A Sahlqvist theorem for subordination algebras

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Subordination algebras have been studied under several different names: precontact algebras ([6]), proximity algebras ([7]), strict implications algebras ([3]) or even quasi-modal algebras ([4]). The equivalences between all these definitions are well discussed in [2] and [5]. Modal algebras and subordination algebras share a common characteristic: their duals, in the sense of the Stone Duality, are Boolean topological spaces with a closed relation and, in particular, Kripke frames. Therefore the latter denomination, quasi-modal, is not a surprise. Moreover, since subordination algebras form a more general class than modal algebra, we propose an investigation of modal logic through subordination algebras.

While this problem has already been studied under other perspectives, for instance in [1] or in [3], our approach is slightly different as we are interested in validity of modal formulas in subordinations algebras instead of validity of subordination formulas.

Definition 1 (see [2])**.** A subordination algebra (or quasimodal algebra) is a pair $\mathcal{B} = (B, \prec)$ where B is a Boolean algebra and $\prec$ is a binary relation on B, called subordination, verifying the following properties :

1. $0 \prec 0$ and $1 \prec 1$,
2. $a \prec b, c$ implies $a \prec b \wedge c$,
3. $a, b \prec c$ implies $a \vee b \prec c$,
4. $a \leq b \prec c \leq d$ implies $a \prec d$.

In [2], it is stated that modal algebras are particular subordination algebras. The authors also provide a sufficient condition for subordination algebras to be modal algebras.

Definition 2. A subordination space is a pair $\mathcal{X} = (X, R)$ where X is a Stone space and R is a binary closed relation on X.

Theorem 3 (see [4])**.** *The category **Sub**, whose objects are subordination algebras and whose morphisms are the q-homomorphisms defined in [4], and the category **SubS**, whose objects are subordination spaces and whose morphisms are the q-morphisms defined in [4], are dually equivalent.*

To be seen as models for modal logic, we need to define valuation and validity on subordination algebras. The problem is that we cannot extend freely the valuation for variables to modal formulas, as for instance $\Box p$ may fail to be a clopen set of the dual. In order to resolve this issue, we will focus on the canonical extension of a subordination algebra.

Theorem 4. *If $\mathcal{B} = (B, \prec)$ is a subordination algebra, then its canonical extension $\mathcal{B}^\delta = (\mathcal{P}(X_\mathcal{B}), \prec_R)$, where $(X_\mathcal{B}, R)$ is the subordination space dual to $\mathcal{B}$ and $\prec_R$ is defined by*

$$E \prec_R F \Leftrightarrow R(-, E) \subseteq F, \tag{1}$$

is a complete tense bimodal algebra with $\Diamond E = R(-, E)$ and $\blacklozenge E = R(E, -)$.

Definition 5. Let $\mathcal{B}$ be a subordination algebra. A valuation on $\mathcal{B}$ is a map $v : \mathrm{Var} \longrightarrow B$, where Var is the set of variables. In particular, this map can be considered as a map $v : \mathrm{Var} \longrightarrow B^\delta$ and, as such, extend to a bimodal morphism between the set of all bimodal formulas and B^δ. As usual, we will say that a formula φ is valid in $\mathcal{B}$ under the valuation v, which will be denoted by $\mathcal{B} \models_v \varphi$, if $v(\varphi) = 1$, where 1 is the top element of both $\mathcal{B}$ and $\mathcal{B}^\delta$. The formula φ is valid in $\mathcal{B}$ if $\mathcal{B} \models_v \varphi$ for all valuation v, this is denoted by $\mathcal{B} \models \varphi$.

Definition 6. Let φ be a bimodal formula. It is closed (resp. open) if it is obtained from propositional variables, negation of propositional variables, $\top$ and $\bot$ by applying $\wedge$, $\vee$, $\Diamond$ and $\blacklozenge$ (resp. $\Box$ and $\blacksquare$).

It is positive (resp. negative) if it is obtained from propositional variables (resp. negation of propositional variables)$\top$ and $\bot$ by applying $\wedge$, $\vee$, $\Diamond$, $\Box$, $\blacklozenge$ and $\blacksquare$.

It is strongly positive if it is a conjunction of formulas of the form

$$\Box^{\langle k \rangle} p = \Box^{k_1} \blacksquare^{k_2} \cdots \Box^{k_n} p$$

where p is a propositional variable, $n \in \mathbb{N}$ and $k \in \mathbb{N}^n$.

It is s-positive (resp. s-negative) if it is obtained from closed positive formulas (resp. open negative formulas) by applying $\wedge$, $\vee$, $\Box$ and $\blacksquare$ (resp. $\Diamond$ and $\blacklozenge$).

It is s-untied if it is obtained from strongly positive and s-negative formulas by applying $\wedge$, $\vee$, $\Diamond$ and $\blacklozenge$.

Theorem 7. *Let $\varphi = \Box^{\langle k \rangle}(\varphi_1 \to \varphi_2)$ be a bimodal formula where φ_1 is s-untied and φ_2 s-positive. Then, there exists a first order formula $\mathfrak{f}$ in the language of the accessibility relation such that for a subordination algebra $(B, \prec)$ and its dual $(X_{\mathcal{B}}, R)$ we have*

$$(B, \prec) \models \varphi \textit{ if and only if } (X_{\mathcal{B}}, R) \models \mathfrak{f}.$$

This theorem is relatively similar to the one obtained in [8] with the particularity that in the formula φ there is no $\Box$ within the scope of a $\Diamond$.

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