Reconstructing words from right-bounded-block words

December 7, 2020

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The classical reconstruction problem

Let us consider finite words $u = u_1 \cdots u_n \in \mathcal{A}^*$.

A *subword* of u is a subsequence of the sequence of letters $(u_i)_{i=1}^{n}$, non necessarily contiguous.

aca is a subword of $abcbaba$

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The *binomial coefficient* $\binom{u}{v}$ denotes the number of times that v occurs as a subword in u.

We have

$$\binom{abcbaba}{aca} = 2.$$

Reconstruction of a word: an example

Let $\mathcal{A} = \{a, b\}$. Let $u \in \mathcal{A}^6$ be an unknown word of length 6 such that

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Can you uniquely determine u?

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$$u = \color{red}{bbb}\ a \quad a \quad a \qquad\qquad \binom{u}{ab} = 0$$

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$$u = \quad ab \quad a \quad a\ b \qquad\qquad \binom{u}{ab} = 4$$

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Answer: **NO**. $u_1 = abaabb$ and $u_2 = aabbab$ are two words satisfying the conditions.

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Add the following condition: $\binom{u}{aab} = 5$. Can you uniquely determine u?
YES, $u = aabbab$.

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Reconstruction problem

Let $\mathcal{A}$ be an alphabet, and n an integer. What is the minimal k such that any word from $\mathcal{A}^n$ can be uniquely determined from its k-deck?

Our adaptation

Let $\mathcal{A}$ be an alphabet, u and v two words. We denote by $Q(u, v)$ the following question:

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By **sequentially**, we mean that, for all i, the answers to $Q(u, v_1), \ldots,$ $Q(u, v_i)$ can influe the choice of v_{i+1}.

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A more general concept

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- Square matrices from some of their minors [Manvel, Stockmeyer, 1971]
- Graphs from some of their subgraphs? [Kelly and Ulam's conjecture, 1957; Harary's conjecture, 1963]
- **Words from some of their subwords? [Kalashnik, 1973]**

Recontruction problem for words

Let us define the function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)$ is the minimal k such that any word of length n is uniquely reconstructed from its k-deck.

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Properties of the k-deck

Let $u \in \mathcal{A}^*$ and $k \in \mathbb{N}$ such that the k-deck of u is known.

Therefore, $\binom{u}{v}$ is known for every $v \in \mathcal{A}^k$.

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Proposition: *The 1-deck of u is known from its k-deck.*

Proof: We obviously have $\binom{u}{a^k} = \binom{\binom{u}{a}}{k}$, for any $a \in \mathcal{A}$.

Properties of the k-deck

Proposition: *The* $(k-1)$*-deck of* u *is known from its* k*-deck.*

Proof: Let $x \in \mathcal{A}^{k-1}$. For any $a \in \mathcal{A}$, we have

$$\binom{u}{x}\binom{u}{a} = \sum_{j=0}^{k} \binom{u}{x_1 \cdots x_{j-1} a x_j \cdots x_{k-1}} + \binom{u}{x}\binom{x}{a},$$

Proposition: *The $(k-1)$-deck of u is known from its k-deck.*

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and thus

$$\binom{u}{x} = \frac{1}{\binom{u}{a} - \binom{x}{a}} \left[\sum_{j=0}^{k} \binom{u}{x_1 \cdots x_{j-1} a x_j \cdots x_{k-1}} \right]$$

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Knowing the entire k-deck of u requires to ask $(\# \mathcal{A})^k$ questions $Q(u, v)$ with $v \in \mathcal{A}^k$.

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Recall: Let $<$ be a total order on $\mathcal{A}$. A word u is *Lyndon* if for any factorization $u = x \cdot y \in \mathcal{A}^+ \times \mathcal{A}^+$, we have $xy <_{lex} yx$.

Why Lyndon words are enough

Let $x \in \mathcal{A}^{n_x}, y \in \mathcal{A}^{n_y}$, $n = n_x + n_y$ and $[n] = \{1, \ldots, n\}$.

The *shuffle* $x \sqcup\!\sqcup y$ is the multiset

$$\{w = w_1 \cdots w_n : \exists I_x = \{i_1 < \ldots < i_{n_x}\}, I_y = \{j_1 < \ldots < j_{n_y}\} \text{ a partition of } [n] \text{ s.t. } w_{i_1} \cdots w_{i_{n_x}} = x \text{ and } w_{j_1} \cdots w_{j_{n_y}} = y\}.$$

Example:

$$ab \sqcup\!\sqcup aab = \{abaab, aabab_3, aaabb_6\}.$$

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The *infiltration* $x \downarrow y$ is the multiset

$$\{w = w_1 \cdots w_{n'} : \exists I_x = \{i_1 < \ldots < i_{n_x}\}, I_y = \{j_1 < \ldots < j_{n_y}\}, n' \leq n \text{ s.t. } I_x \cup I_y = [n'] \text{ and } w_{i_1} \cdots w_{i_{n_x}} = x, w_{j_1} \cdots w_{j_{n_y}} = y, \text{ if well defined}\}.$$

Note that $x \sqcup\!\sqcup\, y \subseteq x \downarrow y$.

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Proposition (Reutenauer): *If* v *is non-Lyndon,* $\exists x, y \in \mathcal{A}^+$ *such that* $v = xy$ *and every word in the multiset* $x \sqcup\!\sqcup\, y$ *is* $<_{lex}$ *less than or equal to* v.

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For any multiset S, denote by S_v the multiplicity of v in S.

Let $u \in \mathcal{A}^*$, v be non-Lyndon and x, y as in the previous proposition. We have

$$\binom{u}{v} = \frac{1}{(x \sqcup\!\sqcup\, y)_v} \left[\binom{u}{x}\binom{u}{y} - \sum_{w \in \mathcal{A}^+ \setminus \{v\}} (x \downarrow y)_w \binom{u}{w} \right].$$

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We can apply the formula recursively on all non-Lyndon words.

Why Lyndon words are enough: an example

Let $u \in \mathcal{A}^*$ and $v = abaab$. Words $x = ab$ and $y = aab$ are such that $xy = v$ and $w \in x \sqcup\!\sqcup\, y \Rightarrow w \preceq v$.

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- Knowing $\binom{u}{v}$ for all $v \in \mathcal{A}^k \Rightarrow$ knowing $\binom{u}{v}$ for all $v \in \mathcal{A}^{\leq k}$.
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  Counterexample: $u_1 = babaa, u_2 = bbaaa$.
  We have $\binom{u_1}{aab} = \binom{u_2}{aab}, \binom{u_1}{abb} = \binom{u_2}{abb}, \binom{u_1}{ab} \neq \binom{u_2}{ab}$.

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- We don't want to restrict to words v having all the same length.

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- $Q(u, a^2 b)?$ $= 2.$

$$s_3 + 3s_4 + 6s_5 = 2.$$

The unique solution is: $(4, 0, 2, 0, 0)$ and $u = bbbbaabbaa$.

Binary case: using right-bounded-block words

First question: $Q(u,b)$? Assume $\binom{u}{b} \geq \frac{|u|}{2}$.

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Maximal number of asked questions

The number of questions is at most

$$\begin{cases} |u|_a + 1 & \text{if } |u|_a \leq \frac{|u|}{2} \\ |u|_b + 1 & \text{if } |u|_b \leq \frac{|u|}{2} \end{cases}$$

Hence

$$\left\lfloor \frac{|u|}{2} \right\rfloor + 1$$

questions are enough.

Reconstructing words from right-bounded-block words

The idea: projections on binary alphabets

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Let $\{a, b\} \subset \mathcal{A}$. Denote by $\pi_{a,b}(u)$ the projection of u on the binary alphabet $\{a, b\}$:

$$\pi_{a,b} : \begin{cases} a \mapsto a \\ b \mapsto b \\ c \mapsto \varepsilon \text{ if } c \in \mathcal{A} \setminus \{a, b\} \end{cases}$$

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2. Combine all projections $\{\pi_{a,b}(u)\}$ to reconstruct u.

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We get $u = banana$.

Remaining questions

Let $\mathcal{A} = \{a_1, \dots, a_q\}$.

1. Is u always uniquely determined from $\{\pi_{a_i,a_j}(u) : \{a_i, a_j\} \subset \{a_1, \dots, a_q\}\}$? How to reconstruct it?

2. Compare the maximal number of questions with the bound of the classical reconstruction problem.

K-markings

Let $K = (k_a)_{a\in\mathcal{A}}$ be a sequence of natural numbers. Let $u^{(1)}, \dots, u^{(\ell)}$ be words of $\mathcal{A}^*$.
A *K-marking* of $u^{(1)}, \dots, u^{(\ell)}$ is a mapping

$$\psi : \{(j, i) : j \in [\ell], i \in [|u^{(j)}|]\} \to \mathbb{N}$$

such that, $\forall j \in [\ell], \forall i, m \in [|u^{(j)}|], a \in \mathcal{A}$, there holds

- if $u_i^{(j)} = a$ then $\psi(j, i) \leq k_a$,
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Example:

$\mathcal{A} = \{a, b, c\}$, $K = (2, 3, 2)$
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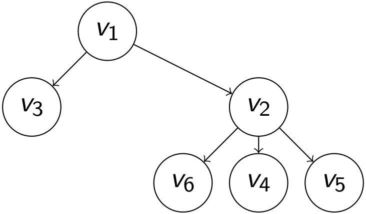
j	1				2		
i	1	2	3	4	1	2	3
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi(j,i)$	1	2	1	3	1	1	2

Topological sorting of a directed graph

Let $G = (V, E)$ be a directed graph. A *topological sorting* of G is a linear ordering $v_1 < \ldots < v_n$ of V such that every edge in G is of the type (v_i, v_j) with $i < j$.



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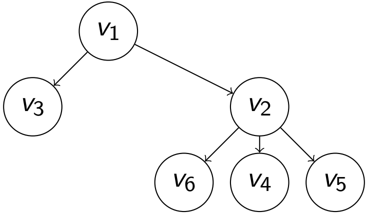

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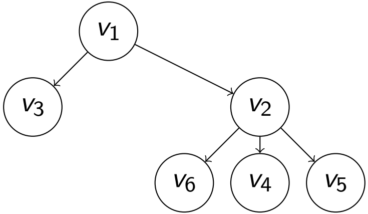

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The topological sorting of an acyclic graph is unique if and only if there is a path going through every vertex.

Graph G_ψ associated to a K-marking ψ

Let ψ be a K-marking of $u^{(1)}, \dots, u^{(\ell)}$. Define the graph $G_\psi = (V_\psi, E_\psi)$ such that:

$$V_\psi = \{(a)_i : a \in \mathcal{A}, 1 \leq i \leq k_a\}$$

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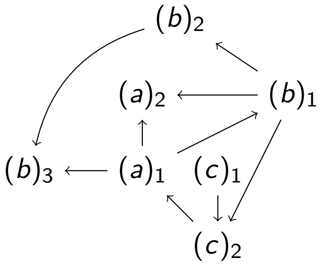

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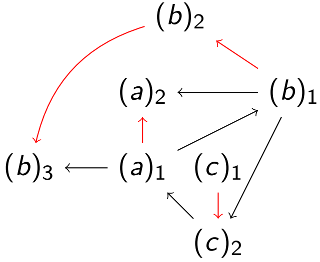

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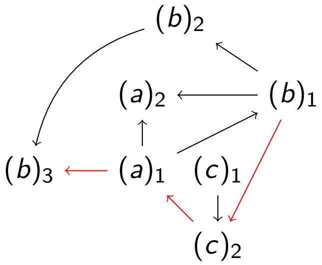

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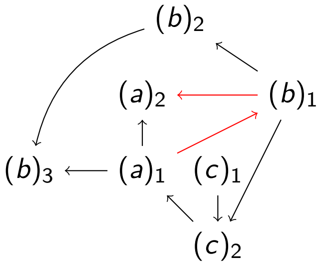

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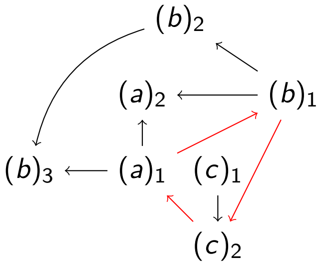

ψ is a K-marking $\not\Rightarrow$ G_ψ admits a topological sorting

Reconstructing a word from subwords

Theorem: Let $u^{(1)}, \dots, u^{(\ell)} \in \mathcal{A}^*$. Let $K = (k_a)_{a \in \mathcal{A}}$.
There exists a word $u \in \mathcal{A}^*$ such that $u^{(j)}$ is a subword of u for all $j \in [\ell]$ and $|u|_a = k_a$ for all $a \in \mathcal{A}$
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there exists a K-marking ψ of the words $u^{(1)}, \dots, u^{(\ell)}$ and a topological sorting of G_ψ.

Moreover, if the values of K are *minimal*, if there is a unique K-marking ψ such that G_ψ admits a topological sorting and if this topological sorting is unique, then u is unique.

Values of K are minimal means that

$$\forall a \in \mathcal{A}, k_a = \max_{j\in[\ell]} |u^{(j)}|_a$$

Illustrating the theorem

$\mathcal{A} = \{a, b, c\}, K = (2, 3, 2)$

j	1				2		
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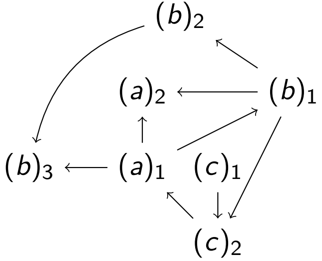
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi(j, i)$	1	2	1	3	1	1	2



No valid topological sorting G_ψ for this K-marking ψ of $u^{(1)}, u^{(2)}$.

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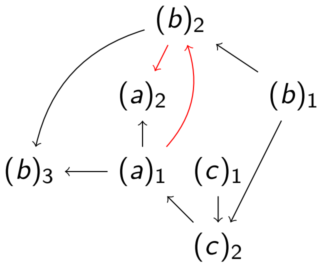
$\mathcal{A} = \{a, b, c\}, K = (2, 3, 2)$

j	1				2		
i	1	2	3	4	1	2	3
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi'(j, i)$	1	2	1	3	1	2	2



Another K-marking ψ' of $u^{(1)}, u^{(2)}$. $G_{\psi'}$ admits a topological sorting.

There exists $u \in \{a, b, c\}^*$ having $u^{(1)}$ and $u^{(2)}$ as subwords, and such that $|u|_a = 2$, $|u|_b = 3$, $|u|_c = 2$.

$$u \in \{bcababc, bcabacb, bcbcaba, cbbcaba, \ldots\}$$

Illustrating the theorem: taking the minimal K

Since $u^{(1)} = bcab$ and $u^{(2)} = aba$, the minimal K is $(2, 2, 1)$.

j	1				2		
i	1	2	3	4	1	2	3
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi(j, i)$	1	1	1 or 2	2	1	1 or 2	2

There are 4 possible K-markings.

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j	1				2		
i	1	2	3	4	1	2	3
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi_1(j, i)$	1	1	1	2	1	1	2

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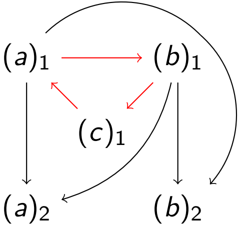

G_{ψ_1} does not have a topological sorting.

Since $u^{(1)} = bcab$ and $u^{(2)} = aba$, the minimal K is $(2, 2, 1)$.

j	1				2		
i	1	2	3	4	1	2	3
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi_2(j, i)$	1	1	1	2	1	2	2

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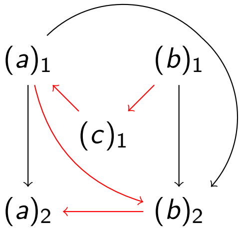

Unique topological sorting of G_{ψ_2}. Reconstructing $u = bcaba$, having $u^{(1)}$ and $u^{(2)}$ as subwords.

Illustrating the theorem: taking the minimal K

Since $u^{(1)} = bcab$ and $u^{(2)} = aba$, the minimal K is $(2, 2, 1)$.

j	1				2		
i	1	2	3	4	1	2	3
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi_3(j,i)$	1	1	2	2	1	1	2

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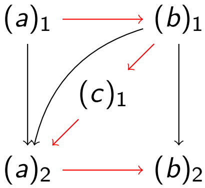

Unique topological sorting of G_{ψ_3}. Reconstructing $u = abcab$, having $u^{(1)}$ and $u^{(2)}$ as subwords.

Illustrating the theorem: taking the minimal K

Since $u^{(1)} = bcab$ and $u^{(2)} = aba$, the minimal K is $(2, 2, 1)$.

j	1				2		
i	1	2	3	4	1	2	3
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi_4(j,i)$	1	1	2	2	1	2	2

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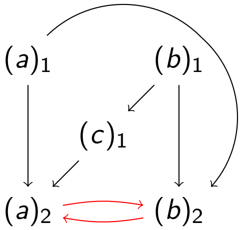

G_{ψ_4} does not have a topological sorting.

Illustrating the theorem: taking the minimal K

Since $u^{(1)} = bcab$ and $u^{(2)} = aba$, the minimal K is $(2, 2, 1)$.

j	1				2		
i	1	2	3	4	1	2	3
$u_i^{(j)}$	b	c	a	b	a	b	a
$\psi(j,i)$	1	1	1 or 2	2	1	1 or 2	2

Since there exist two K-markings ψ_2 and ψ_3 admitting a topological sorting of their associated graph, the word u is not unique.

Reconstructing a word from its binary projections

Let $\mathcal{A}$ be an alphabet of size q. Assume that $u^{(1)}, \dots, u^{(\ell)}$ are all the projections of an unknown word $u \in \mathcal{A}^*$ over binary subalphabets.

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2. It can be proven that G_ψ is acyclic and has a path going through every vertex, hence the topological sorting exists and is unique.

Reconstructing a word from its binary projections

Let $\mathcal{A}$ be an alphabet of size q. Assume that $u^{(1)}, \ldots, u^{(\ell)}$ are all the projections of an unknown word $u \in \mathcal{A}^*$ over binary subalphabets.

Take $K = (k_a)_{a \in \mathcal{A}}$ where $k_a = |u^{(j)}|_a$ if $u^{(j)}$ is the projection of u over a subalphabet containing a. Then K is minimal.

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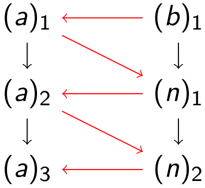
Taking the letters of vertices of G_ψ following the topological sorting gives the unique word u.

Example: reconstructing "banana" from its binary projections.

$u^{(1)} = baaa$, $u^{(2)} = bnn$, $u^{(3)} = anana$.

Then the minimal K is $(k_a, k_b, k_n) = (3, 1, 2)$.

j	1				2			3				
i	1	2	3	4	1	2	3	1	2	3	4	5
$u_i^{(j)}$	b	a	a	a	b	n	n	a	n	a	n	a
$\psi(j,i)$	1	1	2	3	1	1	2	1	1	2	2	3



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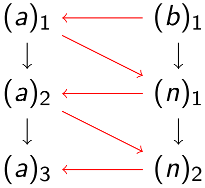

Reconstructing a word from its binary projections can be done in linear time w.r.t. the total length of the projections.

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  Ask questions $Q(u, a_{\sigma(i)} a_{\sigma(j)}), Q(u, a_{\sigma(i)}^2 a_{\sigma(j)}), \ldots, Q(u, a_{\sigma(i)}^{|u|_{a_{\sigma(i)}}} a_{\sigma(j)})$.
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Number of asked questions:

$$(q-1) + \sum_{i=1}^{q} |u|_{a_{\sigma(i)}} (q-i)$$

Comparing to the classical reconstruction problem

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questions. This bound is strictly greater than

$$(q-1) + \sum_{i=1}^{q} |u|_{a_{\sigma(i)}} (q-i)$$

for every q and $n \geq q-1$.

Thank you!