

Synchronization in Networks of Identical Linear Systems^{*}

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Abstract

The paper investigates the synchronization of a network of identical linear state-space models under a possibly time-varying and directed interconnection structure. The main result is the construction of a dynamic output feedback coupling that achieves synchronization if the decoupled systems have no exponentially unstable mode and if the communication graph is uniformly connected. The result can be interpreted as a generalization of classical consensus algorithms. Stronger conditions are shown to be sufficient – but to some extent, also necessary – to ensure synchronization with the diffusive static output coupling often considered in the literature.

Key words: Synchronization; Consensus; Multi-agent systems; Weak coupling; Oscillator analysis

1 Introduction

In these last years, consensus, coordination and synchronization problems have been popular subjects in systems and control, motivated by many applications in physics, biology, and engineering. These problems arise in multi-agent systems with the collective objective of reaching agreement about some variables of interest.

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In the *consensus* literature, the emphasis is on the communication constraints rather than on the individual dynamics: the agents exchange information according to a communication graph that is not necessarily complete, nor even symmetric or time-invariant, but, in the absence of communication, the agreement variables usually have no dynamics. It is the exchange of information only that determines the time-evolution of the variables, aiming at asymptotic synchronization to a common value. The convergence of such consensus algorithms has attracted much attention in the recent years. It only requires a weak form of connectivity for the communication graph [1–4].

In the *synchronization* literature, the emphasis is on the individual dynamics rather than on the communication limitations: the communication graph is often assumed to be complete (or all-to-all), but in the absence of communication, the time-evolution of the systems' variables can be oscillatory or even chaotic. The system dynamics can be modified through the information exchange, and, as in the consensus problem, the goal of the interconnection is to reach synchronization to a common solution of the individual dynamics [5–8].

Coordination problems encountered in the engineering world can often be rephrased as consensus or synchronization problems in which both the individual dynamics and the limited communication aspects play an important role. Designing interconnection control laws that can ensure synchronization of relevant variables is therefore a control problem that has attracted quite some attention in the recent years [9–13].

The present paper deals with a fairly general solution of the synchronization problem in the linear case. Assuming N identical individual agents dynamics each described by the linear state-space model (A, B, C) , the main result is the construction of a dynamic output feedback controller that ensures exponential synchronization to a solution of the linear system $\dot{x} = Ax$ under the following assumptions: (i) A has no exponentially unstable modes, (ii) (A, B) is stabilizable and (A, C) is detectable, and (iii) the communication graph is uniformly connected. The result can be interpreted as a generalization of classical consensus algorithms, studied recently, corresponding to the particular case $A = 0$ [1,2]. The generalization includes the non-trivial examples of synchronizing harmonic oscillators and chains of integrators. On their turn, these models, are applicable to distributed coordination of interconnected mobile systems. Models of vehicles often require to take into account second-order dynamics that can be feedback linearized as double integrators. Furthermore the clocks synchronization problems can be reduced to synchronization of double integrator models [14].

The proposed dynamic controller structure proposed in this paper differs from the static diffusive coupling often considered in the synchronization literature, which requires more stringent assumptions on the communication graph.

For instance, the results in the recent papers [15], [16] and [14] on synchronization of harmonic oscillators and double integrators, assume a time-invariant communication topology. The idea of designing dynamic controllers for synchronization of networked systems was recently proposed in [9] and has been applied to stabilize planar and three-dimensional collective motion [10,11].

The paper also provides sufficient conditions for synchronization by static diffusive coupling and illustrates on simple examples that synchronization may fail under diffusive coupling when the stronger assumptions on the communication graph are not satisfied.

The paper is organized as follows. In Section 2 the notation used throughout the paper is summarized, some preliminary results are reviewed and the synchronization problem is introduced and defined. In Section 3 the main result is presented. In Section 4 we derive the discrete-time counterpart of the main result. Finally, in Section 5, two-dimensional examples are reported to illustrate the role of the proposed dynamic controller in situations where static diffusive coupling fails to achieve synchronization.

2 Preliminaries

Throughout the paper we will use the following notation. Given N vectors x_1, x_2, \dots, x_N we indicate with x the stacking of the vectors, i.e. $x = [x_1^T, x_2^T, \dots, x_N^T]^T$. We denote with I_N the diagonal matrix of dimension $N \times N$ and we define $1_N \triangleq [1, 1, \dots, 1]^T \in \mathbb{R}^N$. Given two matrices A and B we denote their Kronecker product with $A \otimes B$. For notational convenience, we use the convention $\tilde{A}_N = I_N \otimes A$ and $\hat{A}_N = A \otimes I_N$. For a comprehensive list of properties of the Kronecker product the reader is referred to [17].

2.1 Communication Graphs

Given a set of interconnected systems the communication topology is encoded through a *communication graph*. The convention is that system j receives information from system i iff there is a directed link from node j to node i in the communication graph. Let $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), A_d(t))$ be a time-varying weighted digraph (directed graph) where $\mathcal{V} = \{v_1, \dots, v_N\}$ is the set of nodes, $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $A_d(t)$ is a weighted adjacency matrix with nonnegative elements $a_{kj}(t)$. In the following we assume that $A_d(t)$ is piece-wise continuous and bounded and $a_{kj}(t) \in \{0\} \cup [\eta, \gamma], \forall k, j$, for some finite scalars $0 < \eta \leq \gamma$ and for all $t \geq 0$. Furthermore $\{v_k, v_j\} \in \mathcal{E}(t)$ if and only if $a_{kj}(t) \geq \eta$. The set of neighbors of node v_k at time t is denoted by $\mathcal{N}_k(t) \triangleq \{v_j \in \mathcal{V} : a_{kj}(t) \geq \eta\}$. A path is a

sequence of vertices such that for each of its vertices v_k the next vertex in the sequence is a neighbor of v_k . Assume that there are no self-cycles i.e. $a_{kk}(t) = 0$, $k = 1, 2, \dots, N$, and for any t .

The Laplacian matrix $L(t)$ associated to the graph $\mathcal{G}(t)$ is defined as

$$l_{kj}(t) = \begin{cases} \sum_{i=1}^N a_{ki}(t), & j = k \\ -a_{kj}(t), & j \neq k. \end{cases}$$

The in-degree (respectively out-degree) of node v_k is defined as $d_k^{in} = \sum_{j=1}^N a_{kj}$ (respectively $d_k^{out} = \sum_{j=1}^N a_{jk}$). The digraph $\mathcal{G}(t)$ is said to be *balanced* at time t if the in-degree and the out-degree of each node are equal, that is,

$$\sum_{j=1}^N a_{kj} = \sum_{j=1}^N a_{jk}, \quad k = 1, \dots, N.$$

Balanced graphs have the particular property that the symmetric part of their Laplacian matrix is nonnegative: $L + L^T \geq 0$ [18]. We recall some definitions that characterize the concept of connectivity for time-varying graphs.

Definition 1 *The digraph $\mathcal{G}(t)$ is connected at time t if there exists a node v_k such that all the other nodes of the graph are connected to v_k via a path that follows the direction of the edges of the digraph.*

Definition 2 *Consider a graph $\mathcal{G}(t)$. A node v_k is said to be connected to node v_j ($v_j \neq v_i$) in the interval $I = [t_a, t_b]$ if there is a path from v_k to v_j which respects the orientation of the edges for the directed graph $(\mathcal{V}, \cup_{t \in I} \mathcal{E}(t), \int_I A_d(\tau) d\tau)$.*

Definition 3 *$\mathcal{G}(t)$ is said to be uniformly connected if there exists a time horizon $T > 0$ and an index k such that for all t all the nodes v_j ($j \neq k$) are connected to node v_k across $[t, t + T]$.*

2.2 Convergence of consensus algorithms

Consider N agents exchanging information about their state vector x_k , $k = 1, \dots, N$, according to a communication graph $\mathcal{G}(t)$. A classical consensus protocol in continuous-time is

$$\dot{x}_k = \sum_{j=1}^N a_{kj}(t)(x_j - x_k), \quad k = 1, \dots, N. \tag{1}$$

In discrete-time the analogous dynamics write

$$x_k(t + 1) = x_k(t) - \epsilon_k \sum_{j=1}^N l_{kj}(t)x_j(t), \quad k = 1, \dots, N \tag{2}$$

where $\epsilon_k \in (0, 1/d_k^{in})$. Using the Laplacian matrix, (1) and (2) can be equivalently expressed as

$$\dot{x} = -\hat{L}_n(t)x. \quad (3)$$

and

$$x(t+1) = \left(I_{nN} - \hat{\epsilon}\hat{L}_n(t)\right)x(t), \quad (4)$$

where $\hat{\epsilon} = \epsilon \otimes I_n$ and $\epsilon = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_N)$, and $\hat{L}_n(t) = L(t) \otimes I_n$.

Algorithms (3) and (4) have been widely studied in the literature and asymptotic convergence to a consensus value holds under mild assumptions on the communication topology. The following theorem summarizes the main results in [1] and [2].

Theorem 1 *Let $x_k, k = 1, 2, \dots, N$, belong to a finite-dimensional Euclidean space W . Let $\mathcal{G}(t)$ be a uniformly connected digraph and $L(t)$ the corresponding Laplacian matrix bounded and piecewise continuous in time. Then the equilibrium sets of consensus states of (3) and (4) are uniformly exponentially stable. Furthermore the solutions of (1) and (2) asymptotically converge to a consensus value β for some $\beta \in W$. \square*

2.3 Problem statement

Consider N identical linear dynamical systems

$$\dot{x}_k = Ax_k + Bu_k, \quad (5a)$$

$$y_k = Cx_k \quad (5b)$$

for $k = 1, 2, \dots, N$, where $x_k \in \mathbb{R}^n$ is the state of the system, $u_k \in \mathbb{R}^m$ is the control vector, and $y_k \in \mathbb{R}^p$ is the output.

According to the graph-theoretic interpretation of the interconnected system described in Section 2.1, two systems are coupled at time t if there exists an edge connecting them in the associated (time-varying) communication graph $\mathcal{G}(t)$ at time t . We assume that the systems, when connected, can exchange relative information only. Thanks to this assumption the network does not need an external common reference nor a leader to synchronize. The coupling is taken into account through the inputs $u_k, k = 1, 2, \dots, N$. We will call a control law *dynamic* if it depends on

an internal (controller) state, otherwise it is called *static*. The goal of the (coupling) control law is to achieve (asymptotically) state synchronization.

Synchronization Problem: Given N identical systems described by the model (5) and a uniformly connected communication graph $\mathcal{G}(t)$, find a (distributed) control law, that depends on relative information only, such that the solutions of (5) asymptotically synchronize. \square

It is worth noting that in the particular case $A = 0$ and $B = C = I_n$, the problem is equivalent to the consensus problem described in Section 2.2 and the solution is well-known (see Theorem 1): uniform connectivity of the graph is a sufficient condition for synchronization (or consensus) under the control law (1) or (2). The present paper addresses the generalization for linear time invariant systems.

3 Main result

Before stating the main result we introduce a preliminary Lemma that is a direct extension of Theorem 1 when, in (5), C and B are nonsingular (square) matrices.

Lemma 1 *Consider the linear systems (5). Let B and C be $n \times n$ nonsingular matrices and assume that all the eigenvalues of A belong to the imaginary axis. Assume that the communication graph $\mathcal{G}(t)$ is uniformly connected and the corresponding Laplacian matrix $L(t)$ piecewise continuous and bounded. Then the control law*

$$u_k = B^{-1}C^{-1} \sum_{j=1}^N a_{kj}(t)(y_j - y_k), \quad k = 1, 2, \dots, N, \quad (6)$$

uniformly exponentially synchronizes all the solutions of (5) to a solution of the system $\dot{x}_0 = Ax_0$. \square

Proof: Control law (6) applied to system (5) yields the closed-loop system

$$\dot{x}_k = Ax_k + \sum_{j=1}^N a_{kj}(t)(x_j - x_k), \quad k = 1, 2, \dots, N.$$

The change of variable

$$z_k = e^{-A(t-t_0)}x_k, \quad k = 1, 2, \dots, N,$$

leads to

$$\dot{z}_k = -Ae^{-A(t-t_0)}x_k + e^{-A(t-t_0)}Ax_k + e^{-A(t-t_0)} \sum_{j=1}^N a_{kj}(t)(x_j - x_k) = \sum_{j=1}^N a_{kj}(t)(z_j - z_k)$$

or, in a compact form,

$$\dot{z} = -\hat{L}_n(t) z.$$

From Theorem 1 the solutions $z_k(t)$, $k = 1, 2, \dots, N$, exponentially converge to a common value $x_0 \in \mathbb{R}^n$ as $t \rightarrow \infty$, that is, there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that for all t_0 ,

$$\|z_k(t) - x_0\| \leq \delta_1 e^{-\delta_2(t-t_0)} \|z_k(t_0) - x_0\|, \quad \forall t > t_0.$$

In the original coordinates, this means

$$\|x_k(t) - e^{A(t-t_0)} x_0\| \leq \delta_1 e^{-\delta_2(t-t_0)} \|e^{A(t-t_0)}\| \|x_k(t_0) - x_0\|,$$

for every $t > t_0$. Because all the eigenvalues of the matrix A lie on the imaginary axis, there exists a constant $\delta_3 > 0$ such that

$$\|x_k(t) - e^{A(t-t_0)} x_0\| \leq \delta_1 e^{-\delta_3(t-t_0)} \|x_k(t_0) - x_0\|,$$

for every $t > t_0$, which proves that all solutions exponentially synchronize to a solution of the open loop system. ■

Remark 1 *The result is of course unchanged if A also possesses eigenvalues with a negative real part. Exponentially stable modes synchronize to zero, even in the absence of coupling. In contrast, the situation of systems with some eigenvalues with a positive real part can be addressed in a similar way but it requires that the graph connectivity is sufficiently strong to dominate the instability of the system. This is clear from the last part of the proof of Lemma 1 where the exponential synchronization in the z coordinates must dominate the divergence of the unstable modes of A .*

We are now ready to present the main result of the paper. The assumption of *square* (nonsingular) matrices B and C in Lemma 1 are now weakened to stabilizability of the pair (A, B) and detectability of the pair (A, C) . Then, for an arbitrary stabilizing feedback matrix K and observer matrix H , the (dynamic) control law

$$\begin{aligned} \dot{\eta}_k &= (A + BK)\eta_k + \sum_{j=1}^N a_{kj}(t)(\eta_j - \eta_k + \hat{x}_k - \hat{x}_j) \\ \dot{\hat{x}}_k &= A\hat{x}_k + Bu_k + H(\hat{y}_k - y_k) \end{aligned} \tag{7}$$

$$u_k = K\eta_k$$

for $k = 1, 2, \dots, N$, where $\hat{y}_k = C\hat{x}_k$, solves the Synchronization Problem.

Theorem 2 *Assume that the system (5) is stabilizable and detectable and that all the eigenvalues of A belong to the closed left-half complex plane. Assume that the communication graph is uniformly connected and the Laplacian matrix is piecewise continuous and bounded. Then for any gain matrices K and H such that $A + BK$ and $A + HC$ are Hurwitz, the solutions of (5) with the dynamic controller (7) uniformly exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$. \square*

Proof: Define $s_k = \hat{x}_k - \eta_k$ and $e_k = x_k - \hat{x}_k$, and rewrite the closed loop system as

$$\dot{x} = \left(\tilde{A}_N + \tilde{B}_N \tilde{K}_N \right) x - \tilde{B}_N \tilde{K}_N (e + s) \quad (8)$$

$$\dot{s} = \tilde{A}_N s - \hat{L}_n(t)s + \tilde{H}_N \tilde{C}_N e \quad (9)$$

$$\dot{e} = \left(\tilde{A}_N + \tilde{H}_N \tilde{C}_N \right) e, \quad (10)$$

where $\tilde{C}_N = I_N \otimes C$. This system is the cascade of the system

$$\dot{x} = \left(\tilde{A}_N + \tilde{B}_N \tilde{K}_N \right) x - \tilde{B}_N \tilde{K}_N (e + s) \quad (11)$$

with the exponentially stable estimation error dynamics (10) and the (consensus) dynamics (9). Subsystem (9) obeys to the dynamics analyzed in Lemma 1 with an extra input that vanishes exponentially. The assumptions of Lemma 1 are satisfied for the sub-system (9) with zero input. Therefore when $e = 0$, (9) has a uniformly exponentially invariant subspace that is the set where s_k , $k = 1, 2, \dots, N$, are synchronized. Then it is possible to choose a linear coordinate change to decouple the in-manifold dynamics and the off-manifold dynamics. Regarding the off-manifold dynamics, the reduced system has a uniformly exponentially stable isolated equilibrium, forced by an input that vanishes exponentially. From Corollary 1 in [19] we conclude that s_k , $k = 1, 2, \dots, N$ uniformly asymptotically synchronize to a solution of $\dot{s}_0 = As_0$. Subsystem (9) can be rewritten as

$$\dot{\eta} = \left(\tilde{A}_N + \tilde{B}_N \tilde{K}_N \right) \eta + \hat{L}_n(t)s. \quad (12)$$

Since the pair (A, B) is stabilizable and K is chosen to be a stabilizing matrix, the subsystem (12) is an exponentially stable system driven by an input $\hat{L}_n(t)s(t)$ that asymptotically converges to zero. As a consequence, its solution $\eta(t)$ exponentially converges to zero as well. Since $x_k = s_k + e_k + \eta_k$ we conclude that the solutions of (5) with the dynamic controller (7) exponentially synchronize to a solution of $\dot{x}_0 = Ax_0$. \blacksquare

Some comments on Theorem 2 are now in place. The dynamic control law (7) requires that the systems exchange only relative information regarding their controller state and state estimate. This fact implies that an absolute reference frame is not required. The dynamic control law (7) solves the Synchronization Problem for system (5) in its generality. In the particular case of state coupling, i.e. when $C = I_n$, the Luenberger observer in (7) is not required and the dynamic control law simplifies to

$$\dot{\eta}_k = (A + BK)\eta_k + \sum_{j=1}^N a_{kj}(t) (\eta_j - \eta_k + x_k - x_j) \quad (13)$$

$$u_k = K\eta_k,$$

for $k = 1, 2, \dots, N$.

4 Discussion and extensions

For the sake of completeness, in this section we extend the obtained results to address synchronization of discrete-time linear systems. Then we briefly discuss existing results (providing also a simple extension) about synchronization under (static) diffusive output coupling.

4.1 Discrete-time systems

Consider the discrete-time linear systems

$$x_k(t+1) = Ax_k(t) + Bu_k(t), \quad (14a)$$

$$y_k(t) = Cx_k(t), \quad (14b)$$

for $k = 1, 2, \dots, N$, and $t = 1, 2, \dots$.

Now consider system (14) with the dynamic control law (discrete-time version of (7))

$$\eta_k(t+1) = (A + BK)\eta_k(t) + \epsilon_k A \sum_{j=1}^N l_{kj}(t) (\hat{x}_j(t) - \eta_j(t)), \quad (15)$$

$$\hat{x}_k(t+1) = A\hat{x}_k(t) + H(y_k(t) - \hat{y}_k(t)),$$

for $k = 1, 2, \dots, N$, where $\hat{y}_k(t) = C\hat{x}_k(t)$.

Theorem 2 can be easily adapted to the case of discrete-time linear systems. The proof is based on an analogous version of Lemma 1 for discrete-time systems and on an adaptation of the proof of Theorem 2 (see the paper [20], available online, for more details).

Theorem 3 *Assume that the system (14) is stabilizable and detectable and that all the eigenvalues of A belong to the closed unitary disk in the complex plane. Assume that the communication graph $\mathcal{G}(t)$ is uniformly connected and the Laplacian matrix is piecewise continuous and bounded. Then for any gain matrices K and H such that $A + BK$ and $A + HC$ are Schur matrices, the solutions of (14) with the dynamic controller (15) uniformly exponentially synchronize to a solution of $x_0(t + 1) = Ax_0(t)$. \square*

Remark 2 *The results presented in the present paper straightforwardly extend to (time-varying) periodic linear systems. Periodic linear systems, naturally arise in a number of contexts in engineering, physics, and biology [21], furthermore they emerge when linearization of a nonlinear system along a periodic solution is analyzed. For the sake of compactness we refer the interested reader to the paper [20] (available online) for the relative extension.*

4.2 Dynamic coupling versus static coupling

Theorem 2 and 3 provide a general synchronization result for linear systems but the solution requires a dynamic controller. For the sake of comparison, we provide a set of sufficient conditions to prove synchronization under a simple static output feedback (diffusive) interconnection. These sufficient conditions require stronger assumptions on the interconnection and assume a passivity property for the system (A, B, C) , that is, the existence of a symmetric positive definite matrix $P > 0$ that verifies

$$PA + A^T P \leq 0, \quad B^T P = C. \quad (16)$$

Passivity conditions have been considered previously in [22] (where it is assumed that the communication topology is bidirectional and strongly connected) and in [7] (where synchronization is studied for a class of (nonlinear) oscillators assuming that the communication topology is time-invariant and balanced). Assumptions A1 and A2 below lead to a time-varying extension of the results in [7] and [22] in the special case of linear systems.

Theorem 4 *Consider system (5) with the static output feedback control laws*

$$u_k = \sum_{j=1}^N a_{kj}(t)(y_j - y_k), \quad k = 1, 2, \dots, N.$$

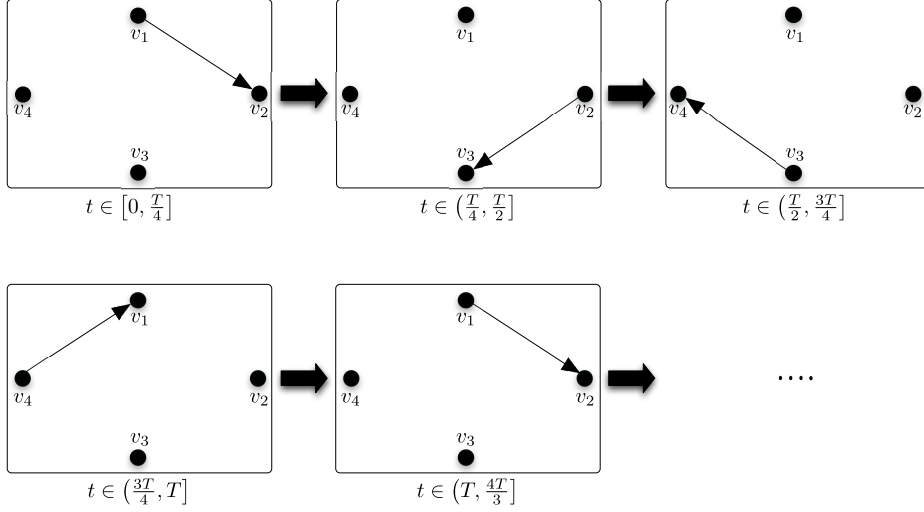


Fig. 1. The time-varying communication topology used in Example 1 and Example 2.

Let the graph Laplacian matrix $L(t)$ be piecewise continuous and bounded. Then exponential synchronization to a solution of $\dot{x}_0 = Ax_0$ is achieved under either one of the following assumptions:

A1. The system (A, B, C) is passive and observable, the communication graph is connected and balanced at each time;

A2. The system (A, B, C) is passive and observable, the communication graph is symmetric, i.e. the Laplacian matrix can be factorized as $L = DD^T(t)$, and the pair $(\tilde{A}_N, \hat{D}_p^T(t)\tilde{C}_N)$, is uniformly observable, where $\hat{D}_p = D \otimes I_p$. \square

Proof: due to the space constraints, we refer to the paper [20] available online for a complete proof of the theorem.

■

5 Examples

The conditions of Theorem 4 are only *sufficient* conditions for exponential synchronization under diffusive coupling. We provide two simple examples to illustrate that these conditions are not far from being necessary when considering time-varying and directed graphs and that the *internal model* of the dynamic controller (7) plays an important role in such situations.

Example 1: Synchronization of harmonic oscillators

Consider a group of N harmonic oscillators

$$\begin{aligned}
\dot{x}_{1k} &= x_{2k} \\
\dot{x}_{2k} &= -x_{1k} + u_k, \\
y_k &= x_{2k}
\end{aligned} \tag{17}$$

for $k = 1, 2, \dots, N$, which corresponds to system (5) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (0 \ 1).$$

The assumptions of Theorem 4 are satisfied: A is Lyapunov stable, (A, B) is stabilizable and (A, C) is detectable. Choosing the stabilizing gain $K = (0 \ -1)$ and the observer matrix $H = (0 \ -1)^T$, the dynamic control law (7) particularizes to

$$\begin{aligned}
u_k &= -\eta_{2k} \\
\dot{\eta}_{1k} &= \eta_{2k} + \sum_{j=1}^N a_{kj}(t)(\eta_{1j} - \eta_{1k} + \hat{x}_{1k} - \hat{x}_{1j}) \\
\dot{\eta}_{2k} &= -\eta_{1k} - \eta_{2k} + \sum_{j=1}^N a_{kj}(t)(\eta_{2j} - \eta_{2k} + \hat{x}_{2k} - \hat{x}_{2j}) \\
\dot{\hat{x}}_{1k} &= \hat{x}_{2k} \\
\dot{\hat{x}}_{2k} &= -\hat{x}_{1k} - \eta_{2k} + y_k - \hat{y}_k
\end{aligned} \tag{18}$$

for $k = 1, 2, \dots, N$, where $\hat{y}_k = \hat{x}_{2k}$ and $u_k = -\eta_{2k}$. Theorem 4 ensures exponential synchronization of the oscillators to a solution of the harmonic oscillator if the graph is uniformly connected. Fig. 2 illustrates the simulation of a group of 4 oscillators coupled according to the time-varying communication topology shown in Fig. 1 (the period T is set to 7 sec). The dynamic control law (18) ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection

$$u_k = \sum_{j=1}^N a_{kj}(t)(x_{2j} - x_{2k}). \tag{19}$$

The system (A, B, C) is nevertheless passive, meaning that stronger assumptions on the communication graph would ensure synchronization with the diffusive coupling (19). We mention the recent result [15] that proves (in

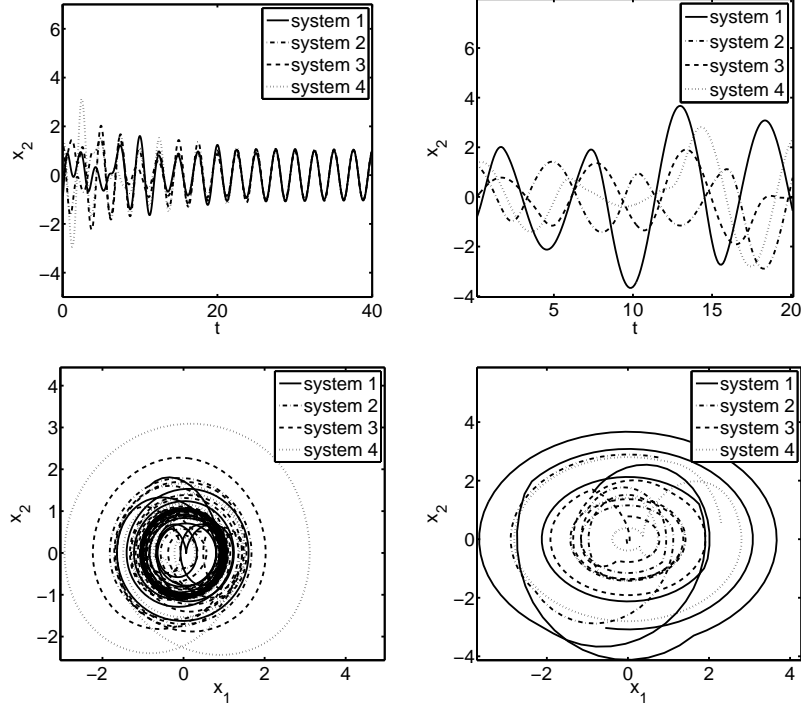


Fig. 2. Evolution of the outputs and of the solutions for the closed loop harmonic oscillators resulting from system (17) with the dynamic control law (18) (to the left) and the static control law (19) (to the right). The dynamic control ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection.

discrete-time) synchronization of harmonic oscillators with diffusive coupling under the assumption that the graph is time-invariant and connected.

Remark 3 *The harmonic oscillators considered in this example are passive. It is worth noting that Theorem 2 does not require passivity and analogous results are obtained, e.g., when $C = (1 \ 0)$ picking $H = (0 \ -1)^T$ (i.e. when the harmonic oscillators are not passive).*

The following example illustrates an analog scenario with unstable dynamics.

Example 2: Consensus for double integrators

Consider a group of N double integrators

$$\begin{aligned}
 \dot{x}_{1k} &= x_{2k} \\
 \dot{x}_{2k} &= u_k, \\
 y_k &= x_{1k} + x_{2k}
 \end{aligned} \tag{20}$$

for $k = 1, 2, \dots, N$, which corresponds to system (5) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1 \ 1).$$

The assumptions of Theorem 4 are satisfied: the two eigenvalues of A are zero and (A, B) is stabilizable. Choosing the stabilizing gain $K = (-1 \ -1)$ and the observer matrix $H = (0 \ -1)^T$, the dynamic control law (13) yields closed-loop system

$$\begin{aligned} u_k &= -\eta_{1k} - \eta_{2k} \\ \dot{\eta}_{1k} &= \eta_{2k} + \sum_{j=1}^N a_{kj}(t)(\eta_{1j} - \eta_{1k} + x_{1k} - x_{1j}) \\ \dot{\eta}_{2k} &= -\eta_{1k} - \eta_{2k} + \sum_{j=1}^N a_{kj}(t)(\eta_{2j} - \eta_{2k} + x_{2k} - x_{2j}) \\ \dot{\hat{x}}_{1k} &= \hat{x}_{2k} \\ \dot{\hat{x}}_{2k} &= -\eta_{1k} - \eta_{2k} + y_k - \hat{y}_k \end{aligned} \tag{21}$$

for $k = 1, 2, \dots, N$, where $\hat{y}_k = \hat{x}_{1k} + \hat{x}_{2k}$. Theorem 4 ensures exponential synchronization to a solution of the double integrator if the graph is uniformly connected. Fig. 3 illustrates the simulation of a group of 4 double integrators coupled according to the time-varying communication topology shown in Fig. 1 (the period T is set to 7 sec). The dynamic control (21) ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection

$$u_k = \sum_{j=1}^N a_{kj}(t)(y_j - y_k), \quad y_k = x_{1k} + x_{2k}. \tag{22}$$

The matrix $A - \alpha BC$ is nevertheless stable for every $\alpha > 0$, suggesting that stronger assumptions on the communication graph would ensure synchronization.

6 Conclusions

In this paper the problem of synchronizing a network of identical linear systems described by the state-space model (A, B, C) has been addressed. A dynamic controller ensuring exponential convergence of the solutions to a synchronized solution of the decoupled systems is provided assuming that (i) A has no exponentially unstable modes, (ii)

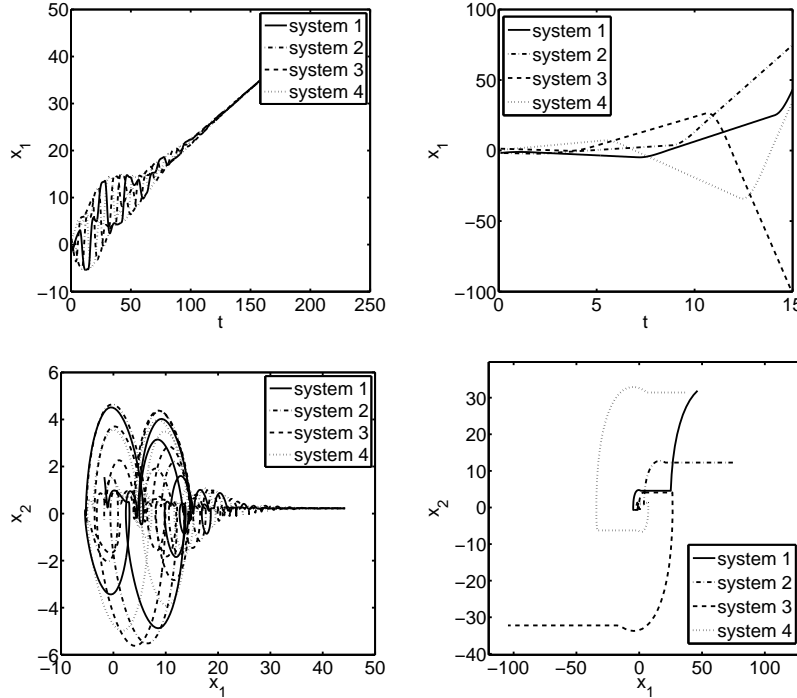


Fig. 3. Evolution of the first component of the solutions and of the solutions for the closed loop double integrators resulting from system (20) with the dynamic control law (21) (to the left) and the static control law (22) (to the right). The dynamic control ensures exponential synchronization. In contrast, synchronization is not observed with the diffusive interconnection. (A, B) is stabilizable and (A, C) is detectable, and (iii) the communication graph is uniformly connected. Stronger conditions are shown to be sufficient (and, to some extent, also necessary) to ensure synchronization with the often considered static diffusive output coupling.

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