

# Active nonlinear energy sink using force feedback under transient regime

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## Abstract

In this paper, an active nonlinear energy sink (ANES) based on force feedback is investigated. The proposed device is composed of a pair of collocated actuator and force sensor. The control law is implemented by feeding back the output of the force sensor, through one single integrator and one double integrator of its cube. Its working principle can be understood by an equivalent mechanical network which consists of a linear dashpot, linear spring and a cube root inerter. Although the nonlinear assignment between the spring and mass or inerter quantities is different from that of traditional nonlinear energy sinks (NESs), it is found that ANES and NES behave similarly in terms of their slow-scale dynamics and the vibration mitigation effectiveness. Closed-form expressions for properly tuning the feedback gains are derived. Numerical simulations are performed to validate the analytical analysis. The damping mechanism of ANES through targeted energy transfer and resonance capture cascade is demonstrated.

**Keywords:** Inerter, force feedback, nonlinear energy sink, targeted energy transfer, resonance capture cascade

## 1. Introduction

Lightweight materials have been more and more used for system constructions in many engineering applications for the sake of fuel efficiency and reduction of environmental pollution [1, 2]. However, this will often make these structures lightly damped and responses could be unacceptably amplified around the resonance, causing many problems such as reduction of structural integrity, compromise of instrument functionality and even threat to human lives. In this sense, proper damping techniques need to be considered in parallel with the future design of lightweight structures. A tuned mass damper (TMD) [3] which typically consists of a proof mass and a spring-dashpot pair is often employed for such purpose. It acts as an auxiliary system to the host structures where additional damping is needed. The natural frequency of the added TMD is often suggested to be roughly equal to one of the resonance frequencies of the host structure [4]. In this way, the vibration energy associated with the considered mode can be quickly transferred and localised in the TMD where it is eventually dissipated. Due to this nature, TMDs are only effective around one particular vibration mode. Deviation from the desired settings can also degrade their performance [5]. In addition, one single TMD is only capable of dealing with one specified vibration mode to which it is tuned.

An interesting nonlinear damping alternative i.e. a nonlinear energy sink (NES) was proposed aiming to overcome these limitations [6, 7]. A NES is often realised by a proof mass, a dashpot and a cubic nonlinear spring. The working principle of NESs is a bit different from that of TMDs, the host structure energy is in fact irreversibly transferred to NESs and dissipated therein. As there are only nonlinear resilient elements within NESs, the natural frequency is not essentially fixed but is rather dependent on the vibration amplitude. This allows NESs to interact basically with every mode of the host structure and also makes them insensitive to parameter variations of the host structure. Substantial work has been done to better understand the underlying dynamics of NESs, see for example [8–13]. In the transient regime, it has been illustrated that there exists a threshold in terms of the vibration energy level of the host structure above which NESs start to work efficiently i.e. where targeted energy transfer (TET) occurs. TET is defined as a one-way irreversible transfer of energy from the host structure to the NES. Some design guidelines for properly choosing the parameters, namely the nonlinear stiffness and the damping coefficient to initiate TET have been established [12, 14]. Other possibilities for implementing NESs than using a cubic nonlinear spring have been also explored [15, 16]. As for the last parameter of NESs, a heavier proof mass is preferable, similarly to that for TMDs. However, the added mass may be penalizing in light weight applications, e.g. automotive and aerospace structures. Zhang et al. [17] and Javidialesaadi and Wierschem [18] proposed to integrate inerters into NESs aiming to boost the performance as the inertance of inerters can be significantly greater than their actual mass [19, 20]. Although the potential of NESs is promising, it is quite challenging to realise them in practice

because attaching them to the host structure with a pure cubic spring without any linear resilient parts remains a challenge.

In order to bypass the aforementioned practical issues, an active nonlinear sink using a novel force feedback controller is proposed in this work. A force sensor is chosen since the control plant for the corresponding underlying linear active system always possesses alternating poles and zeros, which would ease the design concern on the stability of the closed loop [21]. The control law is formed by feeding back the output of the force sensor through one single integrator and one double integrator of its cube. The control concept is actually built upon the previous developments [22] where the linear double integrator is removed from the chain. In this way, the proposed ANES can be understood to play the same role as a pure mechanical system which consists of a cube root inerter, a linear dashpot and a linear spring according to previous derivations in [22–24]. Although the nonlinear assignment of the proposed ANES is different from that of traditional NESs, it is found that ANES and NES behave similarly in terms of vibration mitigation effectiveness. Some other interesting work dealing with control of nonlinear systems can be found in [25–27]. The principal contributions of the work presented are: a) the development of the equivalent mechanical model which enables a straightforward interpretation of the physics behind the active control law, b) the derivation of the ANES tuning law in closed-form, and c) the equivalence examination between an ANES and a mechanical NES which opens the door for the real-time tuning of control parameters of an active nonlinear energy sink.

The rest of paper is organised as follows. In the next section, the mathematical model of the system under consideration is developed. In Section 3, the tuning law of ANES for a single-degree-of-freedom (SDOF) system is derived and numerically verified. This tuning law is extended to a multi-degree-of-freedom system (MDOF) in Section 4. Conclusions are drawn in Section 5.

## 2. Mechanical representation of an ANES

The system under investigation is shown in Fig. 1 (a). It represents a linear MDOF system which is equipped with a massless actuator whose stiffness is denoted by  $k_a$ . A collocated force sensor which measures the transmission force denoted by  $F_s$  is installed between the actuator and the primary structure. The control loop is implemented by feeding the output of the force sensor  $F_s$  through a nonlinear controller  $u(F_s)$  to drive the actuator. In this work, the transient behaviour of the system is of interest and no external disturbance force is applied.

The governing equations of the coupled system can be written as:

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{b}F_s(t) \quad (1)$$

$$F_s(t) = u(F_s(t)) - k_a \mathbf{b}^T \mathbf{X}(t) \quad (2)$$

where  $\mathbf{M} \in \mathfrak{R}^{n \times n}$ ,  $\mathbf{C} \in \mathfrak{R}^{n \times n}$  and  $\mathbf{K} \in \mathfrak{R}^{n \times n}$  represent the mass, damping and stiffness matrix, respectively,  $\mathbf{X} \in \mathfrak{R}^{n \times 1}$  the displacement vector,  $\mathbf{b} = [0 \ \dots \ -1_l \ 0 \ \dots \ 1_k \ \dots \ 0] \in \mathfrak{R}^{n \times 1}$  the actuator connection vector and  $F_s$  a scalar representing the force that acts on the structure (measured by the force sensor). Note that in the considered sketch the actuator is mounted to the primary structure through two attachment points where a pair of control forces are injected into the primary structure. Alternatively, the actuator can be also placed between the ground and the primary structure such that there is only one attachment point. In this case, a point control force is applied by the actuator. The two configurations make no difference in terms of the working principle of the ANES, but the connection vector  $\mathbf{b}$  has to be adapted accordingly such that the ANES parameters can be correctly configured.

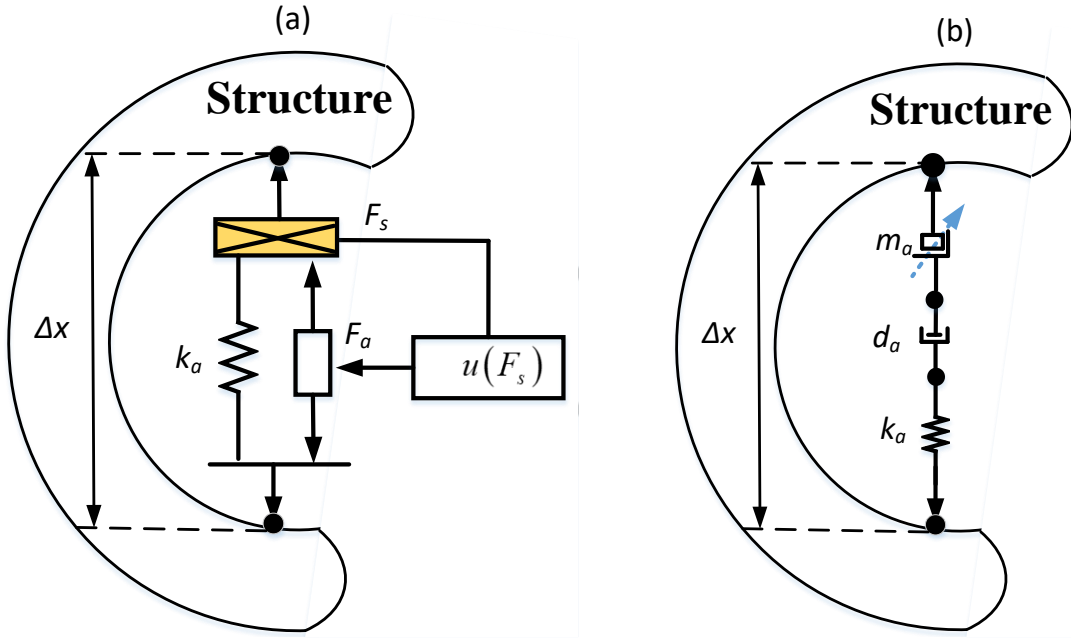
Note that the mass of the transducer is tacitly neglected in the paper. As a result, the validity of Eq. (1) should be restricted to frequencies well below the resonance frequencies of the transducer. In addition, lumped masses of the transducer can be included at the associated DOFs to compensate statically the shift of the resonance frequencies when the mass of the transducer is relatively large.

The nonlinear controller  $u(F_s)$  is built upon the controllers proposed in [22, 23], but modified to have a dynamic behaviour similar to that of NESs. The controller  $u(F_s)$  reads:

$$u(F_s) = -g_s \int_0^t F_s dt - g_{d3} \int_0^t \int_0^t F_s^3 dt dt \quad (3)$$

The working principle of the proposed controller can be better understood from a pure mechanical point of view. As illustrated in Appendix A, the function of the component with the single integrator in Eq. (3) behaves as a mechanical dashpot, and that with the double integrator resembles a cube root inerter. Together with the inherent spring of the actuator, the active system can be represented by a pure mechanical network composed by a linear spring, a linear dashpot and a cube root inerter connected in series. As such, the equivalent mechanical representation of the considered system is shown in Fig. 1 (b). The equivalent damping coefficient and the inertance are linked with the feedback gains according to:

$$d_a = k_a / g_s; \quad m_{a3} = \sqrt[3]{k_a / g_{d3}} \quad (4)$$



**Fig. 1 (a) the sketch of the coupled system and (b) its equivalent mechanical representation**

Because of the full analogy with a mechanical network, the stability of the active system is guaranteed given idealised force sensors and actuators are employed. According to the Lyapunov's linearisation theory [28], the actual nonlinear system is asymptotically stable at the equilibrium point if the corresponding linearised system is strictly stable. For the proposed ANES, the linearised system around the trivial state corresponds to the primary structure coupled with a classical integral force feedback controller. As demonstrated in [24, 29], the theoretical gain margin is infinity and the phase margin is  $\pi/2$  for the linearised system. However, the Lyapunov's linearisation theory is only valid for small range of motions around the equilibrium points (a local stability theorem) and it is not yet clear what are the boundary conditions for the linearisation approximations to hold (global stability theorem is needed). It is left for the subject of future work.

Although it is referred to as 'active nonlinear energy sink', it is actually implemented in a way different than that for NES. Within ANES, the mass-like quantity i.e. the cube root inerter is nonlinear and coupled with a linear spring and a dashpot in series, while a classical NES features a linear mass supported by a nonlinear spring and a linear dashpot. Despite this difference, it is found later in the paper that the working principle and the corresponding control effectiveness of the two devices are similar.

Substituting Eq. (3) into Eq. (2), and transforming the resulting governing equations into modal coordinates according to  $\mathbf{x}(t) = \mathbf{E}\mathbf{Q}(t)$  where  $\mathbf{E} = [\mathbf{e}_1 \ \mathbf{e}_2 \ \dots \ \mathbf{e}_n] \in \mathfrak{R}^{n \times n}$  is the matrix of mode shapes and  $\mathbf{Q} \in \mathfrak{R}^{n \times 1}$  the associated modal coordinates yields:

$$\mathbf{M}_q \ddot{\mathbf{Q}}(t) + \mathbf{C}_q \dot{\mathbf{Q}}(t) + \mathbf{K}_q \mathbf{Q}(t) = \mathbf{E}^T \mathbf{b} F_s(t) \quad (5)$$

$$F_s(t) = -g_s \int_0^t F_s dt - g_{d3} \int_0^t \int_0^t F_s^3 dt dt - k_a \mathbf{b}^T \mathbf{E} \mathbf{Q}(t) \quad (6)$$

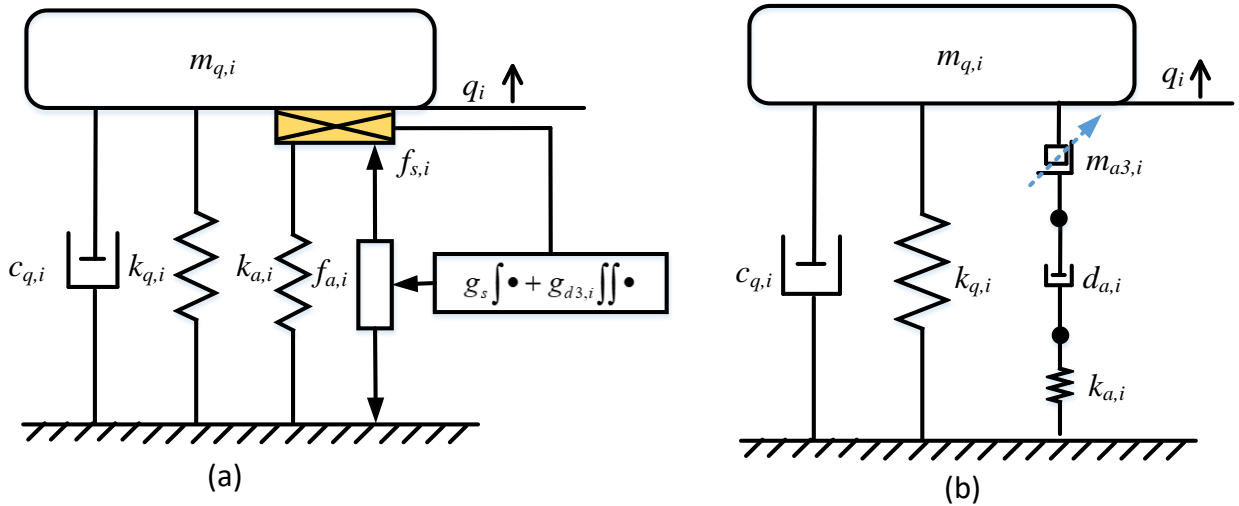
where  $\mathbf{M}_q = \mathbf{E}^T \mathbf{M} \mathbf{E}$  is the diagonal modal mass matrix,  $\mathbf{K}_q = \mathbf{E}^T \mathbf{K} \mathbf{E}$  the diagonal modal stiffness matrix and  $\mathbf{C}_q = \mathbf{E}^T \mathbf{C} \mathbf{E}$  the diagonal modal damping matrix assuming that the primary structure is lightly damped and  $\mathbf{C}$  is considered to be proportional to  $\mathbf{M}$  and  $\mathbf{K}$  ( $\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}$ ,  $\alpha, \beta \in \mathbb{R}_+$ ).

Consider now that the system only vibrates at the resonance frequency associated with mode  $i$  such that  $\mathbf{X}(t) = \mathbf{E} \mathbf{Q}(t) = \sum_{m=1}^n \mathbf{e}_m q_m(t) \approx \mathbf{e}_i q_i(t)$  is valid. Eqs. (5) and (6) can be thus rewritten as:

$$m_{q,i} \ddot{q}_i(t) + c_{q,i} \dot{q}_i(t) + k_{q,i} q_i(t) = f_{s,i}(t) \quad (7)$$

$$f_{s,i}(t) = -g_s \int_0^t f_{s,i} dt - g_{d3,i} \int_0^t \int_0^t (f_{s,i})^3 dt dt - k_{a,i} q_i(t) \quad (8)$$

where  $\Delta \mathbf{e}_i = \mathbf{e}_i(l) - \mathbf{e}_i(k)$ ,  $f_{s,i} = \Delta \mathbf{e}_i^T F_s(t)$ ,  $g_{d3,i} = g_{d3} / \Delta \mathbf{e}_i^2$  and  $k_{a,i} = k_a \Delta \mathbf{e}_i^2$ .



**Fig. 2 (a) The representative model of the whole system for one vibration mode and (b) its equivalent mechanical model.**

The characteristic equations defined by Eqs. (7) and (8) actually correspond to a system as depicted in Fig. 2 (a). This system is defined through a lumped mass  $m_{q,i}$  suspended from the ground by a linear spring  $k_{q,i}$ , a dashpot  $c_{q,i}$  and the actuator whose stiffness is denoted by  $k_{a,i}$ . The variables  $m_{q,i}$ ,  $k_{q,i}$  and  $c_{q,i}$  represent the modal mass, stiffness and damping coefficient of the primary structure for mode  $i$ , while  $f_{s,i}$ ,  $g_{d3,i}$ ,  $g_{d3,i}$  and  $k_{a,i}$  are the corresponding equivalent transmission force, the nonlinear feedback gain, the linear feedback gain and the actuator stiffness of the ANES for mode  $i$ . The corresponding equivalent mechanical model is given in Fig. 2 (b). In the forthcoming section, the simplified system in Fig. 2 (a) will be used to study the ANES characteristics with the aim to derive practical guidelines for the design of an effective ANES.

### 3. ANES Tuning for SDOF systems

#### 3.1 Semi-analytic reduction

The parameters:

$$\begin{aligned} x_1 &= q_i, & x_2 &= f_{s,i}/k_{a,i}, & \omega_i &= \sqrt{k_{q,i}/m_{q,i}}, & \tau &= \omega_i t, \\ \varepsilon &= k_{a,i}/k_{q,i}, & \xi_1 &= c_{q,i}\omega_i/k_{a,i}, & g^{sn} &= g_s/\omega_i, & g^{3n} &= g_{d3,i}(k_{a,i})^2/\omega_i^2 \end{aligned} \quad (9)$$

are introduced to normalise the governing equations Eqs. (7) and (8):

$$x_1'' + \varepsilon\xi_1 x_1' + x_1 - \varepsilon x_2 = 0 \quad (10)$$

$$x_2'' + g^{sn} x_2' + g^{3n} x_2^3 + x_1'' = 0 \quad (11)$$

where ' denotes derivation with respect to (w.r.t.) the time scale  $\tau$ .

Under the assumption of 1:1 resonance and the system vibrates along mode  $i$ , complex variables are introduced to reduce the order of the differential equations [30]:

$$\phi_1(\tau)e^{j\tau} = x_1' + jx_1; \quad \phi_2(\tau)e^{j\tau} = x_2' + jx_2 \quad (12)$$

where  $j = \sqrt{-1}$ .

Substituting Eq. (12) into Eqs. (10) and (11) yields:

$$\phi_1' e^{j\tau} - \varepsilon \frac{\phi_2 e^{j\tau} - \phi_2^* e^{-j\tau}}{2j} + \varepsilon \xi_1 \frac{\phi_1 e^{j\tau} + \phi_1^* e^{-j\tau}}{2} = 0 \quad (13)$$

$$\phi_1' e^{j\tau} + \phi_2' e^{j\tau} - \frac{\phi_1 e^{j\tau} - \phi_1^* e^{-j\tau} + \phi_2 e^{j\tau} - \phi_2^* e^{-j\tau}}{2j} + g^{sn} \frac{\phi_2 e^{j\tau} + \phi_2^* e^{-j\tau}}{2} + g^{3n} j \frac{(\phi_2 e^{j\tau} - \phi_2^* e^{-j\tau})^3}{8} = 0 \quad (14)$$

where \* denotes complex conjugate.

Each complex variable is decomposed into power series of the stiffness ratio  $\varepsilon$  ( $\varepsilon \ll 1$ ) as follows:

$$\phi_i = \phi_{i0} + \varepsilon \phi_{i1} + O(\varepsilon^2), \quad \forall i = 1, 2 \quad (15)$$

where  $\phi_{i0}$  is the dominant term and  $\phi_{i1}$  the small perturbation term. Here, the expansions are truncated for  $O(\varepsilon^2)$ .

The multiple scales method [31] is applied to Eq. (15) to better approximate the system dynamics. This method assumes that the dynamics behave according to several discrete time scales. In this paper,  $\tau_0 = \tau$  and  $\tau_1 = \varepsilon\tau$  representing the fast and the slow time scales are considered; they are related to the original time scale as:

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_1} \quad (16)$$

Substituting Eqs. (15) and (16) into Eqs. (13) and (14), one obtains:

$$\frac{\partial \phi_{10}}{\partial \tau_0} + \varepsilon \frac{\partial \phi_{10}}{\partial \tau_1} + \varepsilon \frac{\partial \phi_{11}}{\partial \tau_0} + \varepsilon^2 \frac{\partial \phi_{11}}{\partial \tau_1} + \frac{j\varepsilon(\phi_{20} + \varepsilon\phi_{21})}{2} + \frac{\varepsilon\xi_1(\phi_{10} + \varepsilon\phi_{11})}{2} = 0 \quad (17)$$

$$\begin{aligned} & \frac{\partial \phi_{10}}{\partial \tau_0} + \varepsilon \frac{\partial \phi_{10}}{\partial \tau_1} + \varepsilon \frac{\partial \phi_{11}}{\partial \tau_0} + \varepsilon^2 \frac{\partial \phi_{11}}{\partial \tau_1} + \frac{\partial \phi_{20}}{\partial \tau_0} + \varepsilon \frac{\partial \phi_{20}}{\partial \tau_1} + \varepsilon \frac{\partial \phi_{21}}{\partial \tau_0} + \varepsilon^2 \frac{\partial \phi_{21}}{\partial \tau_1} + \\ & + \frac{j(\phi_{10} + \varepsilon\phi_{11} + \phi_{20} + \varepsilon\phi_{21})}{2} + \frac{g^{sn}(\phi_{20} + \varepsilon\phi_{21})}{2} \end{aligned} \quad (18)$$

$$\frac{3g^{3n}j\left(|\phi_{20}|^2\phi_{20} + \varepsilon\left(2\phi_{21}|\phi_{20}|^2 + \phi_{20}^2\phi_{21}^*\right) + \varepsilon^2\left(2\phi_{20}|\phi_{21}|^2 + \phi_{20}^*\phi_{21}^2\right) + \varepsilon^3|\phi_{21}|^2\phi_{21}\right)}{8} = 0$$

where the secular terms w.r.t. the time scale  $\tau_0$  are eliminated.

Collecting the terms according to the different orders of the stiffness ratio  $\varepsilon$  in Eqs. (17) and (18), one obtains:

$$\frac{\partial \phi_{10}}{\partial \tau_0} = 0 \Rightarrow \mathcal{O}(\varepsilon^0) \quad (19)$$

$$\frac{\partial \phi_{10}}{\partial \tau_0} + \frac{\partial \phi_{20}}{\partial \tau_0} + \frac{j(\phi_{10} + \phi_{20})}{2} + \frac{g^{sn}\phi_{20}}{2} - \frac{3g^{3n}\phi_{20}|\phi_{20}|^2j}{8} = 0 \Rightarrow \mathcal{O}(\varepsilon^0) \quad (20)$$

$$\frac{\partial \phi_{10}}{\partial \tau_1} + \frac{\partial \phi_{11}}{\partial \tau_0} + \frac{j\phi_{20}}{2} + \frac{\xi_1\phi_{10}}{2} = 0 \Rightarrow \mathcal{O}(\varepsilon^1) \quad (21)$$

Eq. (19) means that the leading variable  $\phi_{10}$  does not depend on  $\tau_0$ . This indicates that  $\phi_{10}(\tau_0, \tau_1)$  can be approximated using only its slowly-varying component  $\phi_{10}(\tau_1)$ . Substituting Eq. (19) into Eq. (20), the resulting expression describes how  $\phi_{20}$  evolves w.r.t. the time scale  $\tau_0$ . For this one-dimensional system, it was shown in [32] that  $\phi_{20}$  ultimately converge to the steady-state solutions of Eq. (20) as:

$$\frac{j(\phi_{10} + \phi_{20})}{2} + \frac{g^{sn}\phi_{20}}{2} - \frac{3g^{3n}\phi_{20}|\phi_{20}|^2j}{8} = 0 \quad (22)$$

Note that the solutions of Eq. (22) neglect the transient dynamics of  $\phi_{20}(\tau_0, \tau_1)$  at the fast time scale, which also means that they are sought by taking the limit  $\tau_0 \rightarrow \infty$ .

Proceeding now to Eq. (21) and taking the above ansatz  $\tau_0 \rightarrow \infty$ , the secular term w.r.t. the time scale  $\tau_0$  i.e.  $\frac{\partial \phi_{11}}{\partial \tau_0}$  is omitted and one obtains:

$$\frac{\partial \phi_{10}}{\partial \tau_1} + \frac{j\phi_{20}}{2} + \frac{\xi_1\phi_{10}}{2} = 0 \quad (23)$$

Up to now, it is possible to solve the slowly-varying components  $\phi_{10}(\tau_1)$  and  $\phi_{20}(\tau_1) = \lim_{\tau_0 \rightarrow \infty} \phi_{20}(\tau_0, \tau_1)$  from Eqs. (22) and (23). Specifically,  $\phi_{20}(\tau_1)$  represents the trend of  $\phi_{20}$  after eliminating the fast oscillations.



In order to further simplify the analysis, a polar decomposition of the complex variables is introduced:

$$\phi_{10} = R_1(\tau_1)e^{j\delta_1(\tau_1)}, \quad \phi_{20} = R_2(\tau_1)e^{j\delta_2(\tau_1)} \quad (24)$$

where  $R_i \in \mathfrak{R}_+$  is the amplitude of  $\phi_i$ , and  $\delta_i$  the corresponding phase.

Substituting Eq. (24) into Eqs. (22) and (23) yields:

$$\frac{\partial R_1}{\partial \tau_1} - \frac{R_2 \sin(\delta_2 - \delta_1)}{2} + \frac{\xi_1 R_1}{2} = 0 \quad (25)$$

$$\frac{\partial \delta_1}{\partial \tau_1} R_1 + \frac{R_2 \cos(\delta_2 - \delta_1)}{2} = 0 \quad (26)$$

$$\frac{R_1 \cos(\delta_1 - \delta_2) + R_2}{2} + \frac{3R_2^3 g^{3n}}{8} = 0 \quad (27)$$

$$\frac{-R_1 \sin(\delta_1 - \delta_2)}{2} + \frac{R_2 g^{sn}}{2} = 0 \quad (28)$$

Eliminating the phase difference  $\delta_1 - \delta_2$  from Eqs. (27) and (28) gives:

$$R_1^2 = R_2^2 \left( (g^{sn})^2 + \left( 1 - \frac{3}{4} g^{3n} R_2^2 \right)^2 \right) \quad (29)$$

Taking the expression for  $\sin(\delta_1 - \delta_2)$  in Eq. (28) and substituting it into Eq. (25), one obtains:

$$\frac{\partial R_1}{\partial \tau_1} R_1 = -\frac{g^{sn} R_2^2}{2} - \frac{\xi_1 R_1^2}{2} \quad (30)$$

Further simplifications can be made by taking  $E_k = R_k^2$ ,  $\forall k = 1, 2$  which gives:

$$E_1 = |\phi_{10}|^2 = (x_1')^2 + x_1^2 + O(\varepsilon^1) \quad (31)$$

$$E_2 = |\phi_{20}|^2 = \lim_{\tau_0 \rightarrow \infty} \left( (x_2')^2 + x_2^2 + O(\varepsilon^1) \right) \quad (32)$$

Variables  $E_1$  and  $E_2$  play a similar role similar to the normalised energy of the primary structure and the attached active system, respectively.

Substituting Eqs. (31) and (32) into Eqs.(30) and (29), the system dynamics are governed by:

$$\frac{\partial E_1}{\partial \tau_1} = -\xi_1 E_1 - g^{sn} E_2 \quad (33)$$

$$E_1 = E_2 \left( (g^{sn})^2 + \left( 1 - \frac{3}{4} g^{3n} E_2 \right)^2 \right) \quad (34)$$

Eq. (33) describes how the energy-like quantities  $E_1$  and  $E_2$  dynamically evolve w.r.t. the slow time scale  $\tau_1$ , while Eq. (34) regulates them statically. Because the normalised damping factors  $\xi_1$

and  $g^{sn}$  as well as  $E_1$  and  $E_2$  are positive variables, the gradient of the primary structure energy  $E_1$  w.r.t. the slow time scale is always negative, as seen from Eq. (33). This means that the primary structure's energy will always be dissipated over the slow flow time. On the other hand, the evolution of  $E_2$  is governed by  $E_1$  following a nonlinear algebraic expression Eq. (34) where only the control parameters  $g^{sn}$  and  $g^{3n}$  are present. Thanks to this nonlinear relation, TET from the primary structure to the ANES can occur. The semi-analytic governing equations (33) and (34) are identical to those obtained in [14] for a mechanical NES provided the time scale in [14] is changed to a dimensionless quantity w.r.t. the resonance frequency of the primary structure. From this perspective, the normalised feedback gains  $g^{sn}$  and  $g^{3n}$  play the same role as the damping ratio and the nonlinear frequency ratio associated with a mechanical NES. This is logical as they correspond to a normalised dashpot and cube root inerter, respectively. We also note that switching nonlinear assignments between the spring and mass or inerter quantities does not significantly change the control effectiveness of the resulting device if one compares the active nonlinear inerter damper in [22] with the nonlinear tuned mass damper in [33]. However, it should be noticed that  $E_1$  and  $E_2$  in this paper correspond to the absolute motion of the primary mass and the transmission force induced by an ANES, while for the mechanical NES they are related to the motion of the center of mass and the relative motion between the primary and NES masses. Despite this difference, the tuning law of an ANES that triggers TET based on Eqs. (33) and (34) is foreseen to be the similar to that obtained as in [14].

Two new variables  $Z_1 = g^{3n}E_1$  and  $Z_2 = g^{3n}E_2$  are now introduced. In addition, the damping factor of the primary structure is omitted. Eqs. (33) and (34) are thus rewritten as:

$$\frac{\partial Z_1}{\partial \tau_1} = -g^{sn} Z_2 \quad (35)$$

$$Z_1 = Z_2 \left( (g^{sn})^2 + \left(1 - \frac{3}{4} Z_2\right)^2 \right) \quad (36)$$

In order to better study the underlying dynamics, Eq. (36) is differentiated w.r.t.  $Z_2$ , which gives:

$$\begin{aligned} \frac{\partial Z_1}{\partial Z_2} &= \frac{27}{16} Z_2^2 - 3Z_2 + 1 + (g^{sn})^2 \\ &= 3 \left( \frac{3}{4} Z_2 - \frac{2}{3} \right)^2 - \frac{1}{3} + (g^{sn})^2 \end{aligned} \quad (37)$$

Eq. (37) is a quadratic equation which is always positive if  $g^{sn} > \sqrt{1/3}$ . If this is the case,  $Z_1$  will monotonically increase/decrease with an increase/decrease of  $Z_2$ . Because Eq. (35) indicates that  $Z_1$  is always reduced w.r.t. the slow time scale,  $Z_2$  will follow the same trend whereby no energy is fed into the ANES and thus no TET occurs. If  $g^{sn} < \sqrt{1/3}$ , Eq. (37) has two roots  $Z_2^{1*}$  and  $Z_2^{2*}$  ( $Z_2^{1*} < Z_2^{2*}$ ,  $\forall Z_2^{i*} \in \mathfrak{R}_+$ ,  $i=1,2$ ). It remains positive when  $Z_2$  is either smaller than  $Z_2^{1*}$  or greater than  $Z_2^{2*}$ . Otherwise, Eq. (37) becomes negative meaning that the energy in the primary structure and in the ANES can flow in opposite ways. In this case, when plotted against  $Z_2 : 0 \rightarrow \infty$ ,  $Z_1$  exhibits a

maximum followed by a minimum due to the changing sign of its derivative w.r.t.  $Z_2$ ; it is initially positive, then negative and finally positive again. In addition, three solutions of  $Z_2$  can be found for  $Z_1$  bounded between the two extrema. In summary, TET can occur as discussed in [14, 15] provided that:

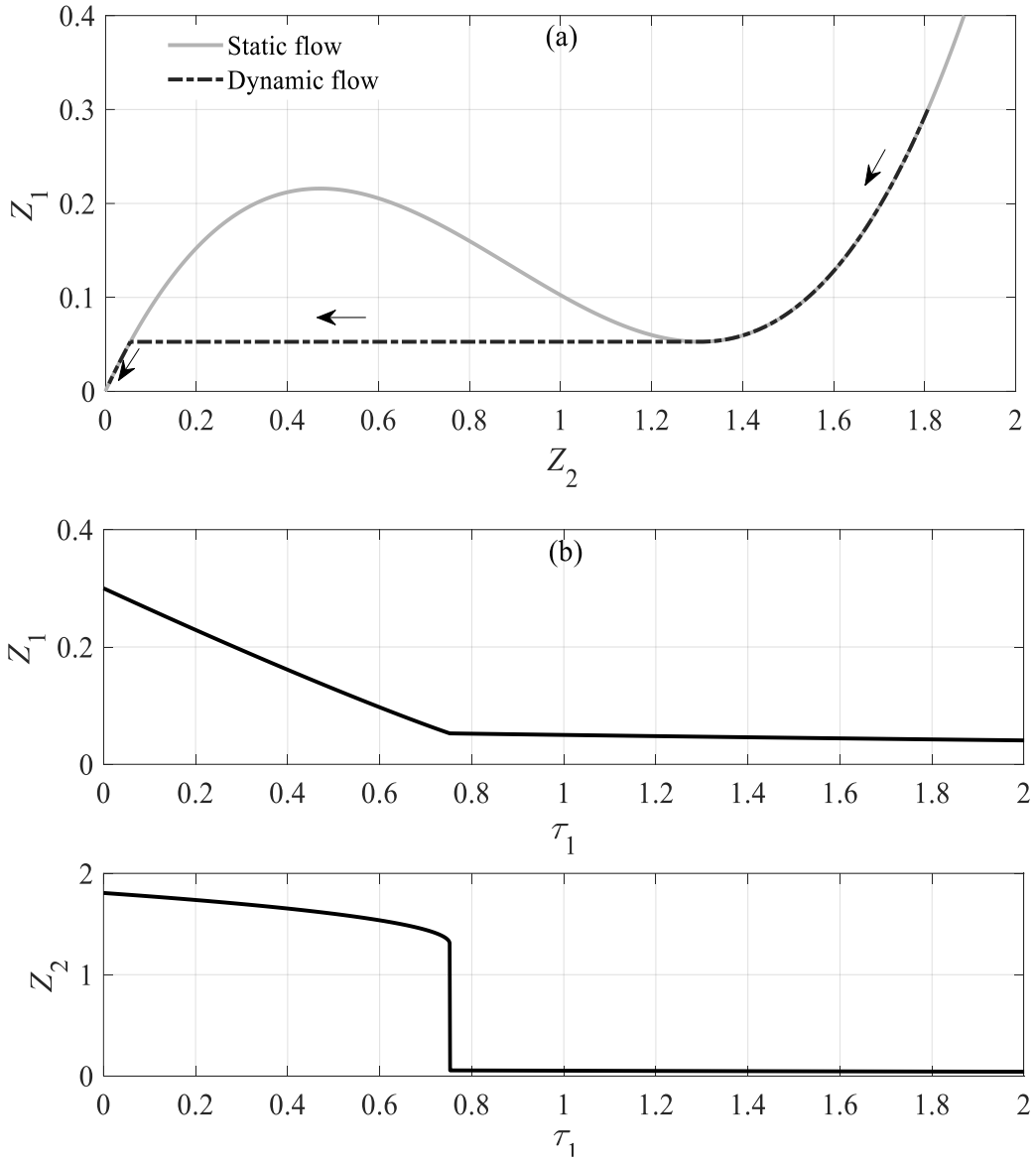
$$g^{sn} < \sqrt{\frac{1}{3}} \quad (38)$$

The solutions to Eq. (37), i.e.  $Z_2^{1*}$  and  $Z_2^{2*}$ , and the corresponding extreme values can be derived:

$$Z_1^{\max} = \frac{8}{81} \left[ 4 - \left( 1 - 3(g^{sn})^2 \right) \left( 3 - \sqrt{1 - 3(g^{sn})^2} \right) \right], \quad Z_2^{1*} = \frac{8}{9} - \frac{4\sqrt{1 - 3(g^{sn})^2}}{9} \quad (39)$$

$$Z_1^{\min} = \frac{8}{81} \left[ 4 - \left( 1 - 3(g^{sn})^2 \right) \left( 3 + \sqrt{1 - 3(g^{sn})^2} \right) \right], \quad Z_2^{2*} = \frac{8}{9} + \frac{4\sqrt{1 - 3(g^{sn})^2}}{9} \quad (40)$$

Besides the condition imposed on  $g^{sn}$ , it was shown that the initial conditions of the system also play an important role in the TET mechanism [10, 14, 15]. Specifically, the initial energy of the primary structure  $Z_1^{\text{init}}$  has to be greater than  $Z_1^{\max}$  given in Eq. (39). Fig. 3a depicts the slow invariant manifold between  $Z_1$  and  $Z_2$  when the condition  $Z_1^{\text{init}} > Z_1^{\max}$  is satisfied. As can be seen,  $Z_1$  and  $Z_2$  smoothly decrease from their initial conditions until  $Z_1$  reaches its local minimum  $Z_1^{\min}$ . At that moment,  $Z_2$  undergoes a jump, and is suddenly reduced from  $Z_2^{2*}$  to the smallest solution of  $Z_2$  corresponding to  $Z_1^{\min}$ . This jump phenomenon, however, is not observed for  $Z_1$  which continues to decrease after  $Z_1^{\min}$  but following a different branch, i.e. the leftmost branch of the invariant manifold. The evolution of  $Z_1$  and  $Z_2$  against the slow time scale  $\tau_1$  is shown in Fig. 3b. It is noted that  $Z_1$  decays with a quasi-linear slope before the critical point  $Z_1^{\min}$  instead of an exponential rate typically associated with linear systems. This clearly indicates the occurrence of TET. After the critical point  $(Z_1^{\min}, Z_2^{2*})$ , TET vanishes and  $Z_1$  decreases much slower compared to the case when TET is active. This drastically decreased rate is understood to be due to the low post-jump value of  $Z_2$ , which by virtue of Eq. (35) entails a low dissipation rate for  $Z_1$ . When the condition  $Z_1^{\text{init}} > Z_1^{\max}$  is not valid, TET is not robustly triggered or not triggered at all, as demonstrated in [14, 15].



**Fig. 3 (a) slow invariant manifold of  $Z_1$  and  $Z_2$  when  $g^{sn}$  is set to 0.2 and  $Z_1^{\text{init}}$  is set to 0.3, (b) the corresponding time history of  $Z_1$  and  $Z_2$  at slow time scale**

In order to meet this condition, the minimal normalised control gain  $g^{3n}$  i.e. the normalised cube root inertance can be derived as:

$$\begin{aligned}
 g^{3n} E_1^{\text{init}} = Z_1^{\text{init}} &\geq Z_1^{\text{max}} \\
 \downarrow & \\
 g^{3n} &\geq \frac{8}{81 E_1^{\text{init}}} \left[ 4 - \left( 1 - 3(g^{sn})^2 \right) \left( 3 - \sqrt{1 - 3(g^{sn})^2} \right) \right]
 \end{aligned} \tag{41}$$

### 3.2 Influence of control parameters

Up to now, the tuning laws (38) and (41) for the two control parameters  $g^{sn}$  and  $g^{3n}$  that initiate TET have been derived. However, it is not yet clear what is their impact on the vibration mitigation

performance. In this section, two metrics introduced in [14], namely the energy dissipation ratio and the pumping time are discussed in order to investigate the influence of the control parameters. The energy dissipation ratio is referred to as the ratio between the energy dissipated during TET and the initial energy, whereas the pumping time is defined as the time span of TET. As discussed previously, the point  $(Z_1^{\min}, Z_2^{2*})$  plays an important role, below which the beneficial effect induced by TET disappears. In this context, the energy dissipation ratio can be calculated as:

$$E_{\text{TET}} = \frac{Z_1^{\text{init}} - Z_1^{\min}}{Z_1^{\text{init}}} = 1 - \frac{8 \left[ 4 - (1 - 3(g^{sn})^2) \left( 3 - \sqrt{1 - 3(g^{sn})^2} \right) \right]}{81 g^{3n} E_1^{\text{init}}} \quad (42)$$

When the two control parameters  $g^{sn}$  and  $g^{3n}$  are chosen according to Eqs. (38) and (41) i.e. TET can be initiated,  $E_{\text{TET}}$  is bounded between  $[0, 1]$ . The closer  $E_{\text{TET}}$  is to unity, the more the energy is dissipated during TET.

The corresponding time duration of TET can be derived by rewriting Eq. (35) as:

$$\frac{\partial Z_1}{\partial \tau_1} = -g^{sn} Z_2 \Rightarrow \frac{\partial Z_1}{\partial Z_2} \frac{\partial Z_2}{\partial \tau_1} = -g^{sn} Z_2 \quad (43)$$

assuming that  $Z_1$  is still greater than  $Z_1^{\min}$  such that the derivative of  $Z_1$  against  $Z_2$  is continuous.

Substituting Eq. (37) into Eq. (43), one obtains:

$$\frac{\frac{27}{16} Z_2^2 - 3Z_2 + 1 + (g^{sn})^2}{Z_2} dZ_2 = -g^{sn} d\tau_1 \quad (44)$$

Integrating on both sides, one obtains:

$$f(Z_2) = C - g^{sn} \tau_1 \quad (45)$$

where  $f(Z_2) = \frac{27}{32} Z_2^2 - 3Z_2 + \left( 1 + (g^{sn})^2 \right) \ln(Z_2)$ .

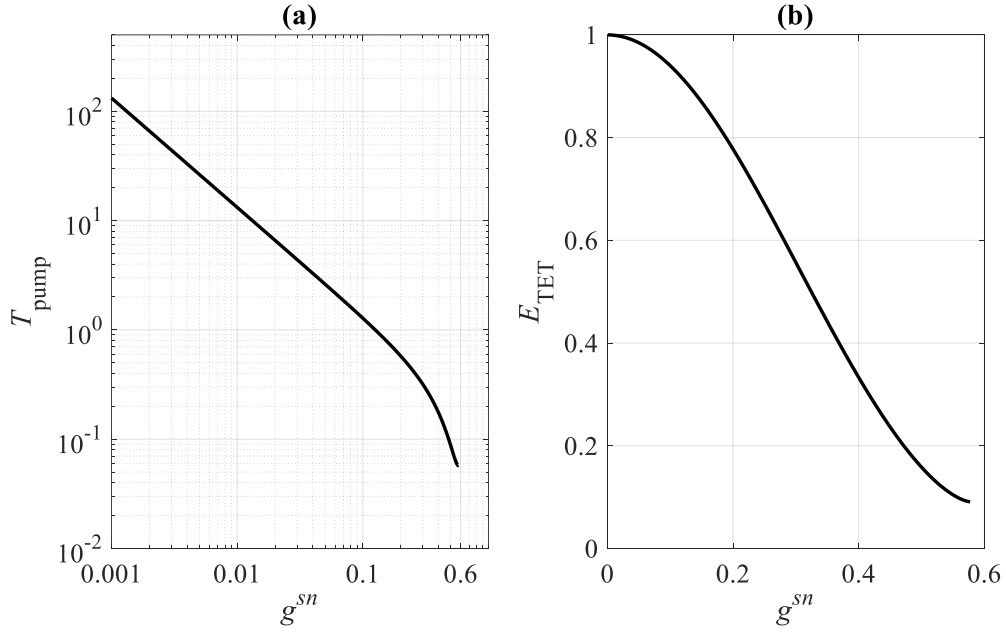
The normalised pumping time at slow time scale can be thus derived from Eq. (45) as:

$$T_{\text{pump}} = \frac{f(Z_2^{\text{init}}) - f(Z_2^{2*})}{g^{sn}} \quad (46)$$

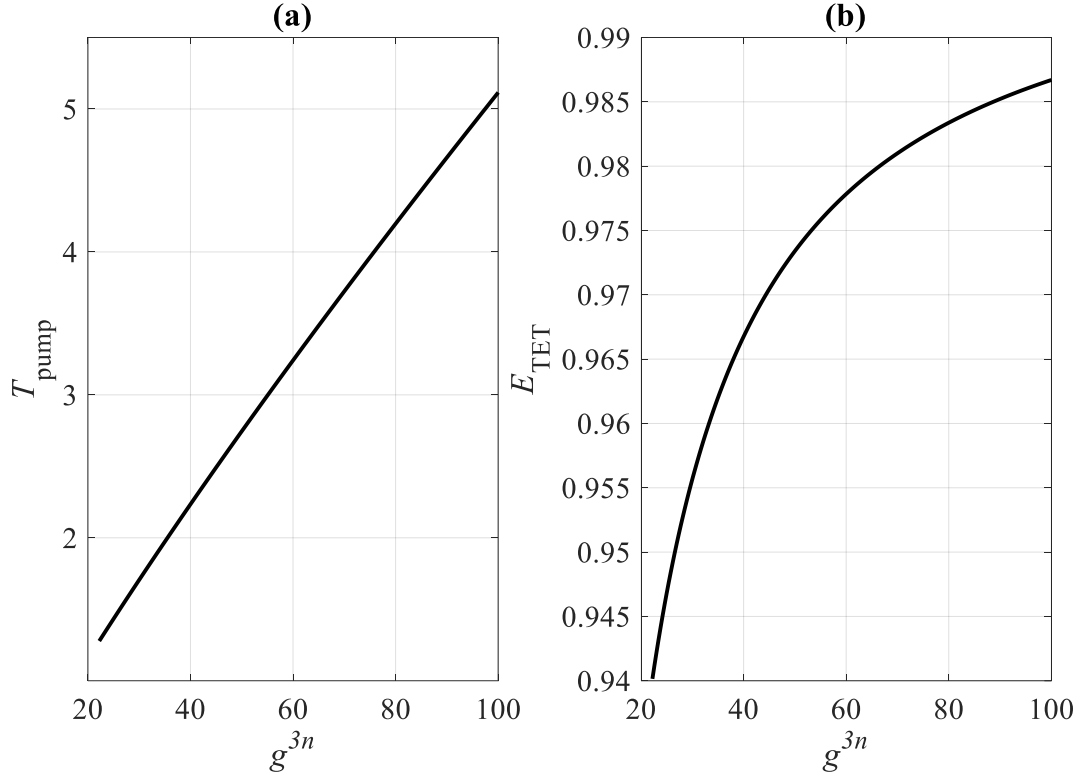
Note that  $T_{\text{pump}}$  is defined in this equation through the slow time scale  $\tau_1$ .

Fig. 4 (a) and (b) depict the evolution of the pumping time and the energy dissipation ratio against different values of  $g^{sn}$  ranging from 0.001 to the maximally allowed value for TET, i.e.,  $\sqrt{1/3}$ . For each value of  $g^{sn}$ ,  $g^{3n}$  is chosen such that the normalised initial energy  $Z_1^{\text{init}}$  is 1.1 times greater than the threshold value  $Z_1^{\text{max}}$  where  $E_1^{\text{init}}$  is set to 0.01. As shown, both metrics monotonically decrease with an increase of  $g^{sn}$ . This means that a large amount of energy is dissipated during TET when  $g^{sn}$  is set to a relatively small value, but this comes at the cost of a longer pumping time. Therefore, there is a trade-off in choosing a suitable value of  $g^{sn}$  for the considered performance metrics.

Fig. 5 (a) and (b) illustrate the same quantities but against  $g^{3n}$  when  $g^{sn}$  is set to 0.1. The range of  $g^{3n}$  is defined such that the normalised initial energy  $Z_1^{\text{init}}$  varies from  $1.1 \times Z_1^{\text{max}}$  to  $5 \times Z_1^{\text{max}}$ . Contrary to the previous case, both metrics monotonically increase with an increase of  $g^{3n}$ . This means that more energy is dissipated for a greater value of  $g^{3n}$ , but again it needs more time to complete. Therefore, the same compromise is observed for the choice of  $g^{3n}$ . In the light of Fig. 5 (a) and (b), it is recommended to set it slightly above the threshold value as the pumping time  $T_{\text{pump}}$  degrades more than the energy dissipation rate  $E_{\text{TET}}$  benefits from an increase of  $g^{3n}$ .



**Fig. 4 (a) Pumping time and (b) energy dissipation ratio against  $g^{sn}$  when  $g^{3n}$  is chosen so that the initial energy  $Z_1^{\text{init}}$  is 1.1 times greater than  $Z_1^{\text{max}}$**



**Fig. 5 (a) Pumping time and (b) energy dissipation ratio against  $g^{3n}$  when  $g^{sn}$  is set to 0.1**

### 3.3 Numerical validation

Numerical studies are performed in order to validate the derived semi-analytic equations and to examine the control effectiveness of the proposed ANES. The MATLAB built-in integration function *ode45* is used to solve the original governing equations (10) and (11). The solutions are then substituted into Eqs. (31) and (32) in order to compute the energy-like variables  $E_1$  and  $E_2$ , and eventually  $Z_1$  and  $Z_2$ . The system parameters are listed in Table 1. Two cases are considered, i.e.  $g^{3n}$  is chosen to be 1.2 and 0.8 times greater than the minimum value required for triggering TET.

**Table 1 parameters used for numerical simulations**

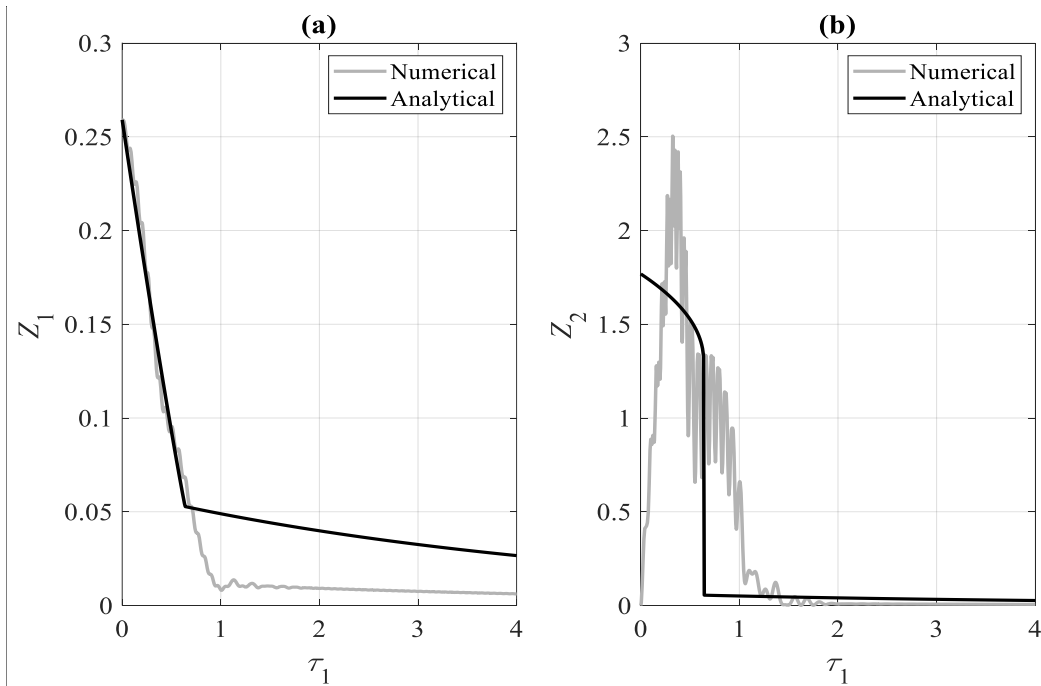
System parameters	$g^{sn}$	$g^{3n}$	$x_1^{\text{init}}$	$\varepsilon$
TET activated	0.2	$2.6 \times 10^3$	$10^{-2}$	0.02
TET inactivated	0.2	$1.7 \times 10^3$	$10^{-2}$	0.02

Fig. 6 compares the results computed using the numerical and the analytical approach when TET is activated. It can be observed that  $Z_1$  decays with a quasi-linear slope until the critical point at which  $Z_2$  undergoes a jump. In addition, the evolution of  $Z_1$  is shown to be determined only by the slow time scale dynamics as suggested by Eq. (19), whereas  $Z_2$  exhibits both fast and slow scale dynamics. Despite some differences present after the critical point, the semi-analytical result

in terms of  $Z_1$  is found to be in good accordance with that obtained from the numerical calculations. In contrast,  $Z_2$  behaves differently between the numerical and analytical results. This is because the semi-analytical approach neglects the fast time scale contributions such that it fails to predict the actual dynamics of  $Z_2$ . Nevertheless, the semi-analytic solution for  $Z_2$  gives the trend of the response when averaging out the fast components of the actual solutions. It should also be noted that  $Z_2$  exhibits initially some high-frequency oscillations. This is caused by the fact that the initial state of the system does not necessarily lie on the slow invariant manifold such that it takes some time to be attracted to this manifold. This is also referred to as nonlinear beating [10]. The mismatch after the critical point is caused by the fact that the sudden jump of  $Z_2$  is not well followed by the fast dynamics. Overall, the semi-analytical approach is shown to offer a good qualitative description of the system's dynamics.

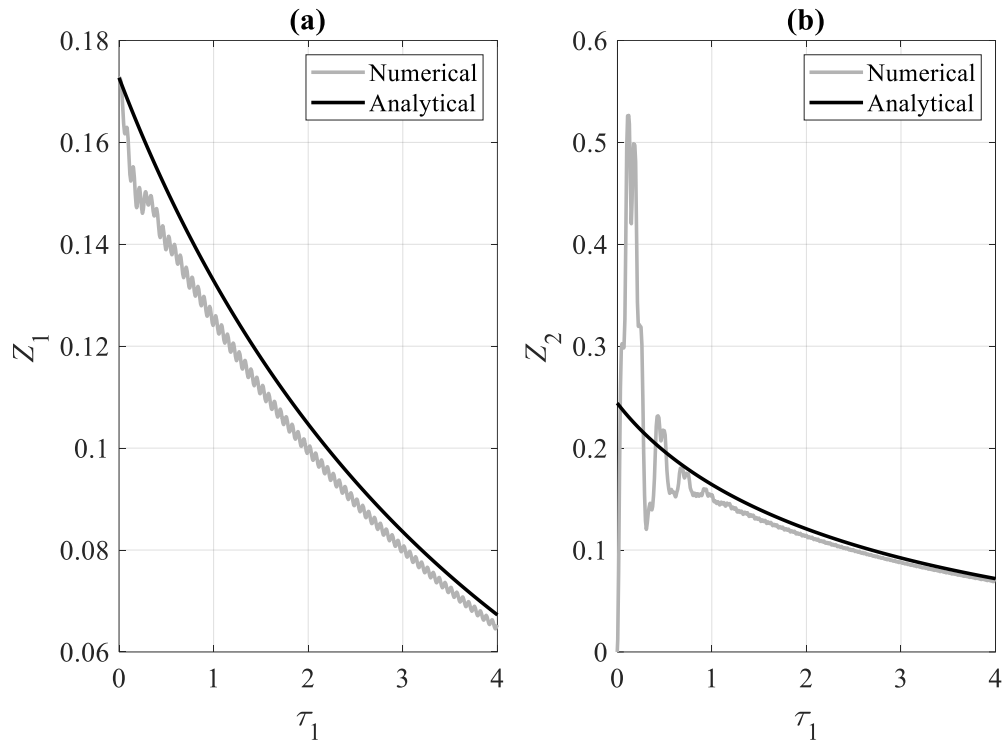
Fig. 7 plots the results obtained when TET is not activated. A fairly good match is observed. Both  $Z_1$  and  $Z_2$  decay with an exponential slope, as linear systems do.

The control performance in terms of the primary structure's displacement is compared in Fig. 8 when TET is activated or not. It is clear that the energy is damped much more quickly when the ANES is tuned to activate TET.

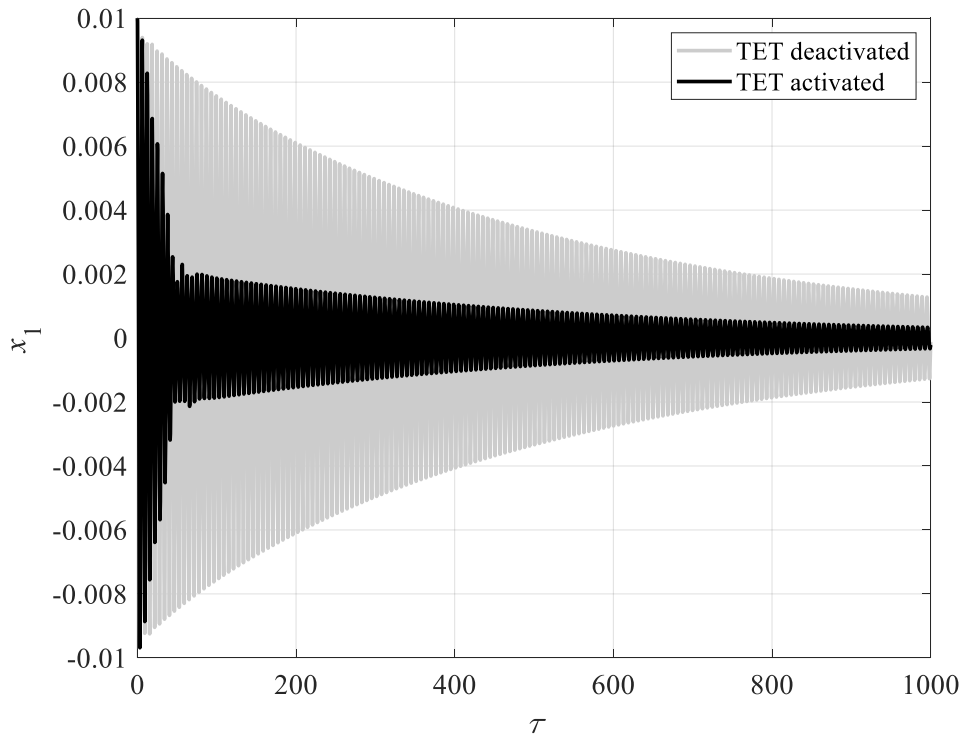


**Fig. 6 Comparison of numerical and analytical results when TET is activated for: (a)  $Z_1$  and (b)  $Z_2$  at the slow time scale**



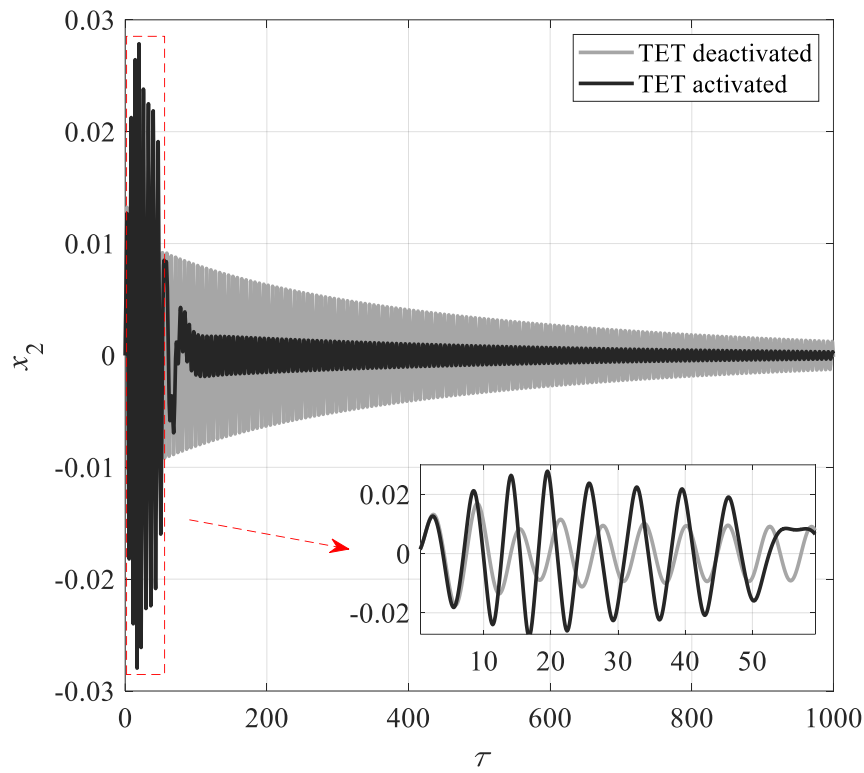


**Fig. 7 Comparison of numerical and analytical results when TET is inactive for: (a)  $Z_1$  and (b)  $Z_2$  at the slow time scale**

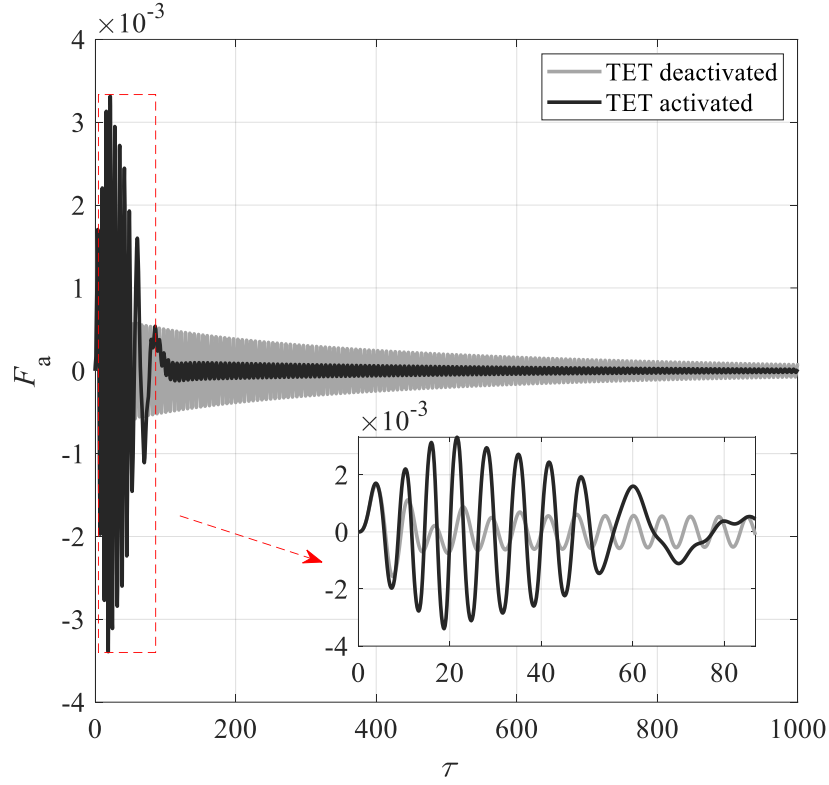


**Fig. 8 Comparison of control effectiveness when TET is activated or deactivated**

Fig. 9 depicts the variable  $x_2$  i.e. the normalised force measured by the force sensor. It can be seen that more energy is pumped into the ANES when TET is activated, as evidenced by the comparison of the oscillation amplitude for the two cases. The corresponding driving forces i.e.  $F_a(t) = g^{sn} \int_{t=0}^t x_2 dt + g^{3n} \int_0^t \int_0^t (x_2)^3 dt dt$  for the two cases are plotted in Fig. 10. Accordingly, more forces need to be delivered when TET is activated. Note that  $x_1$ ,  $x_2$  and  $F_a$  shown in Figs. 8-10 are dimensionless quantities. In order to implement the proposed ANES for practical applications, special care should be given on the specifications of the transducer and the corresponding amplifiers such that the required stroke and force are satisfied.



**Fig. 9 Comparison of the normalised transmission force  $x_2$  when TET is deactivated or activated**



**Fig. 10 Comparison of the normalised driving force when TET is deactivated or activated**

#### 4. ANES tuning for MDOF systems

For MDOF systems, another appealing feature of a mechanical NES is the capability of mitigating the vibrations of multiple structural modes. The underlying mechanism is referred to as resonance capture cascade (RCC). In this section, it is shown that an ANES can damp multiple structural modes through RCC and the tuning law for triggering multi-frequency TET is derived.

The considered system is shown in Fig. 11, where a linear three-degree-of-system is taken as the primary structure and an ANES is placed between  $m_2$  and  $m_1$ . The parameters of the primary structure are listed in Table 2. The question is how to choose the control parameters  $g_s$  and  $g_{d3}$  such that multi-frequency TET can be activated for the this system. Similar to a mechanical NES, the idea is to assume that ANES interacts with a single structural mode at a time. If so, the guidelines for triggering the occurrence of TET for a SDOF system can be directly applied. The values of  $g_s$  and  $g_{d3}$  required for activating TET of each mode can be obtained. Then, one can examine whether TET can be triggered for all the structural modes of interest with one single setting of  $g_s$  and  $g_{d3}$ .

Following upon this idea, the primary system is firstly decomposed into three SDOF systems in modal coordinates. The mass-normalised eigenvector matrix of the primary system and the resonance frequencies are:

$$\mathbf{E} = \begin{bmatrix} 0.328 & -0.737 & 0.591 \\ 0.591 & -0.325 & -0.737 \\ 0.737 & 0.591 & 0.328 \end{bmatrix} \quad (47)$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0.445 \\ 1.247 \\ 1.802 \end{bmatrix} \text{ rad/s} \quad (48)$$

**Table 2 parameters of the primary system**

Parameters	value	Parameters	value	Parameters	value
$m_1$ (kg)	1	$m_2$ (kg)	1	$m_3$ (kg)	1
$k_1$ (N/m)	1	$k_2$ (N/m)	1	$k_3$ (Ns/m)	1
$c_1$ (Ns/m)	0	$c_2$ (Ns/m)	0	$c_3$ (Ns/m)	0

The ANES stiffness  $k_a = 0.057$  is chosen such that the modal stiffness ratio between the ANES and the primary system for mode 1 is equal to 2%. According to Eq. (9), the normalised control parameter  $g^{sn}$  decreases with the resonance frequency. Therefore,  $g_s$  is chosen such that  $g^{sn}$  for mode 1 is equal to 2%. In this way,  $g^{sn}$  for the other modes is smaller than 2% which automatically meets the condition given by (38). The initial condition of the system  $\mathbf{X}^{\text{init}}$  is taken as:

$$\mathbf{X}^{\text{init}} = [x_1, x_2, x_3]^T = [0.01, 0, 0]^T \quad (49)$$

The initial condition in the modal coordinates can be obtained:

$$\mathbf{Q}^{\text{init}} = \mathbf{E}^{-1} \mathbf{X}^{\text{init}} = [3.3 \times 10^{-3} \quad -7.4 \times 10^{-3} \quad 5.9 \times 10^{-3}]^T \quad (50)$$

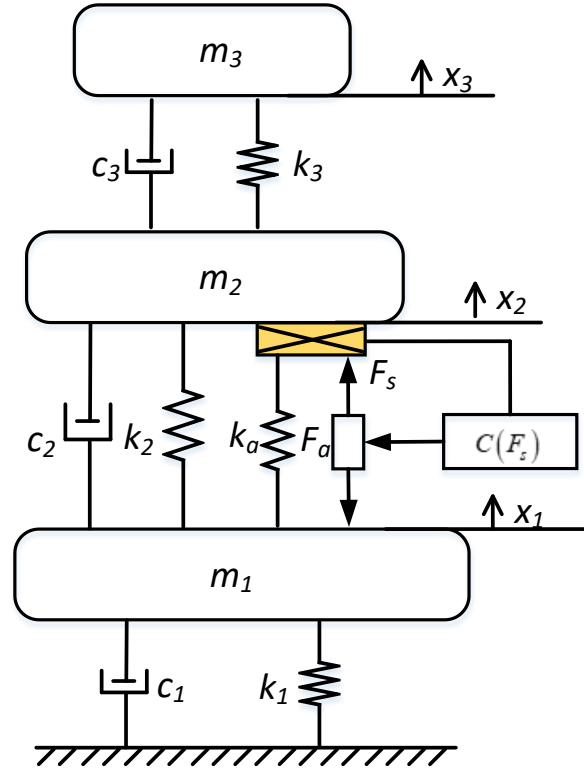
The mode shape difference at the locations of the ANES i.e.  $\Delta e_i$  as in Eq. (8) are calculated by taking the difference between the first two rows of the eigenvector  $\mathbf{E}$ . Grouping them in vector form, one obtains:

$$\Delta \mathbf{e} = [\Delta e_1 \quad \Delta e_2 \quad \Delta e_3]^T = [0.263 \quad 0.409 \quad -1.328]^T \quad (51)$$

The actuator connection vector  $\mathbf{b}$  as defined in Eq. (1) for the considered system is given as:

$$\mathbf{b} = [-1, 1, 0]^T \quad (52)$$

where the indices  $l$  and  $k$  represent the first and the second DOF indices to which the actuator is attached.



**Fig. 11 A linear MDOF system coupled with an ANES**

With the knowledge of  $g_s$ ,  $k_a$ ,  $\mathbf{Q}_{\text{int}}$ ,  $\Delta \mathbf{e}$  and  $\omega$ , the threshold of  $g_{d3}$  for triggering each mode's TET, referred to as  $(g_{d3}^{\text{min}})_i$ ,  $\forall i = 1, 2, 3$ , can be derived by substituting these parameters into Eqs. (41), (9) and (8). The results are given in Table 3. It is found that  $(g_{d3}^{\text{min}})_1$  for mode 1 is greater than that for the other two modes. This implies that TET can be triggered for all modes if  $g_{d3}$  is greater than  $(g_{d3}^{\text{min}})_1$ . The final value of  $g_{d3}$  is thus set to be 1.2 times greater than  $(g_{d3}^{\text{min}})_1$ , completing the determination of the ANES parameters  $g_s$ ,  $g_{d3}$  and  $k_a$ .

The ANES performance is preliminarily assessed by using the two performance indices i.e. the pumping time  $T_{\text{pump}}$  and the energy dropping ratio  $E_{\text{TET}}$ . Their values are listed in Table 3. It is seen that it takes around 415 seconds for pumping out the energy associated with mode 2, while only 71 seconds are required for mode 1. This is mainly because the effective stiffness ratio and the damping ratio for the second mode are relatively low. During TET, the energy possessed by modes 2 and 3 is almost completely dissipated by the ANES, while approximately 80% of the energy is damped out for mode 1. This is caused by the fact that the damping ratio for the first mode is relative high and the corresponding initial value  $Z_1^{\text{init}}$  is close to the threshold  $Z_1^{\text{max}}$ .

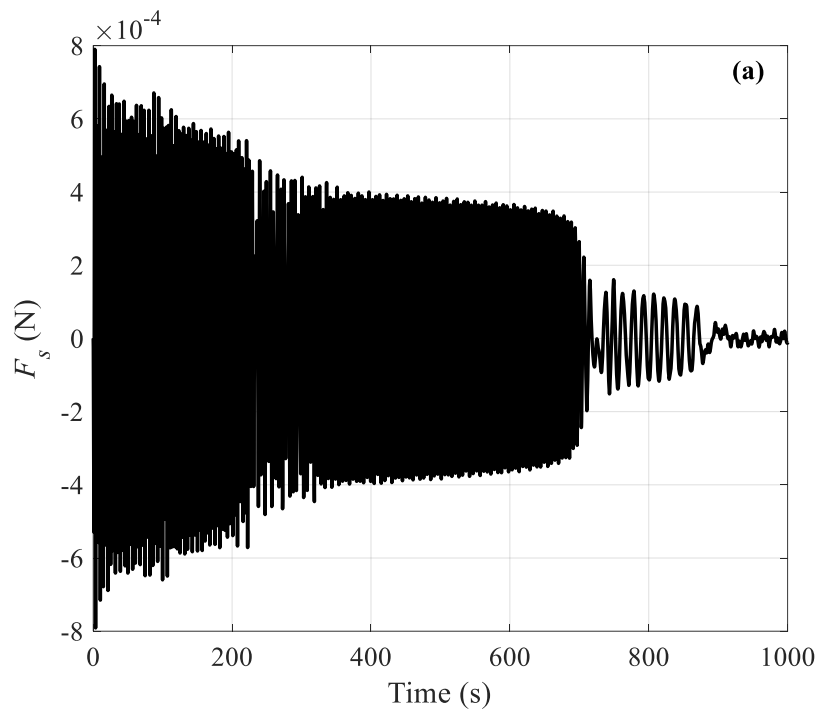
**Table 3 ANES parameters and performance measures**

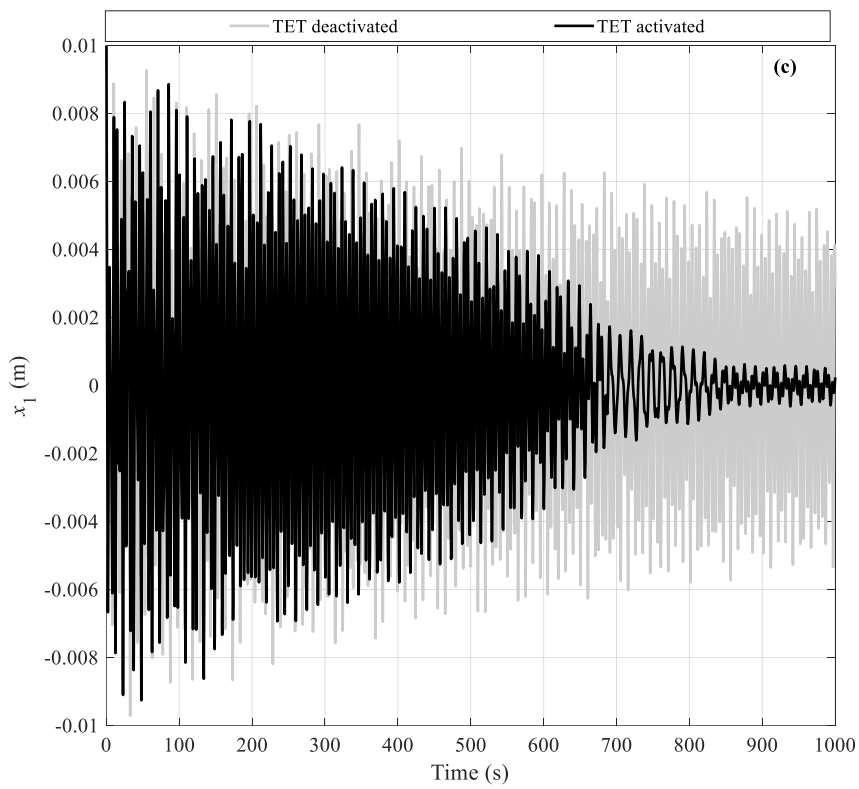
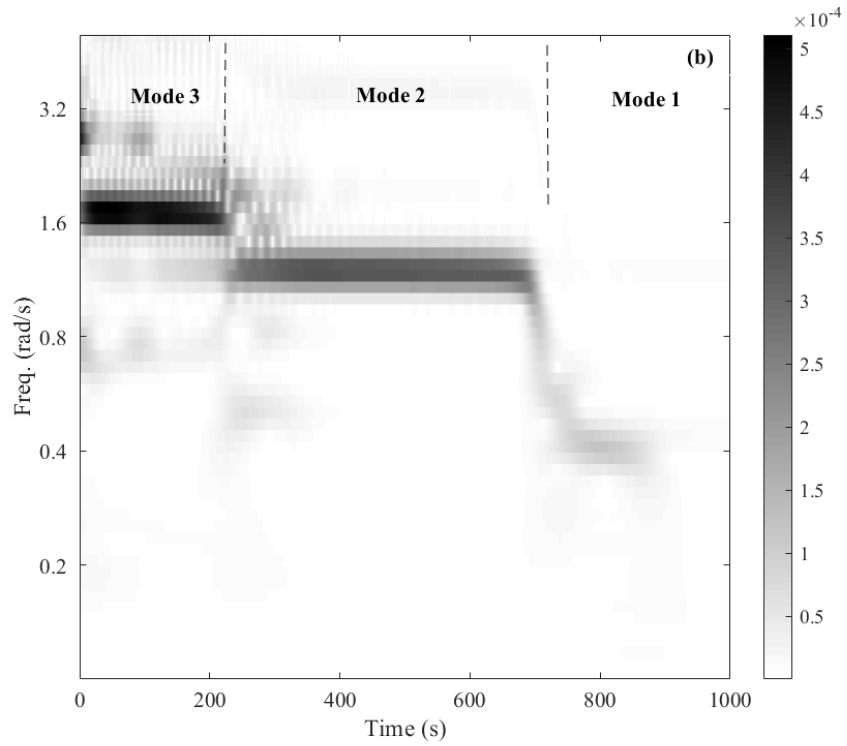
Parameters	$(g^{sn})_i$	$(Z_1^{\max})_i$	$\varepsilon_i$	$(g_{d3}^{\min})_i$	$(g^{3n})_i$	$(Z_1^{\text{init}})_i$	$(T_{\text{pump}})_i$	$(E_{\text{TET}})_i$
mode 1	0.2	0.216	0.02	$1.8 \times 10^7$	$2.4 \times 10^4$	0.259	71.6s	79.6%
mode 2	0.07	0.2	0.006	$1.0 \times 10^7$	$7.4 \times 10^3$	0.403	415.8s	98.3%
mode 3	0.05	0.199	0.03	$3.2 \times 10^6$	$3.7 \times 10^4$	1.308	234.9s	99.7%
ANES	$g_s = 0.09$			$g_{d3} = 2.1 \times 10^7$		$k_a = 0.057$ N/m		
ATID	$g_s = 0.62$			$g_d = 0.32$		$k_a = 0.057$ N/m		

In order to verify the analytical analysis, a numerical simulation is performed for the coupled system. Fig. 12 (a) and (b) plot the time history and the time-frequency analysis of the transmission force  $F_s$ , respectively. As can be seen, the ANES is able to interact with each mode sequentially. Once a sufficient amount of energy has been dissipated, the ANES escapes from the current resonance capture and engages with the next mode, realising in essence a RCC. The comparison between Fig. 12 (a) and Table 3 highlights that the pumping time for each mode is correctly predicted by the analytical results. The vibration mitigation performance is evaluated by the response of  $m_1$ , the results of which are shown in Fig. 12 (c) and (d), respectively. Interestingly, it is seen that the energy associated with each mode is dissipated only when the corresponding TET occurs, again confirming the previous assumption. A benchmark case is considered where  $g_{d3}$  is set to 1% of the current value such that no TETs are triggered. The resulting control performance in terms of the displacement of  $m_1$  is superimposed in Fig. 12 (c). The comparison clearly highlights the vibration mitigation efficiency improved by the well-tuned ANES.

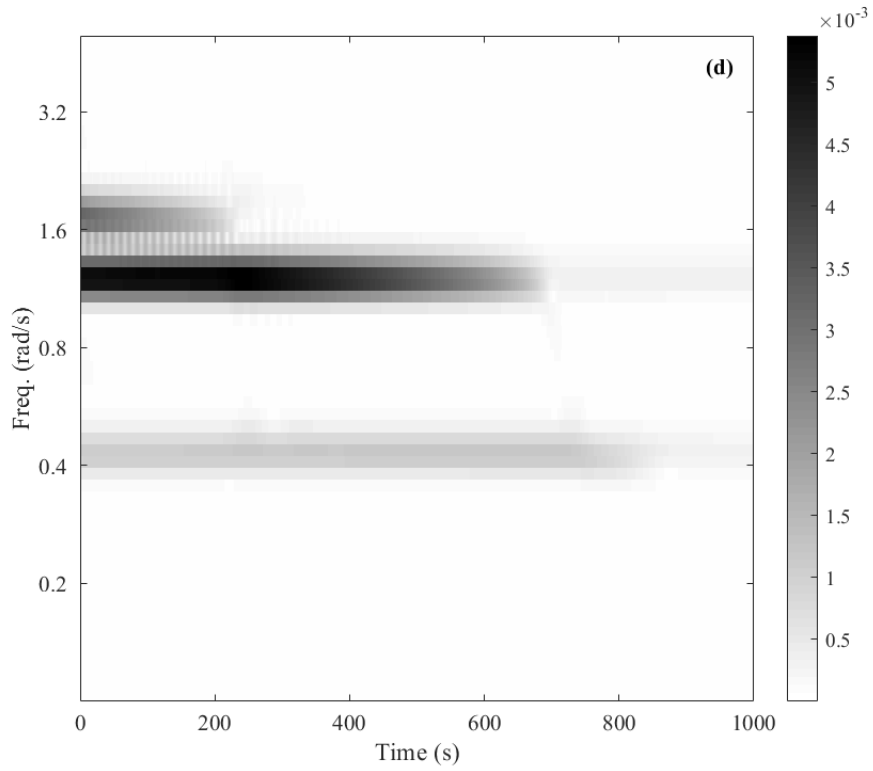
Next, the control effectiveness of the ANES is also compared with that of a linear active tuned inerter-damper (ATID) [23] on the considered primary system. The relationship between an ATID and an ANES can be understood in analogy with a mechanical TMD and a mechanical NES. The ATID is configured such that the effective damping associated with mode 1 is maximised according to the tuning law in [34]. The corresponding control parameters i.e. gains of the single integrator  $g_s$  and the double integrator  $g_d$  are given in Table 3. The resulting control performance is shown in Fig. 13 which is superimposed with that of the ANES. Fig. 14 plots the time-frequency distribution of  $x_1$  when the optimally configured ATID is used. As shown in Fig. 13, the ATID seems to outperform the ANES in the time window between 150 and 600 seconds. In fact, the vibrations when ANES is used during this period are mainly due to two vibration modes (mode 1 & mode 2), as seen from the time-frequency plot in Fig. 12 (d). As ANES takes effect sequentially from high modes to low modes, in this time window, ANES affects only mode 2 and basically leaves mode 1 ‘undamped’. On the other hand, the vibrations associated with mode 1 for ATID

start to be attenuated from the very beginning. When the time reaches 200s, the energy with mode 1 is already well dissipated and the vibrations are basically dominated by mode 2, which is not effectively damped by the ATID. The results show clearly that ANES can effectively damp the energy of all the modes of the system, while ATID can only be tuned to one mode. This broadband efficiency of ANES makes it also much more robust to the presence of uncertainties in the primary system as demonstrated by the comparison between a mechanical NES and a mechanical TMD in [10].

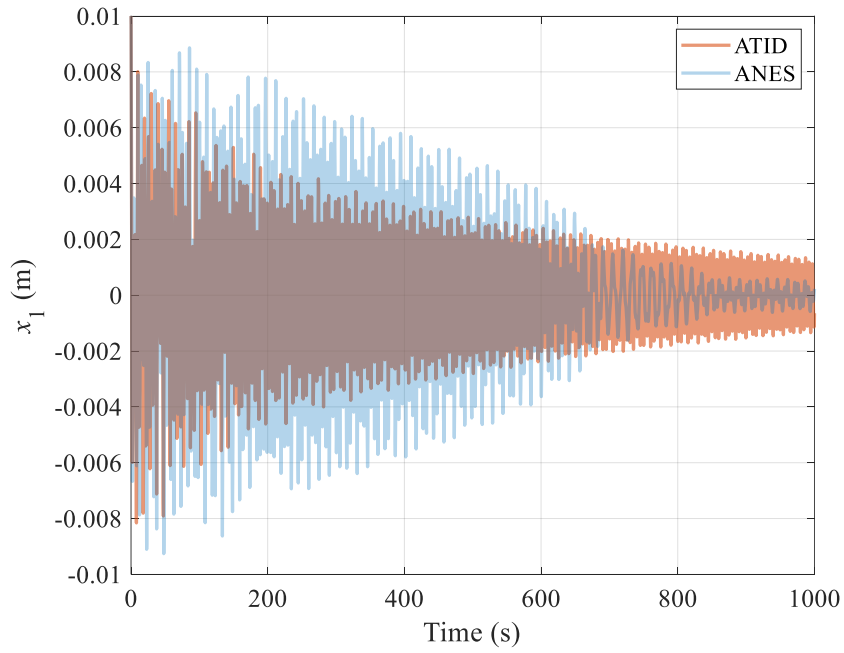




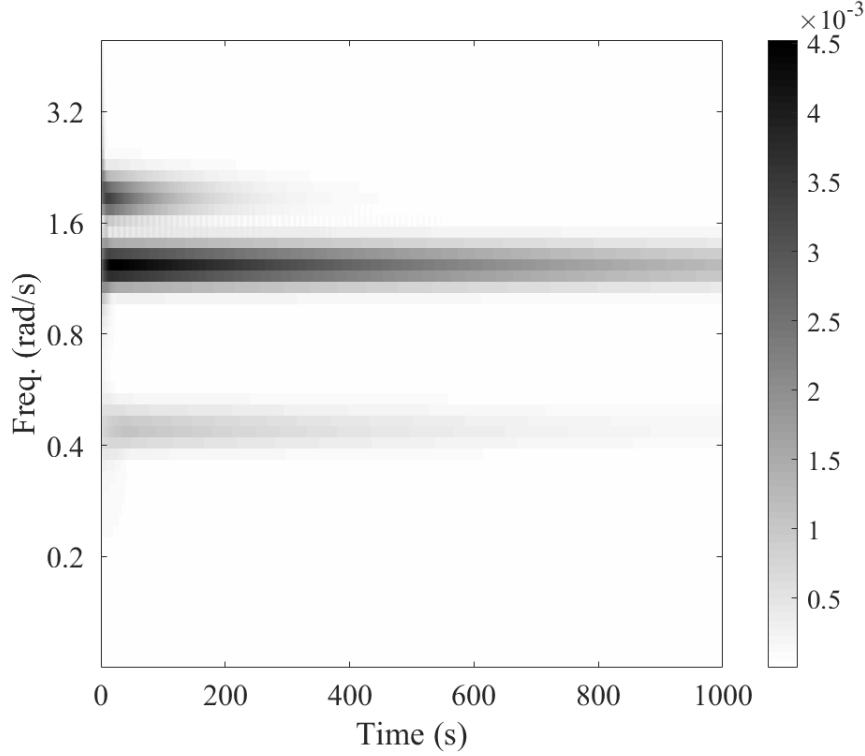




**Fig. 12 (a) time history of the transmission force  $F_s$  ; (b) time-frequency distribution of  $F_s$  ; (c) time history of  $x_1$  ; (d) time-frequency distribution of  $x_1$**



**Fig. 13 Control effectiveness comparison between ANES and ATID**



**Fig. 14 Time-frequency distribution of  $x_1$  when the optimally configured ATID is used**

## 5. Conclusion

This paper discussed an active nonlinear energy sink concept realised by a pair of collocated reactive actuator and force sensor. The equivalent mechanical representation i.e. serially-connected cube root inerter, linear damper and linear spring has been derived to better understand the physics behind the coupled electromechanical system. It is found that an ANES and a NES behave similarly in terms of their slow-scale dynamics and vibration mitigation effectiveness. Closed-form tuning laws for regulating the control parameters of the ANES have been derived for both SDOF and MDOF systems. The well-known threshold in terms of the host structure vibration's energy for triggering TET is converted into constraints on the control parameters which allows a straightforward implementation in practice. The influence of the control parameters on the vibration mitigation performance has also been investigated. More energy is dissipated during TET if the control parameter  $g_s$  is relatively small, but it takes more time to do so. An opposite trend applies for the other control parameter  $g_{d3}$ . Numerical simulations have been performed to verify the analytical developments. From a practical viewpoint, the proposed ANES is preferable over a mechanical NES as it not only features real-time tuning of the control parameters, but also allows for a flexible implementation of various forms of the nonlinear exponent. In addition, one can

envision to develop an analogue electronic control system for the collocated actuator-force sensor pair such that the resulting ANES would be compact enough for smart structure applications.

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## Conflict of Interest

The authors declare that they have no conflict of interest.

## Appendix A

In this appendix, it is shown that the systems sketched in Fig. 1 (a) and (b) are dynamically equivalent, or, in other words, the networks shown in Fig. A1 (a) and (b) are equivalent.

Fig. A1 (b) depicts a pure mechanical system which consists of a cube root inerter, a dashpot and a spring connected in series. A cube root inerter impedes the relative acceleration across its terminals with a force proportional to the cube root of its relative acceleration. Under the excitation force denoted by  $F$ , the governing equations of this system can be written as:

$$F = -m_a \sqrt[3]{\ddot{x}_1 - \ddot{x}_2} \quad (53)$$

$$F = -d_a (\dot{x}_2 - \dot{x}_3) \quad (54)$$

$$F = -k_a (x_3 - x_4) \quad (55)$$

where  $x_i, \forall i=1,2,3,4$  denotes the node displacement of each mechanical component.

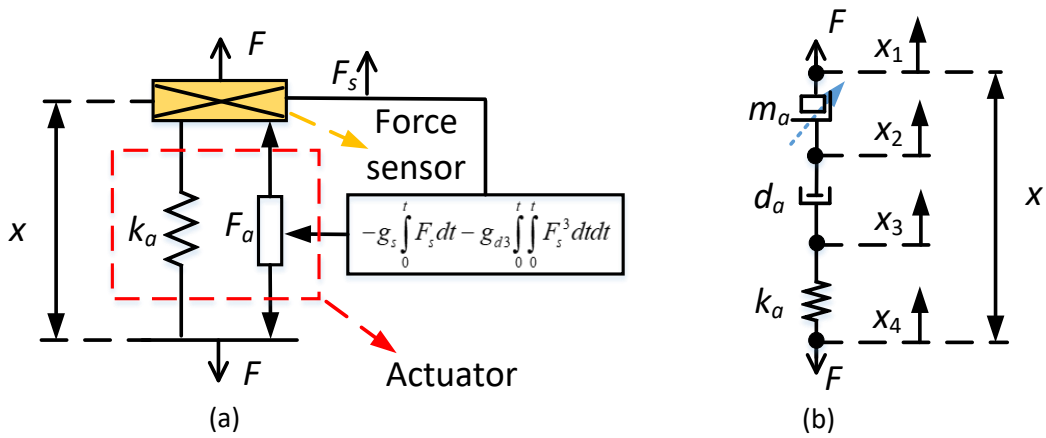


Fig. A1 (a) the sketch of the active system and (b) its mechanical representative

Expressing the relative motion in terms of the transmission force  $F$ , Eqs. (53)-(55) can be rewritten as:

$$-\int_0^t \int_0^t \left( \frac{F}{m_a} \right)^3 dt dt = x_1 - x_2 \quad (56)$$

$$-\int_0^t \frac{F}{d_a} dt = x_2 - x_3 \quad (57)$$

$$-\frac{F}{k_a} = x_3 - x_4 \quad (58)$$

Summing up Eqs. (56)-(58) and multiplying both sides with  $k_a$ , yields:

$$-\frac{k_a \int \int F^3}{m_a^3} - \frac{k_a \int F}{d_a} - F = k_a (x_1 - x_4) \quad (59)$$

According to the control law, the governing equations of the system shown in Fig. A1(a) can be expressed as:

$$F = -g_s \int_0^t F dt - g_{d3} \int_0^t \int_0^t F^3 dt dt - k_a x \quad (60)$$

where  $x = x_1 - x_4$  represents the relative displacement across the ANES.

Comparing Eq. (59) with Eq. (60) and one can find the equivalence between systems shown in Fig. A1(a) and (b). The feedback gains and their corresponding mechanical components are thus related by the following equation:

$$d_a = k_a / g_s; \quad m_{a3} = \sqrt[3]{k_a / g_{d3}} \quad (61)$$

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