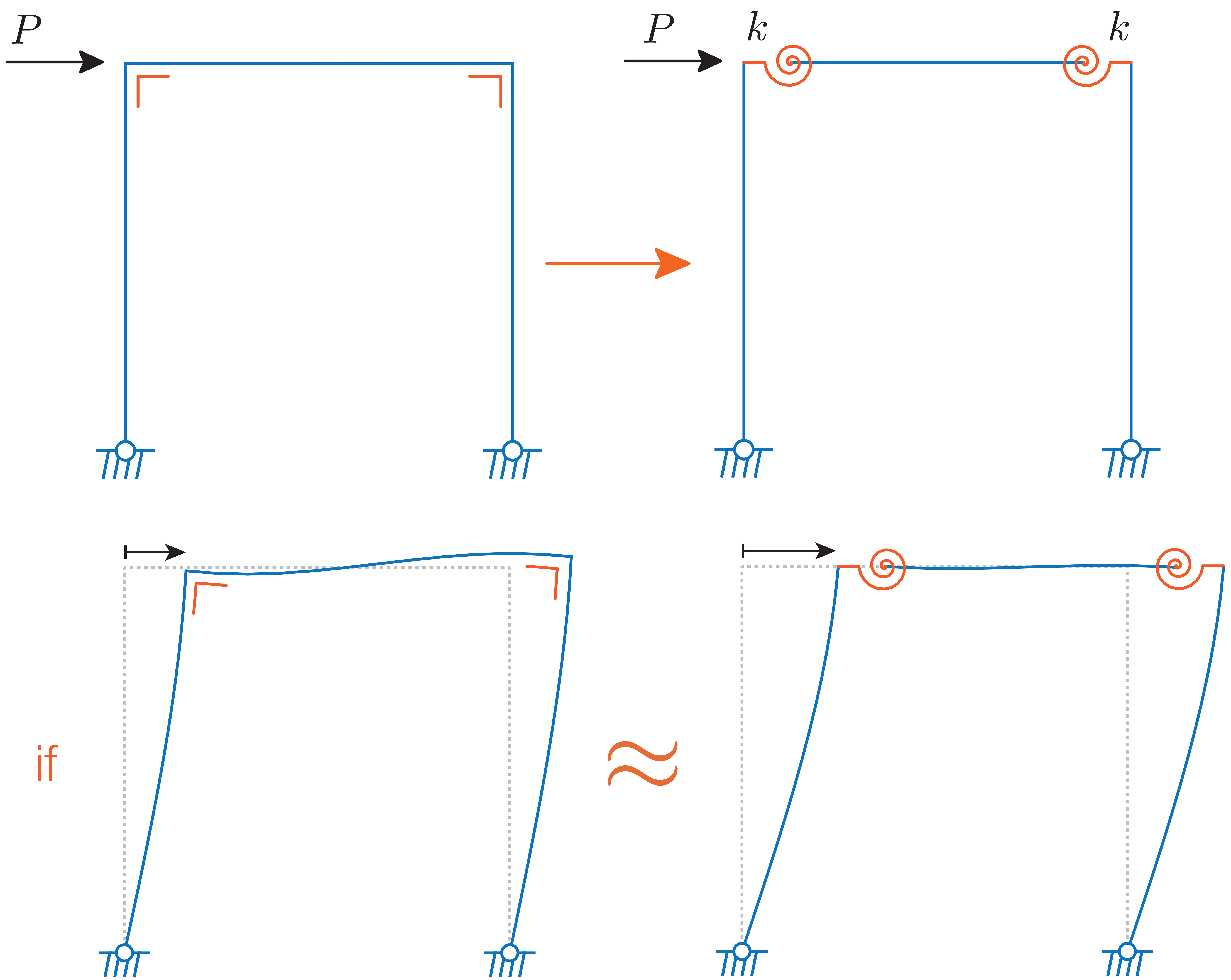
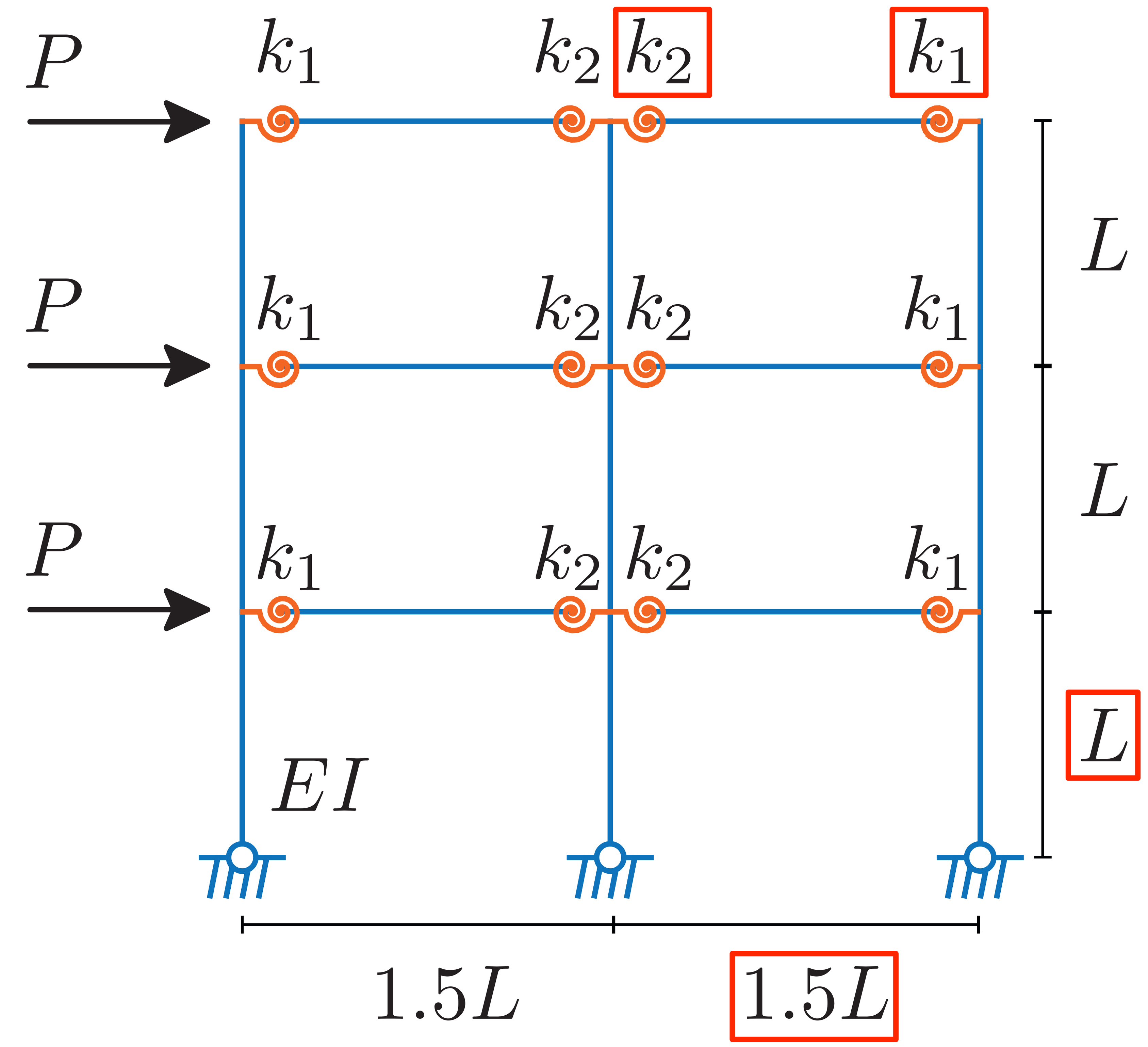
Refining rigid joint classification criteria requires a simplified analysis method !



simple analytical solutions are available



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The small flexibility of almost rigid joints can be treated as initial imperfections are

The small flexibility of almost rigid joints can be treated as initial imperfections are

and corresponds to a small perturbation:

if $\alpha = 0$, the original joints are perfectly rigid

If $\alpha \to 0$, the solution tends to this limit case

$\Leftrightarrow$ the equivalent loads cancel out if $\alpha = 0$, are proportional to α

A brief overview of the complete analysis method

1. Cubic interpolation functions

$$\mathbf{h} = \frac{1}{\Delta_\alpha}\begin{pmatrix} \Delta_\alpha & -6\alpha_1(1+2\alpha_2) & -3(1+2\alpha_2) & 2(1+\alpha_1+\alpha_2) \\ 0 & L(1+4\alpha_2) & -2L(1+3\alpha_2) & L(1+2\alpha_2) \\ 0 & 6\alpha_1(1+2\alpha_2) & 3(1+2\alpha_2) & -2(1+\alpha_1+\alpha_2) \\ 0 & -2L\alpha_1 & -L & L(1+2\alpha_1) \end{pmatrix}\begin{pmatrix} 1 \\ \frac{x}{L} \\ \frac{x^2}{L^2} \\ \frac{x^3}{L^3} \end{pmatrix} \quad ; \quad \Delta_\alpha = 12\alpha_1\alpha_2 + 4\alpha_1 + 4\alpha_2 + 1$$

2. Elementary stiffness matrices

$$\mathbf{K}^{(e)} = \int_0^L EI\,\mathbf{h}''(x)\mathbf{h}''^T(x)\,\mathrm{d}x = \frac{1}{\Delta_\alpha}\frac{EI}{L^3}\begin{pmatrix} 12(1+\alpha_2+\alpha_1) & 6(1+2\alpha_2)L & -12(1+\alpha_2+\alpha_1) & 6(1+2\alpha_1)L \\ 6(1+2\alpha_2)L & 4(1+3\alpha_2)L^2 & -6(1+2\alpha_2)L & 2L^2 \\ -12(1+\alpha_2+\alpha_1) & -6(1+2\alpha_2)L & 12(1+\alpha_2+\alpha_1) & -6(1+2\alpha_1)L \\ 6(1+2\alpha_1)L & 2L^2 & -6(1+2\alpha_1)L & 4(1+3\alpha_1)L^2 \end{pmatrix}$$

3. Elementary load vectors

$$\mathbf{p}^{(e)} = \int_0^L q(x)\,\mathbf{h}(x)\mathrm{d}x$$

4. Global stiffness matrix

$$\mathbf{K} = \sum_{e=1}^{n_e} \mathbf{L}^{(e)^T}\mathbf{K}^{(e)}\mathbf{L}^{(e)}$$

5. Global load vector

$$\mathbf{p} = \sum_{e=1}^{n_e} \mathbf{L}^{(e)^T}\mathbf{p}^{(e)}$$

6. Displacements and rotations

$$\mathbf{K}\ \mathbf{u} = \mathbf{p}$$

7. Internal forces

$$\mathbf{f}_{\text{int}}^{(e)} = \mathbf{K}^{(e)}\mathbf{u}^{(e)}$$

Use perturbation methods to simplify the problem

A. Introduce $\alpha_i = \varepsilon a_i$ with $\varepsilon \ll 1$ and $a_i \sim 1$

B. Expand in series for $\varepsilon \to 0$ and truncate

C. Introduce an Ansatz for the solution

Use perturbation methods to simplify the problem

A. Introduce $\alpha_i = \varepsilon a_i$ with $\varepsilon \ll 1$ and $a_i \sim 1$

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2. Elementary stiffness matrices

$$\mathbf{K}^{(e)} = \mathbf{K}_0^{(e)} + \varepsilon \mathbf{K}_1^{(e)} + \mathcal{O}\left(\varepsilon^2\right)$$

$$\mathbf{K}_0^{(e)} = \frac{EI}{L^3} \begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$$

as if the joints were fully rigid !

$$\varepsilon \mathbf{K}_1^{(e)} = -\frac{EI}{L^3} \begin{pmatrix} 36\left(\alpha_1+\alpha_2\right) & 12L\left(2\alpha_1+\alpha_2\right) & -36\left(\alpha_1+\alpha_2\right) & 12L\left(\alpha_1+2\alpha_2\right) \\ 12L\left(2\alpha_1+\alpha_2\right) & 4L^2\left(4\alpha_1+\alpha_2\right) & -12L\left(2\alpha_1+\alpha_2\right) & 8L^2\left(\alpha_1+\alpha_2\right) \\ -36\left(\alpha_1+\alpha_2\right) & -12L\left(2\alpha_1+\alpha_2\right) & 36\left(\alpha_1+\alpha_2\right) & -12L\left(\alpha_1+2\alpha_2\right) \\ 12L\left(\alpha_1+2\alpha_2\right) & 8L^2\left(\alpha_1+\alpha_2\right) & -12L\left(\alpha_1+2\alpha_2\right) & 4L^2\left(\alpha_1+4\alpha_2\right) \end{pmatrix}$$

proportional to the flexibilities !

4. Global stiffness matrix

$$\mathbf{K} = \mathbf{K}_0 + \varepsilon \mathbf{K}_1 + \mathcal{O}\left(\varepsilon^2\right)$$

$\mathbf{K}_0$ → stiffness matrix of the structure with rigid joints

$\varepsilon \mathbf{K}_1$ → small loss of stiffness due to the joint flexibilities

Use perturbation methods to simplify the problem

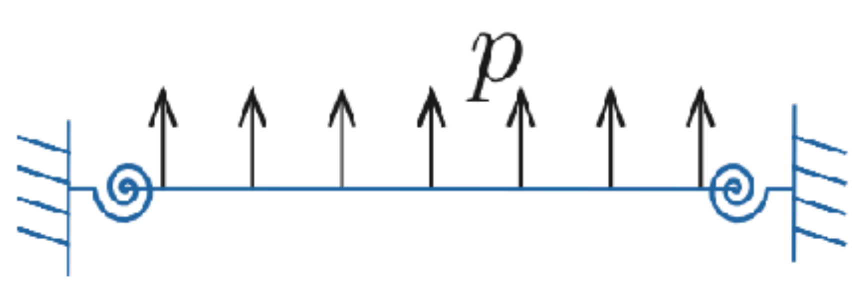
A. Introduce $\alpha_i = \varepsilon a_i$ with $\varepsilon \ll 1$ and $a_i \sim 1$

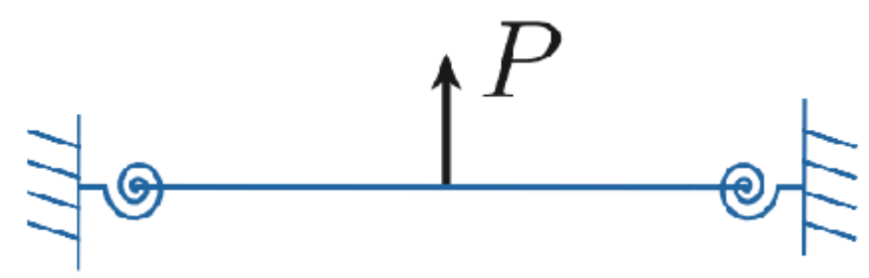
B. Expand in series for $\varepsilon \to 0$ and truncate at first order

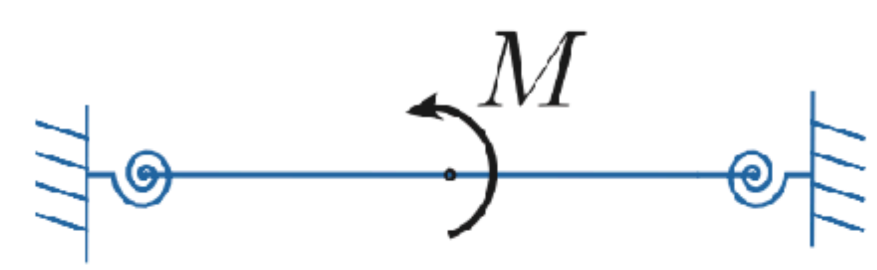
C. Introduce an Ansatz for the solution $\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1$

3. Elementary load vectors

$$\mathbf{p}^{(e)} = \mathbf{p}_0^{(e)} + \varepsilon \mathbf{p}_1^{(e)} + \mathcal{O}\left(\varepsilon^2\right)$$







$\mathbf{p}_0^{(e)}$: as if the joints were fully rigid !

$$\mathbf{p}_0^{(e)} = \begin{pmatrix} \frac{1}{2} pL \\ \frac{1}{12} p L^2 \\ \frac{1}{2} pL \\ -\frac{1}{12} p L^2 \end{pmatrix} \qquad \mathbf{p}_0^{(e)} = \begin{pmatrix} \frac{1}{2} P \\ \frac{1}{8} P L \\ \frac{1}{2} P \\ -\frac{1}{8} P L \end{pmatrix} \qquad \mathbf{p}_0^{(e)} = \begin{pmatrix} -\frac{3}{2} \frac{M}{L} \\ -\frac{M}{4} \\ \frac{3}{2} \frac{M}{L} \\ -\frac{M}{4} \end{pmatrix}$$

$\varepsilon \mathbf{p}_1^{(e)}$: proportional to the flexibilities !

$$\Delta\mathbf{p}^{(e)} = \begin{pmatrix} -\frac{1}{2} pL\left(\alpha_1 - \alpha_2\right) \\ -\frac{1}{12} p L^2 \left(4\alpha_1 - 2\alpha_2\right) \\ \frac{1}{2} pL\left(\alpha_1 - \alpha_2\right) \\ -\frac{1}{12} p L^2 \left(2\alpha_1 - 4\alpha_2\right) \end{pmatrix} \qquad \Delta\mathbf{p}^{(e)} = \begin{pmatrix} -\frac{3}{4} P\left(\alpha_1 - \alpha_2\right) \\ -\frac{1}{8} P L\left(4\alpha_1 - 2\alpha_2\right) \\ \frac{3}{4} P\left(\alpha_1 - \alpha_2\right) \\ -\frac{1}{8} P L\left(2\alpha_1 - 4\alpha_2\right) \end{pmatrix} \qquad \Delta\mathbf{p}^{(e)} = \begin{pmatrix} \frac{3}{2} \frac{M}{L}\left(\alpha_1 + \alpha_2\right) \\ M\left(\alpha_1 + \frac{1}{2}\alpha_2\right) \\ -\frac{3}{2} \frac{M}{L}\left(\alpha_1 + \alpha_2\right) \\ M\left(\frac{1}{2}\alpha_1 + \alpha_2\right) \end{pmatrix}$$

5. Global load vector

$$\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p} + \mathcal{O}\left(\varepsilon^2\right)$$

- $\mathbf{p}_0$ → loading of the structure with rigid joints
- $\Delta\mathbf{p}$ → small perturbation due to the flexibilities

Use perturbation methods to simplify the problem

A. Introduce $\alpha_i = \varepsilon a_i$ with $\varepsilon \ll 1$ and $a_i \sim 1$

B. Expand in series for $\varepsilon \to 0$ and truncate at first order

C. Introduce an Ansatz for the solution $\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1$

6. Displacements and rotations

$$(\mathbf{K}_0 + \varepsilon \mathbf{K}_1)(\mathbf{u}_0 + \overbrace{\varepsilon \mathbf{u}_1}^{\Delta \mathbf{u}}) = (\mathbf{p}_0 + \overbrace{\varepsilon \mathbf{p}_1}^{\Delta \mathbf{p}})$$

Leading order problem $\mathbf{K}_0 \mathbf{u}_0 = \mathbf{p}_0$ analyze the structure **as if the joints were fully rigid**

First order correction $\mathbf{K}_0 \Delta \mathbf{u} = \Delta \mathbf{p} + \hat{\mathbf{p}}$ analyze the structure with rigid joints under an **equivalent loading**

with $\hat{\mathbf{p}} = -\varepsilon \mathbf{K}_1 \mathbf{u}_0$ which depends on the leading order solution and concerns elements with semi-rigid joints only

Use perturbation methods to simplify the problem

A. Introduce $\alpha_i = \varepsilon a_i$ with $\varepsilon \ll 1$ and $a_i \sim 1$

B. Expand in series for $\varepsilon \to 0$ and truncate at first order

C. Introduce an Ansatz for the solution $\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1$

7. Internal forces

$$\mathbf{f}_{\text{int}}^{(e)} = \mathbf{f}_0^{(e)} + \Delta\mathbf{f}^{(e)} - \left(\hat{\mathbf{p}}^{(e)} + \Delta\mathbf{p}\right)$$

a correction appears to ensure the continuity of moments

Leading order problem $\mathbf{f}_0^{(e)} = \mathbf{K}_0^{(e)} \mathbf{u}_0^{(e)}$

First order correction $\Delta\mathbf{f}^{(e)} = \mathbf{K}_0^{(e)} \Delta\mathbf{u}^{(e)}$

The simplified analysis method in a nutshell

A. Analyze the structure with rigid joints under the **original loading**

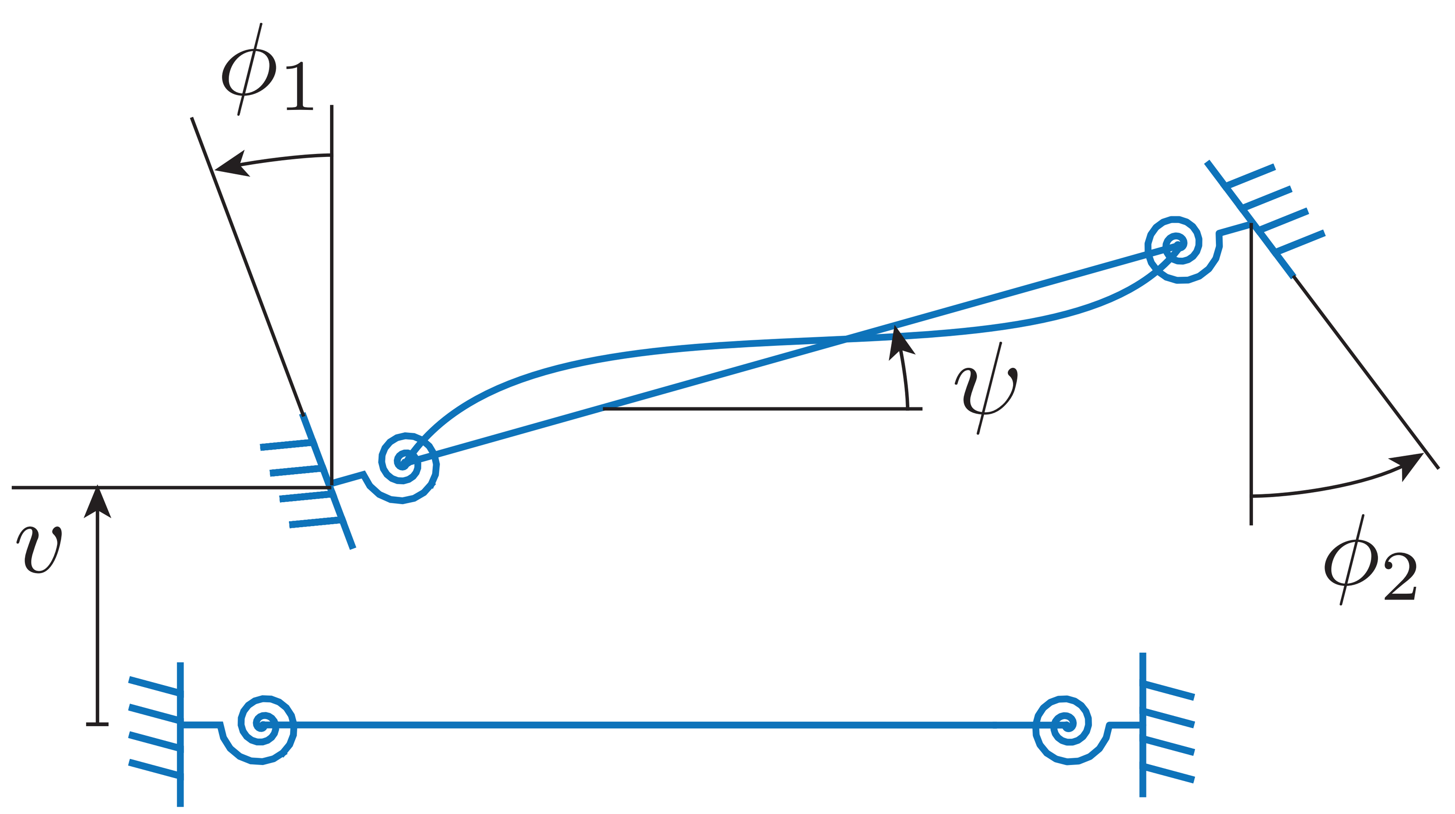
$$\mathbf{K}_0\mathbf{u}_0 = \mathbf{p}_0$$

B. Use rotations and chord drifts to compute the **equivalent loading**

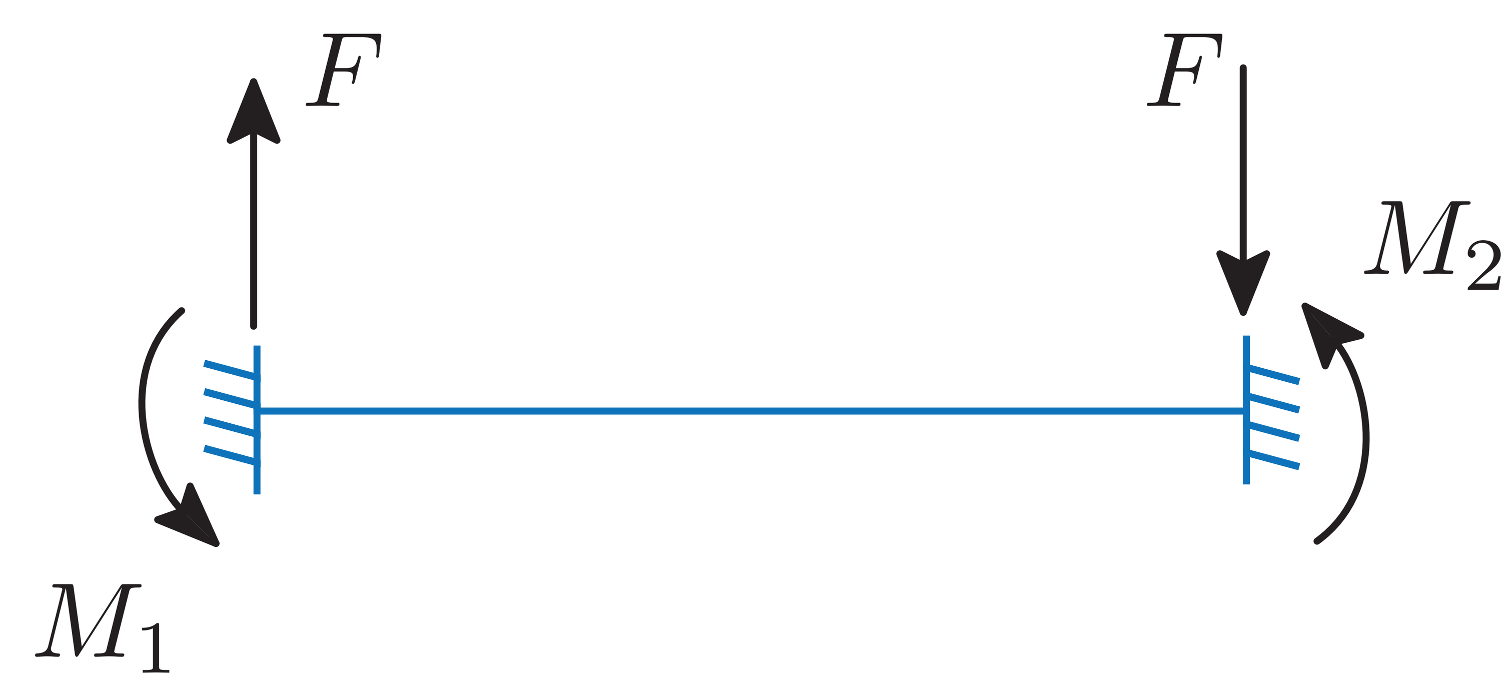
$$\mathbf{u}_0^{(e)} = (v, \phi_1, v + \psi L, \phi_2)^\mathrm{T}$$

$$\hat{\mathbf{p}}^{(e)} = -\varepsilon \mathbf{K}_1^{(e)} \mathbf{u}_0^{(e)}$$

$$\hat{\mathbf{p}}^{(e)} = (F, M_1, -F, M_2)^T$$



= 0 if both $\alpha_i = 0$!



C. Analyze the structure with rigid joints under the **equivalent loading**

$$\mathbf{K}_0\Delta\mathbf{u} = \Delta\mathbf{p} + \hat{\mathbf{p}}$$

D. Add the displacements $\mathbf{u} = \mathbf{u}_0 + \Delta\mathbf{u}$

E. Correct the internal forces $\mathbf{f}_{\mathrm{int}}^{(e)} = \mathbf{f}_0^{(e)} + \Delta\mathbf{f}^{(e)} - \left(\hat{\mathbf{p}}^{(e)} + \Delta\mathbf{p}\right)$

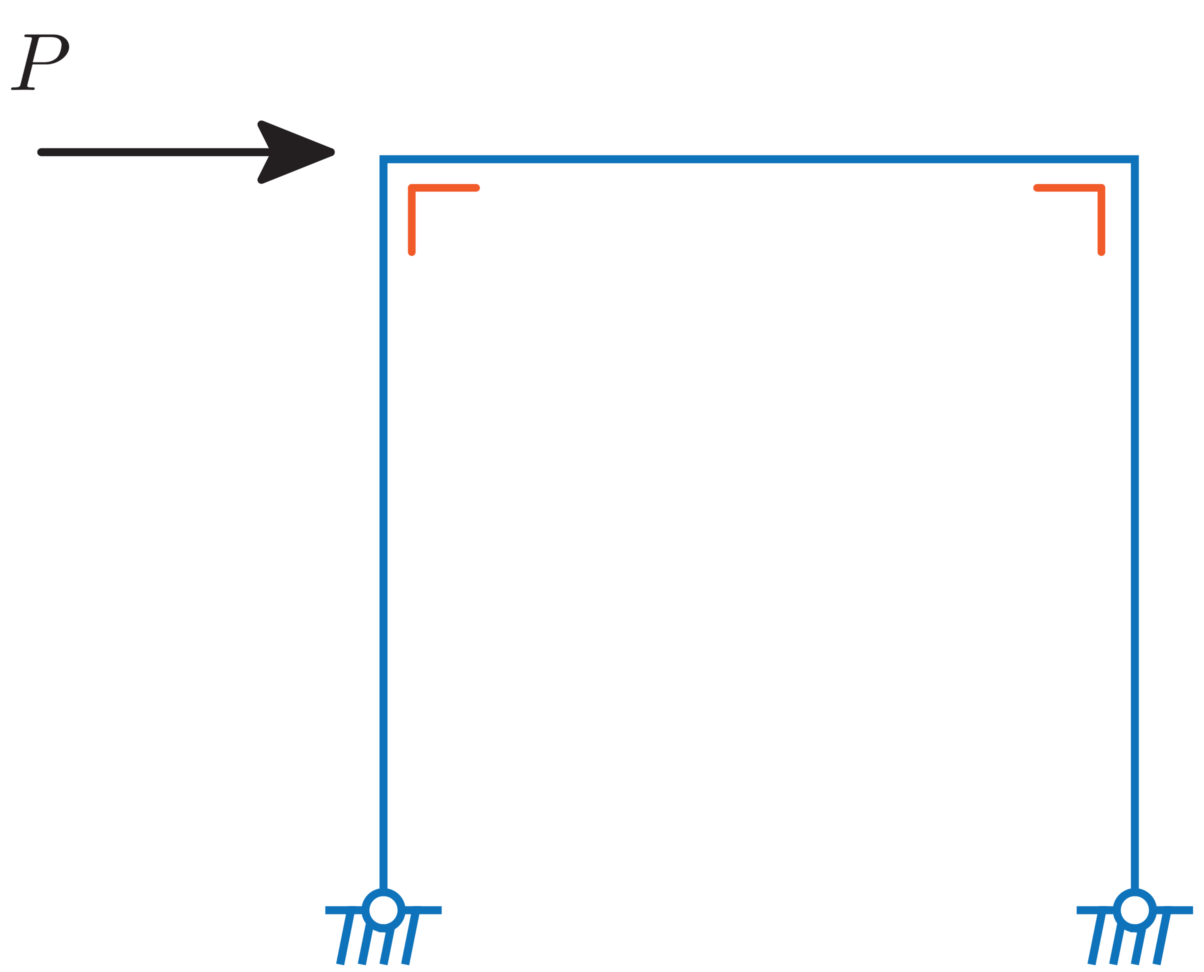
Example 1: one span, one floor, with a concentrated horizontal load

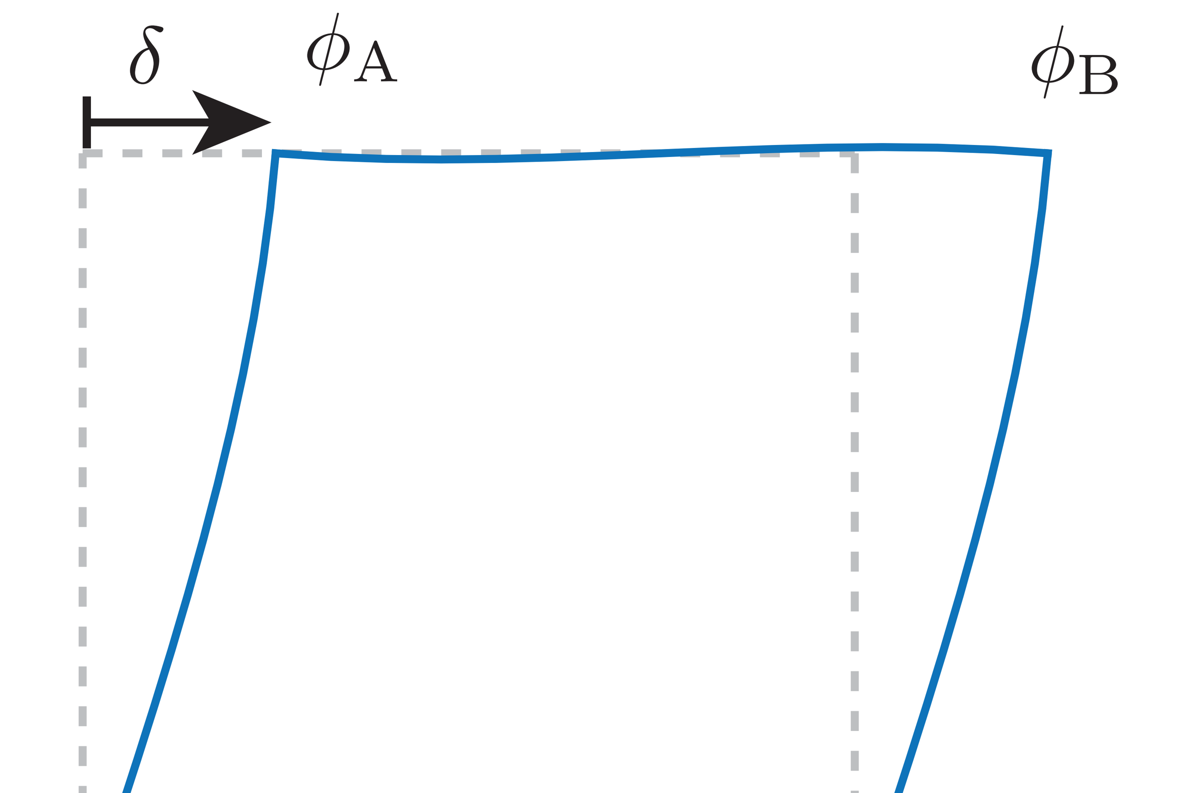
Example 1: one span, one floor, with a concentrated horizontal load

A. Analyze the structure with rigid joints under the **original loading**

Example 1: one span, one floor, with a concentrated horizontal load

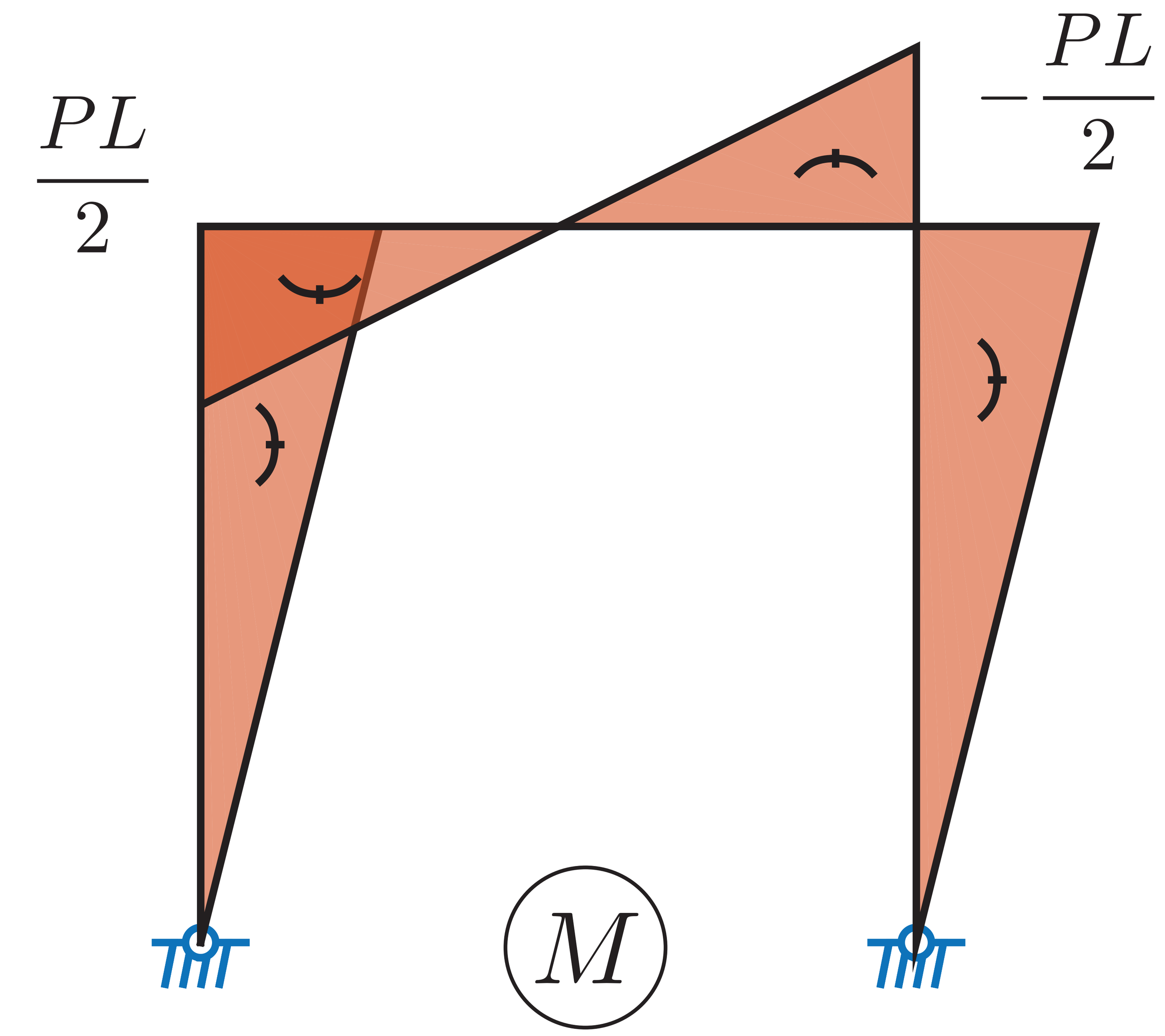
B. Use rotations and chord drifts to compute the **equivalent loading**

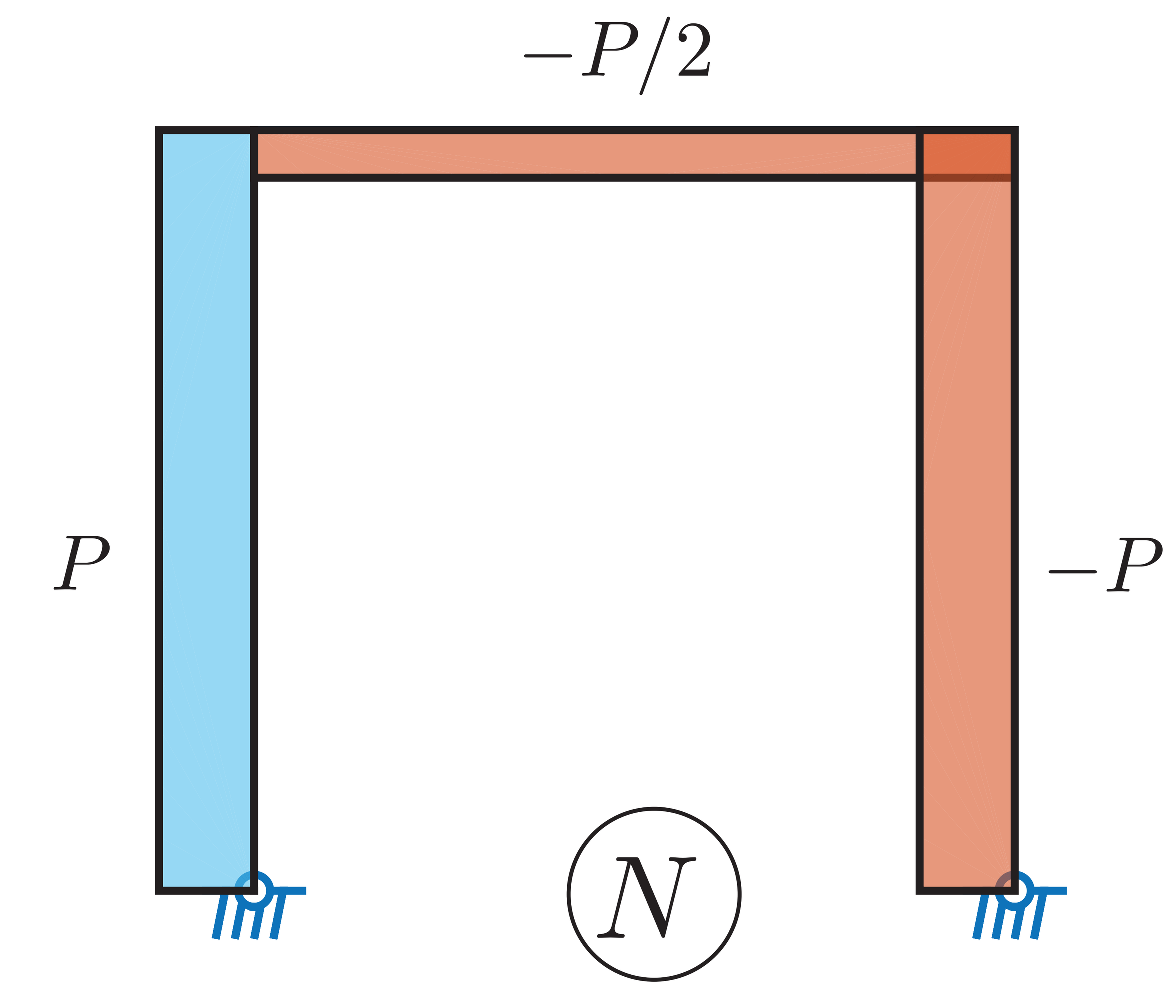


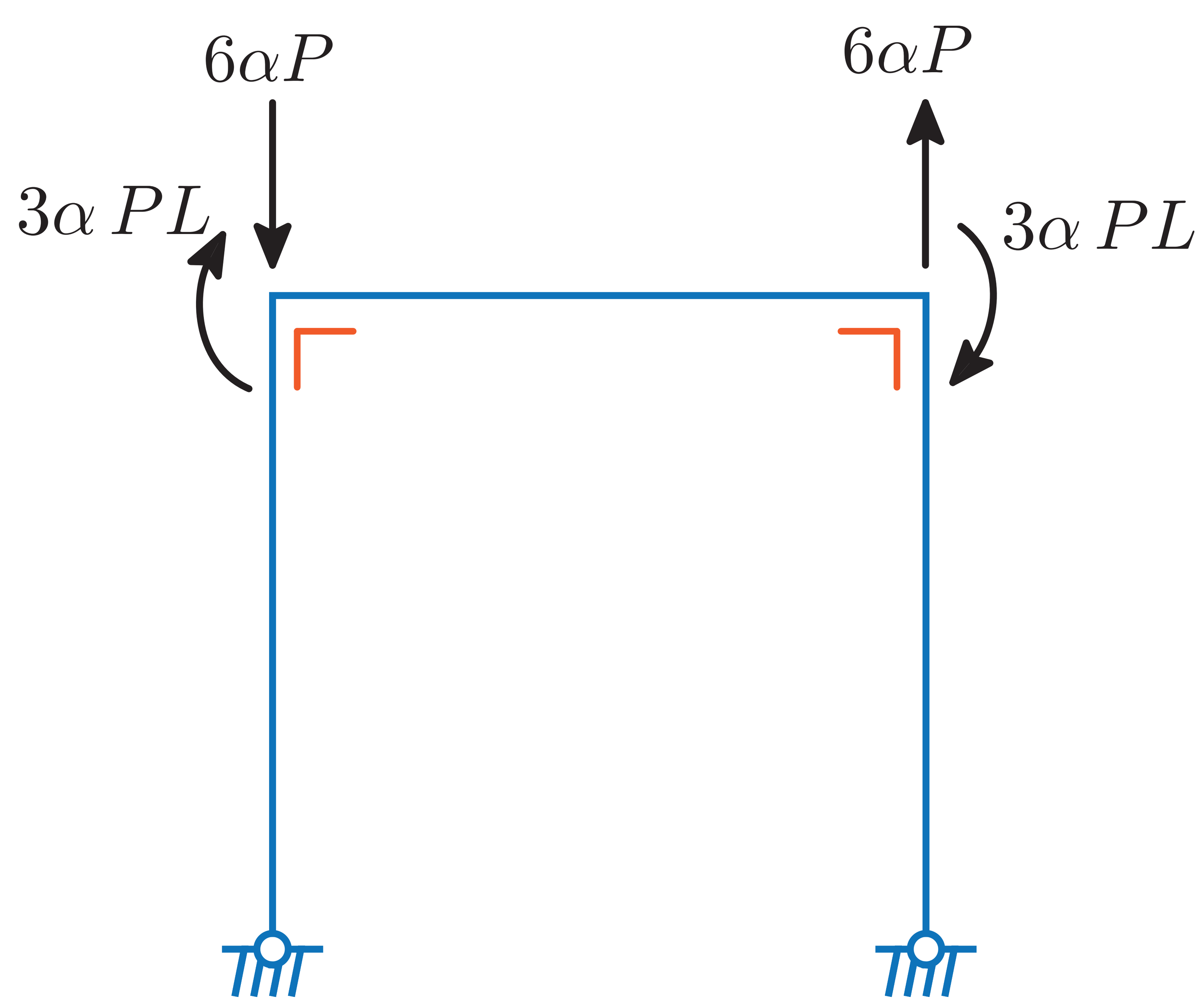


$$\delta = 0.25\frac{PL^3}{EI}$$

$$\boxed{\begin{array}{c}\phi_A = \phi_B \\ \psi = 0\end{array}} = -\frac{1}{12}\frac{PL^2}{EI}$$







$$M_1 = 4\alpha\frac{EI}{L}\left(5\phi_1 + 4\phi_2 - 9\psi\right)$$

$$M_2 = 4\alpha\frac{EI}{L}\left(4\phi_1 + 5\phi_2 - 9\psi\right)$$

$$F = 36\alpha\frac{EI}{L^2}\left(\phi_1 + \phi_2 - 2\psi\right)$$

Example 1: one span, one floor, with a concentrated horizontal load

C. Analyze the <u>structure with rigid joints</u> under the **equivalent loading**

Example 1: one span, one floor, with a concentrated horizontal load

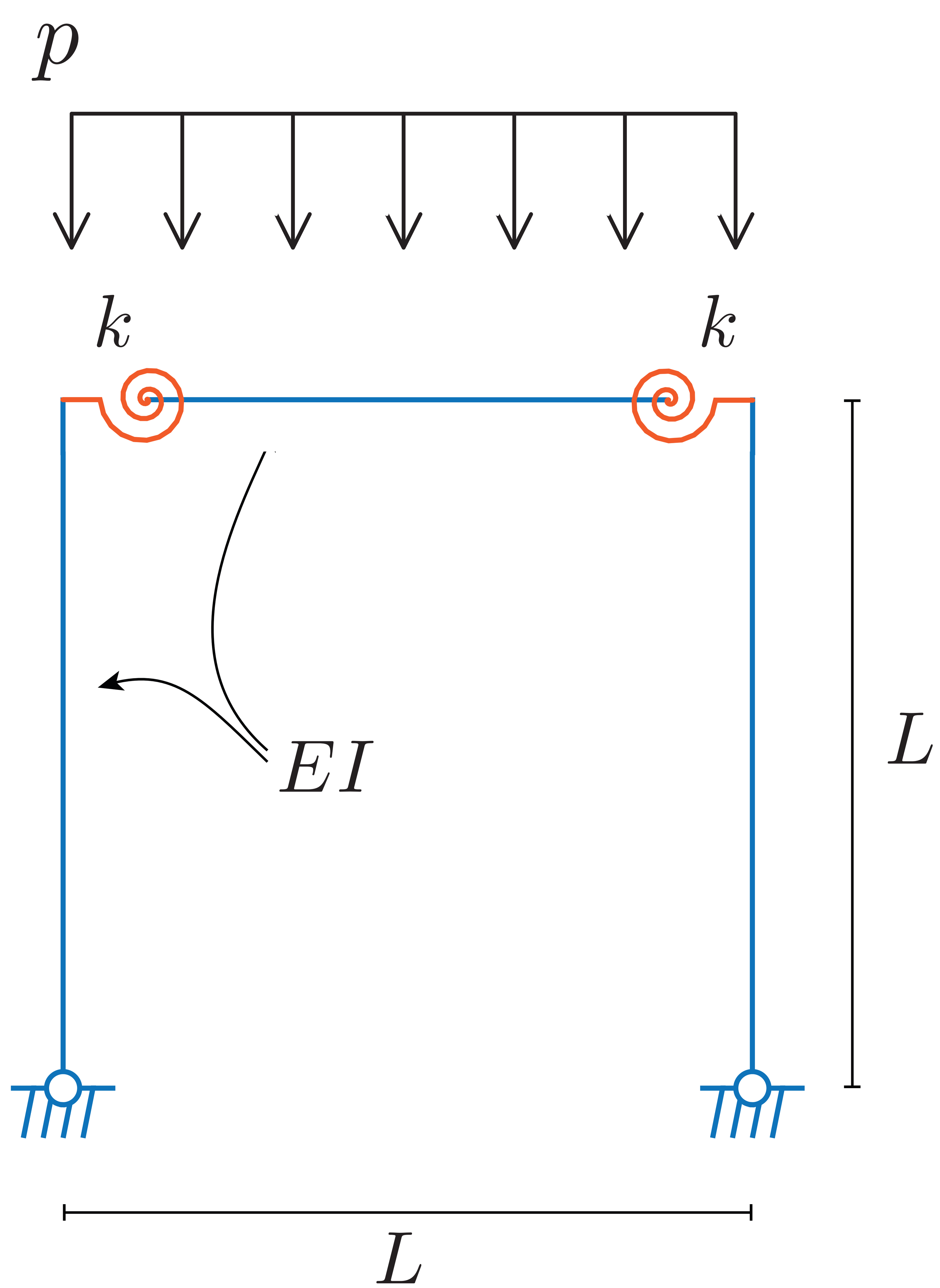
D. Add the displacements

Example 1: one span, one floor, with a concentrated horizontal load

E. Correct the internal forces

Example 1: one span, one floor, with a concentrated horizontal load

Example 2: one span, one floor, with distributed vertical load

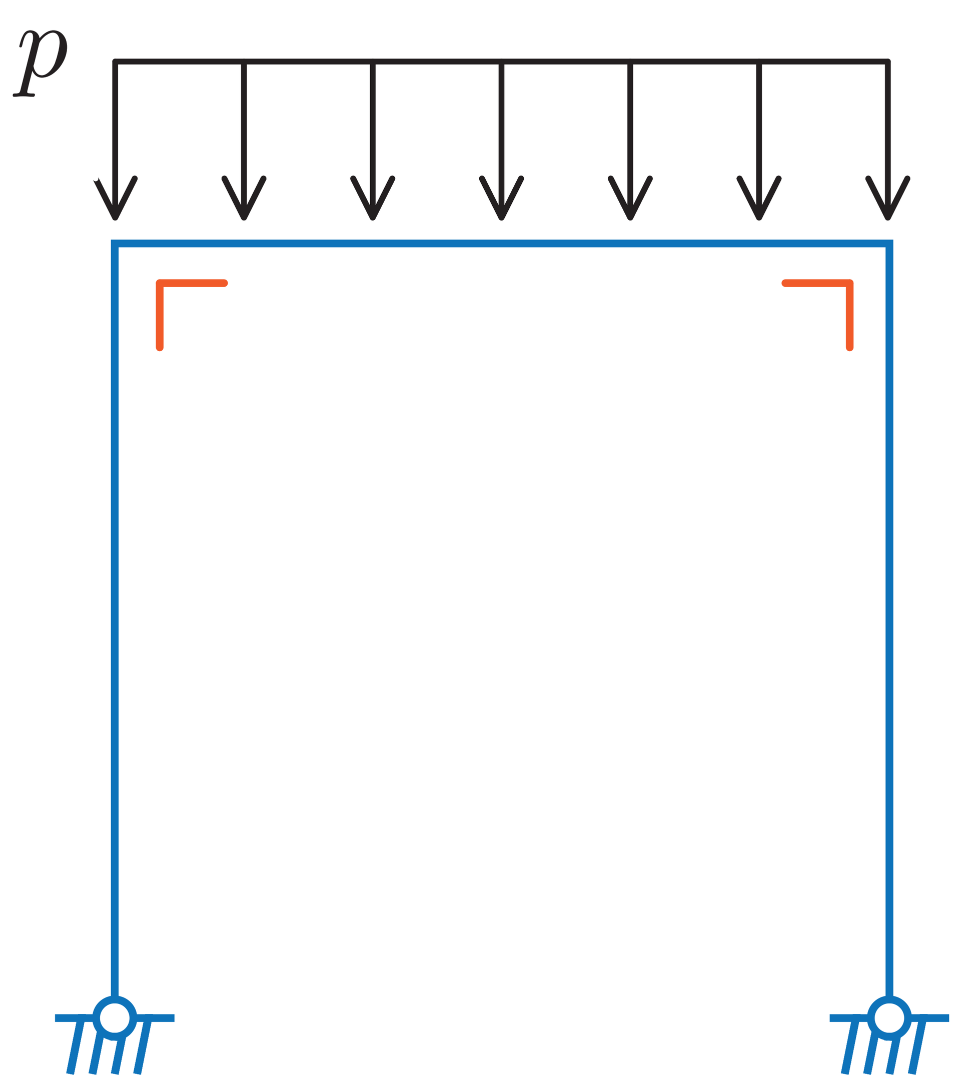


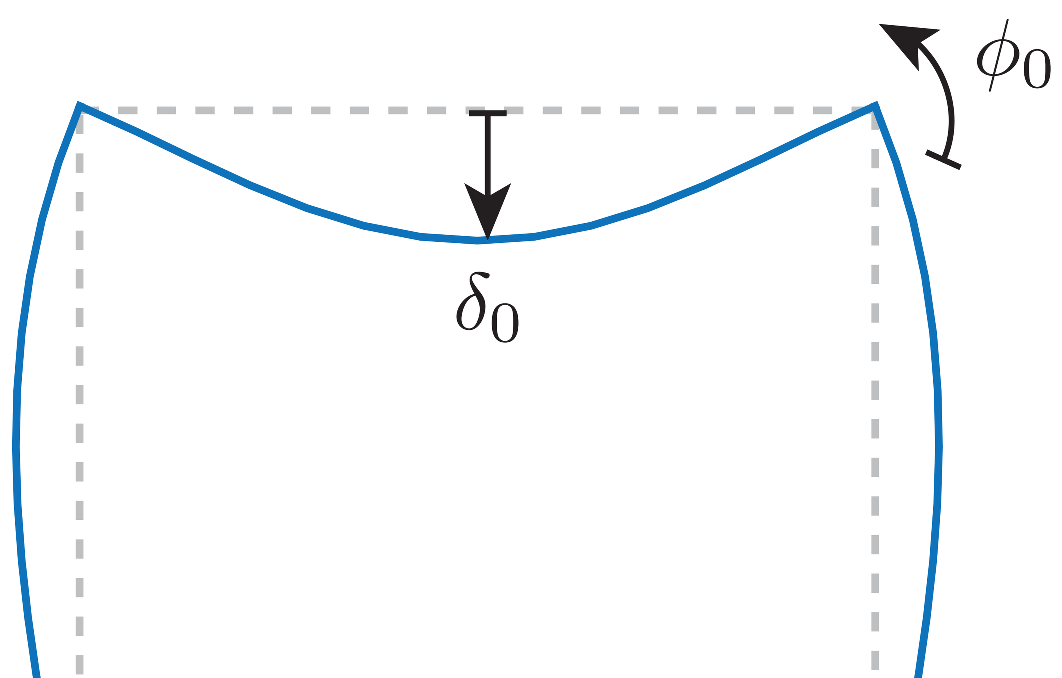
Example 2: one span, one floor, with distributed vertical load

A. Analyze the <u>structure with rigid joints</u> under the **original loading**

Example 2: one span, one floor, with distributed vertical load

B. Use rotations and chord drifts to compute the **equivalent loading**

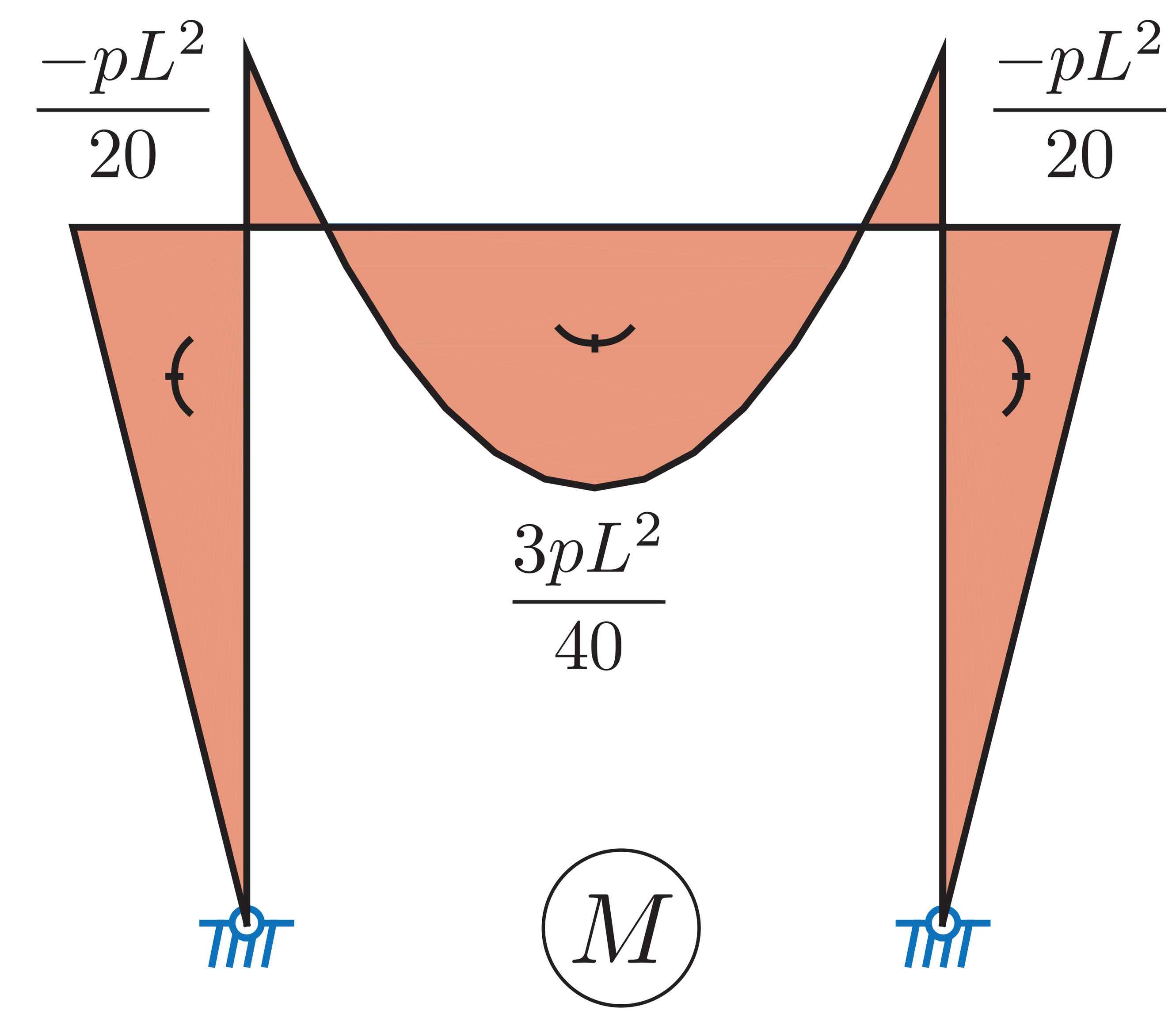


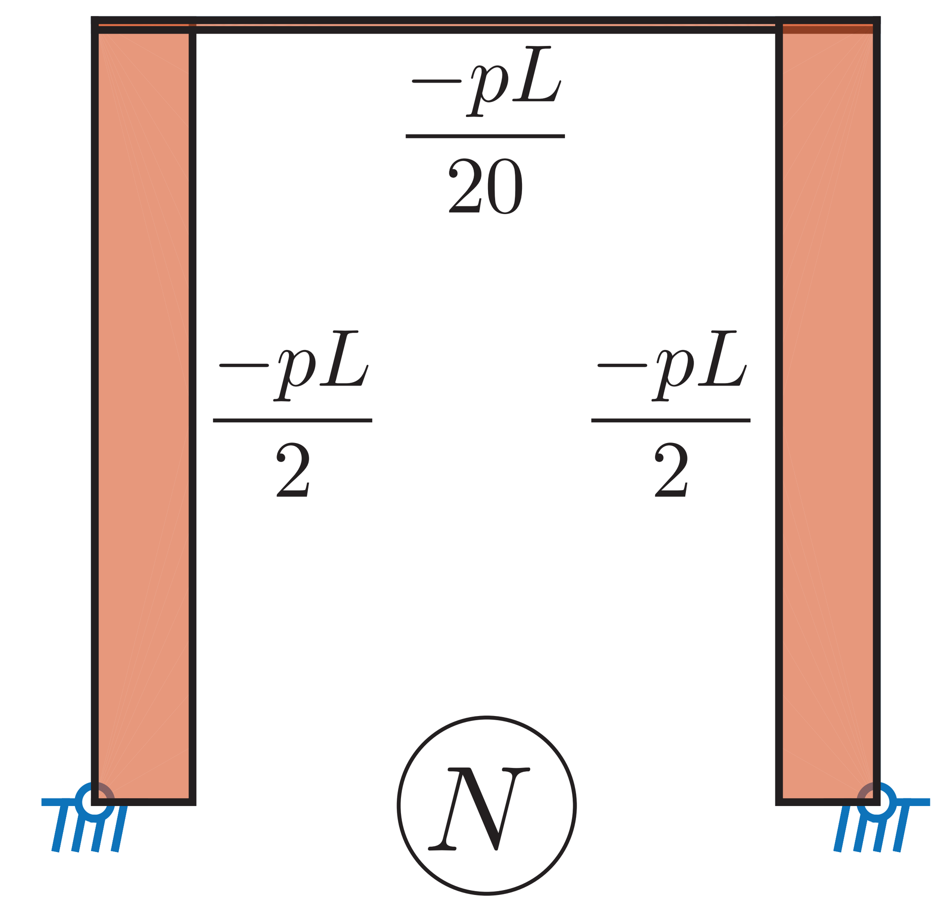


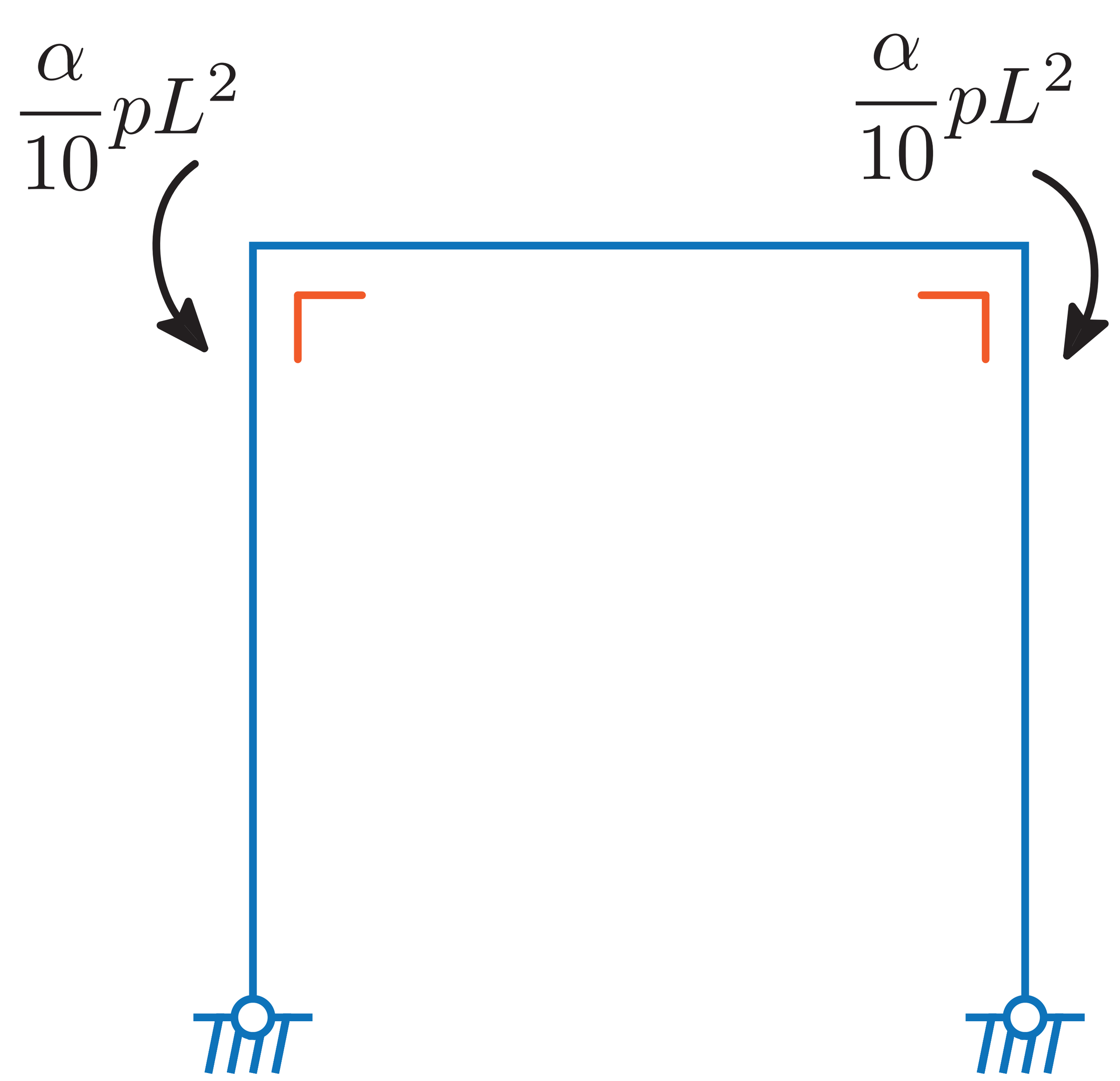
$$\delta_0 = \frac{13}{1920}\frac{pL^4}{EI}$$

$$\phi_0 = \frac{1}{60}\frac{pL^3}{EI}$$

$$\psi = 0$$





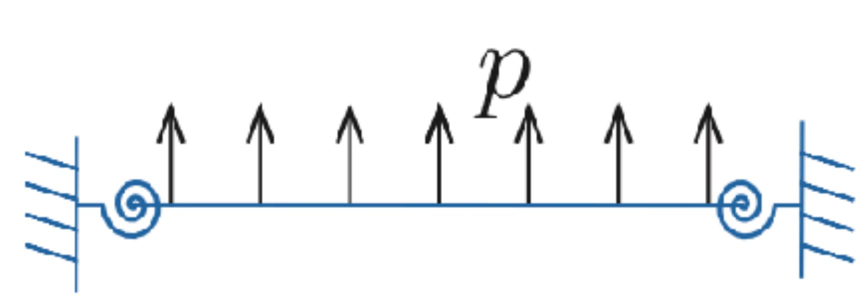


$$M_1 = 4\alpha\frac{EI}{L}\left(5\phi_1 + 4\phi_2 - 9\psi\right)$$

$$M_2 = 4\alpha\frac{EI}{L}\left(4\phi_1 + 5\phi_2 - 9\psi\right)$$

$$F = 36\alpha\frac{EI}{L^2}\left(\phi_1 + \phi_2 - 2\psi\right)$$

$$\hat{\mathbf{p}} = \alpha\frac{EI}{L}\begin{pmatrix} \frac{36}{L}(\phi_1+\phi_2) \\ 4(5\phi_1+4\phi_2) \\ -\frac{36}{L}(\phi_1+\phi_2) \\ 4(4\phi_1+5\phi_2) \end{pmatrix} = \frac{\alpha}{15}pL^2\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$



$$\Delta\mathbf{p}^{(e)} = \begin{pmatrix} -\frac{1}{2}pL(\alpha_1-\alpha_2) \\ -\frac{1}{12}pL^2(4\alpha_1-2\alpha_2) \\ \frac{1}{2}pL(\alpha_1-\alpha_2) \\ -\frac{1}{12}pL^2(2\alpha_1-4\alpha_2) \end{pmatrix} = \frac{\alpha}{6}pL^2\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Example 2: one span, one floor, with distributed vertical load

C. Analyze the structure with rigid joints under the **equivalent loading**

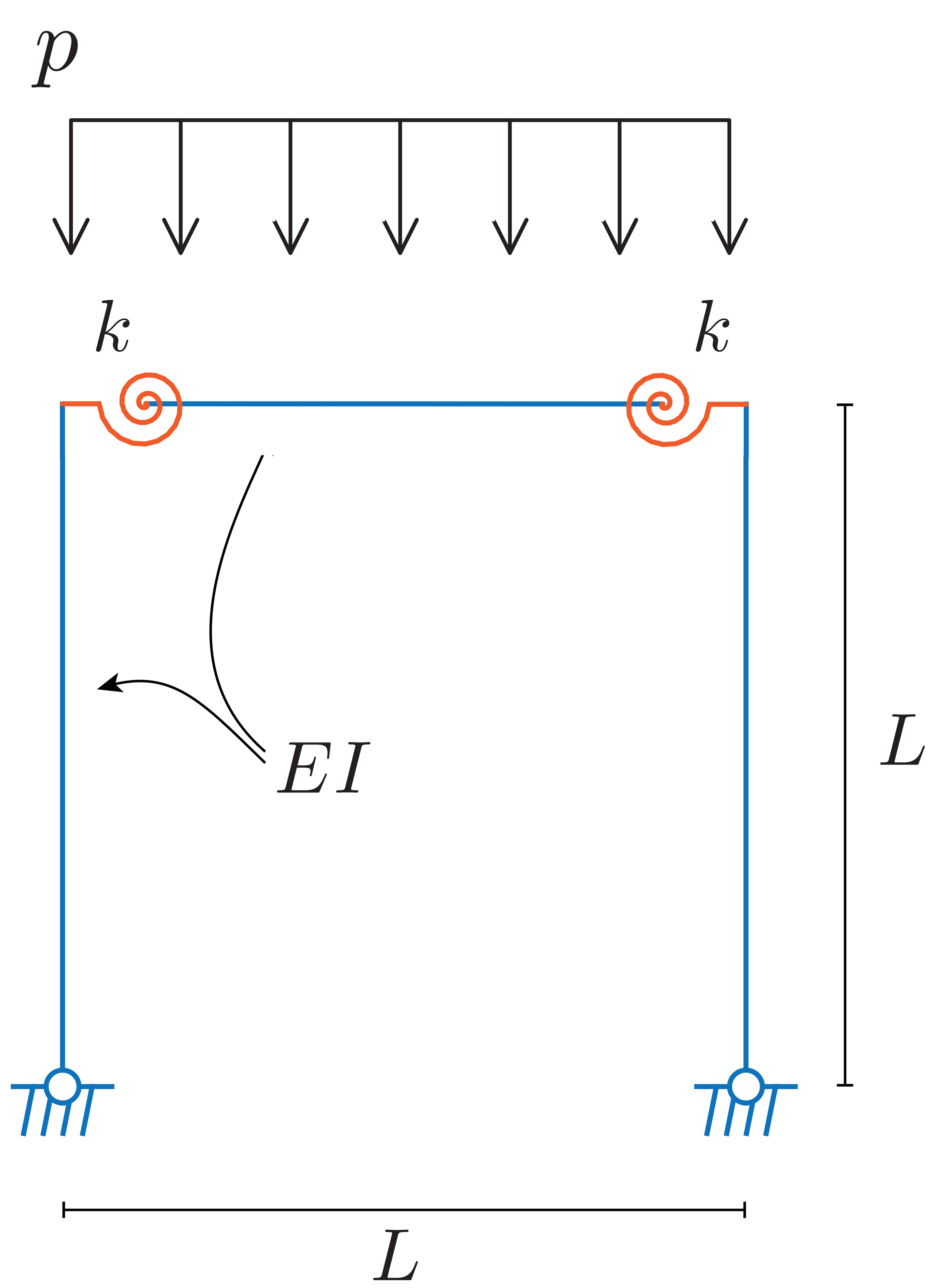
Example 2: one span, one floor, with distributed vertical load

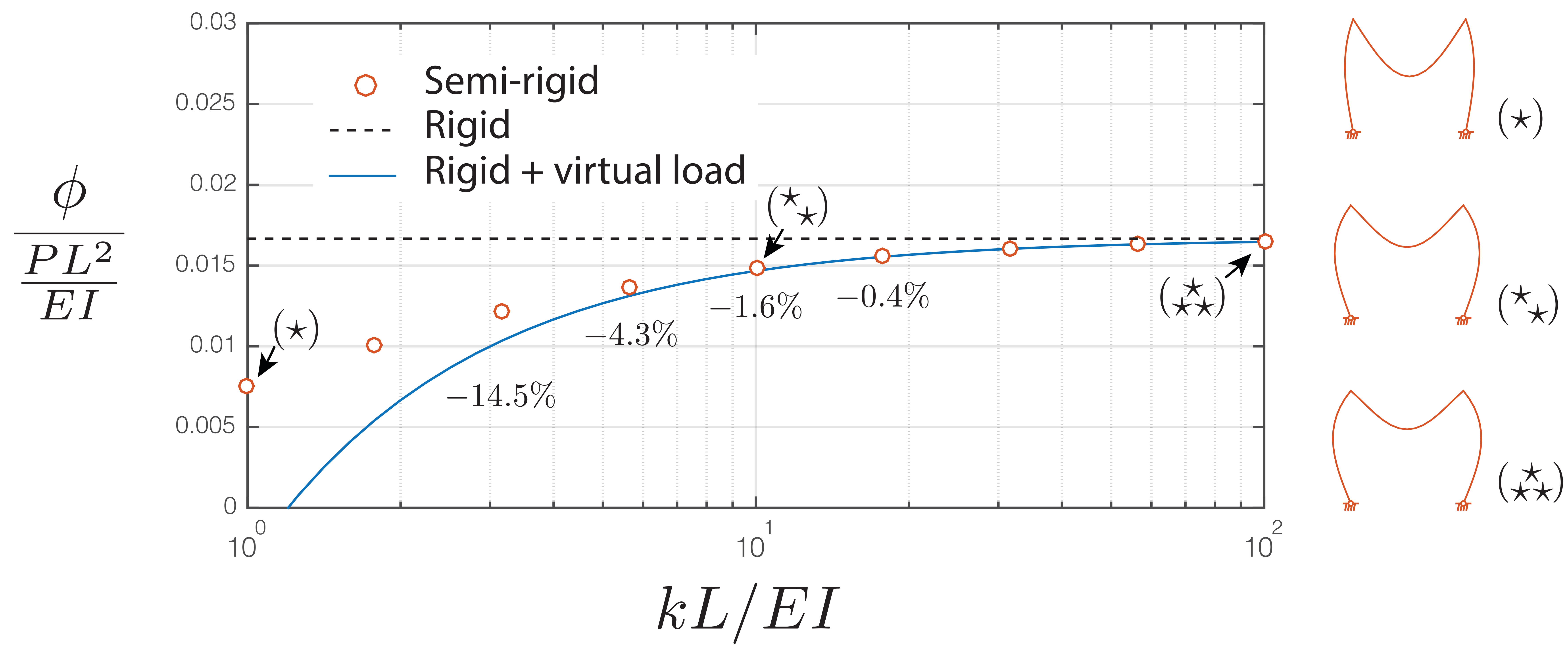
D. Add the displacements

Example 2: one span, one floor, with distributed vertical load

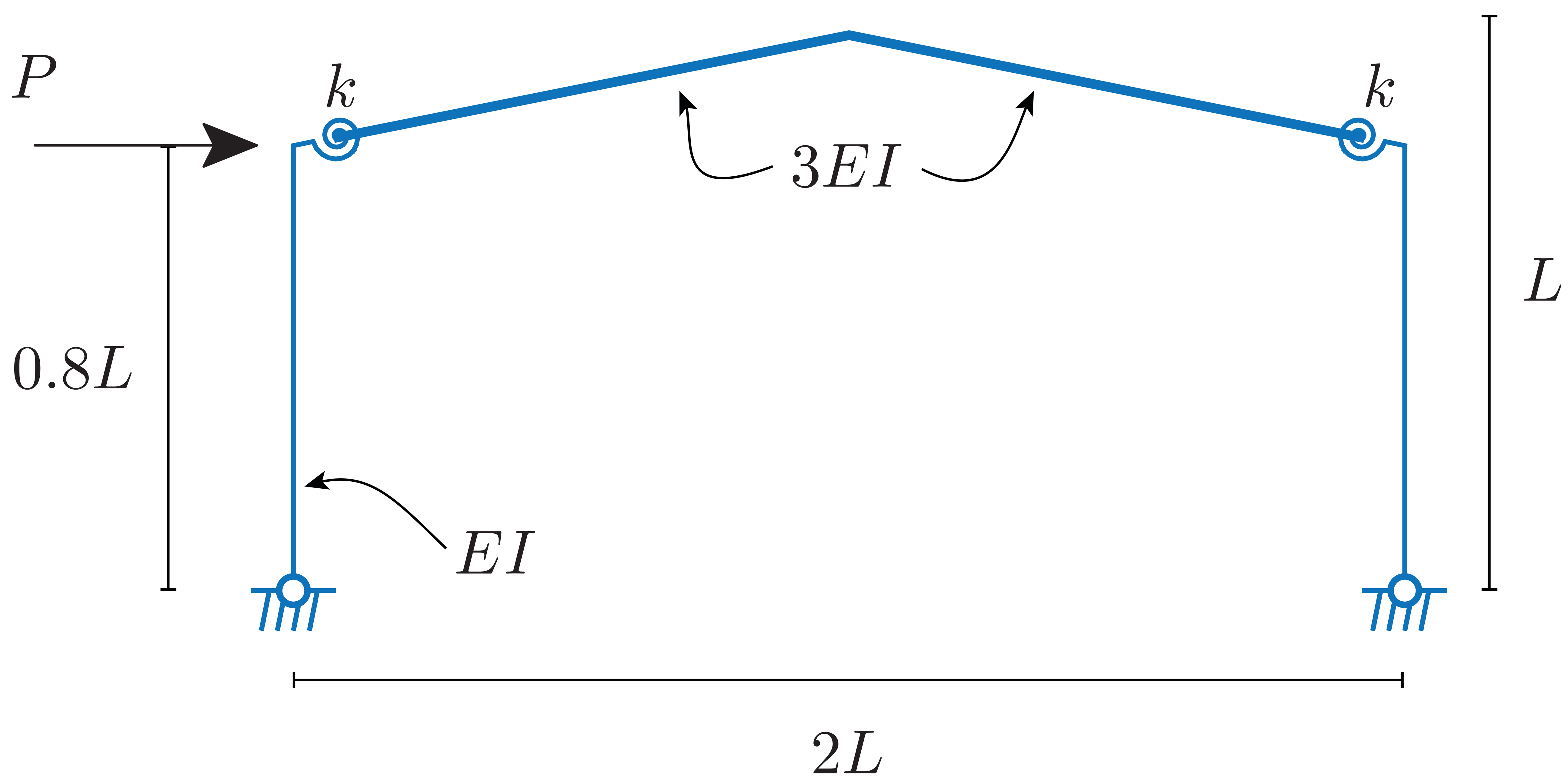
E. Correct the internal forces

Example 2: one span, one floor, with distributed vertical load



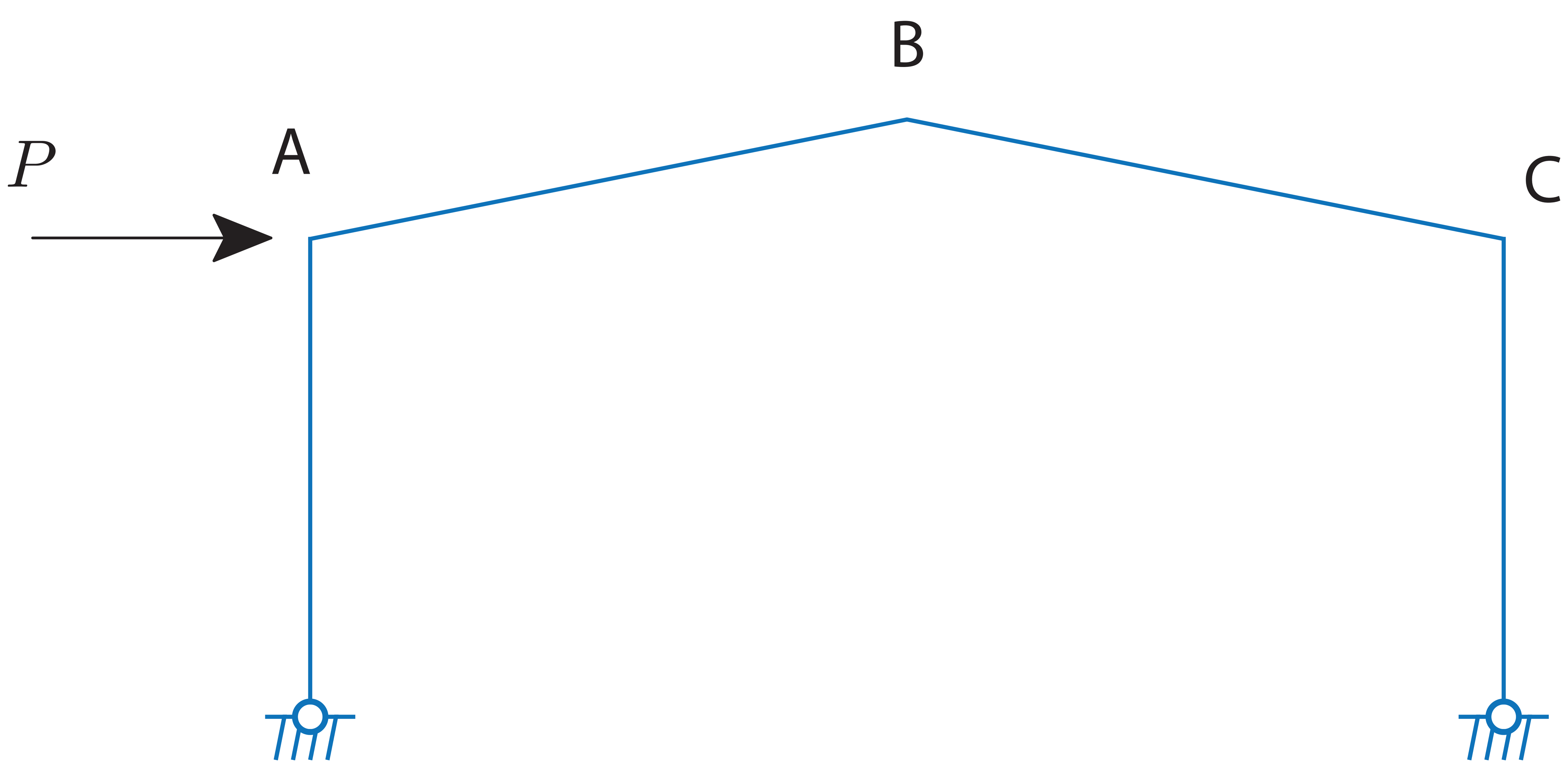


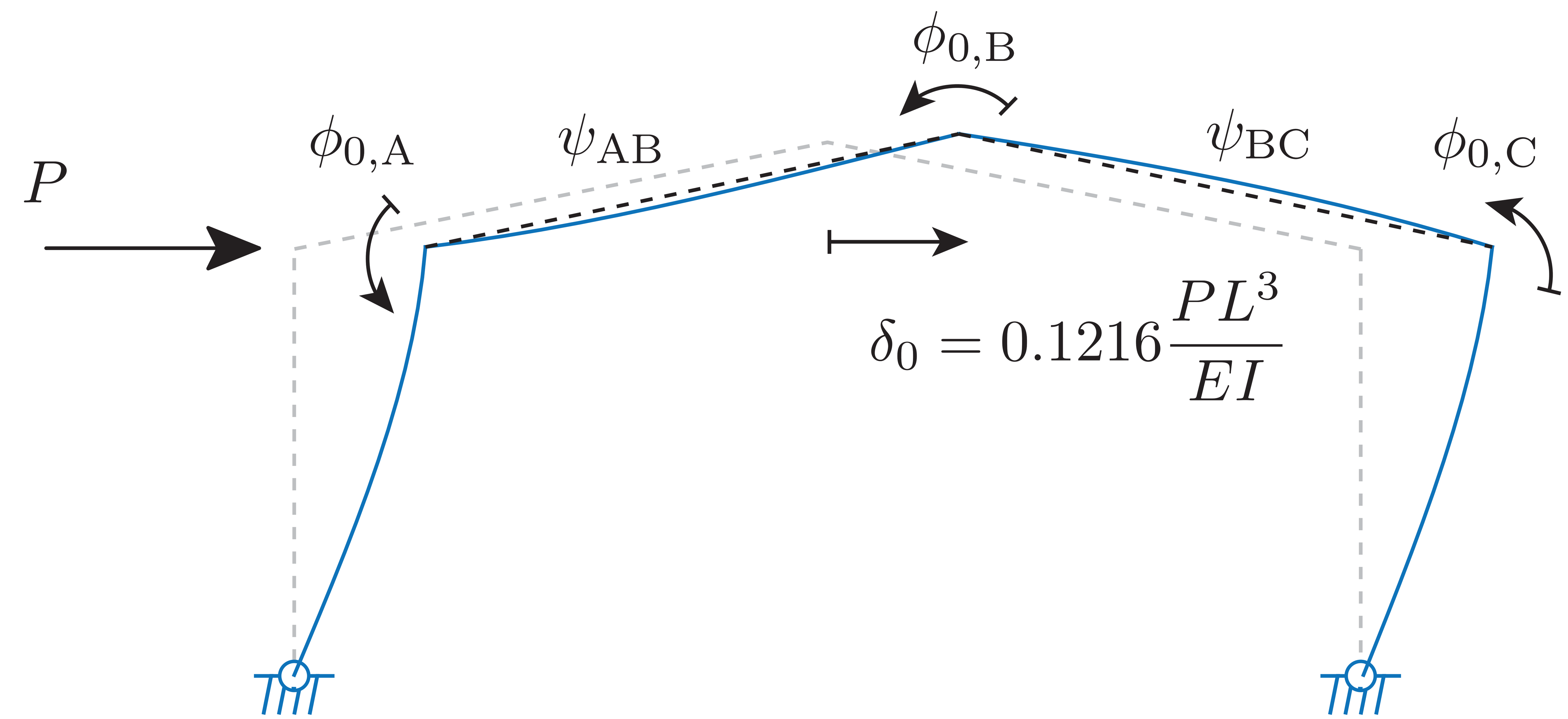
Example 3: portal frame with beam stiffer than columns

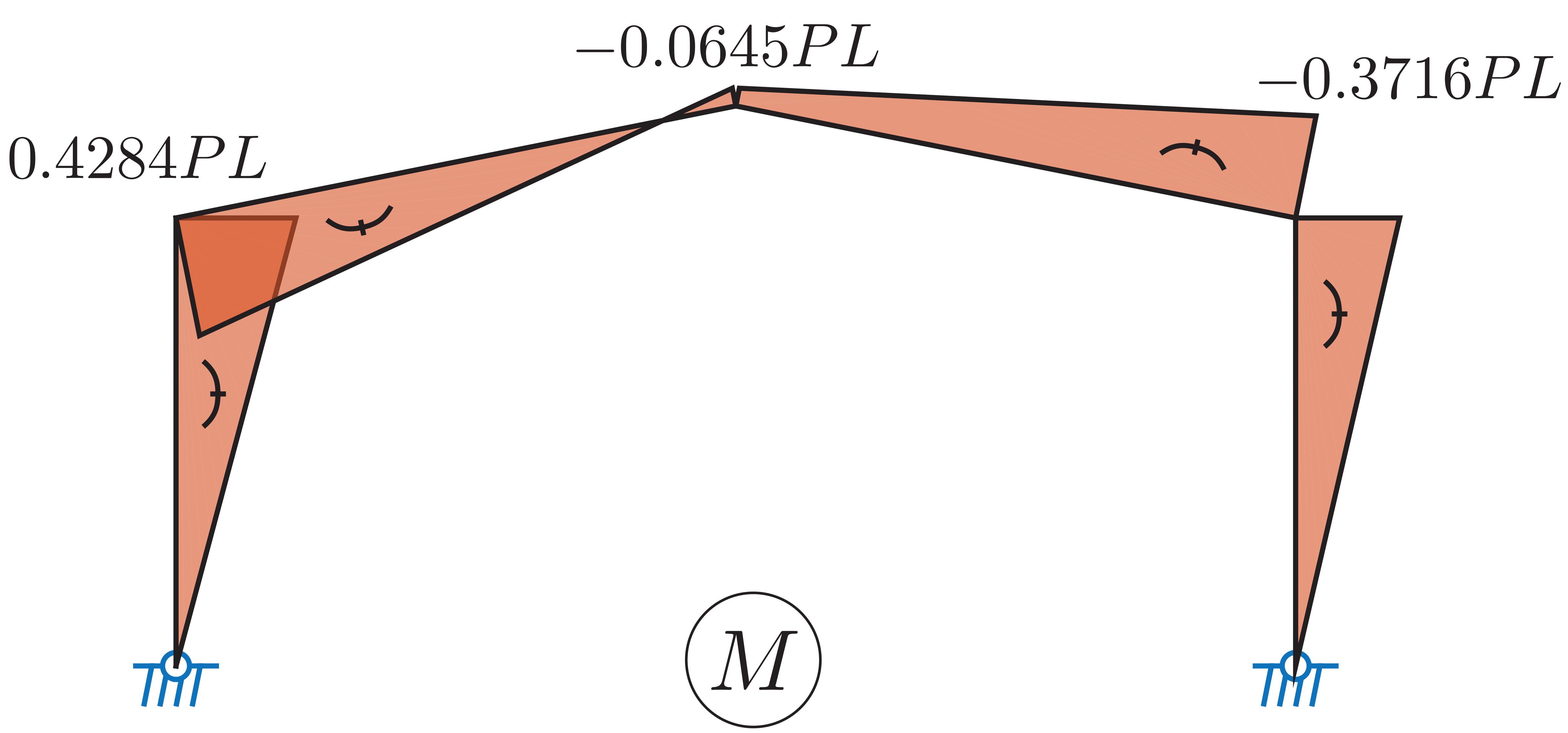


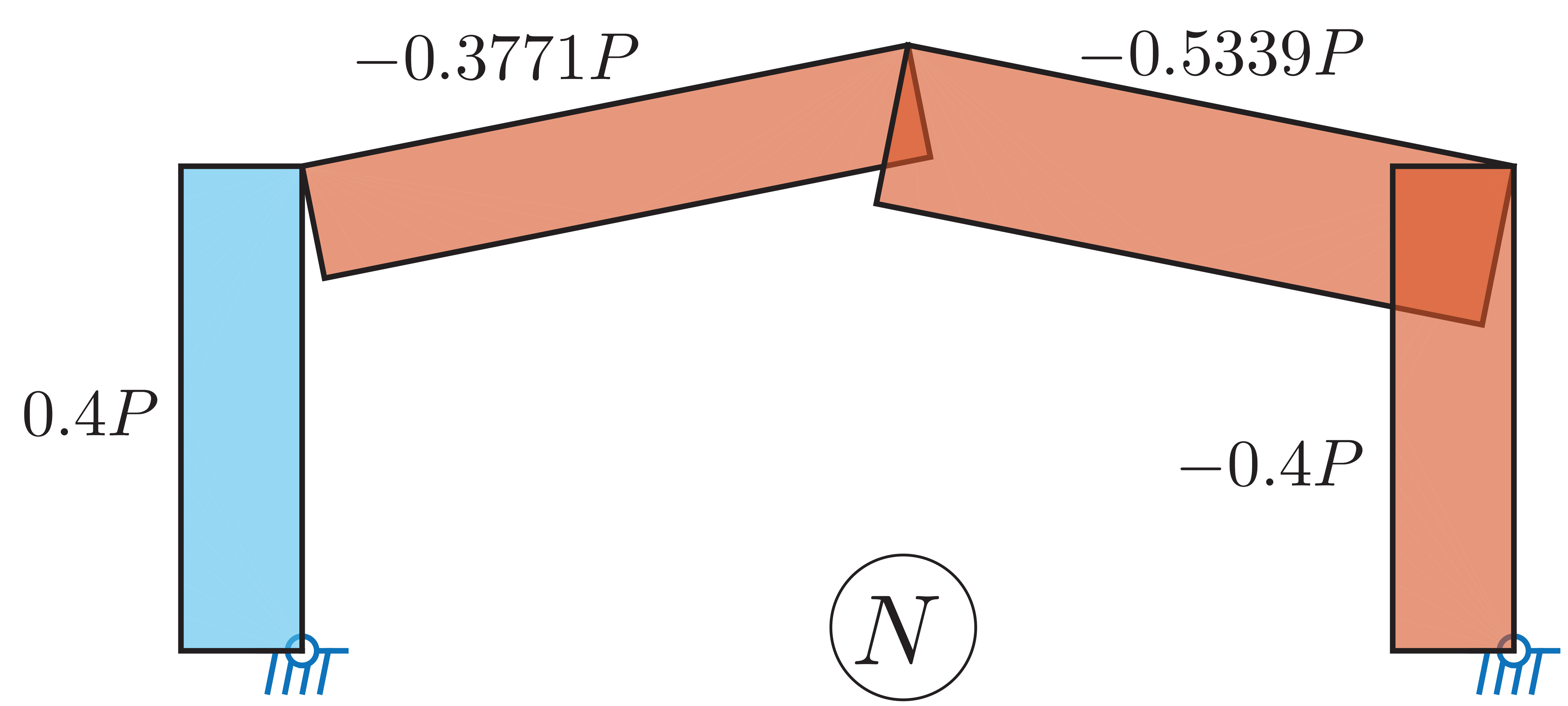
Example 3: portal frame with beam stiffer than columns

A. Analyze the structure with rigid joints under the **original loading**



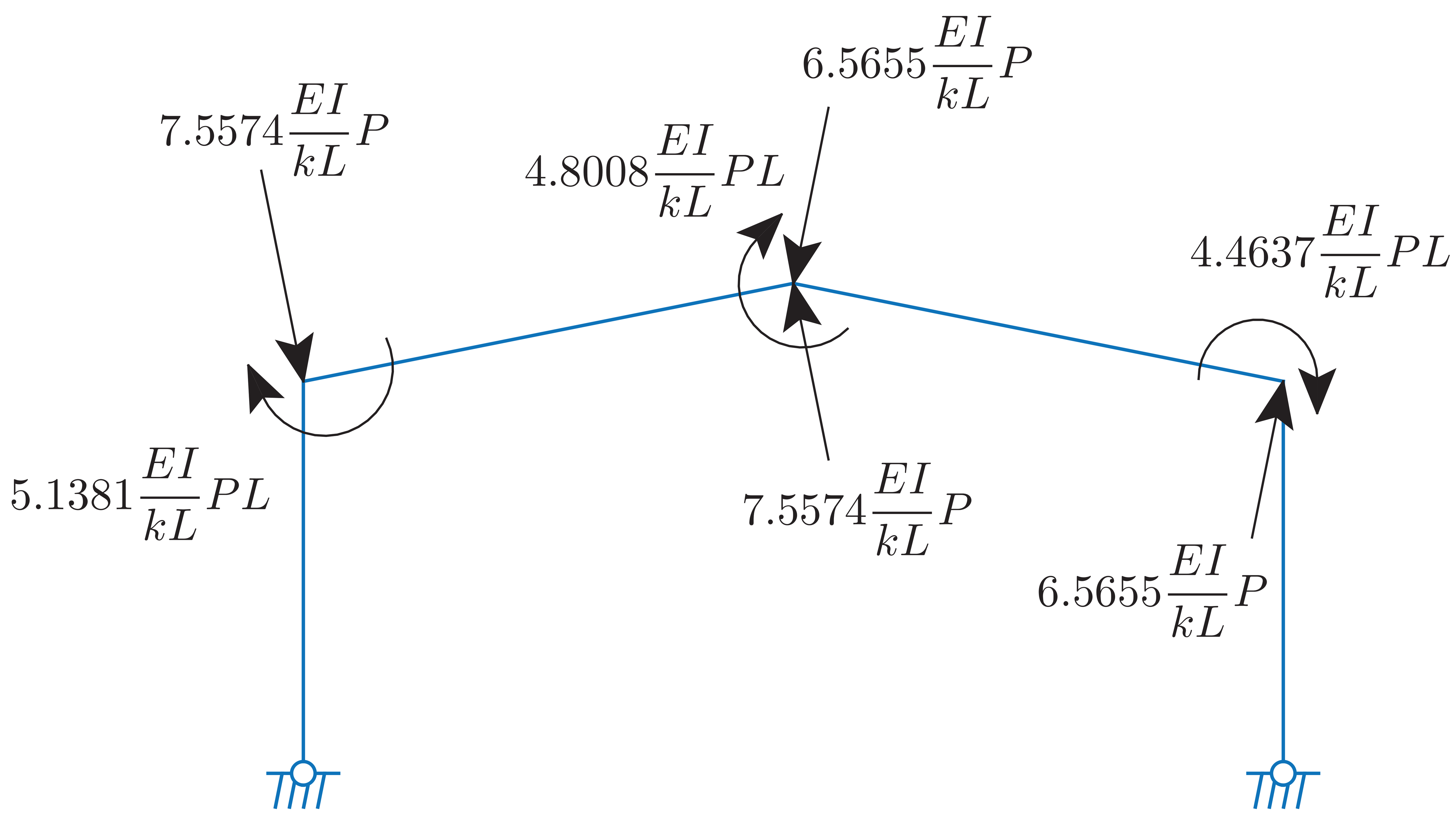


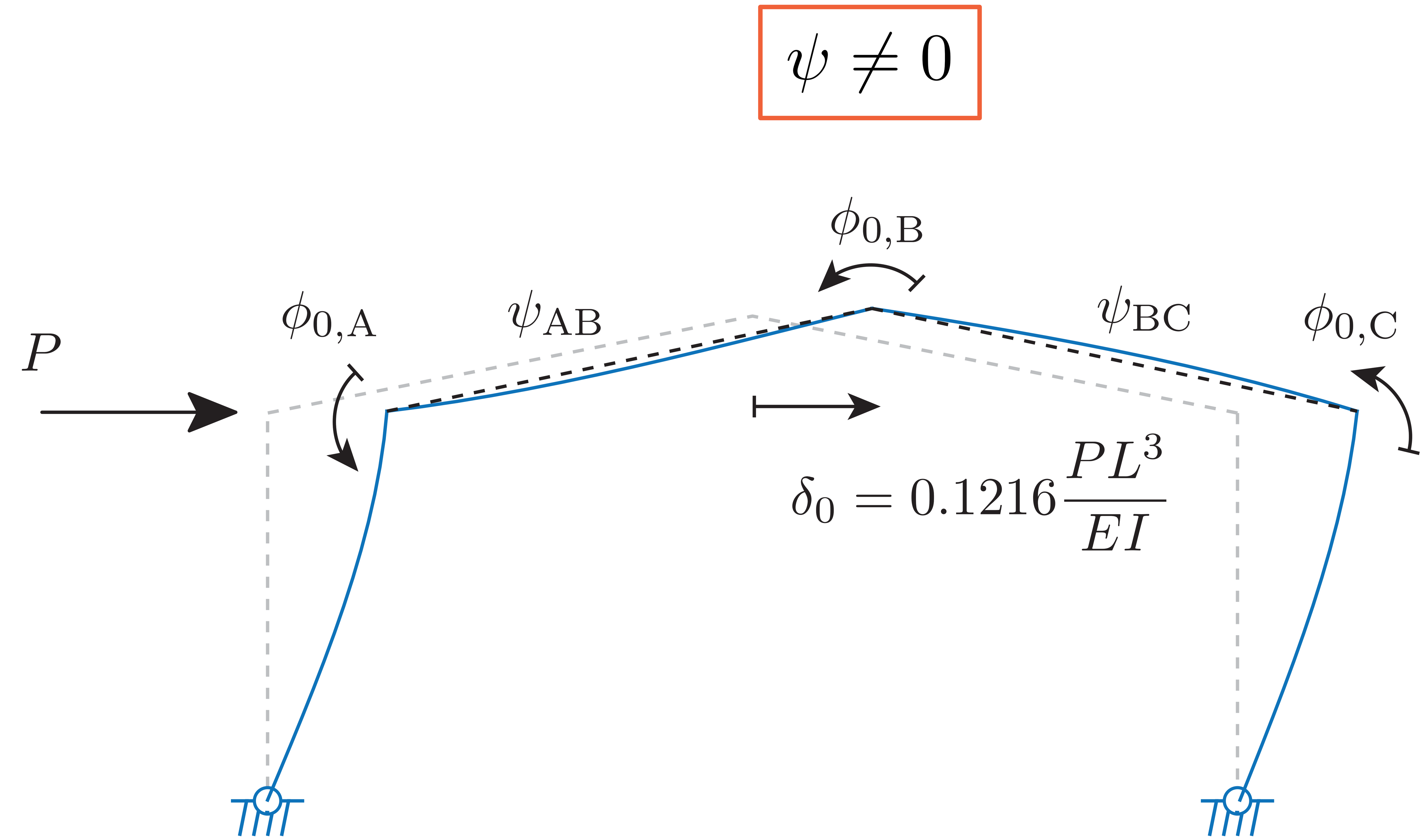




Example 3: portal frame with beam stiffer than columns

B. Use <u>rotations and chord drifts</u> to compute the **equivalent loading**





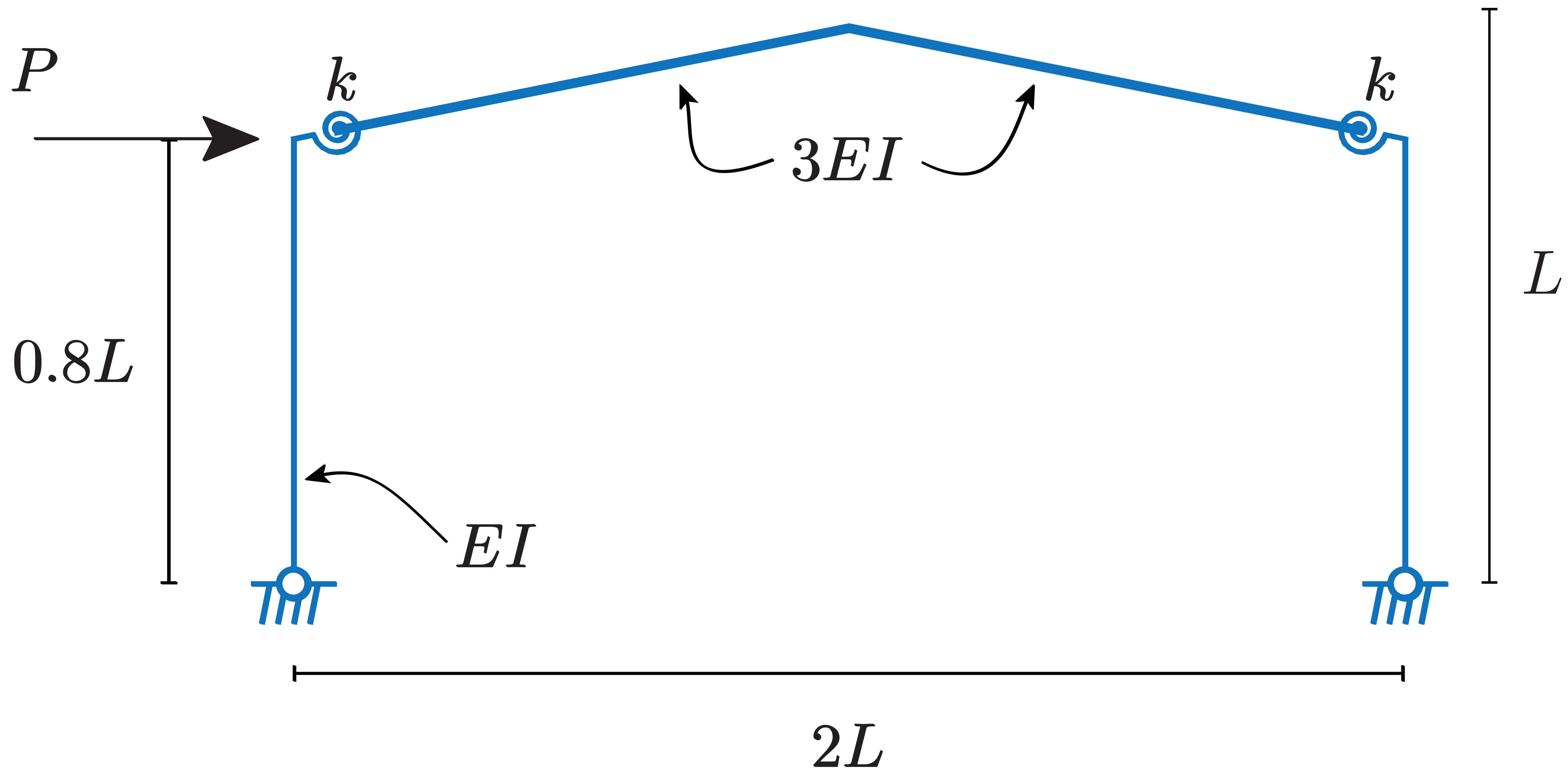
Example 3: portal frame with beam stiffer than columns

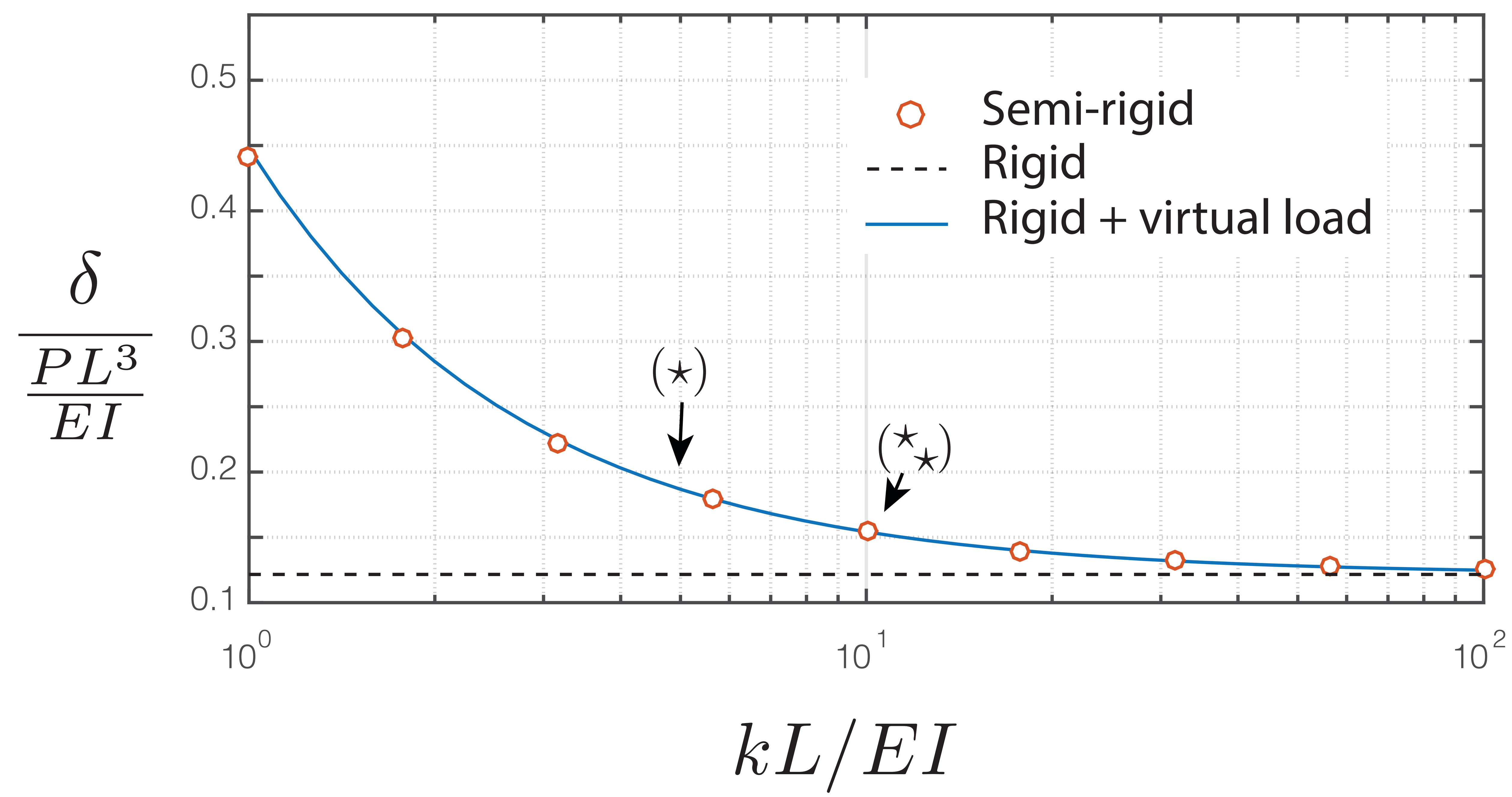
C. Analyze the <u>structure with rigid joints</u> under the **equivalent loading**

Example 3: portal frame with beam stiffer than columns

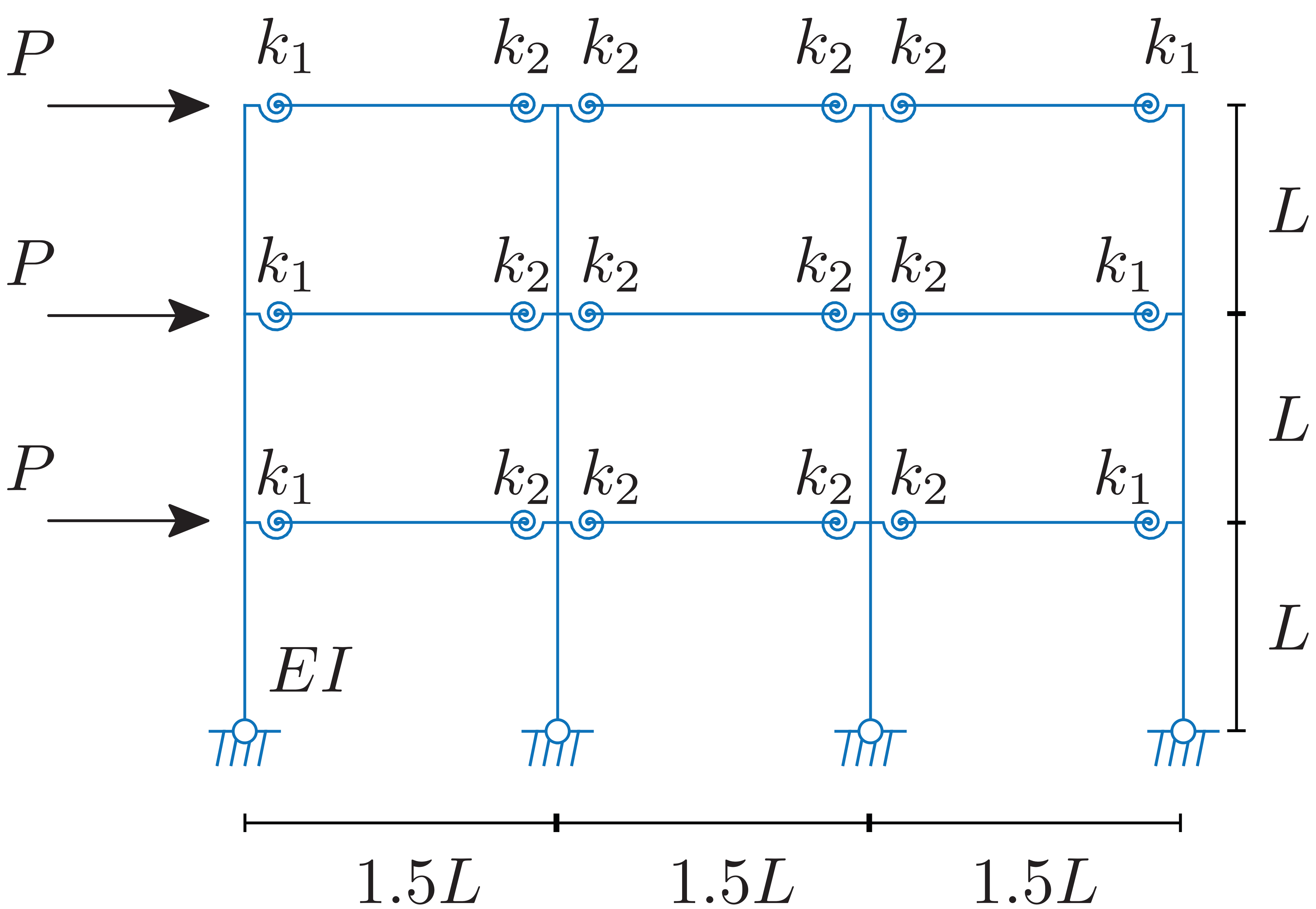
D. Add the displacements

E. Correct the internal forces





Example 4: three spans, three floors

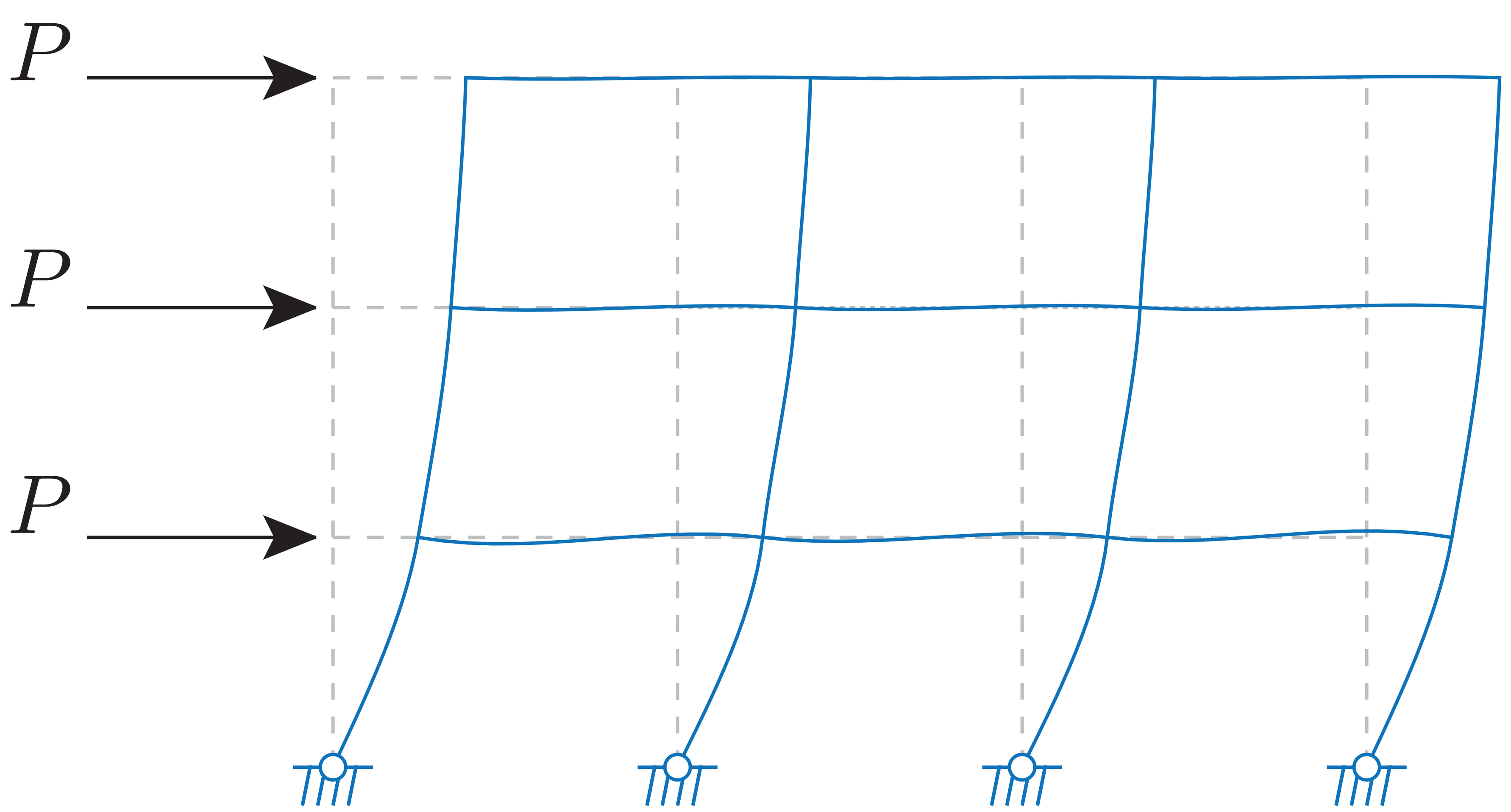


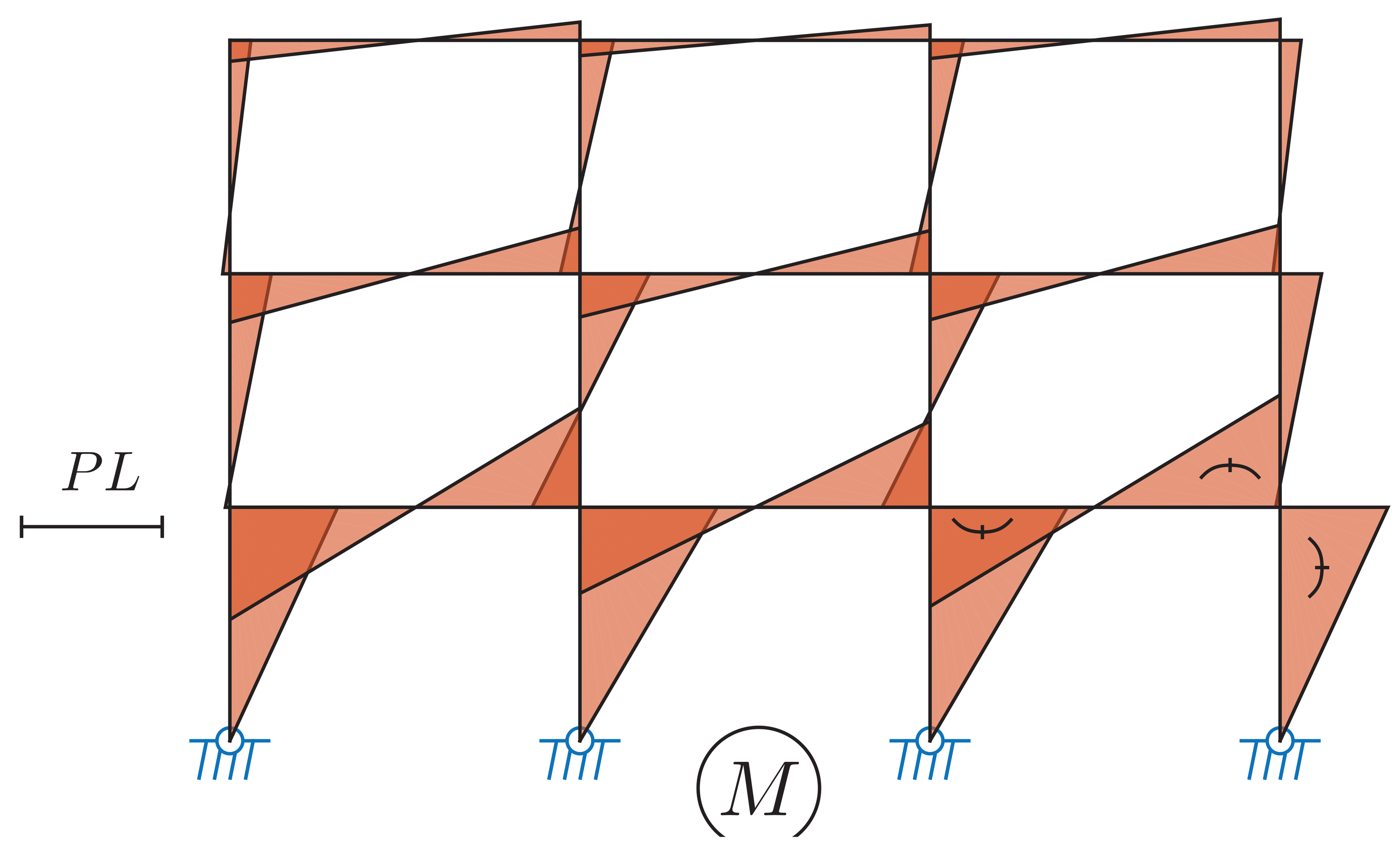
Example 4: three spans, three floors

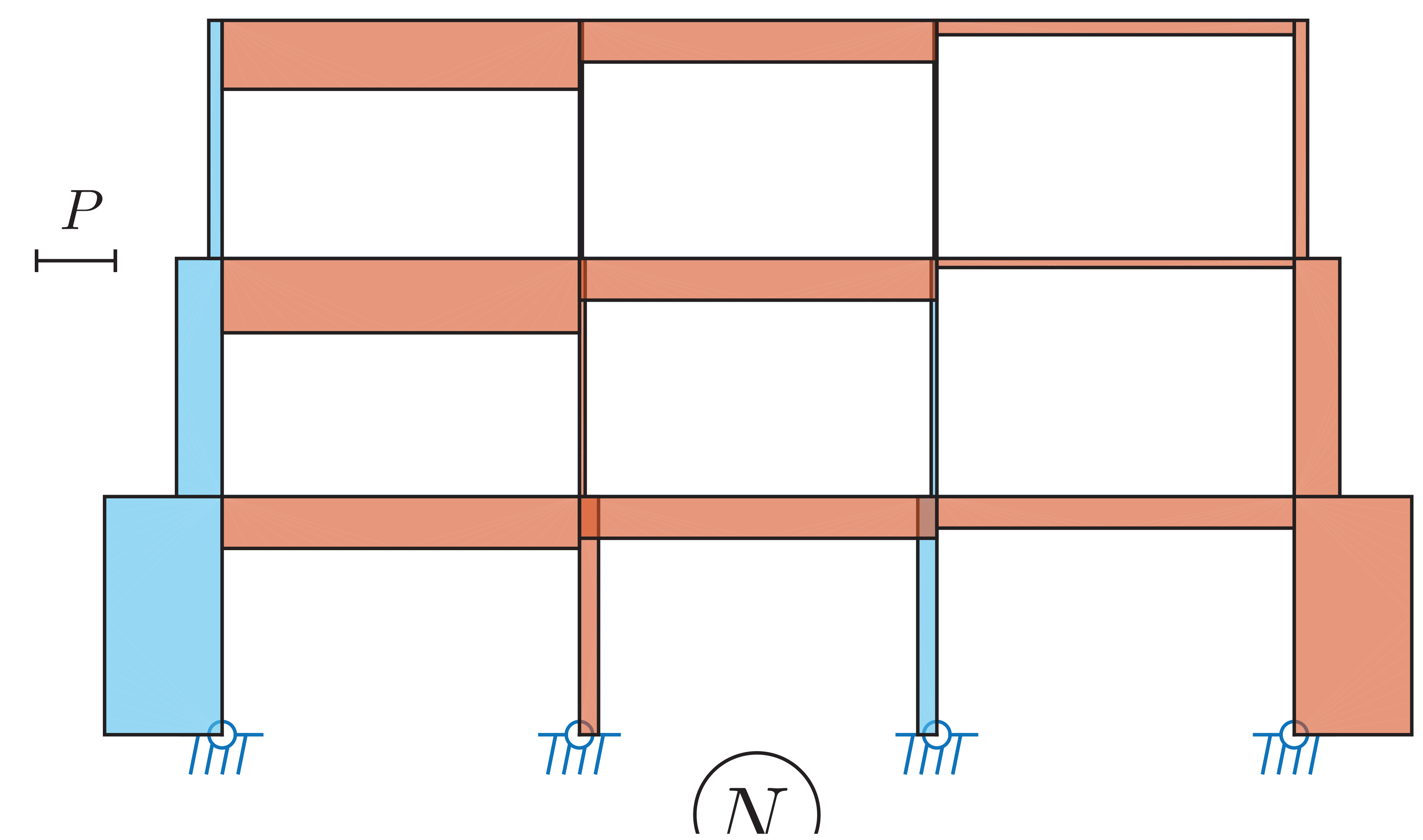
A. Analyze the structure with rigid joints under the **original loading**

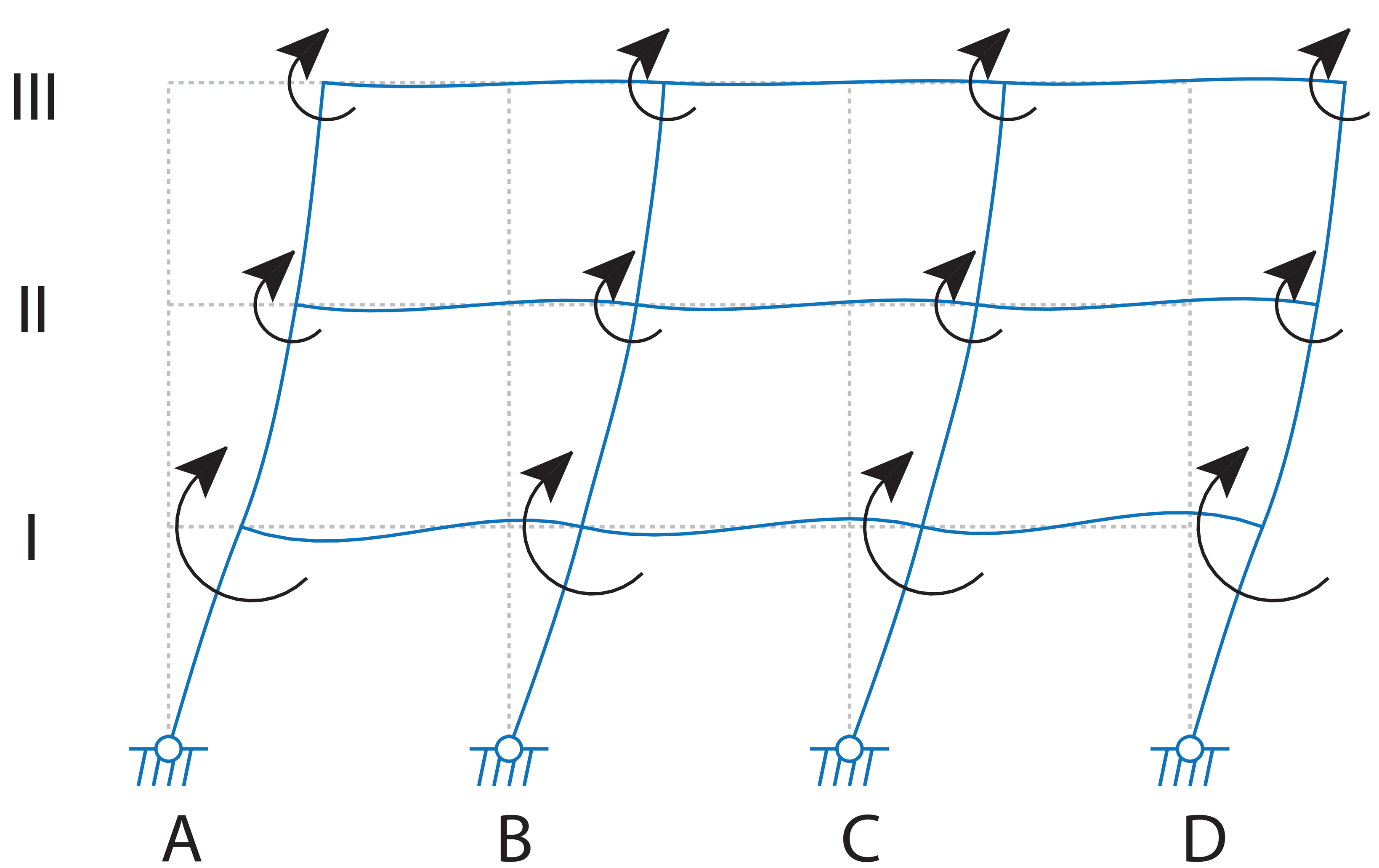
Example 4: three spans, three floors

B. Use rotations and chord drifts to compute the **equivalent loading**









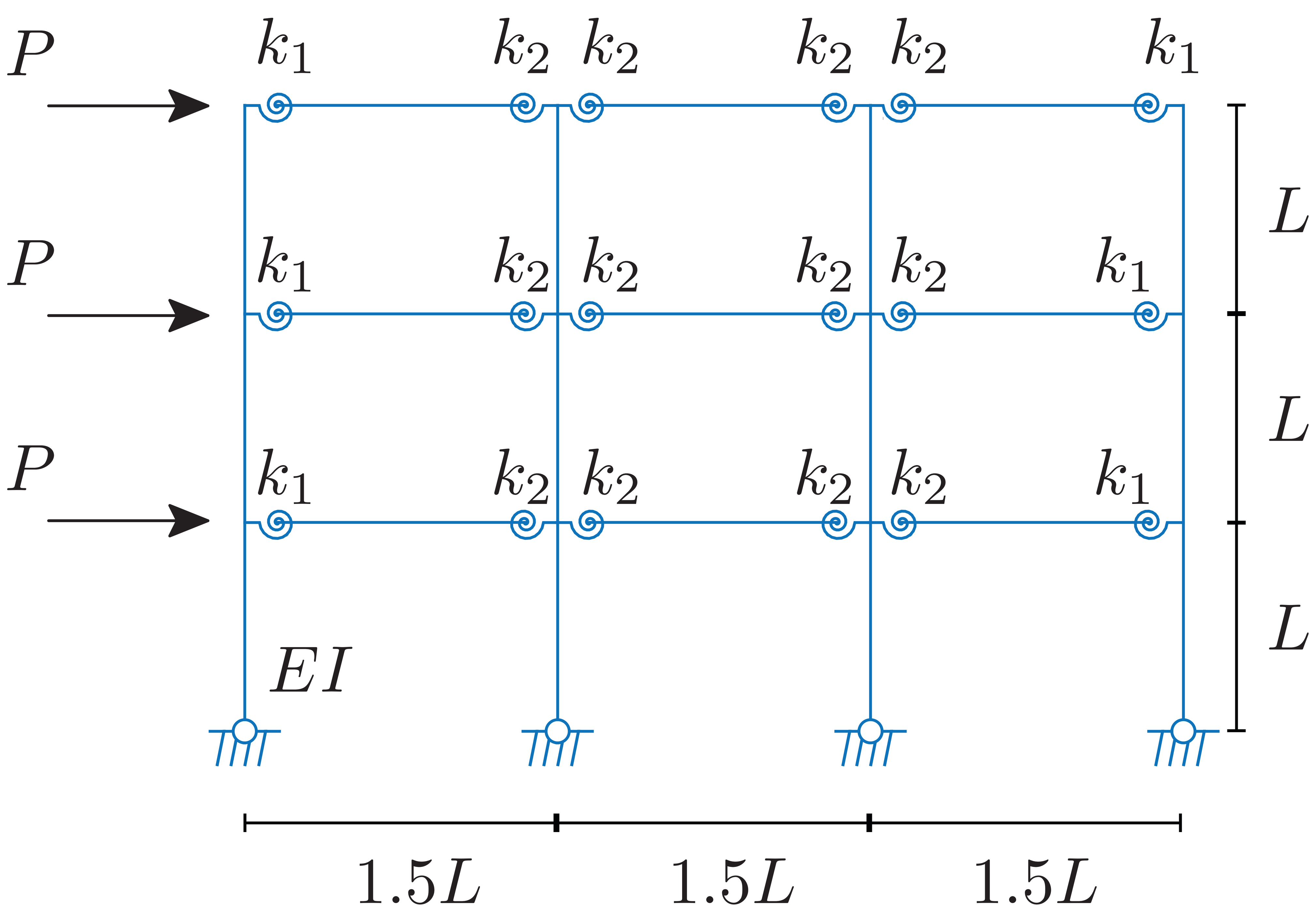
Example 4: three spans, three floors

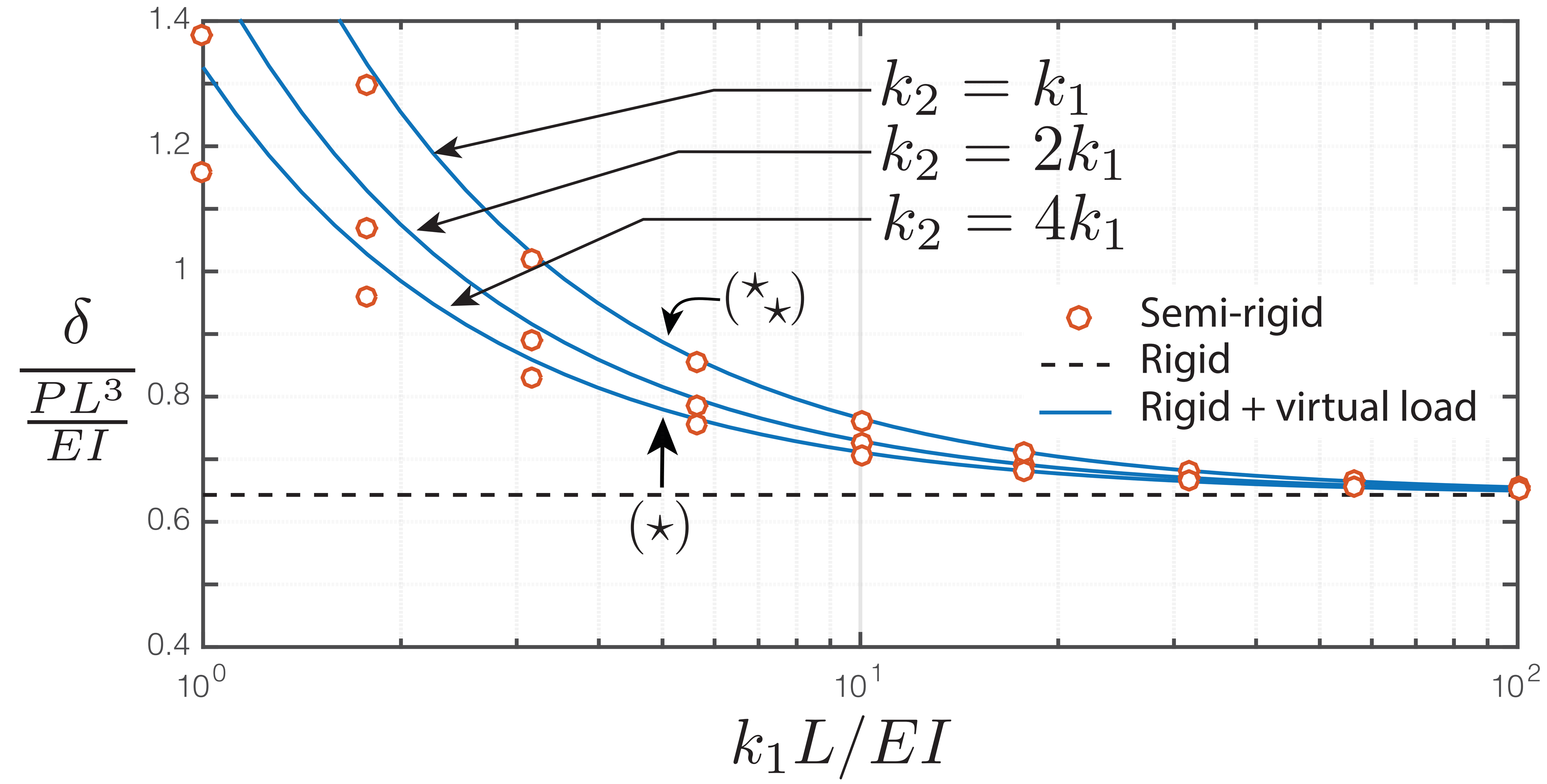
C. Analyze the structure with rigid joints under the **equivalent loading**

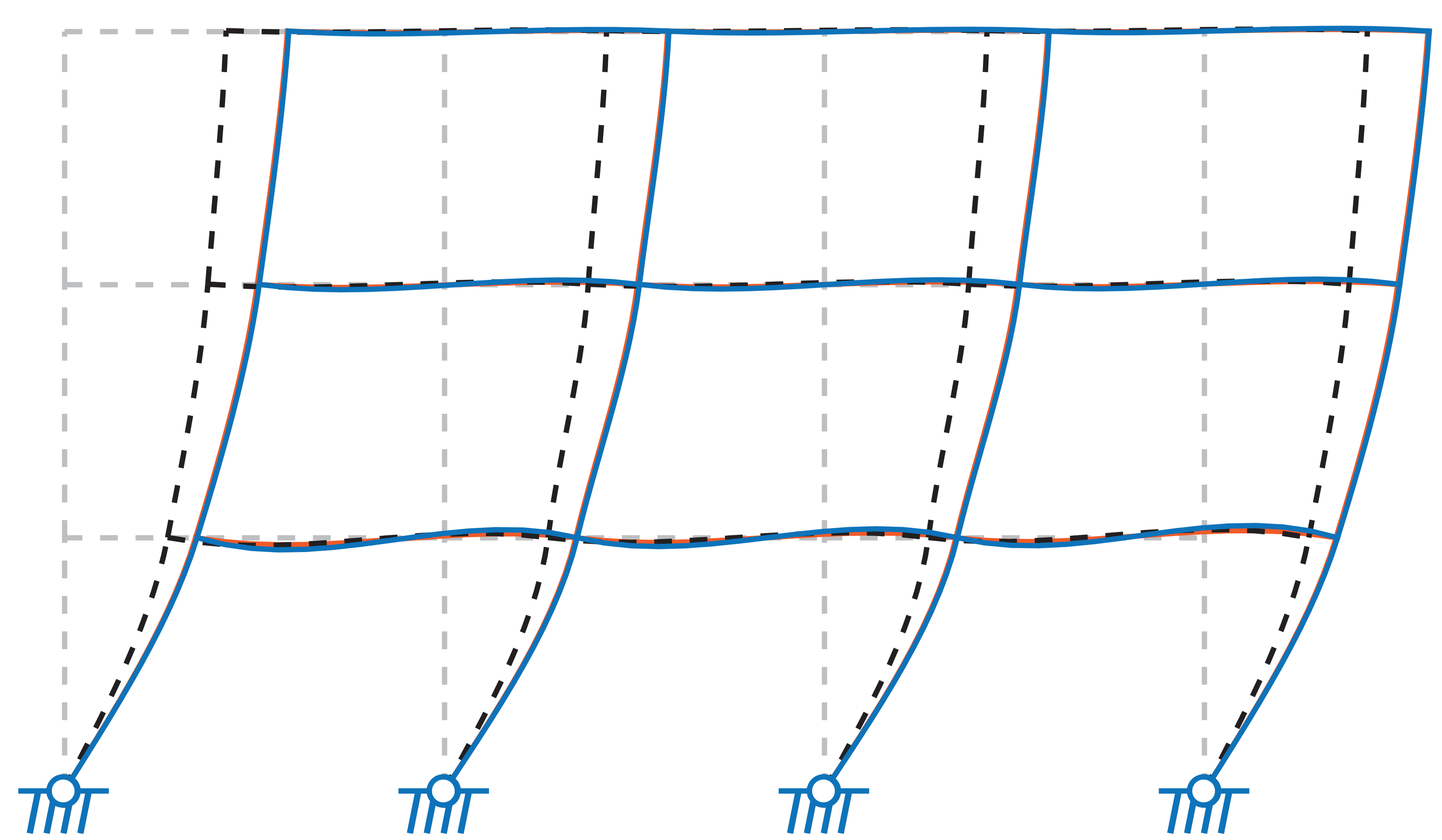
Example 4: three spans, three floors

D. Add the displacements

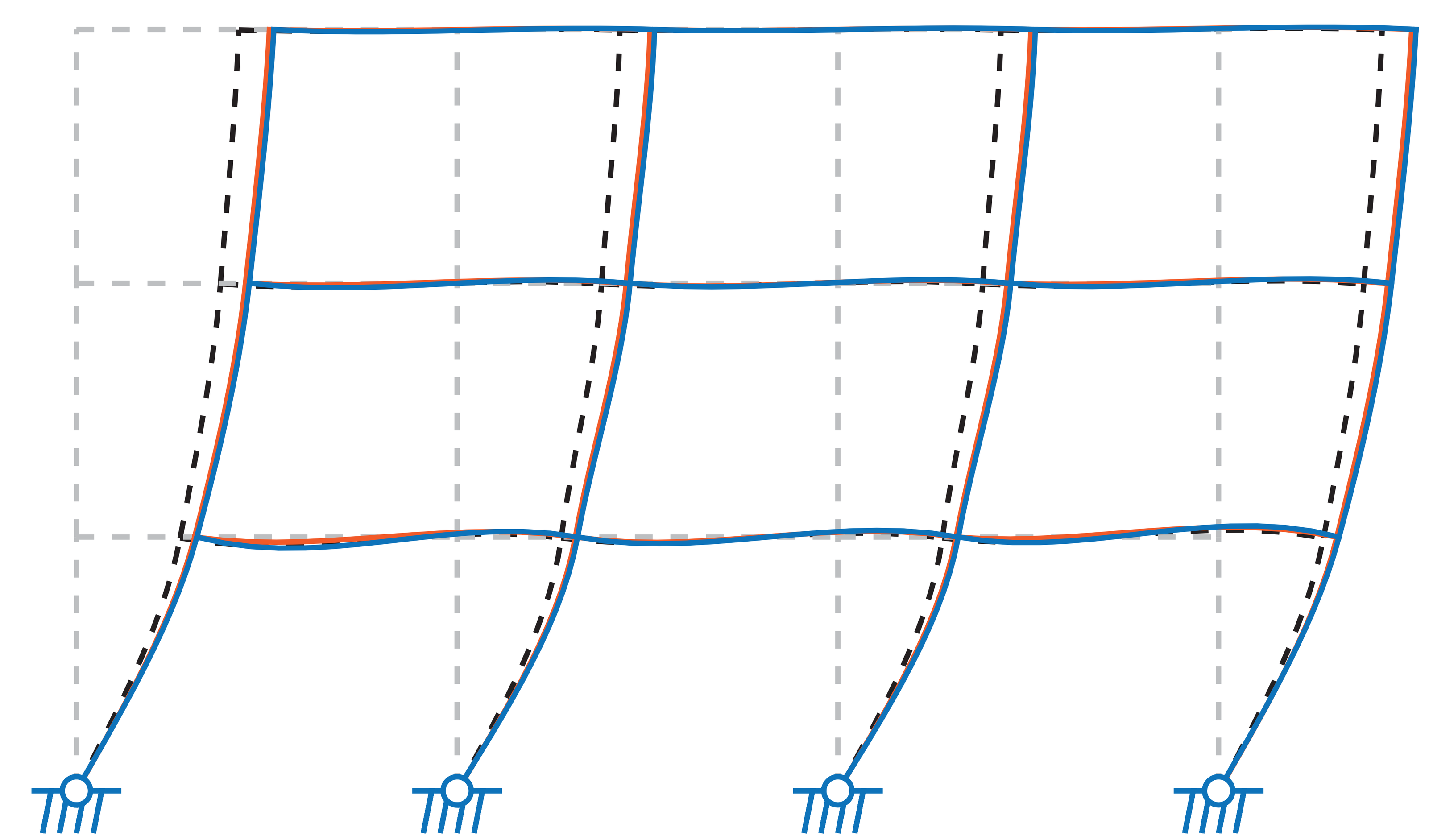
E. Correct the internal forces





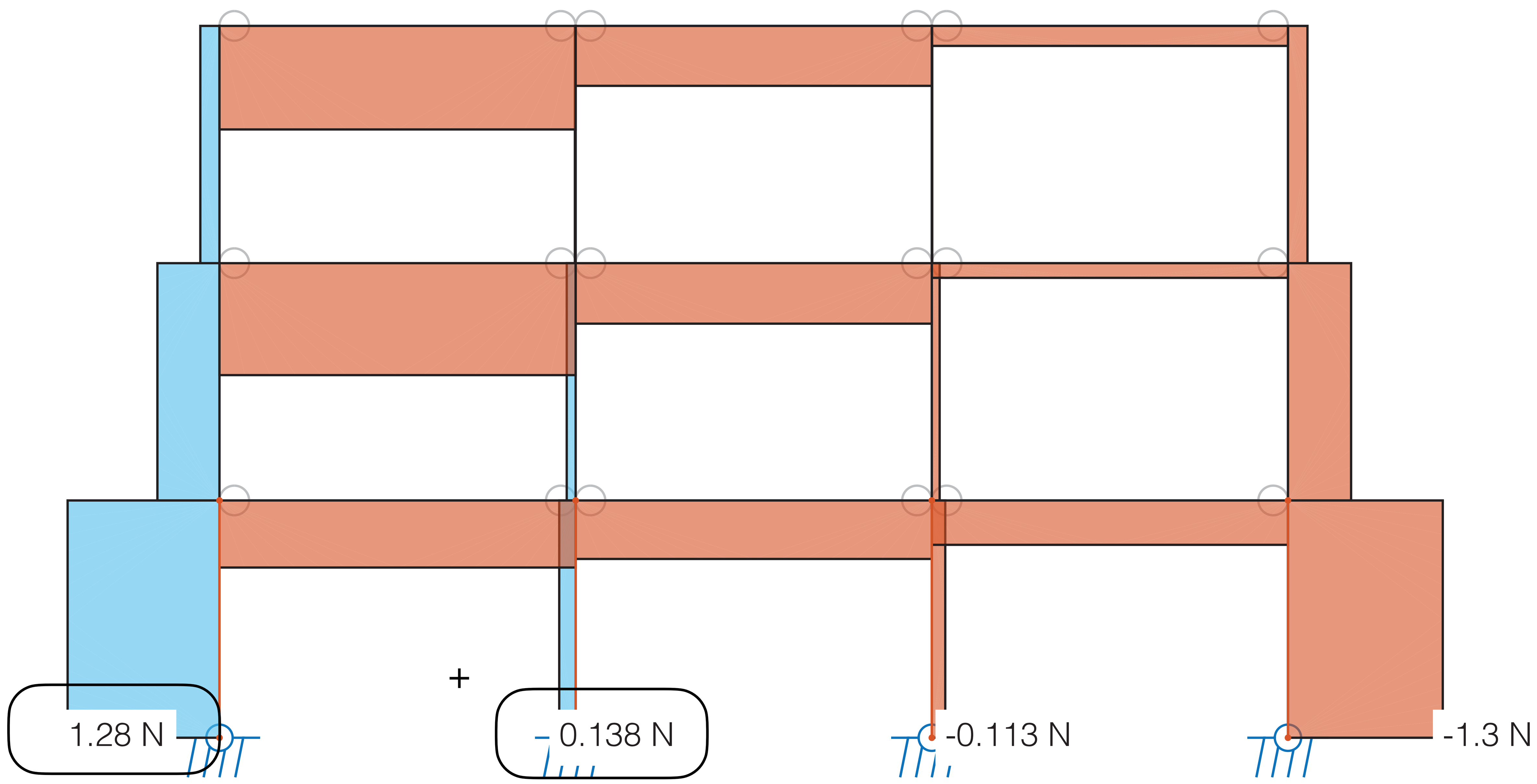


$(\star)\quad k_1 = k_2 = 5\frac{EI}{L}$

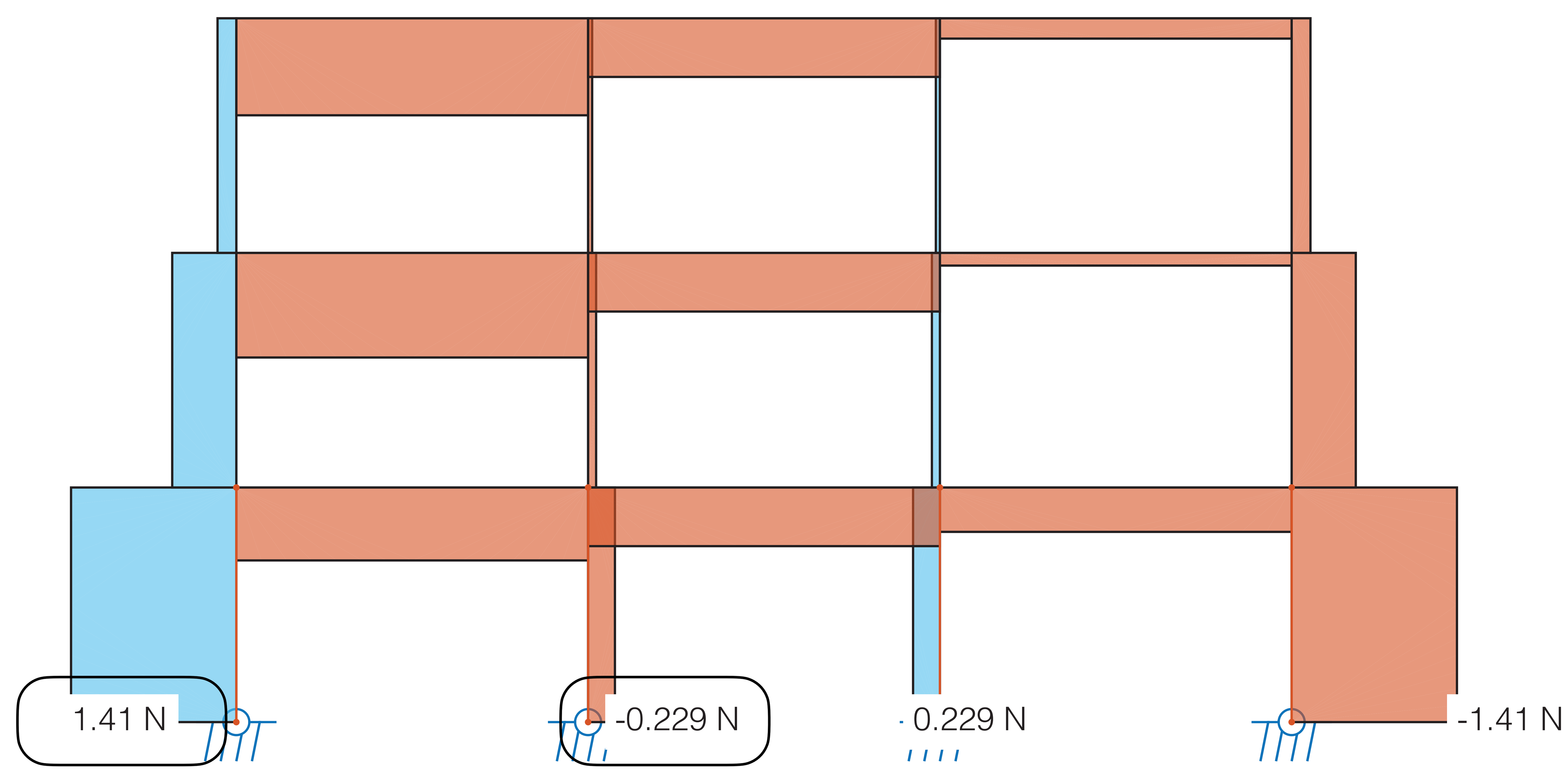


$(^{\star}{}_{\star})\quad k_1 = 5\frac{EI}{L}, k_2 = 20\frac{EI}{L}$

Example 4: three spans, three floors



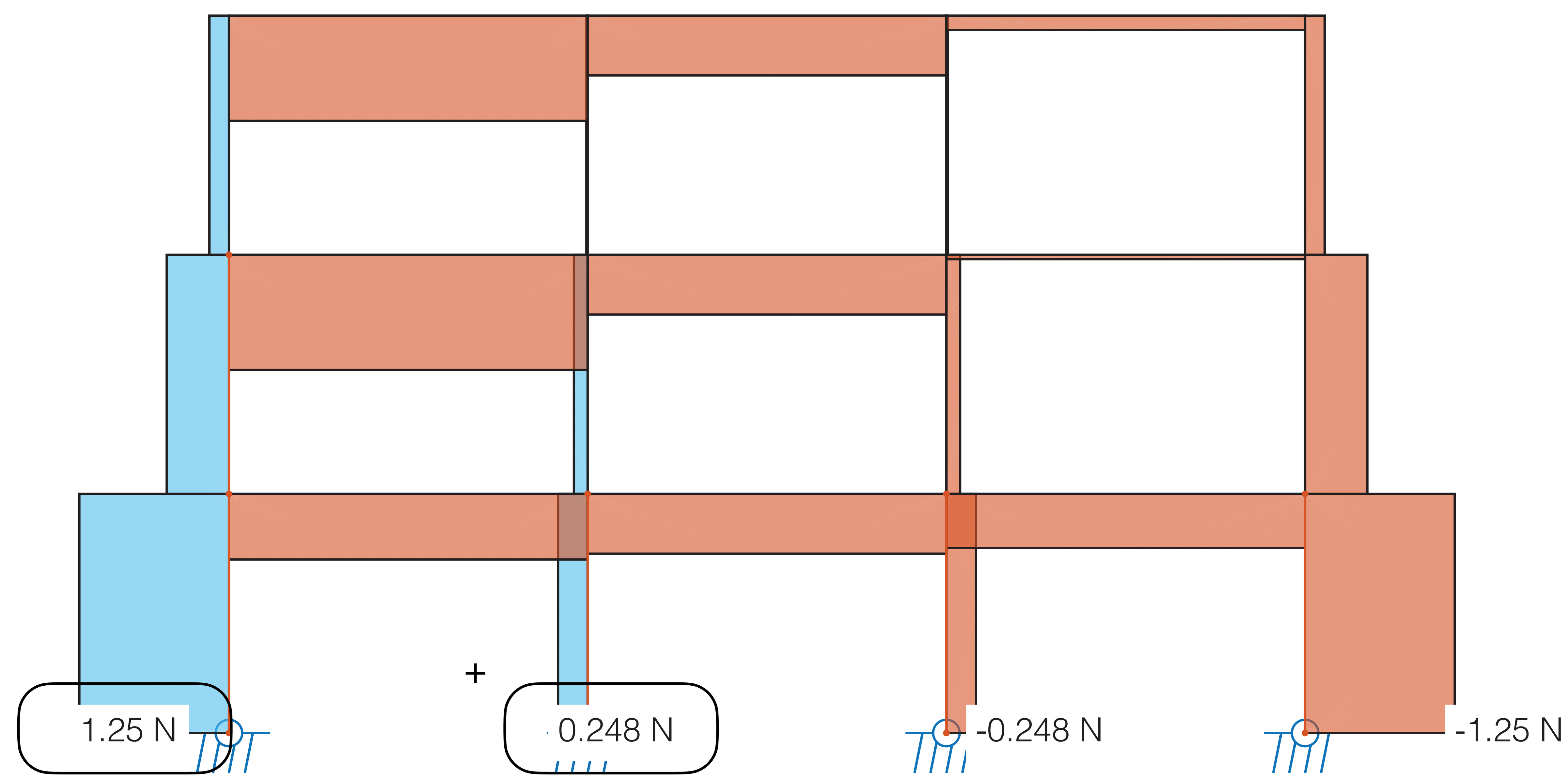
Semi-rigid joints, actual loading



Rigid joints, actual loading

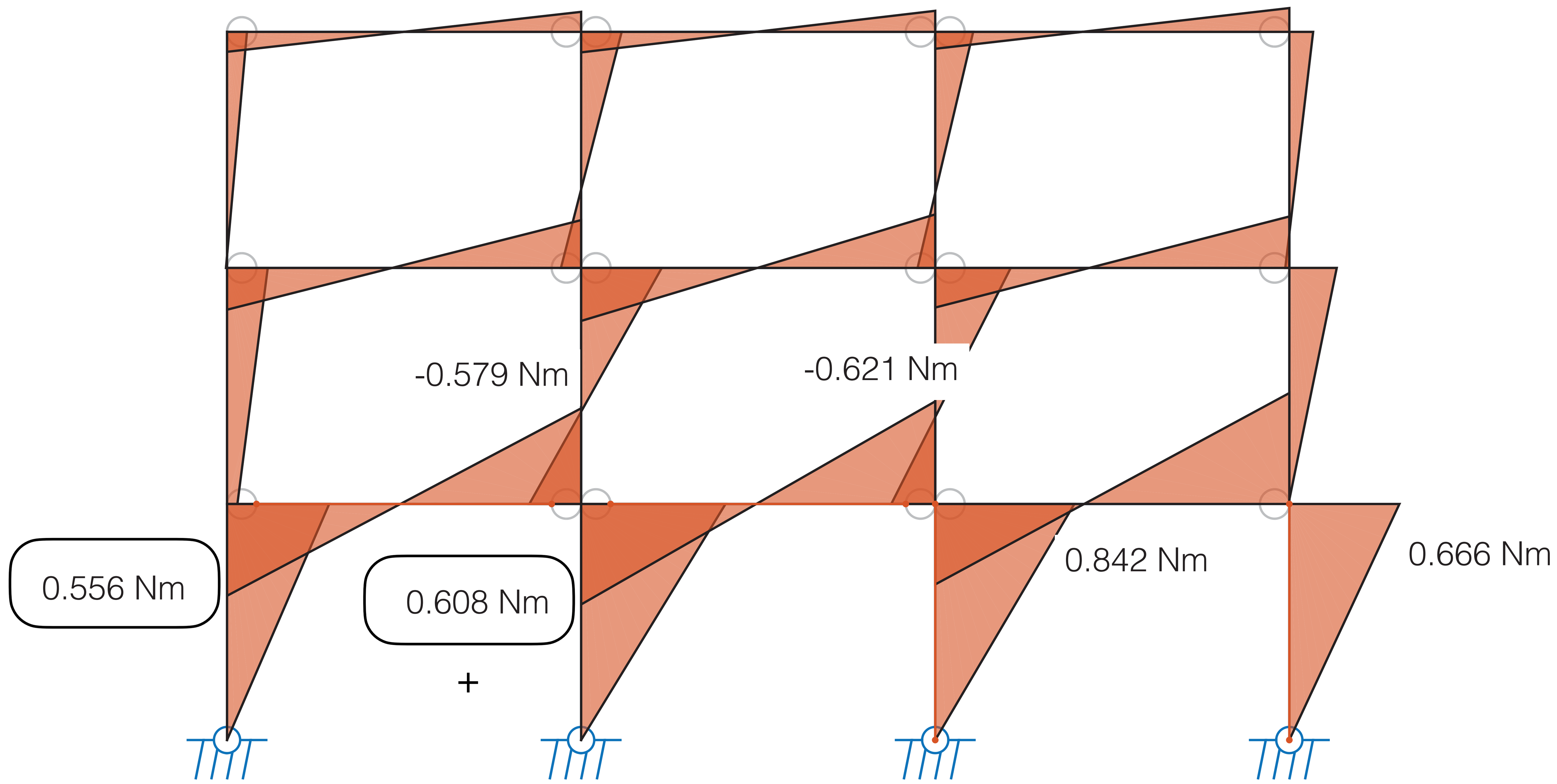
Semi-rigid joints : slightly off from asymmetric

Rigid joints: exactly slightly asymmetric

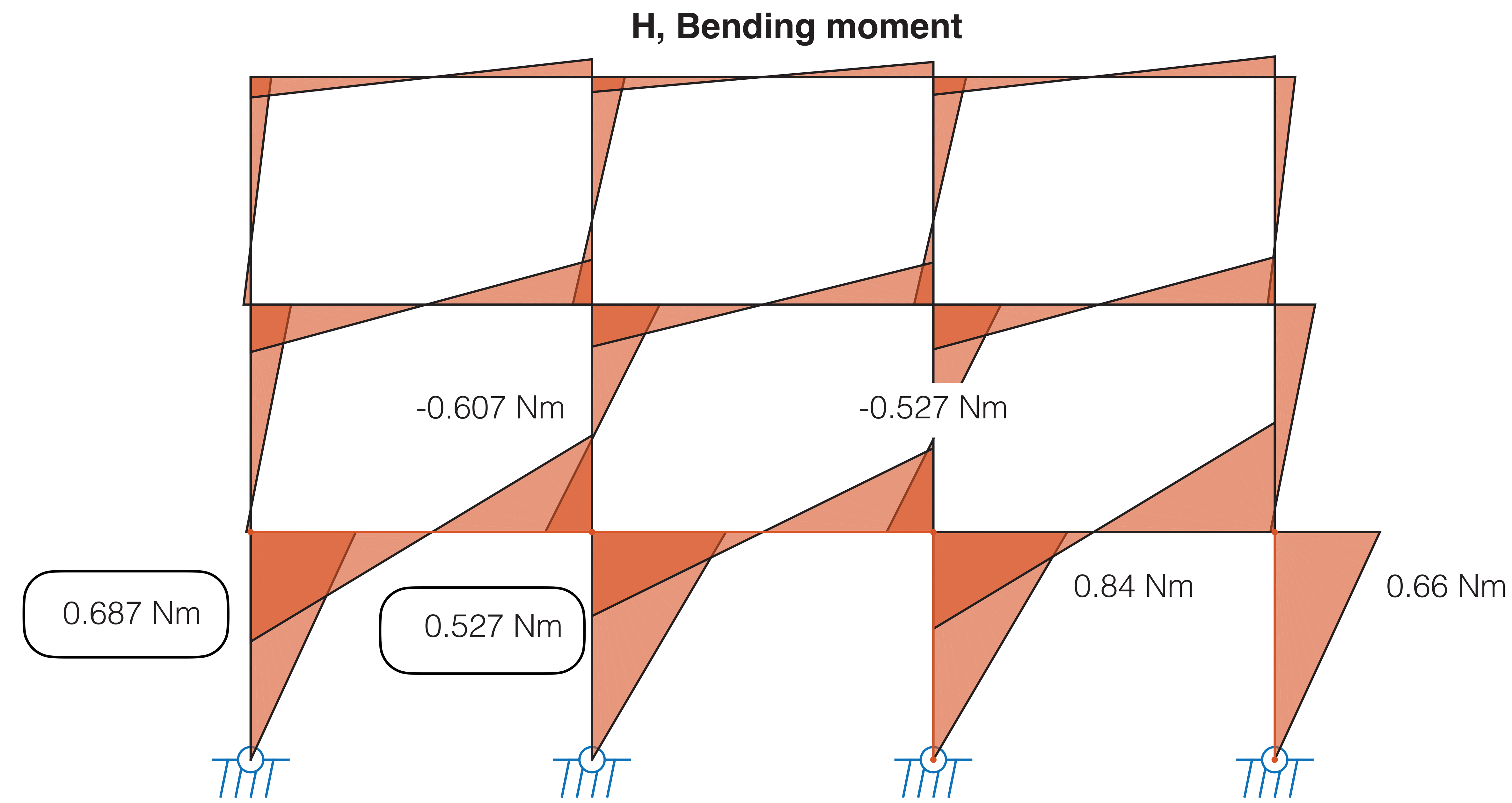


Rigid joints, actual loading+correction

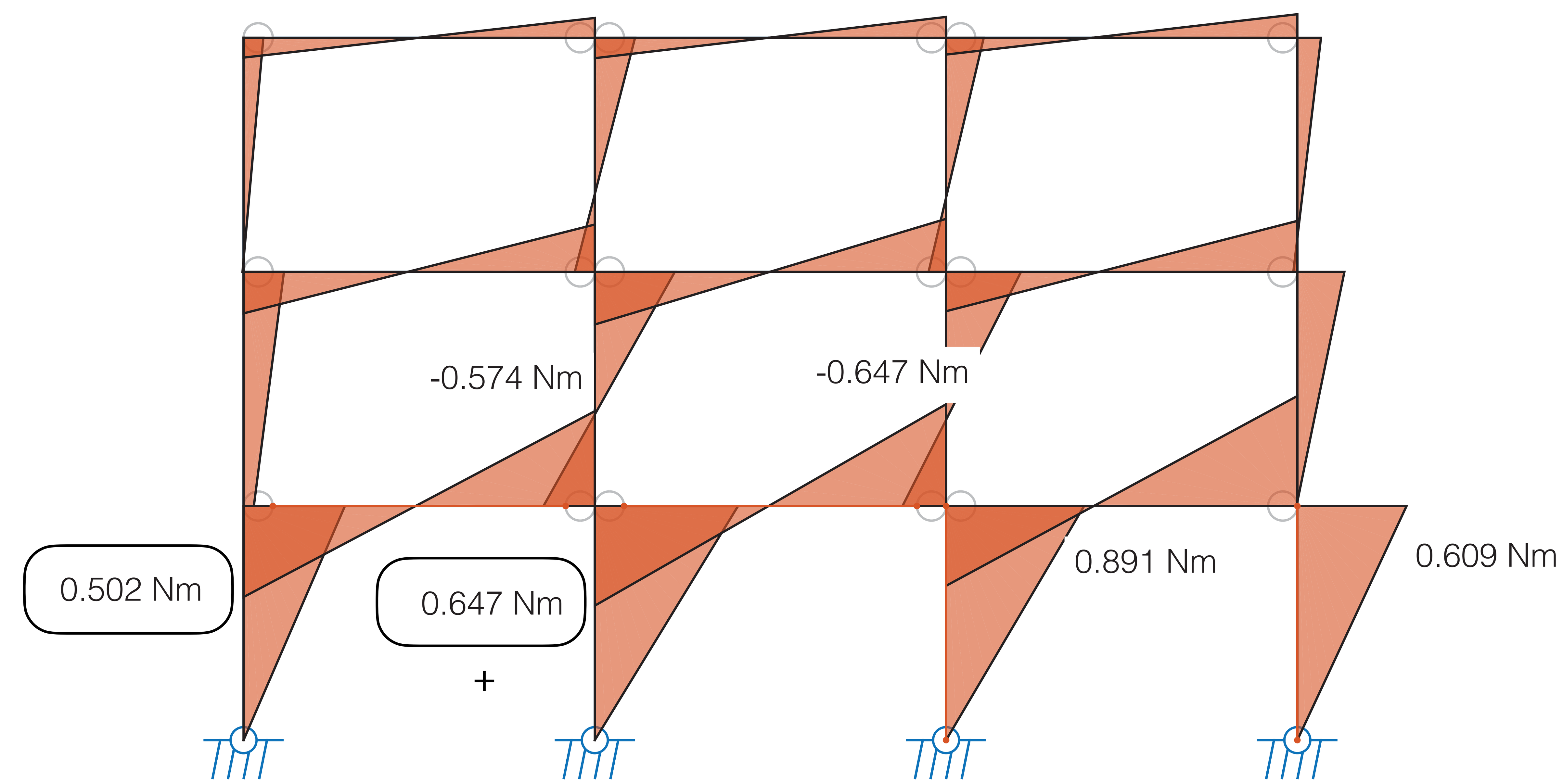
Example 4: three spans, three floors



Semi-rigid joints, actual loading



Rigid joints, actual loading



Rigid joints, actual loading+correction

Conclusions...

The analysis of a frame with very stiff semi-rigid joints provides
the **same asymptotic results** (displacements, internal forces,...)

as

the analysis of a frame with rigid joints under the **same loading**
+ an **equivalent loading** that takes the semi-rigidity into account.

The **equivalent loading** depends on the *nodal rotations* and *drifts*
observed in the frame with rigid joints, under the **original loading**.

...and perspectives !

By simplifying the **stability analysis** of frames with very stiff semi-rigid joints,

we are heading towards a **very general method for the classification of joints** !

+ Analysis of frames with very soft joints ?
+ M-N interaction in very stiff joints ?
+ ... ?

LIÈGE université
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Structural & Stochastic Dynamics
Urban & Environmental Engineering

Thank you !
Questions ? Comments ?

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fnrs
FREEDOM TO RESEARCH

3rd December 2020, Liège, SSD Team Meeting