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Subordination algebras and tense logic

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Mathematics may be defined as the the subject
in which we never know what we are talking about,
not whether what we are saying is true.
BERTRAND RUSSELL

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Introduction

Since the celebrated Stone duality [70] between Boolean algebras and Stone spaces, several other dualities/equivalences joined up in such a way as to create a large graph between numerous categories. For instance, one should consider Priestley duality [56], de Vries duality [26], Gelfand-Neumark duality [37], Esakia duality [32], Jonsson-Tarski duality [48], etc.

Let us note that two orthogonal ways of extending Stone duality are to contemplate in the all but exhaustive list we mentioned. The first one is to consider a "weaker" category on one side of the duality/equivalence and investigate the future of the category on the other side. This is for instance what happened for Priestley duality, where Boolean algebras were weakened to bounded distributive lattices (namely, by dropping the existence of a complement). The second way of generalising Stone duality is, on the contrary, obtained by adding new structures to the pre-existing categories. The Jonsson-Tarski duality is a striking example. Finitary operators are added to Boolean algebras and are translated in a specific family of relations added to Stone spaces.

The motivations behind these dualities and equivalences differ widely. For instance, Stone duality was intended as a way to establish a representation theorem for Boolean algebras, so that it would always be possible to compare an "abstract" Boolean algebra with a "concrete" one. Concerning de Vries duality, the main goal was to develop a theory of compactification (see for instance [77] for a survey on compactifications) of completely regular spaces.

However, a duality may see its use shift through time. To give one example Jonsson-Tarski duality was first intended to be a representation theorem for Boolean algebras with operators. Then, it was rediscovered latter as a bridge between the relational and the algebraic semantic of modal logic, allowing us to co-ordinate their respective advantages.

In accordance with this theme, the main motivation of this thesis is to further extend this ever growing graph of equivalences and dualities, but with different goals.

In this thesis, three specific directions will be considered:

- In Chapter 2, we will explore four dualities that extend the duality between modal algebras and modal spaces. Thanks to these dualities, subordination algebras will have a suited theory to be used as models for standard modal logic.
- In Chapter 3, we push further the generalisation initiated in Chapter 2. First, by noticing that subordination algebras are equivalently presented as unary slanted algebras (that is algebras with operators that do not map element of the algebra to the algebra itself but to closed/open elements of its canonical extension). Then, by moving from Boolean algebras to bounded distributive lattices.
- Finally, in Chapter 5, we will explore a duality that mimics the transition from Stone spaces to Priestley spaces in a Gelfand-Neumark duality setting. This duality is a formalisation of the techniques used by Hansoul in [43] to realise the "Nachbin-Stone-Čech" compactification of a completely regular ordered space.

De Vries algebras

As we mentioned earlier, we extend a tree of dualities and equivalences in several directions. However, these directions share a common "ancestor": de Vries algebras. Indeed, de Vries algebras are nothing but a particular case of subordination algebras treated in Chapter 2 and, morevoer, they are part of a triangle of dualities and equivalences with compact Hausdorff spaces and C^*-algebras, which is extended in Chapter 5. Therefore, it is natural to begin the thesis, in Chapter 1, with the duality developed by de Vries in his thesis [26] and well discussed in [3].

Subordination algebras

The concept of subordination algebra popped up under different names through history: proximity algebras in [30] and [54], pre-contact algebras in [28], strict implication algebras in [5] and quasi-modal algebras in [14]. To describe them shortly, they are hybrid structures, between the algebraic and the relational world.

As announced previously we define, in Chapter 2, four categories whose objects are subordination algebras but whose morphisms differ slightly. Dually, we define four categories whose objects will be subordination spaces, a.k.a. Stone spaces endowed with a binary closed relation and whose morphisms also differ. Of course, these categories are paired in twos, in order to establish four interconnected dualities. We retrieve in particular the dualities of [5] and [14] as two of these four dualities.

Afterwards, we establish four discrete dualities (i.e. non-topological) between complete atomic subordination algebras and Kripke structures. While these dualities are still set in a subordination environment, we remark that they are actually equivalent to dualities, set in a modal environment, between complete atomic modal algebras and Kripke structures.

Having hence noticed that complete atomic subordination algebra and complete atomic modal algebra are isomorphic concepts, we extend the notion of canonical extension of lattices with operators (see for instance [34]) to subordination algebras.

Once proven that the canonical extension of a subordination algebra is a standard modal algebra (and even, more precisely, a tense one), subordination algebras have an appropriate access to a modal structure and we will use this access to promote them as models for modal/tense logic. Moreover, we prove in Section 2.6, that the usual completeness theorems in modal logic can be transported(with a minor variation) into the subordination setting. This last result may be of great importance to show that a formula cannot be proved from a certain set of axioms. Indeed, the pool of frames available to build counterexamples is now larger than since one can use the all set of subordination space.

Subordination algebras as models for tense/modal logics

Let us discuss the challenges behind the promotion of subordination algebras to the rank of model. The most apparent one is that they do not carry a modal structure, but a weaker one. However, their duals, namely subordination spaces, are topological spaces endowed with a closed binary relation. Therefore, it is not hard to see that subordination spaces can be used as models for modal logic, through the usual definitions of valuation (see for instance [16, Section 3.2]). Hence, thanks to the dualities we established previously, we can directly consider subordination algebras as models for the standard tense logic.

Nevertheless, this procedure conceals a less apparent issue: the valuation of a (bi)modal formula on a subordination algebra may fail to be an element of the initial algebra. This is where the canonical extension we introduced previously plays a major role. Indeed, the valuation of a (bi)modal formula is actually an element of the canonical extension. Hence, we have another way, purely algebraic, to consider subordination algebras as models: through their canonical extensions. Fortunately, this second option coincides with the previous one.

A Sahlqvist theorem and canonicity for subordination algebras

We spend the second half of Chapter 2 extending the well known Sahlqvist theorem [63] from the modal/tense to the subordination setting. Recall that this theorem provides a specific family of modal formulas (the Sahlqvist formulas, not to name names) with a first order translation in the language of the accessibility relation. We obtain a family of formulas (the s-Sahlqvist formulas) which is a restriction of the original family of Sahlqvist formulas in the tense case. The reasons behind this restriction are explained infra. We also discuss in Sections 2.7 and 2.10 additional kinds of translations which arise naturally from the subordination setting.

Back to the Sahlqvist theorem, the key Lemma in its proof is the Esakia Lemma ([32]). It was proved in [65] in a modal environment with topological methods. Since it only requires the accessibility relation to be closed, it easily transfers to the subordination setting. Nevertheless, some adjustments have to be done for the key corollary of Esakia Lemma (that we may call Generalised Esakia Lemma). This is due to two facts: the first is that Esakia Lemma requires, in its statement, closed subsets. The second is that, for subordination spaces, the accessibility relation does not send clopen subsets to clopen subsets, but only to closed sets. Therefore, for instance, the box $\Box O$ of a clopen set O is not a closed set, but an open one, while it is impossible to determine whether $\Diamond\Box O$ is closed or open.

However we do not have to entirely forbid the presence of open sets in our Sahlqvist theorem. Indeed, it is sufficient to ensure that they do not appear in a "critical situation", namely in one where the Esakia Lemma is genuinely required.

Besides providing a translation result, the modal Sahlqvist theorem also entails a canonical one: every Sahlqvist formula is canonical, in the sense that it is valid in a modal algebra if and only if it is valid in its canonical extension. This latter outcome is also carried out in the subordination world, where the canonical extension intended is the one we defined earlier.

Subordination and tense/modal languages

We end Chapter 2 with a comparison between the subordination and the tense/modal language. Indeed, since we know that subordination algebras may be used as models for tense/modal logic, we can compare their power of expression in this language with the language that they naturally carry: the subordination one. We lay emphasis on the fact that, when one considers the subordination language, two options arise: one, considered for instance in [1] and [74], which handles the subordination language as an equational language (i.e. without quantifiers) and the other, considered for instance in [4] and [66], which handles it as a first order language (i.e. with quantifiers). The relations between the subordination language and the modal one are of course greatly impacted by the selected option. We discuss this aspect in Section 2.10 and, later on, in Section 4.8.

Slanted duality

Subordination algebras may be presented as a particular case of slanted lattices, namely lattices endowed with operators that do not restrict to clopen elements but can map elements of the original algebra to open or closed elements of its canonical extension. The results presented here answer to the natural question that now arises: what should be the topological counterpart of slanted algebras? Perhaps unsurprisingly, we turn to Stone/Priestley spaces with closed relations, as it was done for standard (i.e. non-slanted) operators in [48] for Boolean algebras and in [67] for distributive bounded lattices. Hence, in short, the slanted duality is a extension of the subordination duality in two different directions: from the Boolean world to the distributive lattice one and from unary operators (or binary relations) to operators of arbitrary arities.

Slanted canonicity

To pursue the generalisation from the subordination algebras world to the slanted lattices one, we extend the Sahlqvist-like result obtained in Chapter 2 from subordination algebras to slanted lattices via the duality established in Chapter 3. Moreover, the fragment of (slanted) canonical formulas obtained is larger that the fragment presented in Chapter 2 since we now have that all analytic inductive formulas are canonical. Note that the concept of analytical formulas arose in [41] in a context apparently uninvolved with the topological restrictions we made. Indeed, they were introduced as a characterisation, in the theory of analytic calculi in structural proof theory, of the logics which can be presented by means of proper display calculi.

A Gelfand duality for compact po-spaces

It is well known that a compact Hausdorff space can be characterised by its ring of complex continuous functions (see [46]) or its ring of real continuous ones (see [7]). Both these characterisation lead to a duality between the category of compact Hausdorff spaces and the C^*-algebras on one side, and the Stone rings on the other side. Interestingly, these dualities belong to a wider frame, which was extended in all directions, but this one, by Bezhanishvili and Harding in [6].

Chapter 5 is dedicated to extend the Gelfand duality to the compact po-spaces. As an order is required in the po-space setting, it is natural to opt for a real ring as in [7] instead of a complex one. As we said earlier, the key observation to obtain this duality is given by Hansoul in [43]: the compactification of completely regular ordered space is obtained via its set of positive real increasing continuous functions. Now, of course, this set is not a ring, as an increasing function clearly lacks an opposite. Therefore, it is natural to turn to a category whose objects are semi-rings.

Chapter 1

De Vries duality

De Vries duality concerns compact Hausdorff spaces and de Vries algebras. The main idea behind this duality is similar to the one behind Stone's one (see Appendix A). Indeed, Stone duality uses the fact that clopen subsets of a Stone space constitute an open basis. Of course, this is not valid anymore for a compact Hausdorff space X which is not zero-dimensional. Nevertheless, X is in particular a regular space and, as such, its regular open sets form a base of X. Therefore, after having studied some properties of regular opens sets given in [39, Chapter 10], we will use them to establish de Vries duality.

1.1 Regular open sets

Notation 1.1.1. Let X be a topological space and $S \subseteq X$. We denote by S° the **interior** of S, that is the largest open set contained in S; by $\overline{S}$, or S^-, the **closure** of S, that is the smallest closed set containing S; by S^c the complement of S and by $S^\perp$ the set $S^{-c} = S^{c\circ}$. Finally, $S^\bullet$ will denote the **boundary** of S, that is $S^- \cap S^{\circ c}$.

Definition 1.1.2. Let X be a topological space, an open set O in X is said to be **regular** if $O = O^{-\circ}$. Dually, a closed set F in X is said to be **regular** if $F = F^{\circ -}$. The set of regular open sets of X will be denoted by $\mathrm{RO}(X)$ while its set of regular closed sets will be denoted by $\mathrm{RC}(X)$.

Lemma 1.1.3. *Let X be a topological space and let $O, U \subseteq X$, then*

1. *O is a regular open set if and only if O^c is a regular closed set,*
2. *$O^{-\circ} = O^{\perp\perp}$,*
3. *if O is open, then $O^{-\circ}$ is the smallest regular open containing O,*
4. *if O and U are open sets, then $(O \cap U)^{-\circ} = O^{-\circ} \cap U^{-\circ}$.*

Proof. See [39, Chapter 10]. □

As a direct corollary of this lemma, we have that $\mathrm{RO}(X)$ is a (complete) Boolean algebra for the operations defined in the next theorem.

Theorem 1.1.4. *Let X be a topological space. The set $\mathrm{RO}(X)$ ordered by inclusion is a complete Boolean algebra such that, for every O and U in $\mathrm{RO}(X)$, we have :*

1. *$1 = X$ and $O = \emptyset$,*
2. *$O \wedge U = O \cap U$,*
3. *$O \vee U = (O \cup U)^{-\circ}$,*
4. *$\neg O = O^{\perp}$,*

Proof. 1. It is clear that $\emptyset$ and X are regular open sets and that they are respectively the bottom and the top element.

2. Since we have
$$O \cap U \subseteq (O \cap U)^{-\circ} \subseteq O^{-\circ} \cap U^{-\circ} = O \cap U,$$
$O \cap U$ is a regular open set and the conclusion follows easily.

3. The conclusion is immediate by Item 3 of Lemma 1.1.3.

4. First, the set $O^{\perp}$ is a regular open set. Indeed, we have
$$O^{\perp-\circ} = O^{-c-\circ} = O^{-\circ c\circ} = O^{c\circ} = O^{\perp}.$$
Then, we have
$$\begin{aligned} O \wedge O^{\perp} &= O \cap O^{c\circ} \\ &= O^{-\circ} \cap O^{c\circ} \\ &\subseteq (O^{-} \cap O^{c})^{\circ} \\ &= O^{\bullet\circ} \subseteq O^{\bullet-\circ} = \emptyset, \end{aligned}$$
and
$$\begin{aligned} O \vee O^{\perp} &= (O \cup O^{\perp})^{-\circ} \\ &= (O^{c} \cap O^{-})^{c-\circ} \\ &= O^{\bullet c-\circ} \\ &= O^{\bullet-c-\circ} \\ &= O^{\bullet-\circ c\circ} = \emptyset^{c\circ} = X. \end{aligned}$$

We still have to show that $\mathrm{RO}(X)$ it complete, but it is just a routine calculation to show that if $S \subseteq \mathrm{RO}(X)$, then $\vee S$ and $\wedge S$ exist and are respectively equal to $(\cup\{U \mid U \in S\})^{-\circ}$ and $(\cap\{U \mid U \in S\})^{-\circ}$. □

Remark 1.1.5. We described here a part of what will be the "de Vries dual" of a compact Hausdorff space. However, to fully establish the duality, more than just a (complete) Boolean algebra will be required. Therefore, we turn to the notion of de Vries algebras introduced by de Vries in [26, Definition 1.1.1.] under the name of compingent Boolean algebras. This notion will be further extended to the more general cases of subordination algebras and contact algebras in Chapter 2 which will be themselves extended to the notion of slanted lattices in Chapter 3.

1.2 The category DeV

Definition 1.2.1. Let B be a Boolean algebra, a binary relation $\prec$ on B is a **de Vries relation** if it satisfies the following properties:

dv1. $0 \prec 0$ and $1 \prec 1$,

dv2. $a \prec b, c$ implies $a \prec b \wedge c$,

dv3. $a \leq b \prec c \leq d$ implies $a \prec d$,

dv4. $a \prec b$ implies $a \leq b$,

dv5. $a \prec b$ implies $\neg b \prec \neg a$,

dv6. $a \prec b$ implies $a \prec c \prec b$ for some c,

dv7. $a \neq 0$ implies $b \prec a$ for some $b \neq 0$.

A **de Vries algebra** is a pair $\mathfrak{B} = (B, \prec)$ where B is a complete Boolean algebra and $\prec$ is a de Vries relation. Note that we will sometimes abuse notations and write B for $\mathfrak{B}$ when it causes no confusion.

Remark 1.2.2. Here are some immediate consequences of the definition of de Vries algebras.

1. In presence of dv4, the following axiom

   dv1' $0 \prec a \prec 1$.

   is equivalent to dv1.

2. In presence of dv1 and dv4, axioms dv6 and dv7 are equivalent to

   dv8 $a \prec b \neq 0$ implies that there exists $c \neq 0$ such that $a \prec c \prec b$.

3. In presence of dv5, axiom dv2 is equivalent to

   dv2' $a, b \prec c$ implies $a \vee b \prec c$.

Lemma 1.2.3. *Let $\mathfrak{B} = (B, \prec)$ be a de Vries algebra.*

1. *For all $a \in B$, we have*
$$a = \vee\{b \mid b \prec a\} = \wedge\{b \mid a \prec b\}.$$

2. *For all $a, b \in B$, we have*
$$a \leq b \Leftrightarrow ((\forall c \in B)(b \prec c \Rightarrow a \prec c)).$$

Proof. 1. We give the proof of the first equality and leave the second one the the reader. By axiom dv2, it is clear that
$$a \geq \vee\{b \mid b \prec a\}.$$

On the other hand, suppose that c is an upper bound of the set $\{b \mid b \prec a\}$ and suppose that $a \not\leq c$. It follows that $a \wedge \neg c \neq 0$. Therefore, by axiom dv7, there exists an element $d \neq 0$ such that $d \prec a \wedge \neg c$. Hence, by dv3 and dv4, we have that $d \in \{b \mid b \prec a\}$ and $d \leq \neg c$. This is absurd, since in particular, it means that $d \leq c \wedge \neg c = 0$.

2. It is clear that the if part follows directly from axiom dv3. For the only if part, suppose that $a \not\leq b$. Using the same trick as in the first item, we find an element $c \neq 0$ such that $c \prec a$ and $c \prec \neg b$, hence, by dv5, $b \prec \neg c$. But we cannot have $a \prec \neg c$ as it would imply that
$$0 \neq c \prec a \wedge \neg a = 0,$$
which is absurd.

$\square$

Example 1.2.4. As previously stated, we intended de Vries algebras to form a category dual to the one of compact Hausdorff spaces. We saw in Theorem 1.1.4 that, for a compact Hausdorff space X, its set of regular open sets $\mathrm{RO}(X)$, ordered by inclusion, is a complete Boolean algebra. To obtain a de Vries algebra, we must define a de Vries relation on it. Consider the binary relation $\prec$ on $\mathrm{RO}(X)$ where
$$U \prec V \iff U^- \subseteq V$$
and let us check that $\prec$ is actually a de Vries relation. Since proving the four first properties is quite straightforward, we will focus on the last three.

dv5 Suppose that $U \prec V$, that is $U^- \subseteq V$. Then, we have that $V^c \subseteq U^{-c} = \neg U$. But, since $V \in \mathrm{RO}(X)$, it follows that
$$V^c = V^{-\circ c} = V^{-c-} = (\neg V)^-$$
and, finally, that $\neg V \prec \neg U$.

dv6 Suppose that $U \prec V$. It follows that $U^- \cap V^c = \emptyset$. Moreover, since X is compact Hausdorff, it is normal. Therefore, there exists an open set ω such that
$$U^- \subseteq \omega \text{ and } \omega^- \cap V^c = \emptyset.$$
Le W denote $\omega^{-\circ}$. Then, by Lemma 1.1.3, W is in $\mathrm{RO}(X)$. Finally, we have
- $U^- \subseteq \omega \subseteq W$, that is $U \prec W$ and
- $W \subseteq \omega^-$, which implies $W^- \subseteq \omega^- \subseteq V$, that is $W \prec V$.

dv7 Let $U \in \mathrm{RO}(X) \setminus \{\emptyset\}$. There exists $x \in U$ and thus, by regularity of X, there exists a closed neighbourhood ν of x such that $\nu \subseteq U$. In particular, there exists an open set ω such that
$$x \in \omega \subseteq \nu \subseteq U.$$
Taking V as $\omega^{-\circ}$ gives us $\emptyset \neq V \prec U$.

Definition 1.2.5. Let $\mathfrak{B} = (B, \prec)$ and $\mathfrak{C} = (C, \prec)$ be de Vries algebras. A map $h : B \longrightarrow C$ is a **de Vries morphism** if it satisfies the following properties:

1. $h(0) = 0$,
2. $h(a \wedge b) = h(a) \wedge h(b)$,
3. $a \prec b$ implies $\neg h(\neg a) \prec h(b)$,
4. $h(a) = \vee\{h(b) : b \prec a\}$.

Lemma 1.2.6. *If h is a de Vries morphism from $\mathfrak{B} = (B, \prec)$ to $\mathfrak{C} = (C, \prec)$, it satisfies the following properties for every $a, b, c, d \in B$:*

1. *$h(a) \leq \neg h(\neg a)$,*
2. *if $a \prec b$ and $c \prec d$, then $h(a \vee c) \prec h(b) \vee h(d)$,*
3. *$a \prec b$ implies $h(a) \prec h(b)$,*
4. *$h(1) = 1$.*

Proof. 1. It follows from $0 = a \wedge \neg a$ that

$$0 = h(0) = h(a \wedge \neg a) = h(a) \wedge h(\neg a)$$

and so that $h(a) \leq \neg h(\neg a)$.

2. From $a \prec b$ and $c \prec d$, we have directly that

$$\neg h(\neg a) \vee \neg h(\neg c) \prec h(b) \vee h(d).$$

The conclusion now follows from the first item, by noticing that

$$h(a \vee c) \leq \neg h(\neg(a \vee c)) = \neg h(\neg a \wedge \neg c) = \neg h(\neg a) \vee \neg h(\neg c).$$

3. Since $a \prec b$ and $0 \prec 0$, we obtain immediately from the second item that

$$h(a) = h(a \vee 0) \prec h(b) \vee h(0) = h(b).$$

4. Since $1 \prec 1$, using the first and third properties of de Vries morphisms, we have $1 \prec h(1)$, which concludes the proof.

□

As noted in [2], the usual composition of two de Vries morphisms may fail to be a de Vries morphism. Indeed, it may not satisfy the fourth property of Definition 1.2.5. Thus, we have to use a slightly different composition in the category of de Vries algebras.

Definition 1.2.7. If $h : B \longrightarrow C$ and $g : D \longrightarrow B$ are de Vries morphisms, their **composition** is defined as

$$h \star g : D \longrightarrow C : a \longmapsto \vee\{h(g(b)) : b \prec a\}. \tag{1.1}$$

To actually use this composition to form a category, it remains to prove that $h \star g$ is a de Vries morphism and also that $\star$ is indeed a categorical composition (see Appendix A.2.1).

Proposition 1.2.8. *Let f, g, h be composable de Vries morphisms and let* id *denote the identity map.*

1. *The map* id *is a de Vries morphism.*
2. *The map $f \star g$ is a de Vries morphism.*
3. *We have $f \star \mathrm{id} = f = \mathrm{id} \star f$ and $f \star (g \star h) = (f \star g) \star h$.*

Proof. 1. Trivial.

2. We have to check one by one the properties of a de Vries morphism.

   (a) Since $b \prec 0$ implies $b = 0$, we have

$$(f \star g)(0) = \vee\{f(g(b) : b \prec 0\} = f(g(0)) = 0.$$

(b) First off all, since $c \prec a \wedge b$ implies $c \prec a$ and $c \prec b$, it is clear that

$$\begin{aligned}(f \star g)(a \wedge b) &= \vee\{f(g(c)) : c \prec a \wedge b\} \\ &\leq \vee\{f(g(c)) : c \prec a\} \wedge \vee\{f(g(c)) : c \prec b\} \\ &= (f \star g)(a) \wedge (f \star g)(b).\end{aligned}$$

On the other hand, $c \prec a$ and $d \prec b$ implies $c \wedge d \prec a \wedge b$. Therefore, it implies also

$$f(g(c)) \wedge f(g(d)) = f(g(c \wedge d)) \leq (f \star g)(a \wedge b).$$

It follows that

$$\begin{aligned}(f \star g)(a) \wedge (f \star g)(b) &= \vee\{f(g(c)) : c \prec a\} \wedge \vee\{f(g(d)) : d \prec b\} \\ &= \vee\{f(g(c)) \wedge f(g(d)) : c \prec a \text{ and } d \prec b\} \\ &\leq (f \star g)(a \wedge b),\end{aligned}$$

as required.

(c) If $a \prec b$, there exist c and d such that $a \prec c \prec d \prec b$. From $a \prec c$, we obtain first $\neg c \prec \neg a$ and then $f(g(\neg c)) \leq (f \star g)(\neg a)$. From $c \prec d$, we obtain successively $\neg d \prec \neg c$, $\neg g(d) \prec g(\neg c)$ and $\neg f(g(d)) \prec f(g(\neg c))$. Finally, from $d \prec b$, we obtain $f(g(d)) \leq (f \star g)(b)$, which is $\neg(f \star g)(b) \leq \neg(f(g(d))$. It suffices now to put together what we just obtained to find

$$\neg(f \star g)(b) \leq \neg(f(g(d)) \prec f(g(\neg c)) \leq (f \star g)(\neg a).$$

Finally, we have that $\neg(f \star g)(\neg a) \prec (f \star g)(b)$.

(d) Since $b \prec a$ implies that there exists c such that $b \prec c \prec a$, which itself implies that $f(g(b)) \leq (f \star g)(c)$, we have that

$$(f \star g)(a) = \vee\{f(g(b)) : b \prec a\} \leq \vee\{(f \star g)(c) : c \prec a\}. \tag{1.2}$$

But, from item (b), we already know that $c \prec a$ implies $(f \star g)(c) \leq (f \star g)(a)$, which means that the inequality (1.2) is actually an equality, as required.

3. It follows directly from the definitions that $f \star \mathrm{id} = f = \mathrm{id} \star f$. Now, to prove associativity, consider the following elements

$$\begin{aligned}a_1 &:= ((f \star g) \star h)(a) = \vee\{(f \star g)(h(b)) : b \prec a\}, \\ a_2 &:= (f \star (g \star h))(a) = \vee\{f((g \star h)(b)) : b \prec a\}, \\ a_3 &:= \vee\{f(g(h(b)) : b \prec a\}.\end{aligned}$$

Let us observe that for every de Vries morphisms f, g and for every element c, we have $f(g(c)) \geq (f \star g)(c)$. It follows therefore that $a_1 \leq a_3$ and $a_2 \leq a_3$.

Moreover, if $b \prec a$, then there exists c such that $b \prec c \prec a$ and, henceforth, such that $h(b) \prec h(c)$. It follows that $f(g(h(b)) \leq (f \star g)(h(c))$ and, consequently, that $a_3 \leq a_1$. With an analogue reasoning, one proves that $a_3 \leq a_1$. In the end, we obtain $a_1 = a_3 = a_2$. $\square$

Corollary 1.2.9. *The de Vries algebras and de Vries morphisms with $\star$ as defined in* (1.1) *as composition form a category, denoted by* **DeV**.

Remark 1.2.10. It is important to note that de Vries morphism have extremely weak properties as they are not even Boolean morphisms. However, Dimov, Ivanova-Dimova and Tholen proved in [27, Theorem 5.13] that a de Vries morphism could be generated as follows

$$V(\varphi)(a) := \vee\{\varphi(b) : b \prec a\},$$

with φ a **Fedorchuk morphism**, that is a Boolean morphism such that $a \prec b$ implies $\varphi(a) \prec \varphi(b)$.

To end this presentation of the de Vries category, we will introduce an equivalent condition for a de Vries morphism to be an isomorphism of the category **DeV**. However, we do not have yet all the required tools to prove the only if part and we will come back to it later, in Theorem 1.5.5.

Theorem 1.2.11. *Let $\mathfrak{B} = (B, \prec)$ and $\mathfrak{C} = (C, \prec)$ be de Vries algebras. If $h : B \longrightarrow C$ is a Boolean isomorphism such that, for every $a, b \in B$, we have*

$$a \prec b \Leftrightarrow h(a) \prec h(b), \tag{1.3}$$

then h is an isomorphism of **DeV**.

Proof. Once again, first of all, we have to check that h is indeed a de Vries morphism. Since h is a Boolean morphism, it is clear that $h(0) = 0$ and $h(a \wedge b) = h(a) \wedge h(b)$ for every $a, b \in B$. Let us now focus on the last two properties of de Vries morphisms.

- As h is a Boolean morphism, we have $h(\neg a) = \neg h(a)$ for every $a \in B$. Then, we obtain by hypothesis that $a \prec b$ implies $\neg h(\neg a) = h(a) \prec h(b)$.
- Using Lemma 1.2.3, we obtain that $h(a) = \vee\{d : d \prec h(a)\}$. Then, since h is bijective, this is equivalent to $h(a) = \vee\{h(b) : h(b) \prec h(a)\}$. The conclusion now follows directly from (1.3).

Secondly, for h to be an isomorphism of **DeV**, it remains to prove that there exists a de Vries morphism $g : C \longrightarrow B$ such that both $g \star h$ and $h \star g$ are identity morphisms. The morphism g we are looking for is of course h^{-1}. Indeed, due to the fact that h^{-1} and h have identical properties, we know that h^{-1} is a de Vries morphism. Finally, it is trivial to prove that $h \star h^{-1} = \mathrm{id}_B$ and that $h^{-1} \star h = \mathrm{id}_C$. □

Definition 1.2.12. We call (temporarily) **strong de Vries isomorphism** a map that verifies the properties of Theorem 1.2.11.

An interesting property of strong de Vries isomorphisms is that the de Vries composition of a strong isomorphism and an arbitrary de Vries morphism corresponds to the usual composition, as we will see in the next proposition.

Proposition 1.2.13. *if $h : B \longrightarrow C$ is a strong de Vries isomorphism and $g : C \longrightarrow D$, $f : A \longrightarrow B$ are de Vries morphisms, then $g \star h = g \circ h$ and $h \star f = h \circ f$.*

Proof. 1. For $a \in B$, we successively have that

$$\begin{aligned}(g \star h)(a) &= \vee\{g(h(b)) : b \prec a\} \\ &= \vee\{g(h(b)) : h(b) \prec h(a)\} &(1.4)\\ &= \vee\{g(c) : c \prec h(a)\} = g(h(a)), &(1.5)\end{aligned}$$

where (1.4) follows from $a \prec b \Leftrightarrow h(a) \prec h(b)$ and (1.5) from the fact that h is onto.

2. For $a \in A$, we have that

$$\begin{aligned} h(f(a)) &= h(\vee\{f(b) : b \prec a\}) \\ &= \vee\{h(f(b)) : b \prec a\} = (h \star f)(a), \end{aligned}$$

where we simply used the fact that h is a Boolean isomorphism.

□

1.3 Round filters

Remember that in Stone duality, the Stone space associated to a Boolean algebra B is the set of B ultrafilters. In de Vries duality, the counterparts of ultrafilters are maximal round filters (or concordant filters in [26, Definition 1.2.1]) . We introduce them in this section and develop some of their properties.

Definition 1.3.1 ([3])**.** Let $\mathfrak{B} = (B, \prec)$ be a de Vries algebra. A subset x of B is a **round filter** if it is a filter such that for every $b \in x$, there exists $a \in x$ such that $a \prec b$. Moreover, an **end** is a round filter maximal among proper round filters ordered by inclusion.

Theorem 1.3.2. *Let $\mathfrak{B} = (B, \prec)$ be a de Vries algebra. If $a \in B \setminus \{0\}$, then there is an end x of B such that $a \in x$.*

Proof. By the properties of de Vries algebras, we have $0 \prec a_1 \prec a$ for some $a_1 \in B \setminus \{0\}$. By induction, we find a sequence $(a_n)_{n\in\mathbb{N}}$ constituted by elements of $B \setminus \{0\}$ such that, for each $n \in \mathbb{N}$, we have $0 \prec a_{n+1} \prec a_n \prec a$. Consider then

$$x = \{b \in B \mid a_n \leq b \text{ for some } n \in \mathbb{N}\}.$$

We have that x is a proper round filter containing a. Using Zorn lemma, it is possible to show that there is a maximal round filter containing x and hence containing a. □

Corollary 1.3.3. *Let a, b be two elements of a de Vries algebra $\mathfrak{B} = (B, \prec)$. We have $a \leq b$ if and only if for every end x of $\mathfrak{B}$, $a \in x$ implies $b \in x$.*

Proof. Since an end x is in particular a filter, it is clear that $a \leq b$ and $a \in x$ implies $b \in x$.

On the other hand, suppose that $a \not\leq b$. Then, we have $a \wedge \neg b \neq 0$ and it follows from Theorem 1.3.2 that there is an end x with $(a \wedge \neg b) \in x$. Trivially, we have that x is an end containing a and not b. □

Just as an ultrafilter x of a Boolean algebra B can be characterised via the condition that for every $a \in B$, x must contain exactly one among a and $\neg a$, it is possible to characterise the ends of a de Vries algebra.

Theorem 1.3.4. *A proper round filter x of a de Vries algebra $\mathfrak{B} = (B, \prec)$ is maximal if and only if for every $a, b \in B$, $a \prec b$ implies $b \in x$ or $\neg a \in x$.*

Proof. Let us begin by assuming that x is not an end. Then, there is a proper round filter y such that $x \subsetneq y$. Now, let us take an element $a \in x$ and an element $b \in y \setminus x$. Since y is a filter, we have $a \wedge b \in y$ and since it is round, there exists an element $c \in y$ such that $c \prec a \wedge b$. Now, as $b \notin x$, we have $a \wedge b \notin x$. But we also have that $\neg c \notin x$, otherwise we would have $\neg c \in y$ and $c \in y$, implying $0 \in y$, which would be nonsense.

Now, suppose that x is a round filter such that there exist $a, b \in B$ with $a \prec b$, $b \notin x$ and $\neg a \notin x$. This implies that

$$(c \in x \text{ and } a \prec d) \Rightarrow c \wedge d \neq 0. \tag{1.6}$$

Indeed, otherwise $c \wedge d = 0$ would imply $c \leq \neg d$ and, consequently, $\neg d \in y$. Since $a \prec d$, this would lead to $\neg d \prec \neg a$ and would imply $\neg a \in x$, contradicting our hypothesis. We now set y as

$$y := \{c \wedge d : c \in x \text{ and } a \prec d\}$$

and prove that y is a round filter. First of all, if $y \ni (c \wedge d) \leq e$, then we have $e = (e \vee c) \wedge (e \vee d)$. Therefore, since $e \vee c \in x$ and $a \prec d \wedge e$, we have that $e \in y$. Thus, y being trivially closed under $\wedge$, we have that y is a filter. Secondly, if $c \wedge d \in y$, then there is $e_1 \in x$ such that $e_1 \prec c$ and $e_2 \in B$ such that $a \prec e_2 \prec d$. So, we have $y \ni (e_1 \wedge e_2) \prec (c \wedge d)$ and y is indeed a round filter. Finally, it is clear that y satisfies the following conditions:

$$x \subseteq y, \; b = 1 \wedge b \in y \text{ and } \; 0 \notin y,$$

which yields that y is a proper round filter strictly containing x. □

Corollary 1.3.5. *If x and y are distinct ends of a de Vries algebra $\mathfrak{B} = (B, \prec)$, then there exists an element $a \in B$ such that $a \in x$ and $\neg a \in y$.*

Proof. Since x and y are distinct, there exists an element $b \in x \setminus y$. Moreover, since x is round, there exists a second element $a \in x$ such that $a \prec b$. By Theorem 1.3.4, it follows that $\neg a \in y$ as required. □

Now, just as Example 1.2.4 foreshadows the functor from **KHaus** to **DeV**, we will give the constructions that will lead to the inverse functor (from **DeV** to **KHaus**).

Definition 1.3.6. Let $\mathfrak{B} = (B, \prec)$ be a de Vries algebra. We denote by $\mathrm{End}(\mathfrak{B})$ its set of ends. Moreover, we define, for $a \in B$

$$\eta(a) = \{x \in \mathrm{End}(\mathfrak{B}) \mid a \in x\}.$$

Lemma 1.3.7. *For $\mathfrak{B} = (B, \prec)$ a de Vries algebra and $a, b \in B$, we have:*

1. *$a \leq b$ if and only if $\eta(a) \subseteq \eta(b)$,*
2. *$\eta(a \wedge b) = \eta(a) \cap \eta(b)$,*
3. *$\eta(0) = \emptyset$ and $\eta(1) = \mathrm{End}(\mathfrak{B})$.*

Proof. The proofs follow directly from the definitions or are just restatements of Corollary 1.3.3. □

Definition 1.3.8. Let $\mathfrak{B} = (B, \prec)$ be a de Vries algebra. We equip the set $\mathrm{End}(\mathfrak{B})$ with the topology whose base is given by $\mathcal{B} = \{\eta(a) \mid a \in B\}$. Note that, by Lemma 1.3.7, we know that $\mathcal{B}$ is indeed a topological base for $\mathrm{End}\,\mathfrak{B}$.

Lemma 1.3.9. *For $\mathfrak{B} = (B, \prec)$ a de Vries algebra and for $a, b \in B$, we have:*

1. *$\eta(a)^{-c} = \eta(\neg a)$,*
2. *$a \prec b$ implies $\eta(a)^{-} \subseteq \eta(b)$.*

Proof. 1. Suppose first that $x \in \eta(\neg a)$. Since $\eta(\neg a) \cap \eta(a) = \emptyset$, we have directly that $x \notin \eta(a)^-$. On the other hand, if $x \notin \eta(a)^-$, there exists $b \in x$ such that $\eta(b \wedge a) = \eta(b) \cap \eta(a) = \emptyset$. It follows that $b \wedge a = 0$ and moreover that $b \leq \neg a$. We then have $x \in \eta(b) \subseteq \eta(\neg a)$, which concludes the proof.

2. Suppose that $a \prec b$ and let x be an element of $\eta(a)^-$. By item 1, we have that $x \notin \eta(\neg a)$. Then, it follows by Theorem 1.3.4 that $b \in x$ or, in other words, that $x \in \eta(b)$.

□

Corollary 1.3.10. *Let $\mathfrak{B} = (B, \prec)$ be a de Vries algebra and let x be in* $\mathrm{End}(\mathfrak{B})$*, then*

$$\mathcal{B}_x = \{\eta(a)^- \mid a \in x\}$$

is a basis of closed neighbourhoods for x.

Proof. Since $x \in \eta(a) \subseteq \eta(a)^-$, it is clear that the elements of $\mathcal{B}_x$ are closed neighbourhoods of x.

Suppose now that $x \in \eta(b)$ for some $b \in B$. As x is a round filter, there exists $a \in x$ such that $a \prec b$ and so such that $x \in \eta(a)^- \subseteq \eta(b)$. □

Lemma 1.3.11. *Let $\mathfrak{B} = (B, \prec)$ be a de Vries algebra. Then, for every subset $S \subseteq B$,*

$$\eta(\vee A) = (\cup\{\eta(a) \mid a \in A\})^{-\circ}.$$

Proof. First of all, if $S \subseteq \mathrm{End}(\mathfrak{B})$, we have that

$$\begin{aligned} S^- &= \{x \mid a \in x \Rightarrow \exists y : y \in S \cap \eta(a)\} \\ &= \{x \mid a \in x \Rightarrow a \in \cup S\} \\ &= \{x \mid x \subseteq \cup S\}. \end{aligned}$$

Thus, we can conclude that

$$S^\perp = \{x \mid x \not\subseteq \cup S\}$$

and that

$$S^{-\circ} = S^{\perp\perp} = \{x \mid x \not\subseteq \cup\{y \mid y \not\subseteq \cup S\}\}.$$

So, in our case, we have

$$(\cup\{\eta(a) \mid a \in A\})^{-\circ} = \{x \mid x \not\subseteq \cup\{y \mid y \not\subseteq \cup(\cup\{\eta(a) \mid a \in A\})\}\}.$$

Moreover, $b \in \cup(\cup\{\eta(a) \mid a \in A\})$ if and only if there are $a \in A$ and $x \in \eta(a)$ such that $b \in x$ which is equivalent to ask that there is an element $a \in A$ such that $a \wedge b \neq 0$. Hence, we have $y \not\subseteq \cup(\cup\{\eta(a) \mid a \in A\})$ if and only if there exists some $b \in y$ such that $a \wedge b = 0$ for all $a \in A$, or equivalently, such that $\vee A \wedge b = 0$. It follows that

$$\cup\{y \mid y \not\subseteq \cup(\cup\{\eta(a) \mid a \in A\})\} = \{b \in B \mid (\neg \vee A) \wedge b \neq 0\}.$$

Indeed, if $\neg(\vee A) \wedge b \neq 0$, then there exists and end y such that $\neg(\vee A) \wedge b \in y$. Hence, since $\vee A \wedge \neg(\vee A) \wedge b = 0$, it follows that $y \not\subseteq \cup(\cup\{\eta(a) \mid a \in A\})$. On the other hand, if b is an element of $\cup(\cup\{\eta(a) \mid a \in A\})$, then there is an end y and an element c such that $c \in y$, $b \in y$ and $\vee A \wedge c = 0$. Suppose that $\neg(\vee A) \wedge b = 0$. Then we would have

$$y \ni c \wedge b \leq \neg(\vee A) \wedge \vee A = 0,$$

which is absurd since y is an end.

In brief, we proved that

$$(\cup\{\eta(a) \mid a \in A\})^{-\circ} = \{x \mid x \nsubseteq \{b \in B \mid \neg(\vee A) \wedge b \neq 0\}\},$$

which means that $x \in (\cup\{\eta(a) \mid a \in A\})^{-\circ}$ if and only if there is $b \in x$ such that $\neg(\vee A) \wedge b = 0$. The last condition being equivalent to $b \leq \vee A$, it follows that

$$x \in (\cup\{\eta(a) \mid a \in A\})^{-\circ} \Leftrightarrow \vee A \in x,$$

as requested. $\square$

Theorem 1.3.12. *Let* $\mathfrak{B} = (B, \prec)$ *be a de Vries algebra. The set* $\mathrm{End}(\mathfrak{B})$ *equipped with the topology whose base is given by* $\mathcal{B} = \{\eta(a) \mid a \in B\}$ *is a compact Hausdorff space.*

Proof. Suppose that x and y are distinct elements of $\mathrm{End}(\mathfrak{B})$. Using Corollary 1.3.5, we have that $x \in \eta(a)$ and $y \in \eta(\neg a)$ for some $a \in B$. Since $\eta(a) \cap \eta(\neg a) = \emptyset$, it follows that $\mathrm{End}(B)$ is a Hausdorff space.

To prove that $\mathrm{End}(\mathfrak{B})$ is compact, let $\mathcal{F}$ be a family of closed subsets with the finite intersection property and let x_0 be defined as

$$\begin{aligned} x_0 = \{a \in B \mid & \exists F_a^1, ..., F_a^n \in \mathcal{F} \text{ for some } n \in \mathbb{N} \text{ and} \\ & \exists b_a \in B : F_a^1 \cap \cdots \cap F_a^n \subseteq \eta(b_a) \text{ and } b_a \prec a\}. \end{aligned}$$

It is clear that x_0 is a round filter and, due to $\mathcal{F}$ having the finite intersection property, it is a proper subset of B. Therefore, there exists an end x such that $x_0 \subseteq x$.

We now prove that $x \in \cap\{F \mid F \in \mathcal{F}\}$ and so that $\mathrm{End}(B)$ is compact. If this is not be the case, there exists $F \in \mathcal{F}$ such that $x \notin F$. Thanks to Corollary 1.3.10, we know that $\mathrm{End}(B)$ is a regular space, which implies that $x \in \eta(a)$ and $\eta(a)^- \cap F = \emptyset$ for some $a \in B$. It follows that $F \subseteq \eta(a)^{-c} = \eta(\neg a)$ and, since x is round, there exists $b \in x$ such that $\neg a \prec \neg b$. In particular, it means that $\neg b \in x_0 \subseteq x$ which is impossible as we already have $b \in x$. $\square$

Corollary 1.3.13. *For a de Vries algebra* $\mathfrak{B} = (B, \prec)$ *and for* $a, b \in B$*, we have*

$$a \prec b \Leftrightarrow \eta(a)^- \subseteq \eta(b).$$

Proof. The if part was already proven in Lemma 1.3.9. Now suppose that $\eta(a)^- \subseteq \eta(b)$. If $x \in \eta(a)^-$, since it is round, there exists $d_x \in x$ such that $d_x \prec b$. In particular, we have

$$\eta(a)^- \subseteq \cup\{\eta(d_x) \mid x \in \eta(a)^-\}.$$

But, $\eta(a)^-$ is closed and thus compact, so that there exists $d_1, ..., d_n \in x$ such that $d_1 \vee \cdots \vee d_n \prec b$ and

$$\eta(a) \subseteq \eta(a)^- \subseteq \eta(d_1) \cup \cdots \cup \eta(d_n) \subseteq \eta(d_1 \vee \cdots \vee d_n).$$

From Lemma 1.3.7, we obtain $a \leq d_1 \vee \cdots \vee d_n \prec b$, and this leads us to the conclusion.

$\square$

1.4 The functors End and RO

In this section, we describe the functors that go from **DeV** to **KHaus** and from **KHaus** to **DeV**. We also prove that they are indeed (contravariant) functors. The duality will be fully realised in Section 1.5.

Lemma 1.4.1. *Let h be a morphism of* $\mathbf{DeV}(B, C)$ *and x be an end of C.*

1. *The set*
$$\prec(h^{-1}(x), -) := \{a \in B \mid \exists b \in h^{-1}(x) : b \prec a\}$$
is an end of B.

2. *The map*
$$\mathrm{End}(h) : \mathrm{End}(C) \longrightarrow \mathrm{End}(D) : x \longmapsto \prec(h^{-1}(x), -)$$
is a continuous function.

Proof. 1. It is clear, using the properties dv3 and dv2 of de Vries algebras, that $\prec(h^{-1}(x), -)$ is a filter of B while the property dv6 assures us that it is round. Moreover, since 0 cannot be in $\prec(h^{-1}(x), -)$, which would imply that 0 is an element of x, $\prec(h^{-1}(x), -)$ is proper. Hence, it remains to show that it is maximal.

Suppose that $a, b \in B$ are such that $a \prec b$. Then, we have $a \prec c \prec d \prec b$ for some $c, d \in B$. As h is a de Vries morphism, it follows that $\neg h(\neg c) \prec h(d)$ and, since x is an end, it follows that $h(\neg c) \in x$ or $h(d) \in x$. The first possibility implies that $\neg a \in \prec(h^{-1}(x), -)$ while the second one implies that $b \in \prec(h^{-1}(x), -)$. This leads to the conclusion.

2. From the previous point, we know that $\mathrm{End}(h)$ is well-defined, so that it remains to prove that $\mathrm{End}(h)$ is continuous. Let $a \in B$, we have

$$\begin{aligned}
\mathrm{End}(h)^{-1}(\eta(a)) &= \{x \in \mathrm{End}(C) \mid \prec(h^{-1}(x), -) \ni a\} \\
&= \{x \in \mathrm{End}(C) \mid \exists b \in h^{-1}(x) : b \prec a\} \\
&= \{x \in \mathrm{End}(C) \mid \exists b \in B : x \in \eta(h(b)) \text{ and } b \prec a\} \\
&= \cup\{\eta(h(b)) \mid b \prec a\}.
\end{aligned}$$

Hence, $\mathrm{End}(h)$ is a continuous function.

□

Lemma 1.4.2. *If h_1 is a morphism of* $\mathbf{DeV}(A, B)$ *and h_2 is a morphism of* $\mathbf{DeV}(B, C)$*, then* $\mathrm{End}(h_2 \star h_1) = \mathrm{End}(h_1) \circ \mathrm{End}(h_2)$.

Proof. Let us take x in $\mathrm{End}(C)$, we have

$$a \in \mathrm{End}(h_2 \star h_1)(x) \Leftrightarrow \exists b \in A : (h_2 \star h_1)(b) \in x \text{ and } b \prec a \tag{1.7}$$

and

$$\begin{gathered}
a \in (\mathrm{End}(h_1) \circ \mathrm{End}(h_2))(x) \\
\Leftrightarrow \\
\exists c \in A \text{ and } d \in B : h_2(d) \in x,\ d \prec h_1(c) \text{ and } c \prec a.
\end{gathered} \tag{1.8}$$

So, first, let us take an element a in $\mathrm{End}(h_2 \star h_1)(x)$ and let b be the element given in (1.7). Then, we have $b \prec d \prec a$ for some $c \in A$ and since $(h_2 \star h_1)(b) \leq h_2(h_1(b))$ and $(h_2 \star h_1)(b) \in x$, it follows that

$$h_2(h_1(b)) \in x,\ h_1(b) \prec h_1(c) \text{ and } c \prec a.$$

Let us set $d = h_1(b)$ and this is exactly the condition of (1.8).

Now, let us take an element a in $(\mathrm{End}(h_1) \circ \mathrm{End}(h_2))(x)$ and let c, d be the elements given in (1.8). From $c \prec h_1(d)$, we draw $h_2(d) \prec h_2(h_1(c))$, and from $c \prec a$, we draw $c \prec b \prec a$ for some $b \in A$ and, in particular, we have $h_2(h_1(c)) \leq (h_2 \star h_1)(b)$. It follows that

$$x \ni h_2(d) \leq (h_2 \star h_1)(b) \text{ and } b \prec a,$$

which is exactly the condition in (1.7).

□

Lemma 1.4.3. *If* id_B *is the identity morphism of* $\mathbf{DeV}(B, B)$*, then* $\mathrm{End}(\mathrm{id}_B)$ *is the identity morphism of* $\mathbf{KHaus}(\mathrm{End}(B), \mathrm{End}(B))$*.*

Proof. If $x \in \mathrm{End}(B)$, then it follows immediately that

$$x = {\prec}(x, -) = \mathrm{End}(\mathrm{id}_B)(x).$$

□

Definition 1.4.4. We denote by End the functor from $\mathbf{DeV}$ to $\mathbf{KHaus}$ which sends a de Vries algebra B to its set of ends $\mathrm{End}(B)$ and a de Vries morphism $h : B \longrightarrow C$ to the continuous function $\mathrm{End}(h) : \mathrm{End}(C) \longrightarrow \mathrm{End}(D)$.

Lemma 1.4.5. *Let* f *be a morphism of* $\mathbf{KHaus}(X, Y)$*. The map*

$$\mathrm{RO}(f) : \mathrm{RO}(Y) \longrightarrow \mathrm{RO}(X) : O \longmapsto (f^{-1}(O))^{-\circ}$$

is a de Vries morphism.

Proof. Let us remark that, by Lemma 1.1.3, we know that $\mathrm{RO}(f)$ is well-defined. Now, we will check that $\mathrm{RO}(f)$ verifies the properties of a de Vries morphism. Let O and U be elements of $\mathrm{RO}(Y)$

1. Trivially, we have $\mathrm{RO}(f)(\emptyset) = \emptyset$.

2. We have

$$\begin{aligned}
\mathrm{RO}(f)(O \wedge U) &= (f^{-1}(O \wedge U))^{-\circ} \\
&= (f^{-1}(O \cap U))^{-\circ} \\
&= (f^{-1}(O) \cap f^{-1}(U))^{-\circ} \\
&= (f^{-1}(O))^{-\circ} \cap (f^{-1}(U))^{-\circ} = \mathrm{RO}(f)(O) \wedge \mathrm{RO}(f)(U).
\end{aligned}$$

3. We have to prove that if $O \prec U$, or in other words if $O^- \subseteq U$, then $\neg\, \mathrm{RO}(f)(\neg O) \prec \mathrm{RO}(f)(U)$. Let us unwind $\neg\, \mathrm{RO}(f)(\neg O)$,

$$\begin{aligned}
\neg\, \mathrm{RO}(f)(\neg O) &= \neg\, \mathrm{RO}(f)(O^{-c}) \\
&= \neg((f^{-1}(O^{-c})^{-\circ}) \\
&= \neg((f^{-1}(O^{-})^{\circ-c}) \\
&= (f^{-1}(O^{-})^{\circ-c})^{c\circ} = f^{-1}(O^{-})^{\circ-\circ}.
\end{aligned}$$

So, we have to prove that $f^{-1}(O^-)^{\circ-\circ} \prec f^{-1}(U)^{-\circ}$, that is

$$f^{-1}(O^-)^{\circ-} = f^{-1}(O^-)^{\circ-\circ-} \subseteq f^{-1}(U)^{-\circ}.$$

Since we have that

$$O^- \subseteq U \Rightarrow f^{-1}(O^-)^{\circ -} \subseteq f^{-1}(O^-) \subseteq f^{-1}(U) \subseteq f^{-1}(U)^{-\circ},$$

and the proof is concluded.

4. We have to prove that $\mathrm{RO}(f)(O) = \vee\{\mathrm{RO}(f)(U) \mid U \prec O\}$, that is

$$(f^{-1}(O))^{-\circ} = (\cup\{f^{-1}(U)^{-\circ} \mid U^- \subseteq O\})^{-\circ}. \tag{1.9}$$

For the sake of simplicity, let us denote by u the set $f^{-1}(O)$ and by v the set

$$\cup\{f^{-1}(U)^{-\circ} \mid U^- \subseteq O\})^{-\circ}.$$

To obtain the equation (1.9), it suffices to prove that $u^- = v^-$.

Let x be an element of v^- and ω an open set containing x. Since $x \in v^-$, there is an element $y \in v \cap \omega$. By definition of v, we have $y \in f^{-1}(U)^{-\circ}$ for some regular open set U such that $U^- \subseteq O$. Hence, it follows that

$$y \in f^{-1}(U)^{-\circ} \subseteq f^{-1}(U)^- \subseteq f^{-1}(U^-) \subseteq f^{-1}(O),$$

which proves the inclusion $v^- \subseteq u^-$.

For the second inclusion, let us consider $x \in u^-$ and an open set ω containing x. As previously, there is an element y such that $y \in \omega \cap u$ and, more precisely, such that $f(y) \in O$. Since Y is compact Hausdorff, we have that $f(y) \in U$ and $O^c \subseteq (U^-)^c$ for some regular open set U of Y. In particular, this means that $U \prec O$ and that $y \in f^{-1}(U) \subseteq f^{-1}(U)^{-\circ}$ or, in other words, that $y \in v$.

□

Lemma 1.4.6. *If f_1 is a morphism of* $\mathbf{KHaus}(X, Y)$ *and f_2 is a morphism of* $\mathbf{KHaus}(Y, Z)$*, then* $\mathrm{RO}(f_2 \circ f_1) = \mathrm{RO}(f_1) \star \mathrm{RO}(f_2)$.

Proof. We need to prove that for every regular open subset O of Z, we have

$$\mathrm{RO}(f_2 \circ f_1)(O) = (\mathrm{RO}(f_1) \star \mathrm{RO}(f_2))(O).$$

Using the same trick as before, we will prove that $u^- = v^-$ for

$$u = f_1^{-1}(f_2^{-1}(O)) \text{ and } v = \cup\{f^{-1}(f_2^{-1}(U)^{-\circ})^{-\circ} \mid U^- \subseteq O\}.$$

Let us consider $x \in u^-$ and ω an open subset of X containing x. There is $y \in u \cap \omega$, that is $f_2(f_1(y)) \in O$. Then, $f_2(f_1(y)) \in U$ and $U^- \subseteq O$ for some regular open set U. It follows that

$$y \in f_1^{-1}(f_2^{-1}(U)) \subseteq f_1^{-1}(f_2^{-1}(U)^{-\circ}) \subseteq f_1^{-1}(f_2^{-1}(U)^{-\circ})^{-\circ}.$$

Hence, y is in v.

On the other hand, suppose that $x \in v^-$ and that ω is an open subset containing x. Hence, there exists $y \in v \cap \omega$, which means that $y \in f_1^{-1}(f_2^{-1}(U)^{-\circ})^{-\circ}$ for some $U^- \subseteq O$. We thus have

$$y \in f_1^{-1}(f_2^{-1}(U)^{-\circ})^{-\circ} \subseteq f_1^{-1}(f_2^{-1}(O)) = u.$$

□

Lemma 1.4.7. *If* id_X *is the identity morphism of* $\mathbf{KHaus}(X,X)$*, then the map* $\mathrm{RO}(\mathrm{id}_X)$ *is the identity morphism of* $\mathbf{DeV}(\mathrm{RO}(X),\mathrm{RO}(X))$.

Proof. If $O \in \mathrm{RO}(X)$, then it follows immediately that

$$O = O^{-\circ} = \mathrm{RO}(\mathrm{id}_x)(O).$$

□

Definition 1.4.8. We denote by RO the functor from **KHaus** to **DeV** which sends a compact Hausdorff space X to $\mathrm{RO}(X)$ and a continuous function $f : X \longrightarrow Y$ to $\mathrm{RO}(f) : \mathrm{RO}(Y) \longrightarrow \mathrm{RO}(X)$.

1.5 Duality

Theorem 1.5.1. *Let* B *be a de Vries algebra. The application*

$$\eta : B \longrightarrow \mathrm{RO}(\mathrm{End}(B)) : a \longmapsto \eta(a)$$

is an isomorphism of $\mathbf{DeV}(B,\mathrm{RO}(\mathrm{End}(B)))$.

Proof. First of all, we have to prove that η is well-defined. For every $a \in B$, we have

$$\eta(a)^{-\circ} = \eta(a)^{\perp\perp} = \eta(\neg a)^{\perp} = \eta(\neg\neg a) = \eta(a).$$

Hence $\eta(a)$ is indeed a regular open set, as required.

Then, by Theorem 1.2.11, we know it is sufficient to prove that η is a Boolean isomorphism such that

$$a \prec b \Leftrightarrow \eta(a) \prec \eta(b). \tag{1.10}$$

We already know by Lemmas 1.3.7 and 1.3.11 that η is injective and respects $\wedge$, $\vee$ 0 and 1. Hence, η is an injective Boolean morphism. Moreover, by Corollary 1.3.13, it is clear that η respects condition (1.10). Therefore, it only remains to prove that η is onto.

Let O be an arbitrary regular open set of $\mathrm{End}(\mathfrak{B})$. Since $\{\eta(a) \mid a \in B\}$ is a base of $\mathrm{End}(\mathfrak{B})$, we know that

$$O = \cup\{\eta(a) \mid a \in A\}$$

for some $A \subseteq B$. Now, since O is regular, we have, by Lemma 1.3.11,

$$O = O^{-\circ} = (\cup\{\eta(a) \mid a \in A\})^{-\circ} = \eta(\vee A),$$

which immediately leads to the conclusion. □

Proposition 1.5.2. *Let* X *be a compact Hausdorff space. A subset* Ξ *of* $\mathrm{RO}(X)$ *is an end if and only if there exist* $x \in X$ *such that*

$$\Xi = \varepsilon(x) := \{O \in \mathrm{RO}(X) \mid x \in O\}.$$

Proof. First let us prove that $\varepsilon(x)$ is an end for all $x \in X$. Since It is clear that $\varepsilon(x)$ is a proper filter, we have to prove that it is round and maximal among round filters. Let O be a regular open set of $\varepsilon(x)$. By regularity of X, there exists an open set ω such that $x \in \omega$ and $\omega^- \subseteq O$. For the same reason, there exists another open set ν such that $x \in \nu$ and $\nu^- \subseteq \omega$. Set now $U := \nu^{-\circ}$. We have that U is a regular open set such that $x \in U$ and $U^- \subseteq O$. Hence, $\varepsilon(x)$ is indeed round. To prove that it is maximal, we use the characterisation of ends given in Theorem 1.3.4. Let O and U be regular open sets such that $O^- \subseteq U$. As $\neg O = O^{-c}$, it is then clear that if $x \notin \neg O$, then $x \in U$.

It remains now to prove that for any end Ξ of $\mathrm{RO}(X)$, there is $x \in X$ such that $\Xi = \varepsilon(x)$. In order to achieve this goal, we will prove that there exists an element x in $\cap\{O \mid O \in \Xi\}$. This will give us $\Xi \subseteq \varepsilon(x)$ and the equality will simply follow from the maximality of Ξ.

Since Ξ is a round filter, for each $O \in \Xi$, we have $U_O^- \subseteq O$ for some $U_O \in \Xi$. In particular, this means that

$$\cap\{U_O^- \mid O \in \Xi\} \subseteq \cap\{O \mid O \in \Xi\}.$$

Let F be a finite family of $\{U_O \mid O \in \Xi\}$. We have

$$(\cap\{U_O \mid U_O \in F\})^- \subseteq \cap\{U_O^- \mid U_O \in F\}).$$

Now, since Ξ is a filter, we have $\cap\{U_O \mid U_O \in F\} \in \Xi$ and, since Ξ is proper, we have that

$$\emptyset \neq \cap\{U_O \mid U_O \in F\} \in \Xi.$$

Using the compactness of X, we can conclude that the intersection $\cap\{U_O^- \mid O \in \Xi\}$ is non-empty, and hence so is $\cap\{O \mid O \in \Xi\}$. $\square$

Theorem 1.5.3. *Let X be a compact Hausdorff space. The function*

$$\varepsilon : X \longrightarrow \mathrm{End}(\mathrm{RO}(X)) : x \longmapsto \varepsilon(x)$$

is an isomorphism of $\mathbf{KHaus}(X, \mathrm{End}(\mathrm{RO}(X)))$.

Proof. We just proved in Proposition 1.5.2 that ε was well-defined and onto. So we just need to prove that it is injective and continuous.

1. Take x and $y \in X$ such that $\varepsilon(x) = \varepsilon(y)$. It means that a regular open set of X contains x if and only if it contains y. Now, since X is Hausdorff and since its regular open subsets form a basis, it follows trivially that $x = y$.

2. Let $\eta(O)$ be an open of $\mathrm{End}(\mathrm{RO}(X))$ basis. Since we have that

$$x \in \varepsilon^{-1}(\eta(O)) \Leftrightarrow x \in O,$$

it is clear that ε is continuous.

$\square$

Theorem 1.5.4. *The categories* **DeV** *and* **KHaus** *are dually equivalent via the functors* End *and* RO.

Proof. Since we already proved Theorems 1.5.1 and 1.5.3, it remains to show that η and ε are natural (in the sense of Definition A.3.1). That is, for every morphism $h \in \mathbf{DeV}(B, C)$ and for every function $f \in \mathbf{KHaus}(X, Y)$, the following diagrams are commutative

$$\begin{array}{ccc} B & \xrightarrow{\eta_B} & \mathrm{RO}(\mathrm{End}(B)) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathrm{RO}(\mathrm{End}(h))} \\ C & \xrightarrow[\eta_C]{} & \mathrm{RO}(\mathrm{End}(C)) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\varepsilon_X} & \mathrm{End}(\mathrm{RO}(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{End}(\mathrm{RO}(f))} \\ Y & \xrightarrow[\varepsilon_Y]{} & \mathrm{End}(\mathrm{RO}(Y)) \end{array}$$

1. We have to show that, for every $b \in B$,

$$(\mathrm{RO}(\mathrm{End}(h)) \star \eta_B)(b) = (\eta_C \star h)(b).$$

   But, since η_B is a strong de Vries isomorphism, that is equivalent to show that

$$\mathrm{RO}(\mathrm{End}(h))(\eta_B(b)) = (\eta_C \star h)(b).$$

   Moreover, since

$$\mathrm{RO}(\mathrm{End}(h)(\eta_B(b))) = (\mathrm{End}(h)^{-1}(\eta_B(b)))^{-\circ}$$

   and

$$(\eta_C \star h)(b) = (\cup\{\eta_C(h(c)) : c \prec b\})^{-\circ},$$

   it is enough to prove that

$$\mathrm{End}(h)^{-1}(\eta_B(b)) = \cup\{\eta_C(h(c)) : c \prec b\}.$$

   Let $x \in \mathrm{End}(C)$, we have

$$\begin{aligned} x \in \mathrm{End}(h)^{-1}(\eta_B(b)) &\Leftrightarrow \mathrm{End}(h)(x) \in \eta_B(b) \\ &\Leftrightarrow b \in {\prec}(h^{-1}(x), -) \\ &\Leftrightarrow \exists c \in h^{-1}(x) : c \prec b \\ &\Leftrightarrow (\exists c)(h(c) \in x \text{ and } c \prec b) \\ &\Leftrightarrow x \in \cup\{\eta_C(h(c)) : c \prec b\}. \end{aligned}$$

2. We have to show that, for every $x \in X$

$$\mathrm{End}(\mathrm{RO}(f))(\varepsilon_X(x)) = \varepsilon_Y(f(x)),$$

   that is, in other words, that for $O \in \mathrm{RO}(Y)$, we have

$$f(x) \in O \Leftrightarrow (\exists U \in \mathrm{RO}(Y))(U^- \subseteq O \text{ and } x \in f^{-1}(U)^{-\circ}).$$

   For the if part, we have the following sequence of inclusions:

$$x \in f^{-1}(U)^{-\circ} \subseteq f^{-1}(U)^- \subseteq f^{-1}(U^-) \subseteq f^{-1}(O).$$

   For the only part, recall that Y is, in particular, a regular space. Therefore, if $f(x) \notin O$, there exists an open set ω such that $f(x) \in \omega$ and $\omega^- \subseteq O$. For analogue reasons, there exists an open set ν such that $f(x) \in \nu$ and $\nu^- \subseteq \omega$. Finally, we have $x \in f^{-1}(\nu) \subseteq f^{-1}(\nu^{-\circ})$ and $\nu^{-\circ-} \subseteq \omega^- \subseteq O$. Hence, we can conclude that $U := \nu^{-\circ}$ fulfils the requirements.

□

We will now use the duality between **DeV** and **KHaus** to give the counterpart of Theorem 1.2.11 about the characterisation of isomorphisms in **DeV**. This counterpart will of course make the denomination of strong isomorphism obsolete.

Theorem 1.5.5. *Every isomorphism h between de Vries algebras is a strong isomorphism.*

Proof. Without loss of generality, we may consider that there exists a homeomorphism $f : X \longrightarrow Y$ between two compact Hausdorff spaces such that $h = \mathrm{RO}(f)$. But now, since f is a homeomorphism, we have that

$$\mathrm{RO}(f)(O) = f^{-1}(O)^{-\circ} = f^{-1}(O^{-\circ}) = f^{-1}(O),$$

for $O \in \mathrm{RO}(Y)$. It is then not hard to prove that $\mathrm{RO}(f)$ is a Boolean isomorphism such that

$$\mathrm{RO}(f)(O) \prec \mathrm{RO}(f)(U) \Rightarrow O \prec U$$

for every $O, U \in \mathrm{RO}(Y)$. □

Remark 1.5.6. In this presentation, we chose to present the duality between **DeV** and **KHaus** via the functor RO, albeit there exists another functor. Indeed, recall that regular opens sets and regular closed sets are in correspondence. Therefore, it is also possible to map a compact Hausdorff space X to its set of regular closed sets $\mathrm{RC}(X)$. Indeed, $(\mathrm{RC}(X), \prec)$, where $\prec$ is the binary relation defined as

$$F \prec G \Leftrightarrow F \subseteq G^{\circ},$$

is a de Vries algebra. Moreover, the functor RC assigns to a morphism f of $\mathbf{KHaus}(X, Y)$ the de Vries morphism

$$\mathrm{RC}(f) : \mathrm{RC}(Y) \longrightarrow \mathrm{RC}(X) : F \longmapsto (f^{-1}(F))^{\circ -}.$$

While the functor End from **DeV** to **KHaus** is identical as in the regular open case, it is important to note that the isomorphism between a de Vries algebra B and $\mathrm{RC}(\mathrm{End}(B))$ is not directly η anymore. Indeed, $\eta(a)$ is a regular open set and, therefore, $\eta(a)^c$ is a regular closed one. It follows that we should rather use the map

$$\eta^c : B \longrightarrow \mathrm{RC}(\mathrm{End}(B)) : a \longmapsto \eta(a)^c = \{x \in \mathrm{End}(B) \mid a \notin x\}$$

to obtain the required isomorphism.

1.6 DeV and ubal

Both **DeV** and **ubal** are categories dually equivalent to **KHaus** (see Appendix B.4). As a direct consequence, they are equivalent categories. In this section, we give a description of the functor between **ubal** and **DeV**. To give a sketch of this description, recall that the functor **ubal** → **KHaus** sends an ubal A to the set of its maximal ℓ-ideals $\mathrm{MaxId}_\ell(A)$. Then the functor **KHaus** → **DeV** sends a compact Hausdorff space to its set of regular open sets or, dually and with the adequate adaptations discussed in Remark 1.5.6, to its set of regular closed sets. Hence, essentially, to describe the functor from **usbal** to **DeV** is to characterise the regular closed sets of $\mathrm{MaxId}_\ell(A)$. This question about regular closed sets in $\mathrm{MaxId}_\ell(A)$ has been treated in [7] to achieved a different objective. Nevertheless, we can use their notations and results to obtain the desired functor.

It isknown (once a gain, see Appendix B.4) that the set $\mathrm{MaxId}_\ell(A)$ of maximal ℓ-ideal of a bal A is naturally endowed with a topology whose base is given by

$$\omega(a) = \{M \in \mathrm{MaxId}_\ell(A) \mid a \notin M\}, \quad \text{for } a \in A.$$

In the next lemma, we give a description of the closed sets of $\mathrm{MaxId}_\ell(A)$ based on the ring ideals of A.

Lemma 1.6.1. *Let A be a bal. The closed sets of* $\mathrm{MaxId}_\ell(A)$ *are exactly the subsets of the form* $Z_\ell(I) := \{M \in \mathrm{MaxId}_\ell(A) \mid I \subseteq M\}$ *for some ring ideal I.*

Proof. First, since

$$Z_\ell(I) = (\cup\{\omega(a) \mid a \in I\})^c,$$

we have that $Z_\ell(I)$ is indeed a closed set.

Secondly, if F is a closed subset of $\mathrm{MaxId}_\ell(A)$, then there exist $S \subseteq A$ such that

$$F = \cap\{\omega(a)^c \mid a \in S\}.$$

It is now just a routine computation to show that $F = Z_\ell(I)$ with I the ideal generated by S. □

Therefore, characterising the regular closed sets of $\mathrm{MaxId}_\ell(A)$ is to determine the ring ideals that generate them. In [7, p. 460], Bezhanishvili, Morandi and Olberding brought forward the notion of annihilator ideals. Using their notations and proofs, we investigate in this direction.

Definition 1.6.2. Let A be a commutative ring and I a ring ideal of A. We say that I is an **annihilator ideal** if

$$I = \mathrm{ann}(J) := \{a \in A \mid aJ = 0\},$$

for some ring ideal J.

It is quite obvious, but still worth noticing, that $\mathrm{ann}(J)$ is indeed a ring ideal. In fact, we have a stronger property: $\mathrm{ann}(S)$ is a ring ideal for all $S \subseteq X$.

Finally, we denote by $\mathrm{ANN}(A)$ the set of all annihilator ideals of A.

Definition 1.6.3. A **Galois connection** between two ordered sets P and Q is a pair of antitone maps $g_1 : P \longrightarrow Q$ and $g_2 : Q \longrightarrow P$ such that for all $p \in P$ and $q \in Q$

$$p \leq g_2(g_1(p)) \text{ and } q \leq g_1(g_2(q)).$$

It is well known (e.g. [10, Section 1.6]) that for $\mathrm{Id}(A)$, the set of ring ideals of a commutative ring A ordered by inclusion, the map

$$\mathrm{ann} : \mathrm{Id}(A) \longrightarrow \mathrm{Id}(A) : I \longmapsto \mathrm{ann}(I)$$

establishes a Galois connection. Therefore, directly from the properties of Galois connections (see for instance [55, pp. 495–496]), we obtain the following lemma.

Lemma 1.6.4. *For a commutative ring A, we have:*

1. $\mathrm{ann}^3 = \mathrm{ann}$,
2. ann^2 *is idempotent,*
3. ann^2 *is a closure operator on* $(\mathrm{Id}(A), \subseteq)$.

As a direct corollary, we have that a ring ideal I is an annihilator if and only if $I = \mathrm{ann}^2(I)$. Indeed, if J denotes the ideal such that $I = \mathrm{ann}(J)$, it follows that

$$\mathrm{ann}^2(I) = \mathrm{ann}^3(J) = \mathrm{ann}(J) = I.$$

Remark 1.6.5. Before we actually get started with the proofs, let us recall three useful facts, for future convenience.

1. If A is a commutative ring and if I is a ring ideal of A then the quotient $A/_I$ is a field if and only if I is a maximal ring ideal.
2. If I is a maximal ℓ-ideal of a bal A, then the quotient $A/_I$ is isomorphic to $\mathbb{R}$ (as proved in [7]). Consequently, every maximal ℓ-ideal is also maximal among ring ideals.
3. The intersection of all maximal ℓ-ideals of a bal is reduced to $\{0\}$ (see [45, Definition II.1.3 and Theorem II.2.11]).

Lemma 1.6.6. *Let A be a bal. For every ring ideal I of A, we have*

$$\operatorname{ann}(I) = \cap(Z_\ell(I)^c)$$

where $Z_\ell(I)$ is defined as in Lemma 1.6.1.

Proof. Let K denote the intersection $\cap Z_\ell(I)^c$. We prove the claim by showing that the two inclusions hold.

First of all, let M be a maximal ℓ-ideals such that $M \in Z_\ell(I)^c$. Notice that we trivially have $I \cdot \operatorname{ann}(I) = 0$. Therefore, since M is in particular a maximal ideal ring (as recalled in Remark 1.6.5), it follows that $I \subseteq M$ or $\operatorname{ann}(I) \subseteq M$. By our hypothesis on M, only the latter inclusion is possible. Since this inclusion is valid for all $M \in Z_\ell(I)^c$, it is clear that $\operatorname{ann}(I) \subseteq \cap Z_\ell(I)^c = K$.

Now, let M be an arbitrary element of $\operatorname{MaxId}_\ell(A)$ and suppose, at first, that $I \not\subseteq M$. Then, by definition of K, we have $K \subseteq M$. Since M is an ideal, this implies that $I \cdot K \subseteq M$. Now, if instead we suppose that $I \subseteq M$, we also have immediately $I \cdot K \subseteq M$. It follows that

$$I \cdot K \subseteq \cap\{M \in \operatorname{MaxId}_\ell(A)\} = \{0\}.$$

Hence, we have $K \subseteq \operatorname{ann}(I)$ and the proof is concluded. □

Lemma 1.6.7. *Let A be a bal.*

1. *For all $S \subseteq \operatorname{MaxId}_\ell(A)$, we have $\overline{S} = Z_\ell(\cap S)$;*
2. *For every ring ideal I of A, we have $Z_\ell(I)^{\circ -} = Z_\ell(\operatorname{ann}^2(I))$.*

Proof.

1. As we already noticed in Lemma 1.6.1, we have that $Z_\ell(\cap S)$ is indeed a closed set of $\operatorname{MaxId}_\ell(A)$. Moreover, for $x \in S$, we have that $\cap S \subseteq x$ and, consequently, that $x \in Z_\ell(\cap S)$. Hence, $S \subseteq Z_\ell(\cap S)$. So, it remains to show that it is the smaller closed set satisfying this property.

   Let F be a closed subset of $\operatorname{MaxId}_\ell(A)$ such that $S \subseteq F$ and let I be the ring ideal with $F = Z_\ell(I)$. Then, for all $M \in S$, we have $I \subseteq M$ and thus $I \subseteq \cap S$, so that $Z_\ell(\cap S) \subseteq Z_\ell(I)$, as required.

2. Using item 1 and Lemma 1.6.6, we have first

   $$Z_\ell(I)^{c-} = Z_\ell(\cap(Z_\ell(I)^c)) = Z_\ell(\operatorname{ann}(I)).$$

   Then,

   $$Z_\ell(I)^\circ = Z_\ell(I)^{c-c} = Z_\ell(\operatorname{ann}(I))^c.$$

   Finally,

   $$Z_\ell(I)^{\circ -} = Z_\ell(\operatorname{ann}(I))^{c-} = Z_\ell(\operatorname{ann}^2(I)).$$

□

We are now ready to characterise the regular closed sets of $\mathrm{MaxId}_\ell(A)$.

Theorem 1.6.8. *Let A be a bal. A closed sets F of $\mathrm{MaxId}_\ell(A)$ is regular if and only if there exists a unique annihilator ideal I with $F = Z_\ell(I)$. Hence, there exists a bijection b between $\mathrm{RC}(\mathrm{MaxId}_\ell(A))$ and $\mathrm{ANN}(A)$.*

Proof. If $F = Z_\ell(\mathrm{ann}(J))$, then

$$F^{\circ -} = (Z_\ell(\mathrm{ann}(J)))^{\circ -} = Z_\ell(\mathrm{ann}^3(J)) = Z_\ell(\mathrm{ann}(J)) = F,$$

that is F is regular. On the other hand, if F is regular and if $F = Z_\ell(I)$, then

$$F = F^{-\circ} = Z_\ell(\mathrm{ann}^2(I))$$

with $\mathrm{ann}^2(I)$ an annihilator ideal.

We now have to prove that if $I = \mathrm{ann}(K)$ and $J = \mathrm{ann}(L)$ are annihilators ideals such that $Z_\ell(I) = Z_\ell(J)$, then $I = J$. By Lemma 1.6.6, we have

$$\mathrm{ann}(I) = \cap Z_\ell(I) = \cap Z_\ell(J) = \mathrm{ann}(J),$$

that is $\mathrm{ann}^2(K) = \mathrm{ann}^2(L)$. By applying ann one more time and using the equality $\mathrm{ann}^3 = \mathrm{ann}$, we can conclude that

$$I = \mathrm{ann}(K) = \mathrm{ann}(L) = J,$$

as required. □

In short, we just established a bijection between $\mathrm{RC}(\mathrm{MaxId}_\ell(A))$, which is a de Vries algebra, and $\mathrm{ANN}(A)$. The point now is to transform this bijection into a (strong) de Vries isomorphism by endowing $\mathrm{ANN}(A)$ with a structure of de Vries algebras.

Proposition 1.6.9. *Let A be a bal and I, J be annihilator ideals of A. In the de Vries algebra $\mathrm{RC}(\mathrm{MaxId}_\ell(A))$, we have the following equalities:*

1. $Z_\ell(I) \cap Z_\ell(J) = Z_\ell(\langle I \cup J\rangle)$, ,
2. $Z_\ell(I) \wedge Z_\ell(J) = Z_\ell(\mathrm{ann}^2(\langle I \cup J\rangle))$,
3. $Z_\ell(I) \vee Z_\ell(J) = Z_\ell(I \cap J)$,
4. $\neg Z_\ell(I) = Z_\ell(\mathrm{ann}(I))$,
5. $Z_\ell(\{0\}) = \mathrm{MaxId}_\ell(A)$ *and* $Z_\ell(A) = \emptyset$.

Proof. 1. We have

$$\begin{aligned} M \in Z_\ell(I) \cap Z_\ell(J) &\Leftrightarrow M \supseteq I \text{ and } M \supseteq J \\ &\Leftrightarrow M \supseteq I \cup J \Leftrightarrow M \supseteq \langle I \cup J\rangle. \end{aligned}$$

2. We have

$$\begin{aligned} Z_\ell(I) \wedge Z_\ell(J) &= (Z_\ell(I) \cap Z_\ell(J))^{\circ -} \\ &= (Z_\ell(\langle I \cup J\rangle))^{\circ -} = Z_\ell(\mathrm{ann}^2(\langle I \cup J\rangle)). \end{aligned}$$

3. We will show that the two inclusions hold.

$\subseteq$ Let $M \in Z_\ell(I) \vee Z_\ell(J) = Z_\ell(I) \cup Z_\ell(J)$, then

$$M \supseteq I \supseteq I \cap J \text{ or } M \supseteq J \supseteq I \cap J,$$

such that $M \in Z_\ell(I \cap J)$.

$\supseteq$ If $M \in Z_\ell(I \cap J)$, we have in particular that $I \cdot J \subseteq I \cap J \subseteq M$. This implies that $I \subseteq M$ or $J \subseteq M$, that is $M \in Z_\ell(I) \cup Z_\ell(J)$.

- We get

$$\neg Z_\ell(I) = Z_\ell(I)^{c-} = Z_\ell(\cap Z_\ell(I)^c) = Z_\ell(\mathrm{ann}(I)).$$

□

For Proposition 1.6.9 to be complete, we need the following lemma.

Lemma 1.6.10. *Let A be a bal and I, J be annihilator ideals of A. The intersection $I \cap J$ is still an annihilator.*

Proof. This follows immediately from

$$I \cap J = \mathrm{ann}(K) \cap \mathrm{ann}(L) = \cap Z_\ell(K)^c \cap \cap Z_\ell(L)^c = \cap Z_\ell(\langle K \cup L \rangle)^c = \mathrm{ann}(\langle K \cup L \rangle),$$

for some ring ideals K and L. □

Therefore, by Proposition 1.6.9, we know how to define the Boolean operators on $\mathrm{ANN}(A)$ to make it isomorphic (as a Boolean algebra) to $\mathrm{RC}(\mathrm{MaxId}_\ell(A))$. To conclude we need to equip it with a de Vries relation satisfying

$$I \prec J \Leftrightarrow Z_\ell(I) \prec Z_\ell(J) \Leftrightarrow Z_\ell(I) \subseteq Z_\ell(J)^\circ.$$

To describe this relation in a purely algebraic way, we have to recall something used in Lemma 1.6.7: we have the equality $Z_\ell(I)^\circ = Z_\ell(\mathrm{ann}(I))^c$. It follows that

$$Z_\ell(I) \prec Z_\ell(J) \Leftrightarrow Z_\ell(I) \subseteq Z_\ell(\mathrm{ann}(J))^c,$$

which is equivalent to

$$(\forall M \in \mathrm{MaxId}_\ell(A))(M \supseteq I \Rightarrow M \not\supseteq \mathrm{ann}(J)).$$

We thus have the following theorem.

Theorem 1.6.11. *Let A be a bal. The set $\mathrm{ANN}(A)$ equipped with the operations*

1. $I \wedge J = \mathrm{ann}^2(\langle I \cup J \rangle)$,
2. $I \vee J = I \cap J$,
3. $\neg I = \mathrm{ann}(I)$,

and with the relation

4. *$I \prec J$ if and only if for every maximal ℓ-ideal M, $M \supseteq I$ implies $M \not\supseteq \mathrm{ann}(J)$,*

is a de Vries algebra whose top and bottom elements are respectively given by $\{0\}$ *and* A. *Moreover, the map*

$$Z_\ell : \mathrm{ANN}(A) \longrightarrow \mathrm{RC}(\mathrm{MaxId}_\ell(A)) : I \longmapsto Z_\ell(I)$$

is a de Vries isomorphism.

Now that the functor is defined on the objects, we can focus on the morphisms.

Theorem 1.6.12. *If* $\alpha : A \longrightarrow B$ *is a bal morphism, then*

$$\mathrm{ANN}(\alpha) : \mathrm{ANN}(A) \longrightarrow \mathrm{ANN}(B) : I \longmapsto \mathrm{ann}^2(\alpha(I))$$

is a de Vries morphism equal to $Z_\ell^{-1} \star \mathrm{RC}(\mathrm{MaxId}_\ell(\alpha)) \star Z_\ell$.

Proof. Let I be an ideal of $\mathrm{ANN}(A)$, we then have

$$\begin{aligned}
Z_\ell(\mathrm{ANN}(\alpha)(I)) &= Z_\ell(\mathrm{ann}^2(\alpha(I)) = Z_\ell(\alpha(I))^{\circ-} \\
&= \{M \in \mathrm{MaxId}_\ell(A) \mid M \supseteq \alpha(I)\}^{\circ-} \\
&= \{M \in \mathrm{MaxId}_\ell(A) \mid \alpha^{-1}(M) \supseteq I\}^{\circ-} \\
&= \{M \in \mathrm{MaxId}_\ell(A) \mid \mathrm{MaxId}_\ell(\alpha)(M) \supseteq I\}^{\circ-} \\
&= \{M \in \mathrm{MaxId}_\ell(A) \mid \mathrm{MaxId}_\ell(\alpha)(M) \in Z_\ell(I)\}^{\circ-} \\
&= ((\mathrm{MaxId}_\ell(\alpha))^{-1}(Z_\ell(I))^{\circ-} \\
&= \mathrm{RC}(\mathrm{MaxId}_\ell(\alpha))(Z_\ell(I))
\end{aligned}$$

Hence, we have $Z_\ell \circ \mathrm{ANN}(\alpha) = \mathrm{RC}(\mathrm{MaxId}_\ell(\alpha)) \circ Z_\ell$. But, Z_ℓ is an isomorphism and hence, by Proposition 1.2.13, this implies

$$Z_\ell \star \mathrm{ANN}(\alpha) = \mathrm{RC}(\mathrm{MaxId}_\ell(\alpha)) \star Z_\ell,$$

as required. □

Thus we have a functor ANN which maps a bal A to the de Vries algebra $\mathrm{ANN}(A)$ and a bal morphism α to the de Vries morphism $\mathrm{ANN}(\alpha)$. By construction, it is clear that this functor is equivalent to the composition of the functors RC and MaxId_ℓ and hence it establishes an equivalence between **usbal** and **DeV** with the functor $C \circ \mathrm{End}$ where C is the functor that maps a compact Hausdorff space to its ring of real continuous functions (see Appendix B.4).

1.7 Stone and de Vries duals

In the previous sections, we have seen that de Vries established a duality between a de Vries algebra $\mathfrak{B} = (B, \prec)$ and its space of ends $Y_B = \mathrm{End}(\mathfrak{B})$. Now, as B is a complete Boolean algebra, it admits a Stone dual $X_B = \mathrm{Ult}(B)$. Then, it is natural to wonder whether the two topological spaces are related to each other.

Looking at a characterization of the ends of $(B, \prec)$ different to the one given in Theorem 1.3.4 may somehow suggest the answer

Proposition 1.7.1. *A filter* x *of a de Vries algebra* $\mathfrak{B} = (B, \prec)$ *is an end if and only there exists an ultrafilter* u *of* B *with* $x = {\prec}(u, -)$.

Proof. For the if part, since x is in particular a proper filter, there exists an ultrafilter u such that $x \subseteq u$. Let us prove that $\prec(u,-)$ is a proper round filter that contains x, the conclusion will then follow from the maximality of x. We have that $0 \notin \prec(u,-)$, as otherwise there would be $a \in U$ with $a \prec 0$ and, hence, with $a = 0$. It is clear that $\prec(u,-)$ is increasing and closed under $\wedge$, hence a filter. And finally, if $a \in e$, then, by definition of roundness, there exists $b \in e \subseteq u$ with $b \prec a$. Therefore, $a \in \prec(u,-)$ as required.

For the only if part, we just have to check that $\prec(u,-)$ is indeed an end. We already now that $\prec(u,-)$ is a round filter, so that we need to prove that it is maximal. We use Theorem 1.3.4. Suppose that $a \prec b$. There exists $c \in B$ with $a \prec c \prec b$. Now, since u is an ultrafilter, we have $\neg c \in u$ or $c \in u$. The first case implies that $\neg a \in \prec(u,-)$ while the second one implies that $b \in \prec(u,-)$, hence, $\prec(u,-)$ is maximal. $\square$

Of course, the map $\mathrm{Ult}(B) \longrightarrow \mathrm{End}(\mathfrak{B}) : u \longmapsto \prec(u,-)$ may not be one-to-one since an end may correspond to several ultrafilters. However, if we consider the equivalence relation on $\mathrm{Ult}(B)$ defined by

$$u \, R \, v \Leftrightarrow \prec(u,-) = \prec(v,-), \tag{1.11}$$

then

$$\sigma_B : \mathrm{Ult}(B)/_R \longrightarrow \mathrm{End}(\mathfrak{B}) : u^R \longmapsto \prec(u,-)$$

becomes a bijection, and, even more, a continuous bijection from a compact space to a Hausdorff one, hence, a homeomorphism.

Theorem 1.7.2. *Let $\mathfrak{B} = (B, \prec)$ be a de Vries algebra.*

1. *The relation* (1.11) *is closed on* $\mathrm{Ult}(B)$*,*
2. $\mathrm{Ult}(B)/_R \cong \mathrm{End}(\mathfrak{B})$.

Proof. In order to avoid confusion, we will denote the elements of the base of $\mathrm{End}(B)$ by $r(a)$ instead of $\eta(a)$, which will be reserved for the base of $\mathrm{Ult}(B)$.

1. Suppose that $(u,v) \notin R$. Then, without loss of generality, we may assume that there exists $a \in B$ such that $a \in \prec(u,-)$ and $a \notin \prec(v,-)$. There exists $b \in u$ with $b \prec a$ and hence, such that $\neg b \in \prec(v,-)$. Consequently, $c \prec \neg b$ for some $c \in v$. We now have

   $$(u,v) \in \eta(b) \times \eta(c) \subseteq R^c.$$

   Indeed, suppose that $(s,t) \in \eta(b) \times \eta(c) \cap R$. Then, we have

   $$\neg c \in \prec(b,-) \subseteq \prec(s,-) = \prec(t,-) \subseteq t,$$

   which contradicts $t \in \eta(c)$.

2. Since $\mathrm{Ult}(B)/_R$ is the quotient of a compact space, it is a compact space. Hence, since $\mathrm{End}(\mathfrak{B})$ is Hausdorff, we just have to show that σ_B is continuous. Let $r(a)$ be an open set of $\mathrm{End}(\mathfrak{B})$ for some $a \in B$ and let π the canonical function associated to R. We have, for $u \in \mathrm{Ult}(B)$

   $$\begin{aligned} & u \in \pi^{-1}(\sigma_B^{-1}(r(a)) \\ \Leftrightarrow\ & u^R \in \sigma_B^{-1}(r(a)) \\ \Leftrightarrow\ & a \in \prec(u,-) \\ \Leftrightarrow\ & (\exists b \in u)(b \prec a) \\ \Leftrightarrow\ & (\exists b)(u \in \eta(b) \text{ and } b \prec a) \\ \Leftrightarrow\ & u \in \cup\{\eta(b) : b \prec a\}. \end{aligned}$$

Therefore, $\pi^{-1}(\sigma_B^{-1}(r(a))$ is an open set of $\mathrm{Ult}(B)$, which implies that $\sigma_B^{-1}(r(a))$ is an open set of $\mathrm{Ult}(B)/_R$.

$\square$

Actually, the definition of the relation R here is quite similar to the one used for the accessibility relation on modal spaces (see Appendix B.3). That is, if $\mathfrak{M} = (B, \Diamond)$ is a modal algebra, we define the relation $S_\Diamond$ on $\mathrm{Ult}(B)$ as follows:

$$u\ S_\Diamond\ v \Leftrightarrow \Diamond v \subseteq u.$$

But the similarity can be pushed one step further thanks to this equivalent definition of R:

$$u\ R\ v \Leftrightarrow \prec(v, -) \subseteq u. \tag{1.12}$$

The implication (1.11) $\Rightarrow$ (1.12) simply follows from the inclusion $\prec(u, -) \subseteq u$. For the other implication, suppose that $a \in \prec(v, -)$. Then $b \prec c \prec a$ for some $b \in v$ and $c \in B$. It follows that $c \in \prec(v, -) \subseteq u$ and, hence, that $a \in \prec(u, -)$. The maximality of $\prec(v, -)$ implies then that $\prec(v, -) = \prec(u, -)$.

Chapter 2

Subordination algebras and tense logic

In Section 1.7, we saw that a de Vries algebra $\mathfrak{B} = (B, \prec)$ gives rise to a Stone space endowed with a closed equivalence relation. Moreover, we also saw that this relation was constructed in a way which is very similar to the one used to construct the accessiblity relation dual to a modal operator (see Theorem B.3.4).

In this chapter, we will investigate a category which is more general, at the objects level, than the de Vries one: the category of subordination algebras. These algebras are Boolean algebras endowed with a (subordination) relation which satisfies minimal requirements to produce a closed binary relation on the Stone dual (not necessarily an equivalence, as in (1.11)).

It is interesting to note that the concept of subordination algebra has cropped up under different names and for different scopes throughout history: pre-contact or proximity algebras in [28], [30] and [54], as a way to establish a region based theory of space, quasi-modal algebras in [14], as a generalisation of modal algebras, subordination algebras in [5], as a generalisation of de Vries algebras or even strict implication algebras in [4], as an algebraic semantic correspondent a compact Hausdorff one. The equivalence between all these concepts is well discussed for instance in [15].

Therefore, according to the selected option, it may seem at first glance that a subordination algebra and a modal algebra have very little in common: one is a purely algebraic structure while the other is a hybrid structure, with algebraic and relational flavours. Nevertheless, both structures share common features. The choice of Celani in [14] perfectly illustrates the situation. Indeed, a quasi-modal algebra is a Boolean algebra B endowed with a map $\blacksquare$ from B to its set of ideals and which satisfies the usual conditions of modal operators. Moreover, we have already mentioned another common feature: the shape of their Stone duals. Namely, these duals are Stone spaces endowed with a closed binary relation (which is also continuous in the modal case). Hence, keeping both these facts in mind, it seems natural to look for a way to use subordination algebras as models for standard modal logic (and even standard tense logic, since the accessibility relation is not "asymmetrical" in the subordination setting).

Before we continue our discussion, two important remarks have to be done. The first one is that, while the quasi-modal option seems the more adequate to our declared goal, we chose to stick with the subordination one. The main reason behind this odd decision is that it allows us to have a rich and unique environment (subordination algebras) which supports three languages (with different expression power): the standard modal/tense language, the subordination (or contact, see Appendix C) language and the accessibility language. Hence, we have perfect structures for a study of the relations between these three languages. These relations will be discussed in Sections 2.7 and 2.10.

The second remark is that our aim is absolutely not to substitute standard modal algebras in the theory of modal logics. We rather want to add a new class of models which can be considered alongside the standard ones. The advantages and disadvantages of these new models will be presented in Section 2.6.

Let us now describe the structure of the current chapter. In the first two sections, we will describe the well known duality between subordination algebras and subordination spaces at the objects levels (see [5] or [14]). Then, we will construct several adequate category-theoretic structures by considering different kinds of morphisms, generalising the morphisms of [5] and [14]. These different kinds of morphisms steer us to four dualities.

Then, we will describe how to consider subordination algebras and subordination spaces as model for the standard tense logic. On the topological side, there is almost no work to accomplish. Since subordination spaces are actually "augmented" Kripke frames, we can use the usual definitions for validity of formulas, restricted to clopen valuations (as it is done for general frames).

As there is an obvious definition for topological validity, we are entitled to expect it has an algebraic counterpart. Unfortunately, the situation here is a bit more critical. Indeed, if we look at Celani's definition, it will be clear that a valuation of a formula on a subordination algebra may certainly fail to be an element of its underlying Boolean algebra, as it is in general an element of its canonical extension.

To overcome this problem, and hence restore the balance between the topological and the algebraic worlds, we will lay emphasis, in Sections 2.3 and 2.4, on the discrete (i.e. non-topological) versions of the dualities we just established and point out that these dualities are the well known dualities between complete atomic modal algebras and Kripke frames (see for instance Chapters 7 and 8 of [16]), so that, from a discrete point of view, there is no differences between subordination and modal algebras.

Once the topological an discrete dualities are established, they will serve as a stepping stone for the constructions of canonical extensions of subordination algebras in Section 2.4. Then, we will use the just established canonical extension to overcome our aforementioned valuation problem. Recall indeed that we want to interpret standard tense formulas in the subordination setting.

We will then be ready to examine the relations between the three available languages in the subordination setting. First, in Section 2.7, we establish a fragment of tense formulas, namely s-Sahqlvist formulas, which admit a first order correspondent in the accessibility language. Let us note that s-Sahlqvist formulas are particular Sahlqvist formulas ([63]). Finally, in Section 2.10, we establish a second fragment of tense formulas which, this time, admit a first order correspondent in the subordination language. We also discuss the possibility of correspondence between tense formulas and subordination formulas (i.e. without quantifiers).

2.1 Basic definitions and properties

In this section, we describe the duality, at the objects level, between subordination algebras and subordination spaces. Moreover, we also describe the translation of additional properties on subordination algebras to subordination spaces. Note that since the objects involved here are the objects of Stone duality with additional structures, almost every results of Stone duality (described in Appendix A) can be transposed here.

These results are adaptations of the results obtained by Celani in [14] for quasi-modal algebras, by Düntsch and Vakarelov in [30] for proximity algebras and by Bezhanishvili et al. in [5] for subordination algebras.

We recall that, for a Boolean algebra B, $\mathrm{Ult}(B)$ denotes its set of ultrafilters with the topology generated by the sets

$$\eta(a) = \{x \in \mathrm{Ult}(b) \mid a \in x\}$$

for $a \in B$ and that, for a Stone space X, $\mathrm{Clop}(X)$ denotes its set of clopen subsets. Also, the unit η and the co-unit ε of Stone duality are given by

$$\eta : B \longrightarrow \mathrm{Clop}((\mathrm{Ult})(B)) : a \longmapsto \eta(a)$$

and

$$\varepsilon : X \longrightarrow \mathrm{Ult}(\mathrm{Clop}(X)) : x \longmapsto \{O \in \mathrm{Clop}(X) \mid x \in O\}.$$

Definition 2.1.1. A **subordination algebra** is a structure $\mathfrak{B} = (B, \prec)$ where B is a Boolean algebra and $\prec$ is a **subordination relation** on B, that is a binary relation on B such that:

S1. $0 \prec 0$ and $1 \prec 1$,

S2. $a \prec b, c$ implies $a \prec b \wedge c$,

S2'. $b, c \prec a$ implies $b \vee c \prec a$,

S3. $a \leq b \prec c \leq d$ implies $a \prec d$.

Note that an arbitrary binary relation $\prec$ on a Boolean algebra B is a subordination relation if and only if for all $a \in B$, the subset $\prec(a, -) := \{b \in B \mid a \prec b\}$ is a filter and the subset $\prec(-, a) := \{b \in B \mid b \prec a\}$ is an ideal.

Definition 2.1.2. A **subordination space** is a topological structure $\mathfrak{X} = (X, R)$ where X is a Stone space and R is a closed binary relation on X. Sometimes we will call the relation R the **accessibility relation** of $\mathfrak{X}$.

Lemma 2.1.3. *Let $\mathfrak{X} = (X, R)$ be a subordination space and F a closed subset of X. Then $R(-, F)$ and $R(F, -)$ are closed subsets of X.*

Proof. Suppose that $y \notin R(F, -)$, i.e. $(x, y) \notin R$ for all $x \in F$. Since R is closed, there exist open sets O_x and U_x such that

$$(x, y) \in O_x \times U_x \subseteq R^c.$$

Now, of course $\{O_x : x \in F\}$ is an open cover of F, which is closed. Hence, there exists $S \subseteq F$ finite with $F = \cup\{O_x \mid x \in S\}$. We just have to remark that

$$y \in \cap\{U_x \mid x \in S\} \subseteq R(F, -)^c$$

to conclude.

Furthermore, $R(-, F)$ is proved to be closed with a symmetric proof. □

The next proposition was first stated by Esakia in [32], an restated by Sambin and Vaccaro in [65] for modal spaces. But, since it only requires the accessibility relation to be closed, it is immediately applicable to subordination spaces.

Proposition 2.1.4 (Esakia Lemma)**.** *Let $\mathfrak{X} = (X, R)$ be a subordination space and $(F_i \mid i \in I)$ a filtered family of non-empty closed sets of X. Then,*

$$R(-, \cap F_i) = \cap R(-, F_i) \text{ and } R(\cap F_i, -) = \cap R(F_i, -).$$

Proof. Let us prove the case $R(-,\cap F_i) = \cap R(-,F_i)$, the other one being exactly symmetrical. Notice that we have immediately $R(-,\cap F_i) \subseteq \cap R(-,F_i)$.

To prove the other inclusion, suppose that $x \in \cap R(-,F_i)$. Then for all $i \in I$, we know that $x\ R\ y_i$ for some $y_i \in F_i$. Therefore, $R(x,-) \cap F_i \neq \emptyset$ for all $i \in I$. Now, since $R(x,-)$ is closed and $(F_i \mid i \in I)$ is filtered, we can conclude that

$$\{R(x,-)\} \cup (F_i \mid i \in I)$$

is a family of closed subsets of X whose every finite intersection is non-empty. By compactness of X, we get that

$$R(x,-) \cap \cap F_i \neq \emptyset.$$

Hence, there exists $y \in \cap F_i$ such that $x\ R\ y$. In other words, we have $x \in R(-,\cap F_i)$ as required. □

Proposition 2.1.5. *1. Let $\mathfrak{B} = (B,\prec)$ be a subordination algebra. Then $\mathfrak{X} = (X_B, R_\prec)$, with $X_B = \mathrm{Ult}(B)$ and $R_\prec$ the binary relation defined by*

$$x\ R_\prec\ y \Leftrightarrow {\prec}(y,-) \subseteq x$$

is a subordination space.

2. Let $\mathfrak{X} = (X,R)$ be a subordination space. Then $\mathfrak{B} = (B_X, \prec_R)$ with $B_X = \mathrm{Clop}(X)$ and $\prec_R$ the binary relation defined by

$$O \prec U \Leftrightarrow R(-,O) \subseteq U$$

is a subordination algebra.

Proof. 1. We have to prove that $R_\prec$ is closed. Suppose that $(x,y) \notin R_\prec$. Then, we have $b \in y$, $b \prec a$ and $\neg a \in x$ for some $a, b \in A$. It follows that

$$(x,y) \in \eta(\neg a) \times \eta(b) \subseteq R^c.$$

2. Immediately from the definition, we have that $\prec_R$ verifies (S1), (S2), (S2') and (S3).

□

Of course, for $\prec_R$ and $R_\prec$ to actually define a duality, we should check if the subordination algebras $(B,\prec)$ and $(\mathrm{Clop}(\mathrm{Ult}(B)), \prec_{R_\prec})$, as well as the subordination spaces (X,R) and $(\mathrm{Ult}(\mathrm{Clop}(X)), R_{\prec_R})$, are somehow "isomorphic". While the definitions of the "isomorphic" concept may not be the cause of surprise, we will discuss them in the next sections. Yet, Proposition 2.1.6 is a first step in this direction.

Proposition 2.1.6. *1. Let $\mathfrak{B} = (B,\prec)$ be a subordination space and η the unit of Stone duality. Then*

$$a \prec b \Leftrightarrow \eta(a) \prec_{R_\prec} \eta(b)$$

2. Let $\mathfrak{X} = (X,R)$ be a subordination space and ε the co-unit of Stone duality. Then

$$x\ R\ y \Leftrightarrow \varepsilon(x)\ R_{\prec_R}\ \varepsilon(y).$$

Proof. 1. Suppose that $a \prec b$ and that $x \in R_{\prec}(-, \eta(a))$. If follows that there exists $y \in \eta(a)$ with $\prec(y, -) \subseteq x$. Thus, $b \in \prec(a, -) \subseteq \prec(y, -) \subseteq x$ and x is an element of $\eta(b)$ as required.

Suppose now that $a \not\prec b$. Hence, we have $\prec(a, -) \cap \downarrow b = \emptyset$. Since $\prec(a, -)$ is a filter and $\downarrow b$ is an ideal, there exists an ultrafilter x such that $\prec(a, -) \subseteq x$ and $\neg b \in x$. From $\prec(a, -) \subseteq x$, we get $\prec(-, x^c) \cap \uparrow a = \emptyset$ and, therefore, $\prec(-, x^c) \subseteq y^c$ and $a \in y$ for some ultrafilter y. Now, $\prec(-, x^c) \subseteq y^c$ is equivalent to $\prec(y, -) \subseteq x$. In short, we have an ultrafilter x such that $x \in \eta(\neg b)$ and $x \in R_{\prec}(-, \eta(a))$, whence $R_{\prec}(-, \eta(a)) \not\subseteq \eta(b)$ as required.

2. Suppose that $x \; R \; y$ and $O \in \prec_R(\varepsilon(y), -)$, i.e. there exists a clopen U such that $y \in U$ and $R(-, U) \subseteq O$. It implies $x \in O$ and, hence, $O \in \varepsilon(x)$.

On the other hand, if $x \not R \; y$, since R is closed, there exists clopen sets O and U with

$$(x, y) \in O \times U \subseteq R^c.$$

In other words, we have $R(-, U) \subseteq O^c$, $y \in U$ and $x \notin O^c$. Henceforward, $\prec_R(\varepsilon(y), -) \not\subseteq \varepsilon(x)$.

□

Remark 2.1.7. In the previous proof, we used a useful equivalence which is worth noticing. Let X be any set equipped with a binary relation R and let O, U be any subsets of X, then

$$R(-, O) \subseteq U \Leftrightarrow R(U^c, -) \subseteq O^c \Leftrightarrow U^c \times O \subseteq R^c.$$

Remark 2.1.8. Actually, if X is a Stone space and if R is any binary relation on X (not necessarily closed), then $(B_x, \prec_R)$ defined as above is still a subordination algebra. However, now we have

$$x \; \overline{R} \; y \Leftrightarrow \varepsilon(x) \; R_{\prec_R} \; \varepsilon(y),$$

where $\overline{R}$ is the adherence of R in X^2.

Definition 2.1.9. An example of subordination algebras is given by **contact algebras** (see for instance [31] and [51]). They are subordination algebras which satisfy the axioms S4 (reflexivity), S5 (symmetry), S7 (extensionality):

S4. $a \prec b$ implies $a \leq b$,

S5. $a \prec b$ implies $\neg b \prec \neg a$,

S7. $\prec(-, a) = \prec(-, b)$ implies $a = b$.

Of course, another example of subordination algebras are given by de Vries algebras: they are Boolean complete contact algebras verifying axiom S6 (transitivity):

S6. $a \prec b$ implies $a \prec c \prec b$ for some c.

Remark 2.1.10. A careful reader might notice that the statements of axioms S7 and dv7 of Definition 1.2.1 differ. Yet, they are equivalent for a subordination relation satisfying S4. Moreover, if $(B, \prec)$ satisfies also S5, then S7 is equivalent to

S7'. $\prec(a, -) = \prec(b, -)$ implies $a = b$.

The names of the additional axioms S4 to S6 are not anodyne. Indeed, we have the following equivalences.

Proposition 2.1.11. *Let $(B, \prec)$ be a subordination algebra. Then:*

1. *$\prec$ satisfies S4 if and only if $R_\prec$ is reflexive,*
2. *$\prec$ satisfies S5 if and only if $R_\prec$ is symmetric,*
3. *$\prec$ satisfies S6 if and only if $R_\prec$ is transitive,*

Proof. 1. If $\prec$ satisfies S4, then we have $\prec(x, -) \subseteq {\uparrow}x = x$.

Now if $\prec$ does not satisfies S4, there exist a, b with $a \prec b$ and $a \not\leq b$. Hence, $a \in x$ and $b \notin x$ for some ultrafilter x. In particular, this implies that $\prec(x, -) \nsubseteq x$ for at least one ultrafilter x and, so, that R is not reflexive.

2. Suppose that $\prec$ satisfies S5 and let x, y be ultrafilters with $\prec(y, -) \subseteq x$ (which is equivalent to $\prec(-, \neg x) \subseteq \neg y$). If $a \in \prec(x, -)$, then $b \prec a$ for some $b \in x$. By S5, we have $\neg a \prec \neg b$ for some $\neg b \in \neg x$, that is $\neg a \in \prec(-, \neg x) \subseteq \neg y$. Hence, we have $a \in y$ as required.

   On the other hand, if $\prec$ does not satisfies S5, then there exist a, b such that $a \prec b$ and $\neg b \not\prec \neg a$. It follows that $\prec(\neg b, -) \cap {\downarrow}\neg a = \emptyset$ and, hence, that $\prec(\neg b, -) \subseteq x$ and $a \in x$ for some ultrafilter x. Since $\prec(\neg b, -) \subseteq x$ is equivalent to ${\uparrow}\neg b \cap \prec(-, \neg x) = \emptyset$, we can draw the existence of an ultrafilter y with $\neg b \in y$ and $\prec(-, \neg x) \subseteq \neg y$, that is $x\ R\ y$. Now, we have that $y \not R\ x$, because $b \in \prec(a, -) \subseteq \prec(x, -)$ but $b \notin y$.

3. Suppose that $\prec$ satisfies S6 and let x, y and z be ultrafilters such that $\prec(z, -) \subseteq y$ and $\prec(y, -) \subseteq x$. We have to prove that $\prec(z, -) \subseteq x$. So, suppose that $a \prec b$ for some $a \in z$. Then, $a \prec c \prec b$ for some c. But $c \in \prec(z, -) \subseteq y$ and hence $b \in \prec(y, -) \subseteq x$.

   Now if $\prec$ does not verify S6, then there exist a and b such that $a \prec b$ and, for all c, we have $a \not\prec c$ or $c \not\prec b$. This means that $\prec(a, -) \cap \prec(-, b) = \emptyset$ and, hence, that there exists an ultrafilter x with $\prec(a, -) \subseteq x$ and $\prec(b, -) \subseteq \neg x$. It follows from the first inclusion that $a \in z$ and $\prec(z, -) \subseteq x$ for some z and, from the second one, that $b \notin y$ and $\prec(x, -) \subseteq y$ for some y. In short, there exist ultrafilters x, y and z such that $y\ R\ x\ R\ z$ and $y \not R\ z$, since $b \in \prec(a, -) \subseteq \prec(z, -)$ and $b \notin y$.

□

Remark 2.1.12. Usually another axiomatisation is used for contact algebras. Indeed, the conventional relation in contact environment is the contact relation $\mathcal{C}$, which can be defined as

$$a\ \mathcal{C}\ b \Leftrightarrow a \not\prec \neg b.$$

In Appendix C, correspondences between subordination axioms and usual contact axioms are given.

Example 2.1.13. We already saw the similarity between the constructions of the accessibility relation in the modal and in the subordination case. But it is time to say that subordination algebras are, in two different ways, a generalisation of modal algebras. Indeed, consider a modal algebra $\mathfrak{B} = (B, \Diamond)$ and set $\prec_\Diamond$ and $\prec_\blacklozenge$ as the following relations:

$$a \prec_\Diamond b \text{ iff } \Diamond a \leq b \text{ and } a \prec_\blacklozenge b \text{ iff } a \leq \Box b.$$

It is not hard to prove that both $(B, \prec_\Diamond)$ and $(B, \prec_\blacklozenge)$ are subordination algebras. The denomination $\prec_\blacklozenge$ may seem surprising at first glance but will be fully realised with the next example.

Example 2.1.14. Let us consider the case of **tense algebras** (or **temporal algebras** ([60]). A tense algebra is a triple $\mathfrak{B} = (B, \Diamond, \blacklozenge)$, where $(B, \Diamond)$ and $(B, \blacklozenge)$ are modal algebras, such that one of the following equivalent conditions is satisfied:

1. for every $a \in B$ and for $\blacksquare$ the dual operator of $\blacklozenge$, we have $a \leq \blacksquare\Diamond a$ and $\Diamond\blacksquare a \leq a$,
2. for every $a, b \in B$, we have $\Diamond a \leq b$ if and only if $a \leq \blacksquare b$,
3. for every $a, b \in B$, we have $\Diamond a \leq b$ if and only if $\blacklozenge\neg b \leq \neg a$.

Of course, tense algebras regroup in a category, denoted by **TensAlg**, whose morphisms are Boolean morphisms that respects both modal operators $\Diamond$ and $\blacklozenge$.

Let $\mathfrak{B} = (B, \Diamond, \blacklozenge)$ be a tense algebra and consider the subordination relation $\prec_\Diamond$ associated to $(B, \Diamond)$ and the one $\prec_\blacklozenge$ associated to $(B, \blacklozenge)$. In this case, we have that, for all $a, b \in B$

$$a \prec_\Diamond b \text{ iff } a \prec_\blacklozenge b.$$

Indeed, we have

$$\begin{aligned} & a \prec_\Diamond b \\ \Leftrightarrow\ & \Diamond a \leq b \\ \Rightarrow\ & \blacksquare\Diamond a \leq \blacksquare b \\ \Rightarrow\ & a \leq \blacksquare b \Rightarrow a \prec_\blacklozenge b. \end{aligned}$$

By a symmetric argument, we obviously have $a \prec_\blacklozenge b$ implies $a \prec_\Diamond b$.

Note that it was shown in [72] that a bimodal algebra $\mathfrak{B} = (B, \Diamond, \blacklozenge)$ was indeed a tense algebra if the accessibility relation associated to $\Diamond$ was the converse of the accessibility relation associated to $\blacklozenge$, that is

$$x\ R_\Diamond\ y \Leftrightarrow y\ R_\blacklozenge\ x.$$

2.2 Subordination morphisms

To investigate on subordination algebras in a categorical environment requires a suitable definition for morphisms, keeping in mind that our goal is to obtain a category that reduces to the category **ModAlg** of modal algebras (described in Appendix B.3). We already saw a possible definition of morphisms in Chapter 1, the one of de Vries morphisms, but, as we already stated, they are extremely weak morphisms, as they are not even Boolean. In [5, Definition 2.7], the authors considered morphisms that respect the subordination relation $\prec$ "weakly", but to actually obtain a category appropriate to our goal, we shall consider morphisms analogous to the ones of Celani in [14, Definition 8].

Definition 2.2.1. Let $\mathfrak{B} = (B, \prec)$ and $\mathfrak{C} = (C, \prec)$ be subordination algebras and $h : B \longrightarrow C$ a Boolean morphism. We consider the following axioms:

(w) $a \prec b$ implies $h(a) \prec h(b)$,

($\Diamond$) $h(a) \prec c$ implies $a \prec b$ and $h(b) \leq c$ for some b,

($\blacklozenge$) $c \prec h(a)$ implies $b \prec a$ and $c \leq h(b)$ for some b.

With these axioms, we can consider four different categories whose objects are subordination algebras:

1. the category **wSubAlg** , whose morphisms are **weak morphism**, that is Boolean morphisms satisfying (w),
2. the category $\Diamond$**SubAlg**, or more simply **SubAlg**, endowed with **white morphisms**, that is Boolean morphisms satisfying (w) and ($\Diamond$),
3. the category $\blacklozenge$**SubAlg**, with **black morphisms** which satisfy (w) and ($\blacklozenge$),
4. the category **sSubAlg**, whose morphisms are **strong morphisms**, i.e. Boolean morphisms that satisfy all three axioms.

Let us have a look at how these categories interact with each other, with **ModAlg** and with **DeV**.

Proposition 2.2.2. *Let $\mathfrak{B} = (B, \prec)$ and $\mathfrak{C} = (C, \prec)$ be subordination algebras and $h : B \longrightarrow C$ be a Boolean morphism. Then, h is an isomorphism in* **wSubAlg**, **SubAlg**, $\blacklozenge$**SubAlg** *and* **sSubAlg** *if and only if h is a Boolean isomorphism satisfying*

$$a \prec b \Leftrightarrow h(a) \prec h(b).$$

Hence, **wSubAlg**, **SubAlg**, $\blacklozenge$**SubAlg** *and* **sSubAlg** *share the same isomorphisms.*

Proof. Suppose first that h is an isomorphism in any of the mentioned categories. Then, there exists a morphism g such that $h \circ g$ and $g \circ h$ are identity morphisms. Hence, h is a Boolean isomorphism and we have

$$h(a) \prec h(b) \Rightarrow a = g(h(a)) \prec g(h(b)) = b.$$

On the other hand, we just have to prove that if $h : \mathfrak{B} \longrightarrow \mathfrak{C}$ is a Boolean isomorphism with $a \prec b \Leftrightarrow h(a) \prec h(b)$, then h and h^{-1} satisfy all three axioms (w), ($\Diamond$) and ($\blacklozenge$).

- (w) By hypothesis, h has the (w) property. Now, let $c, d \in C$ with $c \prec d$. Since h is a bijection, $c = h(a)$ and $d = h(b)$ for some $a, b \in B$. Then, it follows from $h(a) \prec h(b)$ that $a \prec b$, that is $h^{-1}(c) \prec h^{-1}(d)$.
- ($\Diamond$) If $h(a) \prec c$, then there exists $b \in B$ with $h(b) = c$. And, by hypothesis, we have $a \prec b$.

  Now, if $h^{-1}(c) \prec b$, then $c \prec h(b)$ and there exists $a \in B$ with $c = h(a)$. By letting d denote $h(b)$, we obtain from $a \prec b$ that $c \prec d$ and since $h^{-1}(d) = b$, the proof is concluded.
- ($\blacklozenge$) The proof is similar to the white case.

□

Proposition 2.2.3. *The categories* **SubAlg** *and* $\blacklozenge$**SubAlg** *are isomorphic and are subcategories of* **wSubAlg**. *Furthermore,* **sSubAlg** *is a subcategory of both* **SubAlg** *and* $\blacklozenge$**SubAlg**.

Proof. Only the isomorphism between **SubAlg** and $\blacklozenge$**SubAlg** requires a proof. The isomorphic functor maps a subordination algebra $\mathfrak{B} = (B, \prec)$ to $\mathfrak{B} = (B, \prec')$ and a morphism to itself, where $\prec'$ is the dual subordination relation defined by $a \prec' b$ if and only if $\neg b \prec \neg a$. □

Remark 2.2.4. The quasi-modal algebras category defined by Celani in [14] corresponds to our black category (see Appendix C) and the subordination algebra category defined in [5] is exactly our weak category. Note that **DeV** is not a subcategory of any the four categories, of course because it has non Boolean morphisms. For **ModAlg**, we have the following proposition.

Proposition 2.2.5. *The category* **ModAlg** *is a full subcategory of* **SubAlg** *and* ⧫**SubAlg**. *The category* **TensAlg** *is a full subcategory of* **sSubSalg**.

Proof. Suppose that $(B, \Diamond)$ and $(C, \Diamond)$ are modal algebras. First, if $h : B \longrightarrow C$ is a modal morphism, from the definitions of $\prec_\Diamond$ and $\prec_\blacklozenge$, we have immediately that h is a white morphism between $(B, \prec_\Diamond)$ and $(C, \prec_\Diamond)$ and a black morphism between $(B, \prec_\blacklozenge)$ and $(C, \prec_\blacklozenge)$.

Second, suppose that $h : (B, \prec_\Diamond) \longrightarrow (C, \prec_\Diamond)$ is a white morphism. On one hand, we have that $a \prec_\Diamond \Diamond a$ and, by (w), this implies that $h(a) \prec_\Diamond h(\Diamond a)$, that is $\Diamond h(a) \leq h(\Diamond a)$. On the other hand, from $h(a) \prec_\Diamond \Diamond h(a)$ and $(\Diamond)$, we get that $a \prec_\Diamond b$ for some b, that is $\Diamond a \leq b$, and $h(b) \leq \Diamond h(a)$. Now, since h is monotone, we have $h(\Diamond a) \leq h(b) \leq \Diamond h(a)$ and, consequently, $h(\Diamond a) = \Diamond h(a)$.

Third, if $h : (B, \prec_\blacklozenge) \longrightarrow (B, \prec_\blacklozenge)$ is a black morphism, following a path analogue to the one used in the white case, one proves easily that $h(\Box a) = \Box\, h(a)$ for all $a \in B$.

Finally, the case **TensAlg** is a mere consequence of the two previous cases. ☐

Definition 2.2.6. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, r)$ be subordination spaces and $f : X \longrightarrow Y$ a continuous function. Consider the following axioms:

(w) $x \mathrel{R} y$ implies $f(x) \mathrel{R} f(y)$,

$(\Diamond)$ $f(x) \mathrel{R} y$ implies $x \mathrel{R} z$ and $f(z) = y$ for some z,

$(\blacklozenge)$ $y \mathrel{R} f(x)$ implies $z \mathrel{R} x$ and $f(z) = y$ for some z.

As in the algebraic case, we consider four different categories whose objects are subordination spaces:

1. the category **wSubSp**, whose morphisms are **weak functions**, that is continuous functions satisfying axiom (w),
2. the category $\Diamond$**SubSp**, or more simply **SubSp**, whose morphisms, called **white functions**, are continuous functions which satisfy axioms (w) and $(\Diamond)$,
3. the category ⧫**SubSp** whose morphisms are continuous functions verifying (w) and $(\blacklozenge)$ and are called **black functions**,
4. the category **sSubSp** whose morphisms are **strong functions**, continuous functions that satisfy all three axioms.

Remark 2.2.7. In general, a map that satisfies the conditions (w) and $(\Diamond)$ is called a **p-morphism** (see for instance [9]). To prevent any possible confusion between morphisms of subordination algebras and functions of subordination spaces, we chose to not use this usual denomination.

Proposition 2.2.8. *Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be subordination spaces and $f : X \longrightarrow Y$ be a continuous function. Then f is an isomorphism in* **wSubSp**, **SubSp**, ⧫**SubSp** *and* **sSubSp** *if and only if f is a Boolean isomorphism satisfying*

$$x \mathrel{R} y \Leftrightarrow f(x) \mathrel{R} f(y).$$

Hence, **wSubSp**, **SubSp**, ⧫**SubSp** *and* **sSubSp** *share the same isomorphisms.*

Proof. The proof follows the same lines as the one of the algebraic case. ☐

We have the expected dualities between subordination spaces and subordination algebras. It was proved in [5] for the weak case and in [14] for the black one. We extended it to the strong case.

Theorem 2.2.9. *The functors* Ult *and* Clop *of Stone duality establish a duality between* **wSubAlg** *and* **wSubSp**, *that reduces a to duality between* **SubAlg** *and* **SubSp**, **♦SubAlg** *and* **♦SubSp** *and, finally, between* **sSubAlg** *and* **sSubSp**.

Proof. Most of the proof follows immediately from Stone duality and from Proposition 2.1.6 which states that η and ε are isomorphism in their respective categories. Hence, we just need to prove that the algebraic axioms (w), ($\Diamond$) and ($\blacklozenge$) correspond to their respective topological counterparts. Let $\mathfrak{B} = (B, \prec)$ and $\mathfrak{C} = (C, \prec)$ be subordination algebra and $h : B \longrightarrow C$ be a Boolean morphism.

(w) Suppose that h is weak and that x, y are elements of $Y = \mathrm{Ult}(C)$ with $x\ R\ y$. We need to prove that we have $h^{-1}(x)\ R\ h^{-1}(y)$, that is $\prec(h^{-1}(y), -) \subseteq h^{-1}(x)$. Let $a \prec b$ for an a such that $h(a) \in y$. Then, by weakness of h, we have

$$h(b) \in \prec(h(a), -) \subseteq \prec(y, -) \subseteq x,$$

that is $b \in h^{-1}(x)$.

Suppose now that $\mathrm{Ult}(h)$ is weak and that we have $a \prec b$. Then, it follows from Proposition 2.1.6 that $R(-, \eta(a)) \subseteq R(-, \eta(b))$. Let us prove that $R(-, \eta(h(a))) \subseteq \eta(h(b))$. Let x and y be ultrafilters of C such that $\prec(y, -) \subseteq x$ and $h(a) \in y$. Then, by weakness of $\mathrm{Ult}(h)$, we have $\prec(h^{-1}(y), -) \subseteq h^{-1}(x)$. And, since we have $b \in \prec(h^{-1}(y), -)$, it follows that $h(b) \in x$, as required.

($\Diamond$) Suppose that h is a white morphism and that $x \in X = \mathrm{Ult}(B)$ and $y \in Y = \mathrm{Ult}(C)$ are such that $\mathrm{Ult}(h)(y)\ R\ x$, that is $\prec(x, -) \subseteq h^{-1}(y)$. A quick proof shows that this inclusion implies

$$\prec(-, \neg y) \cap \uparrow h(x) = \emptyset. \tag{2.1}$$

Indeed, otherwise there would exist $d \in y$ and $c \in C$ such that we have $h(a) \leq c \prec \neg d$ for some $a \in x$. Since h is white, we would have $a \prec b$ and $h(b) \leq \neg d$ for some $b \in B$ and, therefore, $b \in \prec(x, -) \cap h^{-1}(\neg y)$, which is absurd. Hence, from (2.1), we can conclude that there exists an ultrafilter z of C such that $\uparrow h(x) \subseteq z$, that is $h^{-1}(z) = x$, and $\prec(-, \neg y) \subseteq \neg z$, that is $y\ R\ z$.

Suppose now that $\mathrm{Ult}(h)$ is white and let $a \in B$ and $c \in C$ be such that $h(a) \prec c$. As we already point out, by Proposition 2.1.6, this is equivalent to $R(-, \eta(h(a))) \subseteq \eta(c)$. Let us prove that

$$R(-, \eta(a)) \cap \mathrm{Ult}(h)(\eta(c)^c) = \emptyset.$$

If not, there exist ultrafilters $x \in \eta(c)^c$ and $y \in \eta(a)$ which satisfy $\mathrm{Ult}(h)(x)\ R\ y$. By whiteness, there is an ultrafilter z such that $x\ R\ z$, that is $\prec(z, -) \subseteq x$ and $\mathrm{Ult}(h)(z) = y$, that is $h^{-1}(z) = y$. But we have $h(a) \in z$ and hence $c \in x$, which contradicts $x \in \eta(c)^c$. Hence, there exists a clopen $\eta(b)$ such that $R(-, \eta(a)) \subseteq \eta(b)$, that is $a \prec b$, and $\eta(b) \subseteq h^{-1}(\eta(c)) = \eta(h(c))$, that is $b \leq h(c)$, as required.

($\blacklozenge$) Suppose that h is black and let x and y be ultrafilters of X and Y respectively such that $x\ R\ \mathrm{Ult}(h)(y)$. To prove that there exists z such that $x = \mathrm{Ult}(h)(z)$ and $z\ R\ y$, one can adapt the proof of the white case, considering, this time that

$$\downarrow h(y^c) \cap \prec(x, -) = \emptyset.$$

Now, if $\mathrm{Ult}(h)$ is black and if $a \in B$ and $c \in C$ satisfy $c \prec h(a)$, we can once more exploit the white proof to find an element b with $c \leq h(b)$ and $b \prec a$, by observing, this time, that

$$R(\eta(a)^c, -) \cap \mathrm{Ult}(h)(\eta(c)) = \emptyset.$$

$\square$

Remark 2.2.10. Let us mention that, aside of functors Ult and Clop, there exist other functors that establish equivalence between the algebraic categories and the topological ones. While their actions on morphisms remain unchanged, their behaviours in confront of the objects are different.

First, there is the functor $\blacklozenge$ Ult, which maps a subordination algebra $(B, \prec)$ to the subordination space $(\mathrm{Ult}(B), R^{\blacklozenge}_{\prec})$ with

$$x \; R^{\blacklozenge}_{\prec} \; y \Leftrightarrow \prec(x, -) \subseteq y.$$

Then, there is the functor $\blacklozenge$ Clop, which maps a subordination space (X, R) to the subordination algebra $(\mathrm{Clop}(X), \prec^{\blacklozenge}_R)$ with

$$O \prec^{\blacklozenge}_R \Leftrightarrow R(O, -) \subseteq U.$$

In short, the relations $R^{\blacklozenge}_{\prec}$ and $\prec^{\blacklozenge}_R$ are symmetrical to the relations $R_{\prec}$ and $\prec_R$. This is the way the duality is generally expressed in the contact algebras situation (see for instance [1] or [66]). It is not hard, using symmetry, to be convinced that these functors also establish a duality.

Finally, even though the next proposition could be demonstrated immediately without any difficulties, we also have it as a direct corollary of Theorem 2.2.9 and Propositions 2.2.3 and 2.2.5.

Proposition 2.2.11. *The categories* **SubSp** *and* $\blacklozenge$**SubSp** *are isomorphic and are subcategories of* **wSubSp**. *Furthermore,* **sSubSp** *is a subcategory of both* **SubSp** *and* $\blacklozenge$**SubSp** *and, finally, the category* **ModSp** *is a full subcategory of* **SubSp** *and* $\blacklozenge$**SubSp**.

2.3 Complete atomic subordination algebras

We observed in Proposition 2.2.5 that modal algebras are, in a way, a particular case of subordination algebras. Furthermore, the converse is not true, as, for instance, the dual algebra of a subordination space whose relation is simply closed is not a modal one. Nevertheless, in precise circumstances, a subordination algebra could be endowed with a modal structure. Namely, a subordination algebra $(B, \prec)$ is a modal algebra if and only if for every $a \in B$, the set $\prec(a, -)$ is a principal filter, that is, there exists an element, denoted $\Diamond a$, such that

$$\prec(a, -) = \uparrow \Diamond a.$$

Let us note that a subordination relation that respects (the order-dual of) this condition is rightfully named **modally definable** in [5].

In this section, we will examine a particular subcategory of **SubAlg** whose objects will be modal algebras and that, in addition, yields a discrete duality with the category of Kripke structures.

In order to ease future definitions, we recall the definitions of atoms and atomic Boolean algebras.

Definition 2.3.1. Let B be a Boolean algebra.

1. An element $\alpha \in B$ is an **atom** if $\alpha \neq 0$ and $b < \alpha$ implies $b = 0$. We denote by $\mathrm{At}(B)$ the set of atoms of B.
2. We say that B is **atomic** if for every $a \in B$, $\alpha \leq a$ for some atom α.

We now have these well known properties.

Proposition 2.3.2. *Let B be an atomic Boolean algebra and α an atom of B.*

1. *The set $\uparrow\alpha$ is an ultrafilter.*
2. *For every $a \in B$, $a = \vee\{\alpha : a \geq \alpha \in \mathrm{At}(b)\}$.*
3. *For every $S \subseteq B$ with $\alpha \leq \vee S$, there exist $s \in S$ with $\alpha \leq s$.*

Proof. 1. It is clear that $\uparrow\alpha$ is a filter. Now, if $a \in A$ with $a \notin \uparrow\alpha$, we have

$$0 \neq \alpha \wedge \neg a \leq \alpha.$$

Therefore, $\alpha \wedge \neg a = \alpha$ and $\neg a$ is then an element of $\uparrow\alpha$, which is consequently an ultrafilter.

2. Clearly, a is an upper bound of $\{\alpha : a \geq \alpha \in \mathrm{At}(B)\}$. Now let $b \geq \alpha$ for all $\alpha \leq a$ and suppose that $a \not\leq b$. It follows that $a \wedge \neg b \neq 0$ and, hence, that there exists an atom β with $\beta \leq a \wedge \neg b$. In particular, we draw that $\beta \leq a$, hence, that $\beta \leq b$, and that $\beta \leq \neg b$. This is impossible since it would imply that $\beta = 0$.

3. Suppose that $\alpha \not\leq s$ for all $s \in S$. Since $\uparrow\alpha$ is an ultrafilter, it follows that $\alpha \leq \neg s$ for all $s \in S$. Therefore, we have that

$$\alpha \leq \wedge\{\neg s : s \in S\} = \neg(\vee S),$$

and, consequently, that $\alpha \not\leq \vee S$.

$\square$

Definition 2.3.3. Let $\mathfrak{B} = (B, \prec)$ be a subordination algebra. We say that $\mathfrak{B}$ is **complete atomic** if:

1. B is a complete Boolean algebra,
2. B is atomic,
3. $\prec$ is a **complete subordination relation**, that is it satisfies, for any $S \subseteq B$:

   CS2. $a \prec s$ for all $s \in S$ implies $a \prec \wedge S$,

   CS2'. $s \prec a$ for all $s \in S$ implies $\vee S \prec a$.

As in the non-complete atomic case, we have to consider four categories whose objects are complete atomic subordination algebras: the categories **wCASAlg**, **CASAlg**, **⧫CASAlg** and **sCASAlg** which are constructed alongside the categories **wSubAlg**, **SubAlg**, **⧫SubAlg** and **wSubAlg**, with arrows given by complete Boolean morphisms instead of all Boolean morphisms.

As we foreshadowed in the introduction, complete atomic subordination algebra and complete atomic modal algebra are equivalent concepts. Indeed, axiom CS2 implies that $\prec(a, -)$ is a principal filter, so that there exists $\Diamond a$ with

$$\prec(a, -) = \uparrow\Diamond a,$$

while axiom CS2' implies that the map $a \longmapsto \Diamond a$ is a complete operator, i.e. it commutes with arbitrary infima. Order dually, axiom CS2' signifies that $\prec(-, a)$ is a principal ideal, i.e. there exists an element, denoted by $\blacksquare a$, such that

$$\prec(-, a) = \downarrow\blacksquare a,$$

whereas CS2 means that the map $a \longmapsto \blacksquare a$ is a complete dual operator, i.e. it commutes with arbitrary suprema. On the other hand, it is easy to prove that if $\mathfrak{B} = (B, \Diamond)$ is a complete modal algebra, its associated subordination algebra is complete. Henceforth, we have the following immediate proposition.

Proposition 2.3.4. *The categories* **CASALg**, $\blacklozenge$**CASALg** *and* **CAMALg***, whose objects are complete modal algebras with complete modal morphisms, are isomorphic.*

Remark 2.3.5. Let us note that, if $\mathfrak{B} = (B, \prec)$ is a complete subordination algebra and if $\Diamond$ and $\blacksquare$ are the operators we just obtained from $\prec$, then $(B, \Diamond, \blacklozenge)$ is a tense algebra. Indeed, for all $a \in A$, we have that $\prec(a, -) = \uparrow\Diamond a$ and, hence, $a \prec \Diamond a$. Now, since

$$a \in \prec(-, \Diamond a) = \downarrow(\blacksquare\Diamond a),$$

it is clear that $a \leq \blacksquare\Diamond a$. Similarly, we also have $\Diamond\blacksquare a \leq a$ as required.

As an immediate consequence of Proposition 2.3.4, we get the duality between complete atomic subordination algebras and Kripke structures. This is a well known duality (see for instance [16]) and we recall it here for the sake of completeness.

Definition 2.3.6. A **Kripke structure**[1] is a pair $\mathfrak{X} = (X, R)$ were X is a set and R is a binary relation on X.

Just as in the topological situation, Kripke structures give rise to four different categories: the weak one **wKStr**, whose morphisms are maps that satisfy (w). The white category **KStr**, whose morphisms are maps that satisfy (w) and ($\Diamond$). The black category $\blacklozenge$**KStr** whose morphisms are maps satisfying (w) and ($\blacklozenge$) and finally the strong category **sKStr** whose morphisms are maps which satisfy the three axioms.

Definition 2.3.7. Let $\mathfrak{B} = (B, \prec)$ be a complete atomic subordination algebra. We endow its set of atoms $\mathrm{At}(B)$ with the binary relation R given by

$$\alpha \, R \, \beta \Leftrightarrow \prec(\beta, -) \subseteq \uparrow\alpha.$$

It is clear that $(\mathrm{At}(B), R)$ is a Kripke structure. Remark that, since $\uparrow\alpha$ is an ultrafilter and $\prec(\uparrow\beta, -) = \prec(\beta, -)$, the connection with the relation R we build previously in the topological case is clear as day.

On the other hand, if $\mathfrak{X} = (X, R)$ is a Kripke structure, we define on the power set $\mathcal{P}(X)$ the relation $\prec$ given by

$$E \prec F \Leftrightarrow R(-, E) \subseteq F.$$

With a quick verification, one proves that $(\mathcal{P}(X), \prec)$ is a complete atomic subordination algebra.

[1] Here, we chose to use the denomination Kripke structure instead of the more classical Kripke frame to avoid any possible confusion with frames in the lattice environment. Also, I take advantage of this footnote to make a mandatory reference to a well known science-fiction series. I hope you have your towel.

Definition 2.3.8. We extend the objects mapping $\mathcal{P} : \mathfrak{X} \longmapsto (\mathcal{P}(X), \prec)$ to a functor by defining $\mathcal{P}(f) = f^{-1}$ for all maps between Kripke structures. Likewise, we extend the object mapping $\mathrm{At} : \mathfrak{B} \longmapsto (\mathrm{At}(B), R)$ to a functor by sending a morphism $h : B \longrightarrow C$ between complete atomic subordination algebras to

$$\mathrm{At}(h) : \mathrm{At}(C) \longrightarrow \mathrm{At}(B) : \alpha \longmapsto \wedge\{b \in B \mid \alpha \leq h(b)\}.$$

Remark that the map $\mathrm{At}(h)$ is well-defined as, for every $\alpha \in \mathrm{At}(C)$, $\mathrm{At}(h)(\alpha)$ is an atom of B. Indeed, suppose that a is an element of B such that

$$a < \wedge \underbrace{\{b \in B \mid \alpha \leq h(b)\}}_{=S}.$$

It follows that $a \not\leq h(\alpha)$ and, hence, that $\alpha \geq \alpha \wedge h(\neg a) \neq 0$. Now, due to α being an atom, we have that $\alpha = \alpha \wedge h(\neg a)$. Therefore, $\neg a$ is an element of S and $a \leq \neg a$, which is the case if and only if $a = 0$.

Substantially, we developed here discrete (i.e. without topology) versions of the dualities given in Section 2.2. We give here the statement of the discrete counterpart of Theorem 2.2.9 without proof (see for instance [73]). Of course, it is to note that Propositions 2.2.3, 2.2.5 and 2.2.11 also have their discrete counterparts.

Theorem 2.3.9. *The functors* At *and* $\mathcal{P}$ *establish a dual equivalence between* **wCASALg** *and* **wKStr***, that reduces to a dual equivalence between* **CASALg** *and* **KStr***,* **♦CASALg** *and* **♦KStr** *and, lastly, between* **sCASALg** *and* **sKStr***.*

2.4 Canonical extension and modalisation

In this section, we will set up two constructions based on subordination algebras. The first one, the canonical extension, will play a crucial part in the definition of valuation of (bi)modal formulas on subordination algebras (See Section 2.5). The second one, the modalisation, will have applications in correspondence and Birkhoff-like theorems.

Canonical extensions of po-sets, lattices, Boolean algebras and modal algebras are a well studied concept (see for instance [29], [34], [47] and [76, Section 7]). Here are stated the required definitions and results.

Definition 2.4.1. ([34, Definition 2.1]) A **completion** of a lattice L is a pair (C, e) where C is a complete lattice and $e : L \longrightarrow C$ a lattice embedding.

An element $c \in C$ is say to be **open** (resp. **closed**) if c is the supremum (resp. the infimum) of elements in the images of L. We say that L is **dense** in C if every of its elements is both an infimum of open elements and a supremum of closed elements. We say that L is **compact** in C if for every closed element k and every open element o of C, $k \leq o$ implies $k \leq e(a) \leq o$ for some $a \in L$. Finally, we say that (C, e) is a **canonical extension** of L if L is dense and compact in C.

In the remainder of text, we will sometimes abuse notations and write C for (C, e) when it causes no confusion.

We have now the following theorem.

Theorem 2.4.2. *([34, Propositions 2.6. and 2.7]) Every lattice L has a canonical extension which is unique up to isomorphism. We will denote this unique canonical extension by L^δ.*

Remark 2.4.3. The canonical extension of a Boolean algebra B can be constructed via Stone duality whereas the one of a bounded distributive lattice L via Priestley duality (see Appendix B.1). Indeed, it can be shown (see for instance [39, Chapter 23]) that B^δ is isomorphic to $(\mathcal{P}(\mathrm{Ult}(B)), \eta)$ and L^δ is isomorphic to $(\uparrow\mathcal{P}(\mathrm{Prim}(L)), \eta)$, where $\uparrow\mathcal{P}(\mathrm{Prim}(L))$ denotes the set of the increasing subsets of $\mathrm{Prim}(L)$.

Now, let $\mathfrak{B} = (B, \prec)$ be a subordination algebra. We would like to endow its canonical extension B^δ with a complete subordination relation $\prec^\delta$ satisfying, for all $a, b \in B$,

$$a \prec b \Leftrightarrow \eta(a) \prec^\delta \eta(b). \tag{2.2}$$

Such a relation can be obtained quite easily via the previously established dualities. Indeed, we know that $a \prec b$ if and only if $R(-, \eta(a)) \subseteq \eta(b)$. Moreover, we know that for a Kripke Structure (X, R),

$$E \prec F \Leftrightarrow R(-, E) \subseteq F$$

is a complete subordination relation on $\mathcal{P}(X)$.

What we did here is actually concatenate the functor Ult of Theorem 2.2.9, the forgetful functor from **SubSp** to **KStr** and the functor $\mathcal{P}$ of Theorem 2.3.9. Hence, we have a functor, which will be denoted by $\cdot^\delta$ from **SubAlg** to **CASAlg**. We summarise this construction in the following definition.

Definition 2.4.4. Let $\mathfrak{B} = (B, \prec)$ be a subordination algebra, then its canonical extension is given by $\mathfrak{B}^\delta := (B^\delta, \prec^\delta)$ where $B^\delta = \mathcal{P}(\mathrm{Ult}(X))$ and $E \prec^\delta F$ if and only if $R(-, E) \subseteq F$.

As a natural corollary-definition of the canonical extension, we have the notion of canonicity in the setting of subordination algebras.

Definition 2.4.5. We say that φ is **canonical** if $\mathfrak{B} \models \varphi$ implies $\mathfrak{B}^\delta \models \varphi$ for all subordination algebras $\mathfrak{B}$.

Let us remark that the construction of the relation $\prec^\delta$ given in Definition 2.4.4 is not "unique". Indeed, we know that

$$R(-, E) \subseteq F \Leftrightarrow R(F^c, -) \subseteq E^c \Leftrightarrow E \subseteq R(F^c, -)^c.$$

And, unsurprisingly, this dual definition mirrors the links between the categories **SubAlg** and **♦SubAlg**. Indeed, we have that the map $\Diamond : E \mapsto R(-, E)$ is a diamond, while the map $\blacksquare$ defined as $F \mapsto R(F^c, -)^c$ is a box. Moreover, we have

$$\Diamond\eta(a) \leq \eta(b) \Leftrightarrow a \prec b \Leftrightarrow \eta(a) \leq \blacksquare\eta(b).$$

Proposition 2.4.6. *For each morphism f in* **wSubAlg** *from $\mathfrak{B}$ into a complete atomic subordination algebra $\mathfrak{C}$, there is a unique morphism $g : \mathfrak{B}^\delta \longrightarrow \mathfrak{C}$ in* **wCASAlg** *such that $g \circ \eta = f$.*

Proof. First of all, notice that this assertion does not follow from the functioriality of $\cdot^\delta$ because $\mathfrak{C}^\delta$ does not necessarily coincide with $\mathfrak{C}$ in case the latter is a complete atomic subordination algebra. Anyway, the result is well known at the Boolean level, so that we can focus on the subordination one. To reach it, the easiest way is to notice that g is necessarily the dual (in the discrete duality **wCASAlg** - **wKStr**) of the composition $h = \mathrm{Ult}(f) \circ j$, where j is the natural weak embedding $\mathrm{At}(C) \longrightarrow \mathrm{Ult}(C) : \alpha \longmapsto \alpha\uparrow$, and $\mathrm{Ult}(f)$ is the dual (in the duality **wSub** - **wSubS**) of $f : \mathfrak{B} \longrightarrow \mathfrak{C}$. So g is weak whenever f is weak since j is weak. ☐

Remark 2.4.7. Note that Proposition 2.4.6 does not extend to morphisms in **SubAlg**, meaning that g is not necessarily a morphism in **SubAlg** in this case, as seen in the case $\mathfrak{B} = \mathfrak{C}$ and f is the identity.

It is now the time to recall that, since $\mathfrak{B}^\delta$ is a complete atomic subordination algebra, it can be seen as a modal algebra as shown in Section 2.3. Therefore, it is possible to consider the modal subalgebra generated by $\eta(B)$ in B^δ.

Definition 2.4.8. Let $\mathfrak{B}$ be a subordination algebra and let η be the embedding $\mathfrak{B} \longrightarrow \mathfrak{B}^\delta$. The (white) modal subalgebra of $\mathfrak{B}^\delta$ generated by $\eta(B)$ is called the **modalisation** of $\mathfrak{B}$ and it is denoted by $\mathfrak{B}^m$ and we have the following result.

Proposition 2.4.9. *The object mapping $\mathfrak{B} \longrightarrow \mathfrak{B}^m$ can be extended to a covariant functor $\cdot^m$:* **SubAlg** $\longrightarrow$ **ModAlg**. *The natural map $\eta : \mathfrak{B} \longrightarrow \mathfrak{B}^m$ is an embedding such that $a \prec b$ if and only if $\Diamond a \leq b$.*

Proof. Suppose that $f : \mathfrak{B} \longrightarrow \mathfrak{C}$ is a morphism in **SubAlg**. By the canonical extension functor, f lifts to $f^\delta : \mathfrak{B}^\delta \longrightarrow \mathfrak{C}^\delta$ in **CAMAlg**. Let f^m be the restriction of f^δ to $\mathfrak{B}^m$. It suffices now to show that f^m takes value into $\mathfrak{C}^m$.

If $b \in \mathfrak{B}^m$, there are $b_1, ..., b_n \in B$ and a modal formula φ such that $b = \varphi(b_1, ..., b_m)$. As f^δ is a morphism of **ModAlg**), it follows that

$$f^\delta(b) = f^\delta(\varphi(b_1, ..., b_n)) = \varphi(f^\delta(b_1), ..., f^\delta(b_n)) \in \mathfrak{C}^m$$

as required. $\square$

Remark 2.4.10. In particular, we showed in Proposition 2.4.9 that the map $\eta : \mathfrak{B} \longrightarrow \mathfrak{B}^m$ is a morphism of **wSubAlg**. However, it is not a morphism of **SubAlg**. Indeed, consider X to be a Stone space with an accumulation point x_0 and R to be $\{(x_0, x_0)\}$. Then, $(\mathfrak{X}, R)$ is a subordination space and, now, take $\mathfrak{B}$ to be its dual.

We have that $\Diamond\eta(1) = R(-, X) = \{x_0\}$, such that $\{x_0\}$ is an element of $\mathfrak{B}^m$ which is not open, and, consequently, 0 is the unique element of $\mathfrak{B}$ such that $\eta(0) = \emptyset \leq \Diamond\eta(1)$.

Now, we have $\Diamond\eta(1) \leq \Diamond\eta(1)$, that is $\eta(1) \prec_\Diamond \eta(1)$, and, if η was a morphism of **SubAlg**, then there would be an element $b \in \mathfrak{B}$ such that $1 \prec b$ and $\eta(b) \leq \Diamond\eta(1)$, which is impossible since $1 \not\prec 0$.

Recall that every modal algebra $\mathfrak{B}$ can be seen as a subordination algebra. We have then the following proposition.

Proposition 2.4.11. *If $\mathfrak{B}$ is a modal algebra, then $\mathfrak{B} \cong \mathfrak{B}^m$.*

Remark 2.4.12. Of course, next to the **modalisation functor**, there is the **black modalisation functor** $\cdot^{blm}$: **♦SubAlg** $\longrightarrow$ **ModAlg** (with $\mathfrak{B}^{blm}$ is the "black" modal algebra generated by $\eta(B)$) and the **bimodalisation functor** $\cdot^{bim}$: **sSubAlg** $\longrightarrow$ **TAlg** (where **TAlg** is the category of tense algebras and $\mathfrak{B}^{bim}$ is the least tense algebra generated by $\eta(B)$).

2.5 Validity on subordination algebras

It this section, we will define valuation of (bi)modal formulas on subordination spaces and subordination algebras. While defining valuations on subordination spaces can be done exactly as it is done in modal spaces, a crucial difference emerges. Indeed, the valuation of a formula in a subordination space may fail to be a clopen subset of the latter. While harmless in the topological setting, it will create problems in the algebraic one. Indeed, the elements of a subordination algebra $\mathfrak{B}$ are the clopen subsets of a subordination space. Therefore, the valuation of a modal formula on a subordination algebra may fail to be one of its elements. Nevertheless, we will see that the valuation does not leap to far away from $\mathfrak{B}$, since it lands in the canonical extension $\mathfrak{B}^\delta$.

Definition 2.5.1. The **bimodal language** is constituted by a set of propositional variables $\mathrm{Var} = \{p, q, r, \cdots\}$, a constant symbol $\top$, the classical Boolean operators $\vee$ and $\neg$ and finally two modal operators $\Diamond$ and $\blacklozenge$.

A **bimodal formula** is constructed by induction as follows:

- for all $p \in \mathrm{Var}$, p is a bimodal formula,
- $\top$ is a bimodal formula,
- if ψ and χ are bimodal formulas, then $\psi \vee \chi$, $\neg\psi$, $\Diamond\psi$ and $\blacklozenge\psi$ are bimodal formulas.

As shortcuts, we also have the following operators $\bot$, $\wedge$, $\rightarrow$, $\Box$, $\blacksquare$ defined as follows

$$\bot := \neg\top \mid \psi \wedge \chi := \neg(\neg\psi \vee \neg\chi) \mid \psi \rightarrow \chi := \neg\psi \vee \chi \mid \Box\psi = \neg\Diamond\neg\psi \mid \blacksquare\psi = \neg\blacklozenge\neg\psi.$$

Notation 2.5.2. Let φ be a bimodal formula. We write $\varphi(p_1, \ldots, p_n)$ to indicate that the variables $p_1, \ldots, p_n$ occur in φ. Moreover, for the sake of a compact notation, we will sometimes denote $p_1, \ldots, p_n$ as $\underline{p}$. Also, if p is a variable that occurs only once in a formula φ, we will highlight it through the notation $\varphi(!p)$. Joining these two conventions, we write $\varphi(!\underline{p})$ to indicate that all variables in $\underline{p}$ occur exactly once.

Definition 2.5.3. Let $\mathfrak{X} = (X, R)$ be a subordination space. Then, a **valuation** on $\mathfrak{X}$ is a map $v : \mathrm{Var} \longrightarrow \mathrm{Clop}(X)$.

A valuation v is extended to the set of all bimodal formulas according to the following inductive definition:

- $v(\top) = X$,
- $v(\psi \vee \chi) = v(\psi) \cup v(\chi)$,
- $v(\neg\psi) = (v(\psi))^c$,
- $v(\Diamond\psi) = R(-, v(\psi))$,
- $v(\blacklozenge\psi) = R(v(\psi), -)$.

Note that we have as direct consequence of this definition that:

- $v(\bot) = \emptyset$,
- $v(\psi \wedge \chi) = v(\psi) \cap v(\chi)$,
- $v(\Box\psi) = R(-, v(\psi)^c)^c$,
- $v(\blacksquare\psi) = R(v(\psi)^c, -)^c$.

Notation 2.5.4. Suppose that $\varphi(\underline{p})$ is a bimodal formula and that v is a valuation on a subordination space $\mathfrak{X}$ such that $v(p) = O$ for all $p \in \underline{p}$. Then, for the sake of simplicity, we will sometimes write $\varphi(\underline{O})$ instead of $v(\varphi(\underline{p}))$.

Example 2.5.5. As stated in the introduction, the valuation of a bimodal formula may fail to be a clopen subset. Consider the following example.

Let X be an infinite Stone space with a limit point x_0 and endowed with the relation R defined by

$$x \; R \; y \Leftrightarrow x = y \text{ or } x = x_0.$$

Then R is a closed relation and, consequently, $\mathfrak{X} = (X, R)$ is a subordination space. Indeed, if $(x, y) \notin R$, then $x \neq x_0$ and $x \neq y$. Therefore, there exist clopen sets O and U such that $x \in O \cap U$, $y \in O^c$ and $x_0 \in U^c$ and such that we have $(O \cap U) \times O^c \subseteq R^c$. Indeed, if $(z, t) \in (O \cap U) \times O$, then $z \neq t$, as the first is in O and the second in O^c, and $z \neq x_0$, as the first is in U and the second in U^c.

Now, let O_0 be a clopen set of X that does not contain x_0 and let v be the valuation defined as $v(p) = O_0$. Then, we have

$$v(\Diamond p) = R(-, O_0) = \{x_0\} \cup O_0,$$

which is not clopen, as $\{x_0\}$ would be open, an absurdity.

Definition 2.5.6. Let φ be a bimodal formula, $\mathfrak{X} = (X, R)$ a subordination space and $x \in X$.

1. We say that φ is **valid** in x for a valuation v, which is denoted by $(\mathfrak{X}, x) \models_v \varphi$, if $x \in v(\varphi)$.
2. We say that φ is **valid** in $\mathfrak{X}$ for a valuation v, which is denoted by $\mathfrak{X} \models_v \varphi$, if $(\mathfrak{X}, x) \models_v \varphi$ for all $x \in X$.
3. We say that φ is **satisfied**, which is denoted by $\mathfrak{X} \models \varphi$, if $\mathfrak{X} \models_v \varphi$ for all valuations v.

Now that we have a definition of satisfaction of bimodal formulas on subordination spaces, we could define quite easily satisfaction on a subordination algebra $\mathfrak{B}$ via the duality of Section 2.2, i.e.

$$\mathfrak{B} \models \varphi \text{ iff } \mathrm{Ult}(\mathfrak{B}) \models \varphi. \tag{2.3}$$

Nevertheless, a purely algebraic definition of validity, equivalent to (2.3) can be obtained via the canonical extension defined in Section 2.4.

Definition 2.5.7. Let $\mathfrak{B} = (B, \prec)$ be a subordination algebra. A **valuation** on $\mathfrak{B}$ is a map $v : \mathrm{Var} \longrightarrow B$.

We now go back to Example 2.5.5 and consider $\mathfrak{B}$ to be the dual of $\mathfrak{X}$. It is clear that a valuation cannot be extended to all bimodal formulas, as $v(\Diamond p)$ is not clopen and, hence, not an element of $\mathfrak{B}$. However, $v(\Diamond p) = R(-, v(p))$ is an element of $\mathcal{P}(X)$, the canonical extension of $\mathfrak{B}$. Thus, by composing the valuation $v : \mathrm{Var} \longrightarrow B$ with the canonical embedding $\eta : B \longrightarrow B^\delta$, we obtain a valuation $\eta \circ v : \mathrm{Var} \longrightarrow B^\delta$. Now, since B^δ is a modal algebra, this latter valuation can be extended to a unique modal morphism between the set of formulas and B^δ.

Definition 2.5.8. Let φ be a bimodal formula. We say that φ is **valid** in $\mathfrak{B}$ for a valuation v, which is denoted by $\mathfrak{B} \models_v \varphi$, if $\mathfrak{B}^\delta \models_{\eta \circ v} \varphi$ in the usual sense, that is $(\eta \circ v)(\varphi) = 1_{\mathfrak{B}^\delta} = 1_{\mathfrak{B}}$. We say that φ is **satisfied** in $\mathfrak{B}$ if φ is valid for all valuations.

Finally, we have to check that the equivalence (2.3) is indeed respected.

Theorem 2.5.9. *Let φ be a bimodal formula and $\mathfrak{B} = (B, \prec)$ a subordination algebra whose dual is $\mathfrak{X} = (X, R)$. Then*

$$\mathfrak{B} \models \varphi \Leftrightarrow \mathfrak{X} \models \varphi.$$

Proof. Suppose that $\mathfrak{B} \models \varphi$ and let $v : \mathrm{Var} \longrightarrow \mathrm{Clop}(X)$ be a valuation on $\mathfrak{X}$. Then, in particular, it is a valuation $v : \mathrm{Var} \longrightarrow B^\delta$ on $\mathfrak{B}^\delta$ which comes from the valuation $\eta^{-1} \circ v : \mathrm{Var} \longrightarrow B$. Hence, we have $v(\varphi) = 1_{\mathfrak{B}^\delta} = X$.

On the other hand, suppose that $\mathfrak{X} \models \varphi$ and let $v : \mathrm{Var} \longrightarrow B$ be a valuation on $\mathfrak{B}$. Then, $\eta \circ v : \mathrm{Var} \longrightarrow \mathrm{Clop}(X)$ is a valuation on $\mathfrak{X}$ (and on $\mathfrak{B}^\delta$). Hence, we have $(\eta \circ v)(\varphi) = X = 1_{\mathfrak{B}^\delta}$. □

Remark 2.5.10. For obvious reasons, a valuation $\mathrm{Var} \longrightarrow B^\delta$ which arises from a valuation $\mathrm{Var} \longrightarrow B$ is often called a **clopen valuation**.

We saw that tense algebras and modal algebras were particular examples of subordination algebras. Since both of these structures were already endowed with notions of validity, it is important that the notion of validity we presented here corresponds to them.

Proposition 2.5.11. *1. If $\mathfrak{B}$ is a tense algebra and φ is a bimodal formula, then the validity of φ on $\mathfrak{B}$ qua subordination qua tense algebra correspond.*

2. If $\mathfrak{B}$ is a modal algebra and φ is a modal formula, then the validity of φ on $\mathfrak{B}$ qua subordination qua modal algebra correspond.

Proof. This follows trivially from the fact that $\mathfrak{B}$ is a subalgebra of $\mathfrak{B}^\delta$ whenever $\mathfrak{B}$ is a tense or a modal algebra. □

This is not the only case that we have to consider. Indeed, we saw that complete atomic subordination algebras were actually complete atomic modal algebras and complete atomic tense algebras. Hence, we have a sort of converse proposition to Proposition 2.5.11 to consider. In order to prove it, the following lemma will be required.

Lemma 2.5.12. *Let $\mathfrak{B} = (B, \prec)$ be a complete atomic subordination algebra and $\mathfrak{B}' = (B, \Diamond)$ (resp. $\mathfrak{B}' = (B, \Diamond, \blacklozenge)$) its associated complete atomic modal (resp. tense) algebra, then the topological dual of $\mathfrak{B}$ is the topological dual of $\mathfrak{B}'$.*

Proof. We only prove the modal case and leave the tense one to the reader. Since the underlying topological set of both duals is $\mathrm{Ult}(B)$, we just have to check that $R_\prec$ and $R_\Diamond$ are identical, that is that, for $x, y \in \mathrm{Ult}(B)$, $\prec(y, -) \subseteq x$ if and only if $\Diamond y \subseteq x$.

Hence, suppose first that $\Diamond y \subseteq x$ and let $a, b \in B$ such that $a \prec b$ and $a \in y$. Now, recall from the discussion after Definition 2.3.3 that $\Diamond a = \bigwedge\{c \in B \mid a \prec c\}$. Hence, it is clear that $\Diamond a \leq b$ and since $\Diamond a$ is an element of x, it is also the case for b.

Now, suppose that $\prec(y, -) \subseteq x$ and let $b = \Diamond a$ for some $a \in y$, that is $b = \bigwedge\{c \in B \mid a \prec c\}$. Since $\prec$ is a complete subordination relation, it follows that $a \prec b$ and therefore, that $b \in x$, as required. □

Proposition 2.5.13. *Let $\mathfrak{B}$ be a complete atomic subordination algebra.*

1. *If φ is a bimodal formula, the validity of φ on $\mathfrak{B}$ qua subordination algebra qua tense algebra correspond.*

2. *If φ is a modal formula, the validity of φ on $\mathfrak{B}$ qua subordination algebra qua modal algebra correspond.*

Proof. This is a direct corollary of Lemma 2.5.12. Indeed, since the topological dual (X, R) of $(B, \prec)$ is identical to the topological dual of $(B, \Diamond)$, we have

$$(B, \prec) \models \varphi \iff (X, R) \models \varphi \iff (B, \Diamond) \models \varphi.$$

□

2.6 Completeness theorems

In this section, we consider the logic of a class $\mathcal{K}$ of subordination algebras, that is, the family of modal (resp. bimodal) formulas that are satisfied in all subordination algebras/spaces of a given class. However, we will see that this family is nothing but modal (resp. tense) logic, which we recall the definitions of.

Definition 2.6.1. 1. The **modal logic** K is the smallest family of modal formulas containing all tautologies, the axiom K:

(K) $\Box(p \to q) \to (\Box p \to \Box q)$

and closed under the following induction rules:

(MP) Modus Ponens: $\frac{\varphi, \varphi \to \psi}{\psi}$;

(Sub) Substitution: $\frac{\varphi(p)}{\varphi(\psi)}$;

(RN) Necessitation rule: $\frac{\varphi}{\Box\varphi}$.

It is well known (see for instance [16]) that K is the family of modal formulas that are satisfied by all modal algebras and all Kripke frames.

2. The **tense logic** T is the smallest family of bimodal formulas containing all tautologies, the white and black versions of the axiom K and the axioms

(T_1) $p \to \Box\blacklozenge p$;

(T_2) $\blacklozenge\Box p \to p$,

and which is closed under (MP), (Sub) and black and white (RN). Once again, it is well known (see for instance [60]) that T is the family of bimodal formulas that are satisfied by all tense algebras.

Definition 2.6.2. Let $\mathcal{K}$ be a class of subordination algebras, or spaces. We denote by $\mathrm{Log}(\mathcal{K})$ the logic of $\mathcal{K}$, that is

$$\mathrm{Log}(\mathcal{K}) := \{\varphi \mid \forall \mathfrak{B} \in \mathcal{K} : \mathfrak{B} \models \varphi\}.$$

The first observation about $\mathrm{Log}(\mathcal{K})$ is that it is not a normal logic, while this would be the case for a class $\mathcal{K}$ of modal algebras. Indeed, while it is not hard, using classical methods, to prove that $\mathrm{Log}(\mathcal{K})$ is closed under modus ponens and necessitation, $\mathrm{Log}(\mathcal{K})$ is not necessarily closed under substitutions, as we now prove in the following example.

Example 2.6.3. Let us consider once more the subordination space $\mathfrak{X}$ of Example 2.5.5. We proved in Example 2.9.1 that $\mathfrak{X} \models p \to \Diamond\Box p$. Now consider the formula $\psi(p) = \neg p \wedge \Diamond p$ and O a clopen set not containing x_0. Then, we have $\Diamond O = O \cup \{x_0\}$ and hence $\psi(O) = \{x_0\}$. Now,

$$\Diamond\Box\psi(O) = \Diamond\emptyset = \emptyset.$$

Hence, we have $x_0 \in \psi(O)$ but $x_0 \notin \Diamond\Box\psi(O)$, consequently, $\mathfrak{X} \not\models \psi \to \Diamond\Box\psi$. It follows that $\mathrm{Log}(\mathfrak{X})$ is not closed under substitution.

An immediate observation is that the substitution rule may be replaced by the use of schemes. To distinguish the formula $\varphi(\underline{p})$ from its **associated scheme**, we shall write the latter $\varphi(\underline{\psi})$, this expression denotes the collection of formulas $\varphi(\underline{\psi})$ when $\underline{\psi}$ ranges over all modal (or bimodal if needed) tuples of formulas. The next definition will lead us to completeness results for standard modal logic in subordination algebras.

Definition 2.6.4. Let φ be a bimodal formula. We say that φ is **scheme extensible** if $\mathfrak{B} \models \varphi(\underline{p})$ implies $\mathfrak{B} \models \varphi(\underline{\psi})$ for all subordination algebras $\mathfrak{B}$ and for all k-uples $\underline{\psi}$ of bimodal formulas.

Theorem 2.6.5. *Let L be a set of schemes of modal formulas, and let φ be a modal formula. Then the following are equivalent:*

1. $L \vdash \varphi$*,*

2. *for any modal algebra $\mathfrak{B}$, $\mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$,*

3. *for any subordination algebra $\mathfrak{B}$, $\mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$.*

Proof. The implication 3. $\Rightarrow$ 2. follows from 2.5.11 while 1. $\Leftrightarrow$ 2. is well known. It remains to show the soundness part 1. $\Rightarrow$ 3.. We can proceed by induction on the length of a proof of φ if we prove first that the formulas θ obtained in the said proof are also scheme-extensible. If θ is an axiom, then it is clear by definition. The claim is also immediate if θ is obtained by substitution or the necessitation rule. Hence, it remains to consider the case where θ is obtained by modus ponens χ, $\chi \to \theta$. By induction, we know that there exist formulas $\alpha_1(\underline{p})$,$\alpha_2(\underline{p})$, $\underline{\phi_1}$ and $\underline{\phi_2}$ such that

$$\mathfrak{B} \models \alpha_1(\underline{\psi}), \mathfrak{B} \models \alpha_2(\underline{\psi}), \chi(\underline{p}) = \alpha_1(\underline{\phi_1}(\underline{p})) \text{ and } \alpha_2(\underline{\phi_2}(\underline{p})) = \chi(\underline{p}) \to \theta(\underline{p}).$$

Now, let $\underline{\psi}$ be an arbitrary k-uple of bimodal formulas. We need to prove that $\mathfrak{B} \models \theta(\underline{\psi})$. But, we have $\alpha_1(\underline{\phi_1}(\underline{\psi})) = \chi(\underline{\psi})$ and $\alpha_2(\underline{\phi_2}(\underline{\psi})) = \chi(\underline{\psi}) \to \theta(\underline{\psi})$. Since we have

$$\mathfrak{B} \models \alpha_1(\underline{\phi_1}(\underline{\psi})) \text{ and } \mathfrak{B} \models \alpha_2(\underline{\phi_2}(\underline{\psi})),$$

the conclusion is immediate. □

We also have the tense version of the previous theorem.

Theorem 2.6.6. *Let L be a set of schemes of bimodal formulas containing the least tense bimodal logic, and let φ be a modal formula. Then the following propositions are equivalent:*

1. $L \vdash \varphi$*,*

2. *for any tense algebra, $\mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$.*

3. *for any subordination algebra $\mathfrak{B}$, $\mathfrak{B} \models L$ implies $\mathfrak{B} \models \varphi$.*

Note that both Theorem 2.6.5 and Theorem 2.6.6 are not really new completeness theorems. Indeed, only the part 1. $\Rightarrow$ 3. is really a new result. Nevertheless, we have a direct corollary that, under the hypothesis of the theorem, if there exists a subordination algebra $\mathfrak{B} \models L$ with $\mathfrak{B} \not\models \varphi$, then there exists some modal algebra $\mathfrak{C} \models L$ with $\mathfrak{C} \not\models \varphi$. An analogue observation can be made about Theorem 2.6.6. Hence, the theorems mostly allow to enlarge the pool of algebras and spaces (from modal/tense to subordination) available in the search of counterexample.

Moreover, Theorems 2.6.5 and 2.6.6 highlight that it is the scheme of a formula that bears importance for the validity in the subordination setting.

It is possible to characterise the scheme-extensible formulas via the bimodalisation. Moreover, this characterisation will determine precisely the algebra $\mathfrak{C}$ mentioned earlier.

Proposition 2.6.7. *Let $\mathfrak{B} = (B, \prec)$ be a subordination algebra. Then, for any scheme $\varphi(\underline{\psi})$,*

$$\mathfrak{B} \models \varphi(\underline{\psi}) \Leftrightarrow \mathfrak{B}^{bim} \models \varphi(\underline{\psi}).$$

Proof. Suppose that $\mathfrak{B}^{bim} \models \varphi(\underline{\psi})$ and let $v : \mathrm{Var} \longrightarrow B$ be a valuation on B, extended to $\mathfrak{B}^{\delta}$. In particular, since $\mathfrak{B}^{bim}$ is a tense subalgebra of $\mathfrak{B}^{\delta}$ containing B, we get that $v(\varphi(\underline{\psi})) \in B^{bim}$. Consider now the valuation $v' : \mathrm{Var} \longrightarrow B^{bim}$ such that $v'(\varphi(\underline{\psi})) = v(\varphi(\underline{\psi}))$, then by hypothesis, we have $v'(\varphi(\underline{\psi})) = 1 = v(\varphi(\underline{\psi}))$, as required.

Now, suppose that $\mathfrak{B} \models \varphi(\underline{\psi})$ and let $v : \mathrm{Var} \longrightarrow B^{bim}$ be a valuation on $\mathfrak{B}^{bim}$. If $\underline{\psi}$ breaks down in $(\psi_1, \ldots, \psi_k)$, then for every $i \leq k$ there exists $b^i_1, \ldots, b^i_{n_i} \in B$ and a bimodal formula $\phi_i(p^i_1, \ldots, p^i_{n_i})$ such that $v(\psi_i) = \phi_i(b^i_1, \ldots, b^i_{n_i})$. Set $v' : \mathrm{Var} \longrightarrow B$ as the valuation that maps p^i_j to b^i_j for all i, j. Then, since $\mathfrak{B} \models \varphi(\tilde{\phi})$, we have that

$$\begin{aligned}
& v'(\varphi(\tilde{\phi})) = 1 \\
\Leftrightarrow\ & \varphi(v'(\phi_1(p^1_1, \ldots, p^1_{n_1})), \ldots, v'(\phi_k(p^k_1, \ldots, p^k_{n_k}))) = 1 \\
\Leftrightarrow\ & \varphi(\phi_1(b^1_1, \ldots, b^1_{n_1}), \ldots, \phi_k(b^k_1, \ldots, b^k_{n_k})) = 1 \\
\Leftrightarrow\ & \varphi(v(\psi_1), \ldots, v(\psi_k)) = 1 \\
\Leftrightarrow\ & v(\varphi(\psi_1, \ldots, \psi_k)) = 1,
\end{aligned}$$

as required. $\square$

Remark 2.6.8. Note that, in Proposition 2.6.7, we use schemes of formulas instead of actual formulas. Indeed, the proposition is simply false in the latter case, since a formula may fail to be preserved under modalisation. Consider for instance $\mathfrak{B}$ to be the dual algebra of the subordination space $\mathfrak{X}$ given in Example 2.5.5. We know that $\mathfrak{B} \models p \to \Diamond\Box p$, but it is clear that we also have $\mathfrak{B}^{bim} \not\models p \to \Diamond\Box p$. Indeed, since $\mathfrak{B}^{bim}$ is a tense algebra and, hence, such that its logic is normal, we would have that $\mathfrak{B}^{bim} \models p \to \Diamond\Box p$ implies $\mathfrak{B}^{bim} \models \psi \to \Diamond\Box\psi$. But, by Proposition 2.6.7, it would also imply that $\mathfrak{B} \models \psi \to \Diamond\Box\psi$, which was proven to be false in Example 2.6.3.

Corollary 2.6.9. *Let φ be a bimodal formula. Then, φ is scheme extensible if and only if $\mathfrak{B} \models \varphi$ implies $\mathfrak{B}^{bim} \models \varphi$ for all subordination algebras $\mathfrak{B}$.*

Proof. Suppose that φ is scheme extensible and let $\mathfrak{B}$ be a subordination algebra such that $\mathfrak{B} \models \varphi$. Then, $\mathfrak{B} \models \varphi(\underline{\psi})$ and it follows from Proposition 2.6.7 that $\mathfrak{B}^{bim} \models \varphi(\underline{\psi})$.

On the other hand, suppose that φ is not scheme extensible. Then, there exists a subordination algebra $\mathfrak{B}$ such that $\mathfrak{B} \models \varphi$ and $\mathfrak{B} \not\models \varphi(\psi)$. By Proposition 2.6.7, we hence have that $\mathfrak{B}^{bim} \not\models \varphi(\psi)$. Now, since $\mathfrak{B}^{bim}$ is a tense algebra, this is equivalent to $\mathfrak{B} \not\models \varphi$, as required. $\square$

We conclude this section with a theorem that provides a useful criterion to determine the non-canonicity of a formula.

Theorem 2.6.10. *If a bimodal formula φ is canonical, then it is scheme extensible.*

Proof. Suppose that $\mathfrak{B} \models \varphi$. Then, since φ is canonical, we have $\mathfrak{B}^{\delta} \models \varphi$. Moreover, as $\mathfrak{B}^{bim}$ is a tense subalgebra of $\mathfrak{B}^{\delta}$, we have $\mathfrak{B}^{bim} \models \varphi$. The conclusion is now direct from Corollary 2.6.9. $\square$

2.7 A Sahlqvist theorem for subordination algebras

Correspondence between modal formulas on modal algebras and first order formulas on their dual modal space is a classical problem in modal logic. A well known example is given by the formula $\Box p \to \Box\Box p$ which is satisfied in a modal algebra if and only if the accessibility relation on its dual space is transitive (see for instance [9]). It is easy to show that this correspondence stays true for subordination algebras.

In [63], Sahlqvist gave a family of modal formulas, now called Sahlqvist formulas, that admit a first order translation, translation which is effectively obtainable from the modal formulas. A topological proof of this correspondence was given by Sambin and Vaccaro in [64] and [65] and the results presented here are an adaptation of their work to the subordination algebra case. Finally, the Sahlqvist theorem was extended to Boolean algebras with operators by de Rijcke and Venema in [23].

However, for subordination algebras other kinds of correspondence may be studied. We have seen for instance in Proposition 2.1.11 that a subordination algebra satisfies the formula S6, that is

$$a \prec b \to (\exists c)(a \prec c \prec b)$$

if and only the accessibility relation associated to $\prec$ is transitive. The correspondence between subordination formulas and first order formulas has been studied by Balbiani and Kikot in [1] and by Santoli in [66, Chapter 6].

Therefore, with both this examples, we deduce that a subordination algebra satisfies the modal formula $\Box p \to \Box\Box p$ if and only if it satisfies S6. Hence, we have a third kind of possible correspondence: between (bi)modal and subordination formulas. This last kind of correspondence is studied in [24, Section 3].

We will further expand on the connections between the various kinds of correspondence in Section 2.10.

Now, let us start the study of the correspondence between (bi)modal formulas and first order formulas in the subordination setting.

Definition 2.7.1. Let φ be a bimodal formula. We say that:

1. φ is **closed** (resp. **open**) if it is obtained from constants $\top$, $\bot$, propositional variables and their negations, by applying $\vee$, $\wedge$, $\Diamond$ and $\blacklozenge$ (resp. $\vee$, $\wedge$, $\Box$ and $\blacksquare$).
2. φ is **positive** (resp. **negative**) if it is obtained from constants $\top$, $\bot$ and propositional variables (resp. and negations of propositional variables) by applying $\wedge$, $\vee$, $\Diamond$, $\Box$, $\blacklozenge$ and $\blacksquare$.
3. φ is **s-positive** (resp. **s-negative**) (s for subordination) if it is obtained from closed positive formulas (resp. open negative formulas) by applying $\vee$, $\wedge$, $\Box$ and $\blacksquare$ (resp. $\vee$, $\wedge$, $\Diamond$ and $\blacklozenge$).
4. φ is **strongly positive** if it is a conjunction of formulas of the form
$$\Box^{\epsilon} p := \Box^{\epsilon_1}\Box^{\epsilon_2}\dots\Box^{\epsilon_k} p,$$
where $p \in \mathrm{Var}$, $k \in \mathbb{N}$, $\epsilon \in \{1, \partial\}^k$ and $\Box^1 := \Box$ and $\Box^{\partial} := \blacksquare$.

Remark 2.7.2. Let us make a couple of remarks about the taxonomy of the formulas in Definition 2.7.1. If φ is a closed formula and $v : \mathrm{Var} \longrightarrow X$ is a valuation, then $v(\varphi)$ is a closed set of X (hence the name "closed"). Conversely, if ψ is an open formula, then $v(\psi)$ is an open set of X for all valuations $v : \mathrm{Var} \longrightarrow X$. In general, if $F_1, \dots, F_n$ are closed subsets of X and $O_1, \dots, O_n$ are open subsets of X, we have that $\varphi(F_1, \dots, F_n)$ is closed and $\psi(O_1, \dots, O_n)$ is open. These assertions can be proved rather easily by induction on the length of φ and ψ, using Lemma 2.1.3.

Moreover, if $\varphi(p)$ is a positive formula, then it is monotone, in the sense that, for S_1 and S_2 subsets of X, $S_1 \subseteq S_2$ implies $\varphi(S_1) \subseteq \varphi(S_2)$. Conversely, if $\psi(p)$ is a negative formula, then it is antitone. Both these assertions can be checked quite easily by an induction on the length of $\varphi(p)$ and $\psi(p)$.

Finally, the motive behind the specifications of s-positive and s-negative from positive and negative formula will become clear later, but we can anyway give an alternative definition. A bimodal formula φ is s-positive (resp. s-negative) if and only if it is a positive (resp. negative) formula with no $\Box$ or $\blacksquare$ under the scope of a $\Diamond$ or $\blacklozenge$ (resp. no $\Diamond$ or $\blacklozenge$ under the scope of a $\Box$ or $\blacksquare$).

Definition 2.7.3. Let φ be a bimodal formula without implication. The **dual formula** of φ, denoted by φ^∂, is the formula obtained by interchanging the roles of $\top$ and $\bot$, $\wedge$ and $\vee$, $\Diamond$ and $\Box$ and, finally, the roles of $\blacklozenge$ and $\blacksquare$.

For instance, if $\varphi := p_1 \vee (\Diamond p_2 \wedge \blacksquare p_3)$ then its dual formula φ^∂ is given by $\varphi^\partial = p_1 \wedge (\Box p_2 \vee \blacklozenge p_3)$.

Lemma 2.7.4. *1. A bimodal formula φ is closed if and only if its dual formula φ^∂ is open.*

2. A bimodal formula $\varphi(\underline{p})$ is positive if and only if $\varphi(\neg\underline{p})$ is negative and if and only if $\varphi^\partial(\underline{p})$ is positive.

3. A bimodal formula $\varphi(\underline{p})$ is s-positive if and only if $\varphi^\partial(\neg\underline{p})$ is s-negative.

Lemma 2.7.5. *Let $\mathfrak{X} = (X, R)$ be a subordination space, $\varphi(\underline{p})$ a bimodal formula. Then, for all $\underline{E} \in (\mathcal{P}(X))^n$, we have*

$$\varphi(\underline{E}) = (\neg\varphi^\partial\neg)(\underline{E}).$$

Proof. As usual, it can be proved by induction on the length of φ. As an example, we can consider the case where $\varphi = \Box p$. Then, $\varphi^\partial = \Diamond p$ and, for all $E \subseteq X$, we have $\varphi(E) = R(-, E^c)^c$ and

$$(\neg\varphi^\partial\neg)(E) = \neg\varphi^\partial(E^c) = \neg R(-, E^c) = R(-, E^c)^c.$$

$\Box$

Definition 2.7.6. Let $\mathfrak{X} = (X, R)$ be a subordination space. An **R-expression** on $\mathfrak{X}$ is the empty set or a set of the form

$$R^{\epsilon_1}(-, x_1) \cup \cdots \cup R^{\epsilon_n}(-, x_n)$$

with $n \in \mathbb{N}$, $\epsilon_i \in \{1, \partial\}^{k_i}$ for some $k_i \in \mathbb{N}$, $x_i \in X$ and where $R^{\epsilon_i}(-, x_i)$ is defined recursively on the length of ϵ_i by

$$\begin{aligned} R^{\emptyset}(-, x_i) &= \{x_i\} \\ R^{(1,\epsilon_i)}(-, x_i) &= R(-, R^{\epsilon_i}(-, x_i)) \\ R^{(\partial,\epsilon_i)}(-, x_i) &= R(R^{\epsilon_i}(-, x_i), -). \end{aligned}$$

Remark 2.7.7. Using a notation similar to the one of strongly positive formulas, an R-expression can be rewritten as

$$\Diamond^{\epsilon_1}(\{x_1\}) \cup \cdots \cup \Diamond^{\epsilon_n}(\{x_n\}).$$

As a direct consequence of the definition and Lemma 2.1.3, we have that every R-expression on (X, R) is a closed subset of X.

Lemma 2.7.8. *Let $\mathfrak{X} = (X, R)$ be a subordination space, $\underline{\mathcal{E}}$ R-expressions on $\mathfrak{X}$ and $\varphi(\underline{p})$ a bimodal formula. Then $x \in \varphi(\underline{\mathcal{E}})$ is equivalent to $\Phi(x)$ for some first order formula $\Phi(t)$ where t is a free variable.*

Proof. Let us proceed by induction on the length of φ.

- If $\varphi = p_1$, then $\varphi(\mathcal{E}_1) = \mathcal{E}_1$ and we proceed by induction on the length of $\mathcal{E}_1$. If $\mathcal{E}_1 = \emptyset$, then $x \in \mathcal{E}_1$ if and only if $x \neq x$. If $\mathcal{E}_1 = \{x_1\}$, then $x \in E_1$ if and only if $x = x_1$. Now, the induction. Suppose that
$$\mathcal{E}_1 = R^{(1,\epsilon)}(-, x_1),$$
then $x \in \mathcal{E}_1$ if and only
$$(\exists y)(y \in R^{\epsilon}(-, x_1) \text{ and } x \; R \; y.)$$
Since by induction, $y \in R^{\epsilon}(-, x_1)$ is equivalent to a first order formula, the conclusion is immediate. Of course, the case $x \in R^{(\partial,\epsilon)}(-, x_1)$ is proved similarly. Finally, we consider the case
$$\mathcal{E}_1 = R^{\epsilon_1}(-, x_1) \cup ... \cup R^{\epsilon_n}(-, x_n). \tag{2.4}$$
We have that $x \in R^{\epsilon_i}(-, x_i)$ is equivalent to a first order formula Φ_i. Then, (2.4) is rapidly seen to be equivalent to the disjunction of the formulas Φ_i.

- If $\varphi = \psi \vee \chi$, then $x \in \varphi(\underline{\mathcal{E}})$ is equivalent to
$$x \in \psi(\underline{\mathcal{E}}) \text{ or } x \in \chi(\underline{\mathcal{E}}).$$
The conclusion then follows from the induction on $x \in \psi(\underline{\mathcal{E}})$ and $x \in \chi(\underline{\mathcal{E}})$.

- If $\varphi = \neg\psi$, we have the same proof as the previous case.

- If $\varphi = \Box\psi$, then $x \in \varphi(\underline{\mathcal{E}})$ is equivalent to
$$(\forall y)(x \; R \; y \Rightarrow y \in \psi(\underline{\mathcal{E}})).$$
The induction on $y \in \psi(\underline{\mathcal{E}})$ allows us to conclude. The case $\varphi = \blacksquare\psi$, is treated similarly.

$\square$

Lemma 2.7.9. *Let $\mathfrak{X} = (X, R)$ be a subordination space and let $\varphi(\underline{p})$ be a strongly positive formula. Then there exist R-expressions $\mathcal{E}_1, \ldots, \mathcal{E}_n$ such that for every $S_1, \ldots, S_n \subseteq X$*
$$x \in \varphi(S_1, \ldots, S_n) \Leftrightarrow (\forall i)(\mathcal{E}_i \subseteq S_i).$$

Proof. By induction on the length of φ. It is sufficient to remark that $x \in \Box S$ if and only if $R(x, -) \subseteq S$ and $x \in \blacksquare S$ if and only if $R(-, x) \subseteq S$. $\square$

We now have a generalisation of Esakia Lemma (see Proposition 2.1.4).

Lemma 2.7.10 (Generalised Esakia Lemma)**.** *Let $\mathfrak{X} = (X, R)$ be a subordination space, $\varphi(\underline{p})$ an s-positive formula, $\psi(\underline{p})$ an s-negative formula, $(F_i \mid i \in I)$ a filtered family of closed sets and $C_1, ..., C_{k_1}$ closed sets of X. Then, we have*

1. $\varphi(C_1, ..., \cap F_i, ..., C_{k-1}) = \cap\{\varphi(C_1, ..., F_i, ..., C_{k-1}) \mid i \in I\}$
2. $\psi(C_1, ..., \cap F_i, ..., C_{k-1}) = \cup\{\psi(C_1, ..., F_i, ..., C_{k-1}) \mid i \in I\}$.

Proof. For the sake of readability of the proof, $\varphi(C_1, ..., \star, ..., C_{k-1})$ will be shortened as $\varphi(\star)$, and the same applies of course for ψ. We will nevertheless highlight where the condition C_i *closed* is important.

1. Of course, by monotonicity, the inclusion

$$\varphi(\cap F_i) \subseteq \cap\varphi(F_i)$$

is trivially satisfied, so that we only have to prove $\varphi(\cap F_i) \supseteq \cap\varphi(F_i)$. We proceed by induction on the length of φ.

   (a) If $\varphi = p$, then the inclusion is blatantly true.

   (b) If $\varphi = \chi \wedge \xi$, suppose that $x \notin (\chi \wedge \xi)(\cap F_i)$. Without loss of generality, we may consider that $x \notin \chi(\cap F_i)$. Hence, by induction, there exists $i \in I$ such that $x \notin \chi(F_i)$. Therefore, we have that

$$x \notin (\chi \wedge \psi)(F_i)$$

   and, so, that $x \notin \cap\varphi(F_i)$ as required.

   (c) If $\varphi = \chi \vee \xi$, suppose that $x \notin (\chi \vee \xi)(\cap F_i)$. Then, x is neither an element of $\chi(\cap F_i)$ nor one of $\xi(\cap F_i)$. By induction, there exist $i, j \in I$ such that $x \notin \chi(F_i)$ and $x \notin \chi(F_j)$. Now, since $(F_i \mid i \in I)$ is a filtered family, there exists an element $k \in I$ with $F_k \subseteq F_i \cap F_j$ and, by monotonicity of both χ and ξ, it follows that $x \notin (\chi \wedge \xi)(F_k)$.

   (d) If $\varphi = \Diamond\chi$, then χ is a positive closed formula as, otherwise there would be a $\Box$ or a $\blacksquare$ under the scope of $\Diamond$, which is impossible for an s-positive formula. We have that $\cap\varphi(F_i)$ is equal to $\cap R(-, \chi(F_i))$. Then, since F_i and C_j are closed for all i, j, it is clear that $(\chi(F_i) \mid i \in I)$ is a filtered family of closed sets such that, by Proposition 2.1.4, we have $\cap R(-, \chi(F_i)) = R(-, \cap\chi(F_i))$. Finally, the conclusion follows from the induction hypothesis on χ. The case $\varphi = \blacklozenge\chi$ is treated similarly.

   (e) If $\varphi = \Box\chi$, then we have by induction on χ

$$\varphi(\cap F_i) = R(-, \chi(\cap F_i)^c)^c = R(-, \cup\chi(F_i)^c)^c.$$

   Then, we simply have to rewrite $R(-, \cup\chi(F_i)^c)^c$ as $\cap R(-, \chi(F_i)^c)^c$ to conclude the proof. Of course, the case $\varphi = \blacksquare\chi$ is treated similarly.

2. Using this time antonicity, we only have to prove $\psi(\cap F_i) \subseteq \cup\psi(F_i)$.

   (a) If $\psi = \neg p$, then we have

$$\varphi(\cap F_I) = (\cap F_I)^c = \cup F_i^c = \cup\varphi(F_i).$$

   (b) If $\psi = \chi \wedge \xi$, we have, by induction on χ and ξ,

$$\psi(\cap F_i) = \chi(\cap F_i) \cap \xi(\cap F_i) = (\cup\chi(F_i)) \cap (\cup\xi(F_i))$$

   that is

$$\psi(\cap F_i) = \cup\{\chi(F_i) \cap \xi(F_j) \mid i, j \in I\}.$$

   Now, remember that $(F_i \mid i \in I)$ is a filtered family and that the formulas χ and ξ are antitone. Hence, we have

$$\psi(\cap F_i) = \cup\{\chi(F_i) \cap \xi(F_i) \mid i \in I\} = \cup\psi(F_i).$$

   (c) If $\psi = \chi \vee \xi$, suppose that $x \notin \cup\psi(F_i)$, then we have that

$$x \notin \cup(\chi(F_i) \cup \xi(F_i)).$$

   Hence, for all $i \in I$, we have $x \notin \chi(F_i)$ and $x \notin \xi(F_i)$ and, consequently, we have by induction that $x \notin \chi(\cap F_i)$ and $x \notin \xi(\cap F_i)$. That is $x \notin \chi(\cap F_i) \cup \xi(\cap F_i)$ as required.

(d) If $\psi = \Diamond\chi$, then we have by induction on χ

$$\psi(\cap F_i) = R(-, \chi(\cap F_i)) = \cup R(-, \chi(F_i)) = \cup\psi(F_i).$$

The case $\psi = \blacklozenge\chi$ is treated similarly.

(e) If $\psi = \Box\chi$, then χ is a negative open formula as, otherwise there would be a $\Diamond$ or a $\blacklozenge$ under the scope of $\Box$, which is impossible for an s-negative formula. We have that $\cup\psi(F_i)$ is equal to $\cup R(-, \chi(F_i)^c)^c$ that is $(\cap R(-, \chi(F_i)^c))^c$. Now, since χ is negative open, we have that $(\chi(F_i)^c \mid i \in I)$ is a family of filtered closed sets such that, by Proposition 2.1.4, we have

$$\cup\psi(F_i) = (R(-, \cap\chi(F_i)^c))^c = (R(-, (\cup\chi(F_i))^c))^c.$$

The conclusion now follows immediately from the induction hypothesis on χ.

□

This generalised Esakia Lemma conceals behind its hypothesis a far more simple proposition that will be helpful in the future.

Corollary 2.7.11 (Intersection Lemma)**.** *Let $\mathfrak{X} = (X, R)$ be a subordination space, $\varphi(\underline{p})$ an s-positive formula and $\psi(\underline{p})$ an s-negative formula. For every $S \subseteq X$ and every $C_1, ..., C_{k-1}$ closed sets of X,*

$$\varphi(C_1, ..., \overline{S}, ..., C_{k-1}) = \cap\{\varphi(C_1, ..., O, ..., C_{k-1}) \mid S \subseteq O \in \mathrm{Clop}(X)\}$$

and

$$\psi(C_1, ..., \overline{S}, ..., C_{k-1}) = \cup\{\psi(C_1, ..., O, ..., C_{k-1}) \mid S \subseteq O \in \mathrm{Clop}(X)\}$$

Proof. One has simply to remember that $\mathcal{F} = \{O \in \mathrm{Clop}(X) \mid S \subseteq O\}$ is a filtered family of closed sets such that $\overline{S} = \cap\mathcal{F}$. □

We just encountered the propositions where the extraction of s-positive formulas from the positive ones is needed. It appeared in the proof of 2.7.10, more specifically in the case $\varphi = \Diamond\psi$ of the induction. Indeed, unlike the modal case, where this proposition is valid for all positive formulas, $\Box p$ may fail to be a closed set of X (recall Example 2.5.5). And, hence, $(\chi(F_i) \mid i \in I)$ may fail to be a family of closed sets, making it impossible to use Proposition 2.1.4. Nevertheless, this does not imply that we could not have bypassed this problem with another proof. Later, we will prove that this extraction was indeed needed. Let us now introduce the penultimate family of bimodal formulas required for the Sahlqvist theorem.

Definition 2.7.12. Let φ be a bimodal formula. We say that φ is **s-untied** (with, again, s for subordination) if it is obtained from strongly positive and s-negative formulas using only $\wedge$, $\Diamond$ and $\blacklozenge$.

Proposition 2.7.13. *Let $\mathfrak{X} = (X, R)$ be a subordination space, $\varphi(p_1, \dots, p_k)$ an s-untied formula. Then, for all $O_1, \dots O_k$ clopen sets of X, $x \in \varphi(O_1, \dots, O_k)$ is equivalent to*

$$(\exists y_1, \dots, \exists y_n)(\Psi \wedge \bigwedge_{i \leq k} \mathcal{E}_i \subseteq O_i \wedge \bigwedge_{j \leq m} u_j \in N_j(O_1, \dots, O_k)),$$

where

- $n, m \in \mathbb{N}$,

- *$\mathcal{E}_i$ is an R-expression for all i,*
- *N_j is an s-negative formula for all j,*
- *$(y_i \mid i \leq n)$ are pointwise distinct variables, all distinct from x,*
- *Ψ is a conjunction of formulas of the form $u_i \; R \; u_j$,*
- *$(u_j \mid j \leq n)$ are variables taken from $y_1, ..., y_n, x$.*

Proof. Let us proceed by induction on the length of φ.

- If φ is a strongly positive formula, then, this is simply Lemma 2.7.9.
- If φ is an s-negative formula, then it is sufficient to set $y_1 = u_1 = x$ and $N_1 = \varphi$.
- If $\varphi = \psi \wedge \chi$, the conclusion is immediate by induction.
- If $\varphi = \Diamond\psi$, then x is an element of $\varphi(O_1, \ldots, O_k) = \Diamond\psi(O_1, \ldots, O_k)$ if and only there exists an element y in $\psi(O_1, ..., O_k)$ such that $x \; R \; y$. By induction, we know that $y \in \psi(O_1, ..., O_k)$ is equivalent to

$$(\exists y_1 \; ...\exists y_n)(\Psi \wedge \bigwedge_{i\leq k} \mathcal{E}_i \subseteq O_i \wedge \bigwedge_{j\leq m} u_j \in N_j(O_1, ..., O_k)).$$

  Therefore, we have that

$$[(\exists y)(x \; R \; y)] \wedge \left[(\exists y_1 \; ...\exists y_n)(\Psi \wedge \bigwedge_{i\leq k} \mathcal{E}_i \subseteq O_i \wedge \bigwedge_{j\leq m} u_j \in N_j(O_1, ..., O_k))\right]$$

  is equivalent to $x \in \varphi(O_1, ..., O_k)$.

  Since each y_i is a variable distinct from y, this formula can be rewritten as

$$(\exists y \exists y_1 \; ...\exists y_n)(\Psi' \wedge \bigwedge_{i\leq k} \mathcal{E}_i \subseteq C_i \wedge \bigwedge_{j\leq m} u_j \in N_j(C_1, ..., C_k))$$

  where $\Psi' = (x \; R \; y) \wedge \Psi$.

- If $\varphi = \blacklozenge\psi$, the proof is identical to the white one.

$\square$

Definition 2.7.14. Let φ be a bimodal formula. We say that φ is an **s-Sahlqvist formula** if it is of the form $\Box^\epsilon(\varphi_1 \to \varphi_2)$ with φ_1 an s-untied formula, φ_2 an s-positive formula and $\epsilon \in \{1, \partial\}^k$ for some $k \in \mathbb{N}$ (Recall that we introduced the notation $\Box^\epsilon$ in Definition 2.7.1).

Theorem 2.7.15 (Sahlqvist theorem for subordination algebras)**.** *Let φ be an s-Sahlqvist formula. Then, there exists a first order formula Φ in the language of the accessibility relation, effectively computable from φ, and such that for all subordination algebra $\mathfrak{B} = (B, \prec)$ with dual $\mathfrak{X} = (X, R)$*

$$\mathfrak{B} \models \varphi \textit{ iff } \mathfrak{X} \models \Phi.$$

Proof. We have $\mathfrak{B} \models \varphi$ if and only if $\mathfrak{X} \models \varphi$.

Set $x \in X$, we have successively

$$\begin{aligned}
&\mathfrak{X} \models_x \varphi \\
\Leftrightarrow &(\forall O_1 ... O_k \in \mathrm{Clop}(X))(x \in \Box^\epsilon(\phi_1 \to \phi_2)(O_1, ..., O_k)) \\
\Leftrightarrow &(\forall O_1 ... O_k \in \mathrm{Clop}(X))(\forall y)(x\ R^\epsilon\ y \Rightarrow y \in (\phi_1 \to \phi_2)(O_1, ..., O_k)) \\
\Leftrightarrow &(\forall O_1 ... O_k \in \mathrm{Clop}(X))(\forall y)\,\{x\ R^\epsilon\ y \\
&\Rightarrow \left[\left((\exists y_1 ... y_n)(\Psi \wedge \bigwedge_{i \leq k} \mathcal{E}_i \subseteq O_i \wedge \bigwedge_{j \leq m} u_j \in N_j(O_1, ..., O_k)\right)\right. \\
&\Rightarrow y \in \phi_2(O_1 ... O_k)]\} \\
\Leftrightarrow &(\forall O_1 ... O_k \in \mathrm{Clop}(X))(\forall y)(\forall y_1 ... y_n)\{x\ R^\epsilon\ y \wedge \Psi \\
&\Rightarrow \left.\left[\left(\bigwedge_{i \leq k} \mathcal{E}_i \subseteq O_i \wedge \bigwedge_{j \leq m} u_j \in N_j(O_1, ..., O_k)\right) \Rightarrow y \in \phi_2(O_1 ... O_k)\right]\right\}.
\end{aligned}$$

By Lemmas 2.7.4 and 2.7.5, we know that $\neg\left(\bigwedge u_j \in N_j(O_1, ..., O_k)\right)$ and $\bigvee u_j \in N_j^\partial(O_1^c, ..., O_k^c)$ are equivalent, and that $N_j^\partial(\neg p_1, ..., \neg p_k)$ is an s-positive formula.

Set then $P_j = \overline{N_j}(\neg p_1, ..., \neg p_k)$ for $j \leq m$, $P_{m+1} = \phi_2(p_1, ..., p_k)$, $u_{m+1} = y$ and finally $\Psi' = x\ R^\epsilon\ y \wedge \Psi$, we obtain

$$\begin{aligned}
&(\forall O_1 ... O_k \in \mathrm{Clop}(X))(x \in \Box^\epsilon(\phi_1 \to \phi_2)(O_1, ..., O_k)) \\
\Leftrightarrow &(\forall y_1 ... y_{m+1})\,\{\Psi' \Rightarrow [(\forall O_1 ... O_k \in \mathrm{Clop}(X)) \\
&\left.\left.\left(\bigwedge_{i \leq k} (\mathcal{E}_i \subseteq O_i)\right) \Rightarrow \left(\bigvee_{j \leq m+1} u_j \in P_j(O_1, ..., O_k)\right)\right]\right]\right\}.
\end{aligned}$$

We just have to apply Proposition 2.7.11 to get the equivalence between the previous formula and

$$(\forall y_1 ... y_{m+1})(\Psi' \Rightarrow \bigvee_{j \leq m+1} u_j \in P_j(\mathcal{E}_1, ..., \mathcal{E}_k)). \tag{2.5}$$

Finally, the conclusion follows from the fact that $u_j \in P_j(\mathcal{E}_1, ..., \mathcal{E}_k)$ is equivalent to a first order formula via Lemma 2.7.8. $\square$

To conclude the section, we recall that, in the modal setting, every Sahlqvist formulas is canonical. This characteristic is carried in the subordination setting, as stated in the next theorem.

Theorem 2.7.16. *If a bimodal formula φ is an s-Sahlqvist formula, then it is canonical.*

Proof. Let us denote by Φ the first order formula equivalent to φ. Suppose that $\mathfrak{B} \models \varphi$. Then, for $\mathfrak{X}$ the dual of $\mathfrak{B}$, we have $\mathfrak{X} \models \Phi$ and the topology of $\mathfrak{X}$ does not have any role to play in this latter validity. Therefore, since φ is an s-Sahlqvist formula and, hence, a Sahlqvist one, the discrete dual of $\mathfrak{X}$, that is $\mathfrak{B}^\delta$, satisfies φ. $\square$

2.8 Examples

In this short section, we will give some examples of translations of bimodal formula into first order ones. We will always let $\mathfrak{B} = (B, \prec)$ and $\mathfrak{X} = (X, R)$ be respectively a subordination algebra and a subordination space, duals to each other.

Example 2.8.1. We have
$$\mathfrak{B} \models \blacksquare p \to p \Longleftrightarrow \mathfrak{X} \models x\ R\ x.$$

Let us first note that $\blacksquare p$ is a strongly positive formula, and hence an s-untied one, and that p is an s-positive formula. Therefore, $\blacksquare p \to p$ is an s-Sahlqvist formula.

As for the translation, we have successively:

$$\begin{aligned}
& \mathfrak{B} \models \blacksquare p \to p \\
\Longleftrightarrow\ & \mathfrak{X} \models \blacksquare p \to p \\
\Longleftrightarrow\ & (\forall x \in X)(\forall O \in \mathrm{Clop}(X))(x \in (\blacksquare O \to O)) \\
\Longleftrightarrow\ & (\forall x \in X)(\forall O \in \mathrm{Clop}(X))(x \in \blacksquare O \Rightarrow x \in O) \\
\Longleftrightarrow\ & (\forall x \in X)(\forall O \in \mathrm{Clop}(X))(R(-,x) \subseteq O \Rightarrow x \in O) \text{ (by Lemma 2.7.9)} \\
\Longleftrightarrow\ & (\forall x \in X)(x \in \overline{R(-,x)} = R(-,x)) \text{ (By 2.7.11 and } R(x,-) \text{ closed)} \\
\Longleftrightarrow\ & (\forall x \in X)(x\ R\ x) \\
\Longleftrightarrow\ & \mathfrak{X} \models x\ R\ x.
\end{aligned}$$

Example 2.8.2. We have

$$\mathfrak{B} \models \Diamond\Box p \to \Box\Diamond p \iff \mathfrak{X} \models (x\ R\ y \text{ and } x\ R\ z) \Rightarrow (\exists t)(y\ R\ t \text{ and } z\ R\ t).$$

The formula $\Box p$ is a strongly positive formula, and hence, $\Diamond\Box p$ is an s-untied formula. Moreover, $\Box\Diamond p$ is an s-positive formula. In conclusion, $\Diamond\Box p \to \Box\Diamond p$ is an s-Sahlqvist formula.

Now, we have the following succession of equivalences, where, for the sake of readability, we omit to write
$$(\forall x \in X)(\forall O \in \mathrm{Clop}(X))$$
when it is needed. We have

$$\begin{aligned}
& \mathfrak{X} \models \Diamond\Box p \to \Box\Diamond p \\
\Longleftrightarrow\ & (x \in \Diamond\Box O \Rightarrow x \in \Box\Diamond O) \\
\Longleftrightarrow\ & [(\exists y_1)(x\ R\ y_1 \wedge y_1 \in \Box O)] \Rightarrow [x \in \Box\Diamond O] \\
\Longleftrightarrow\ & (\forall y_1)[(x\ R\ y_1 \wedge y_1 \in \Box O) \Rightarrow (x \in \Box\Diamond O)] \\
\Longleftrightarrow\ & (\forall y_1)[x\ R\ y_1 \Rightarrow \{ \quad \underbrace{(y_1 \in \Box O)}_{\text{Strongly positive formula}} \quad \Rightarrow (x \in \Box\Diamond O)\}] \\
\Longleftrightarrow\ & (\forall y_1)[x\ R\ y_1 \Rightarrow \{R(y_1,-) \subseteq O \Rightarrow x \in \Box\Diamond O\}] \\
\Longleftrightarrow\ & (\forall y_1)[x\ R\ y_1 \Rightarrow x \in \Box\Diamond R(y_1,-)] \text{ (by 2.7.11)}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
& x \in \Box\Diamond R(y_1,-) \\
\Longleftrightarrow\ & (\forall z)(x\ R\ z \Rightarrow z \in \Diamond R(-,y_1)) \\
\Longleftrightarrow\ & (\forall z)(x\ R\ z \Rightarrow (\exists t)(z\ R\ t \wedge y_1\ R\ t)).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathfrak{X} \models \Diamond\Box p \to \Box\Diamond p \\
\iff &(\forall y_1)[x\ R\ y_1 \Rightarrow \{(\forall z)(x\ R\ z \Rightarrow (\exists t)(z\ R\ t \wedge y_1\ R\ t))\}] \\
\iff &(\forall y_1)(\forall z)[(x\ R\ y_1 \wedge x\ R\ z) \Rightarrow \{(\exists t)(y_1\ R\ t \wedge z\ R\ t)\}].
\end{aligned}$$

Renaming y_1 as y concludes the proof.

Example 2.8.3. We have

$$\mathfrak{B} \models \blacksquare((\Diamond\Box\neg p \wedge \Diamond\Box p) \to p) \iff \mathfrak{X} \models x\ R\ y \Rightarrow (\forall t)(t\ R\ y \vee \forall z(y\ R\ z \Rightarrow \exists s : z\ R\ s\ R\ t)).$$

The formula $\Box p$ is a strongly positive one and $\Diamond\Box\neg p$ an s-negative one. Therefore, $\Diamond\Box\neg p \wedge \Diamond\Box p$ is an s-untied formula. Since p is s-positive, we have that $\blacksquare((\Diamond\Box\neg p \wedge \Diamond\Box p) \to p)$ is indeed an s-Sahlqvist formula.

We have

$$\begin{aligned}
&\mathfrak{X} \models \blacksquare((\Diamond\Box\neg p \wedge \Diamond\Box p) \to p) \\
\iff &(\forall y)[y\ R\ x \Rightarrow \{y \in (\Diamond\Box O^c \wedge y \in \Diamond\Box O) \Rightarrow y \in O\}]
\end{aligned}$$

Now, we have

$$\begin{aligned}
&y \in (\Diamond\Box\neg O \wedge y \in \Diamond\Box O) \Rightarrow y \in O \\
\iff &[\{(\exists t)(t \in \Box O \wedge y\ R\ t)\} \wedge (y \in \Diamond\Box\neg O)] \Rightarrow [y \in O] \\
\iff &[(\exists t)(R(t,-) \subseteq O \wedge y\ R\ t)] \Rightarrow [y \in (O \vee \Box\Diamond O)] \text{ (by Lemma 2.7.5)} \\
\iff &(\forall t)(y\ R\ t \Rightarrow y \in (R(t,-) \cup \Box\Diamond R(-,t))).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&y \in (R(t,-) \cup \Box\Diamond R(-,t))) \\
\iff &t\ R\ y \vee \forall z(y\ R\ z \Rightarrow \exists s : z\ R\ s\ R\ t).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\mathfrak{B} \models \blacksquare((\Diamond\Box\neg p \wedge \Diamond\Box p) \to p) \\
\iff &\mathfrak{X} \models x\ R\ y \Rightarrow (\forall t)(t\ R\ y \vee \forall z(y\ R\ z \Rightarrow \exists s : z\ R\ s\ R\ t))
\end{aligned}$$

2.9 Discussion about s-Sahlqvist formulas

By construction s-Sahlqvist formulas form a family strictly included in the one of Sahlqvist formulas, that is bimodal formulas that admit a first order translation for tense algebras. For instance, the formula $p \to \Diamond\Box p$ is a Sahlqvist formula, while it is not an s-Sahlqvist one.

The question that remains now is: is the set of translatable formulas for subordination algebras actually smaller than the one for tense algebras? In order to answer this question, we will consider in more detail the formula $p \to \Diamond\Box p$, formula which contains a forbidden $\Box$ in the scope of a $\Diamond$.

Example 2.9.1. The formula $p \to \Diamond\Box p$ is a Sahlqvist formula (for modal algebras) equivalent to

$$\Phi := (\forall x)(\exists y)((x\ R\ y) \wedge (y\ R\ z \to z = x)). \tag{2.6}$$

This can be proved using Sahlqvist's theorem from [65], which we extended in Theorem 2.7.15. Indeed, let $\mathfrak{X} = (X, R)$ be a modal space. We obtain, for all $O \in \mathrm{Clop}(X)$ and for all $x \in X$

$$\begin{aligned}
&x \in O \to x \in \Diamond\Box O \\
\Leftrightarrow &\{x\} \subseteq O \to x \in \Diamond\Box O \\
\Leftrightarrow &x \in \cap\{\Diamond\Box O \mid O \supseteq \{x\}\} && (2.7)\\
\Leftrightarrow &x \in \Diamond\Box\{x\} && (2.8)\\
\Leftrightarrow &\exists y \in \Box x : x\ R\ y \\
\Leftrightarrow &\exists y\ ((\forall z\ y\ R\ z \to z = x) \wedge x\ R\ y).
\end{aligned}$$

Where the transition from (2.7) to (2.8) is done by the Intersection Lemma, valid only in the modal case.

However, the equivalence between $p \to \Diamond\Box p$ and (2.6) is not true anymore in a subordination environment. Indeed, consider the subordination space $\mathfrak{X} = (X, R)$ of Example 2.5.5, that is an infinite Stone space X with a limit point x_0 endowed with the relation R defined by

$$x\ R\ y \Leftrightarrow x = y \text{ or } x = x_0.$$

Let us show that $\mathfrak{X} = (X, R)$ satisfies $p \to \Diamond\Box p$ but not (2.6).

1. To prove that $\mathfrak{X}$ satisfies $p \to \Diamond\Box p$ is to prove that for all $x \in X$ and for all $O \in \mathrm{Clop}(X)$, $x \in O$ implies $x \in \Diamond\Box O$. More specifically, we have to prove that for all $x \in O$ there exists $y \in X$ with $x\ R\ y$ and $R(y, -) \subseteq O$.
   (a) If $x \neq x_0$. Then, $x\ R\ x$ and $R(x, -) = \{x\} \subseteq O$.
   (b) If $x = x_0$. Then, as x_0 is an accumulation point, there is $y \in O \setminus \{x\}$. We thus have $x_0\ R\ y$ and $R(y, -) = \{y\} \setminus O$.
2. On the other hand, $\mathfrak{X}$ does not verify (2.6). This follows from the fact that, for all $y \in X$, $R(y, -) \neq \{x_0\}$.

In short, we just proved that, for a subordination space $\mathfrak{X}$, $\mathfrak{X} \models p \to \Diamond\Box p$ does not imply $\mathfrak{X} \models (2.6)$. Notice that it is however quite simple to prove the converse implication.

Example 2.9.2. Note that, while $p \to \Diamond\Box p$ may not have a first order correspondent when considered alone, since it is not an s-Sahlqvist formula, this is not the case anymore when it is associated with other bimodal formulas.

1. For a subordination space $\mathfrak{X} = (X, R)$, we have

$$\mathfrak{X} \models p \to \Diamond\Box p, \underbrace{\Box p \to \Box\Box p}_{\text{transivity}}, \underbrace{p \to \Box\Diamond p}_{\text{symmetry}} \tag{2.9}$$

   if and only if

$$\mathfrak{X} \models x\ R\ y \leftrightarrow x = y. \tag{2.10}$$

   It it almost immediate that (2.10) implies (2.9), so let us focus on the other implication. Suppose that there exist $x, y \in X$ such that $x\ R\ y$ and $x \neq y$. Since X is a Stone space, this implies that $x \in O$ and $y \in O^c$ for some clopen set O. Now, by $\mathfrak{X} \models p \to \Diamond\Box p$ and $x \in O$, we have that $x \in \Diamond\Box O$, hence, there exists $z \in X$ such that $x\ R\ z$ and $R(z, -) \subseteq O$. By symmetry and transitivity, we have $y \in R(z, -) \subseteq O$, which is absurd.

2. For a subordination space $\mathfrak{X} = (X, R)$, we have
$$\mathfrak{X} \models p \to \Diamond\Box p,\ \underbrace{\Diamond p \to \Box p}_{functionality} \tag{2.11}$$
if and only if
$$\mathfrak{X} \models (\forall x)(\exists y)(x\ R\ y \text{ and } R(y, -) = \{x\}). \tag{2.12}$$
Once again, we have immediately that (2.12) implies (2.11). For the other implication, set $x \in X$. Since $\mathfrak{X} \models p \to \Diamond\Box p$, it follows that $x \in \Diamond\Box X$ and, hence, in particular, that there exists $y \in \Box X$ such that $x\ R\ y$. Now, by functionality, we have that $R(x, -) = \{y\}$.

   By contradiction, suppose that $y\ R\ z$ for some $z \neq x$. Since X is a Stone space, there exists a clopen set O with $x \in O$ and $z \in O^c$. Moreover, we have $x\ R\ t$ and $R(t, -) \subseteq O$ for some t. But, we know that t is necessarily y, so that $z \in R(y, -) \subseteq O$, which is absurd.

Remark 2.9.3. Let us consider the second item of Example 2.9.2 in a modal setting. We have that the following are equivalent:

- $\mathfrak{X} \models p \to \Diamond\Box p, \Diamond p \to \Box p$,
- $\mathfrak{X} \models (\forall x)(\exists y)(x\ R\ y \text{ and } R(y, -) = \{x\})$,
- $\mathfrak{X} \models p \to \Diamond\Box p$.

In particular, this implies that there exists a syntactical proof of $\Diamond p \to \Box p$ when $p \to \Diamond\Box p$ is considered as an axiom. Moreover, we know that this proof should at least contain one substitution, since modus ponens and necessitation are sound in the subordination setting.

As we already said, we know that $\mathfrak{X} \models p \to \Diamond\Box p$ is not equivalent to
$$\mathfrak{X} \models (\forall x)(\exists y)(x\ R\ y \text{ and } R(y, -) = \{x\}).$$
Moreover, we have to consider another property that the formula $p \to \Diamond\Box p$ lacks. Indeed, recall that we proved that it was not scheme extensible. Therefore, one could ask: what about $\mathfrak{X} \models \varphi \to \Diamond\Box\varphi$? Even if we do not have an answer yet, the previous example gives us the following proposition.

Proposition 2.9.4. *Let $\mathfrak{X}$ be a subordination space. We have*[2]
$$[\mathfrak{X} \models \varphi \to \Diamond\Box\varphi] \Leftrightarrow [\mathfrak{X} \models (\forall x)(\exists y)(x\ R\ y \text{ and } R(y, -) = \{x\})].$$
if and only if $\mathfrak{X} \models \varphi \to \Diamond\Box\varphi$ implies $\mathfrak{X} \models \Diamond p \to \Box p$.

Proof. $\Rightarrow$ We have successively, by Example 2.9.2,
$$\begin{aligned} & \mathfrak{X} \models \varphi \to \Diamond\Box\varphi \\ \Rightarrow\ & \mathfrak{X} \models (\forall x)(\exists y)(x\ R\ y \text{ and } R(y, -) = \{x\}) \\ \Rightarrow\ & \mathfrak{X} \models \Diamond p \to \Box p \end{aligned}$$

$\Leftarrow$ Since, in this case, $\mathfrak{X} \models \varphi \to \Diamond\Box\varphi$ implies both $\mathfrak{X} \models p \to \Diamond\Box p$ and $\mathfrak{X} \models \Diamond p \to \Box p$, the conclusion follows immediately from Example 2.9.2.

$\Box$

[2] Note that the necessary condition of Proposition 2.9.4 is always verified for modal spaces.

2.10 Further correspondences

As stated in the introduction of Section 2.7, subordination algebras (and hence, subordination spaces) are an ideal setting to study the interaction between three languages: the modal one, the subordination one and the accessibility one.

Nevertheless, until now, we did not define precisely the subordination syntax and we actually have two options to consider. The first option is the one we implicitly considered up to this point in this thesis and which also considered in [66] (under the name statements): namely, the subordination syntax is constructed in a similar way to the one of the accessibility relation. Hence, we have first-order formulas/statements such as

$$a \prec b \Rightarrow \exists c : a \prec c \prec b.$$

The second option is to consider the subordination syntax as the modal syntax (that is without quantifiers). This second option is considered for instance in [1] and [74]. Let us have a look at both options.

2.10.1 Subordination statements

Using the accessibility language as a bridge, we know that there exist modal formulas which are equivalent to subordination statements. Indeed, consider for instance the following correspondences.

Tense	Accessibility	Subordination
$\Box p \to \Box\Box p$	Transitivity	$a \prec b \Rightarrow \exists c :\ a \prec c \prec b$
$p \to \Diamond p$	Reflexivity	$a \prec b \Rightarrow a \leq b$
$p \to \Box\Diamond p$	Symmetry	$a \prec b \Rightarrow \neg b \prec \neg a$
$\Box p \to \Diamond p$	Right seriality	$p \neq 0 \Rightarrow p \not\prec 0$
$\blacksquare p \to \blacklozenge p$	Left seriality	$p \neq 1 \Rightarrow 1 \not\prec p$

A natural question that now arises is to determine whether it is possible to bypass the accessibility bridge between bimodal formulas and subordination statements or not. Let us start with an example.

Example 2.10.1. Let $\mathfrak{X} = (X, R)$ be a subordination space such that $\mathfrak{X} \models \Box p \to \Box\Box p$. Hence, for every clopen subset O of X, we have

$$\Box O \subseteq \Box\Box O. \tag{2.13}$$

Now, $\Box O$ is an open subset of X and, as such, it is a union of clopen subsets. Therefore, (2.13) is equivalent to

$$\forall U \in \mathrm{Clop}(X) : U \subseteq \Box O \Rightarrow U \subseteq \Box\Box O. \tag{2.14}$$

Let us have a detailed look at the antecedent and the consequent of (2.14). First, we have

$$U \subseteq \Box O \Leftrightarrow U \subseteq R(-, O^c)^c \Leftrightarrow R(-, O^c) \subseteq U^c \Leftrightarrow \neg O \prec \neg U.$$

Then, for in consequent, we have $U \subseteq \Box\Box O$ if and only if $\blacklozenge U \subseteq \Box O$. Furthermore, since $\blacklozenge U$ is closed and $\Box O$ is open, $\blacklozenge U \subseteq \Box O$ is equivalent to

$$\exists V \in \mathrm{Clop}(X) : \blacklozenge U \subseteq V \subseteq \Box O.$$

With the technique used for the antecedent, we have that $\blacklozenge U \subseteq V \subseteq \Box O$ is equivalent to $\neg O \prec \neg V \prec \neg U$. Renaming $\neg O, \neg U$ and $\neg V$ respectively as O, U and V, we obtain that (2.14) is equivalent to

$$O \prec U \Rightarrow \exists V : O \prec V \prec U,$$

as required.

Hence, we just saw a way to establish a subordination statement correspondent to the modal formula $\Box p \to \Box\Box p$. The key feature in this example is that, for propositional variables p and q, a subordination space $\mathfrak{X}$ and a valuation v, we have $\mathfrak{X} \models_v \Diamond p \to q$ if and only if $\mathfrak{X} \models_v p \prec q$, $\mathfrak{X} \models_v p \to \Box q$ if and only if $\mathfrak{X} \models_v \neg q \prec \neg p$ and $\mathfrak{X} \models_v p \to q$ if and only if $\mathfrak{X} \models_v p \leq q$. Hence, we will now define a fragment of (bi)modal formulas for which these equivalences can be adequately used in order to find a correspondence. Another important fact has to be noted in Example 2.10.1. The variable of $\Box p \to \Box\Box p$ (that is p or, in its topological valuation, O) stays a variable of its correspondent statement and we added variables which were disjoint from p (in this case U and V).

Definition 2.10.2. Let $\varphi(\underline{p})$ be a bimodal formula. We say that

1. φ **has a correspondent subordination statement** (or in short **has a css**) if there exists a subordination statement $\Phi(\underline{p})$, effectively computable from φ, such that, for every subordination space $\mathfrak{X}$, we have

$$\mathfrak{X} \models \varphi(\underline{p}) \text{ if and only if } \mathfrak{X} \models \Phi(\underline{p}).$$

2. φ **has an upper correspondent subordination statement** (or in short **has a ucss**) if there exists a subordination statement $\Phi_u(\underline{p}, q)$, effectively computable from $q^{\pm} \to \varphi$ (where $q^{\pm}$ is a shorthand for q or $\neg q$), such that, for every subordination space $\mathfrak{X}$, we have

$$\mathfrak{X} \models q^{\pm} \to \varphi(\underline{p}) \text{ if and only if } \mathfrak{X} \models \Phi_u(\underline{p}, q).$$

3. φ **has a down correspondent subordination statement** (or in short **has a dcss**) if there exists a subordination statement $\Phi_d(\underline{p}, q)$, effectively computable from $\varphi \to q^{\pm}$, such that, for every subordination space $\mathfrak{X}$, we have

$$\mathfrak{X} \models \varphi(\underline{p}) \to q^{\pm} \text{ if and only if } \mathfrak{X} \models \Phi_d(\underline{p}, q).$$

Before we start to prove the expected correspondence theorem, we have the following useful lemma.

Lemma 2.10.3. *Let φ be a bimodal formula.*

1. *If φ has a ucss, then φ has a css.*
2. *If φ has a dcss, then $\neg\varphi$ has a css.*

Proof. Let $\underline{p}$ denotes the variables of φ and let us prove Item 1 (Item 2 is proved similarly). Suppose that $\mathfrak{X}$ is a subordination space such that $\mathfrak{X} \models \varphi(\underline{p})$. Then, in particular, we have that $\mathfrak{X} \models \top \to \varphi(\underline{p})$. Hence, if q is a variable distinct from $\underline{p}$, we have that $\mathfrak{X} \models_v q \to \varphi(\underline{p})$ for all valuations v' such that $v(q) = 1$. Therefore, since φ has a ucss $\Phi(\underline{p}, q)$, we know that $\mathfrak{X} \models \varphi(\underline{p})$ if and only $\mathfrak{X} \models \Phi(\underline{p}, 1)$ and the proof is complete. $\square$

It turns out that we already have encountered fragments of formulas having css. Indeed, consider the following proposition.

Proposition 2.10.4. *If φ is an open (resp. a closed) formula, then φ and $\neg\varphi$ have both a ucss and a dcss.*

Proof. Let us prove the open case by induction on the length of φ.

1. If φ is a (negation of a) variable, then we already observed that $p^\pm \to q^\pm$ was equivalent to $p^\pm \leq q^\pm$.

2. If $\varphi = \psi \vee \chi$, then ψ and χ are again open. Since the formula $(\psi \vee \chi) \to q^\pm$ is equivalent to $(\psi \to q^\pm) \wedge (\chi \to q^\pm)$, we can use the induction to conclude. Now, let us consider the case $q^\pm \to (\psi \vee \chi)$. We know that a valuation of $q^\pm$ is a clopen set (let O denote this valuation) and that valuations of χ and ψ are open (valuations that we will denote respectively by ω_1 and ω_2). Moreover, we have $q^\pm \to \chi \vee \psi$ if and only if $O \subseteq \omega_1 \cup \omega_2$. Hence, in particular, there exist clopen sets U_1 and U_2 such that $O \setminus \omega_2 \subseteq U_1 \subseteq \omega_1$ and $O \setminus \omega_1 \subseteq U_2 \subseteq \omega_2$. It follows that $V_1 = (U_1 \cup (O \setminus U_2))$ and $V_2 = (U_2 \cup (O \setminus U_1))$ are clopen subsets such that $O \subseteq V_1 \cup V_2$ and $V_i \subseteq \omega_i$. Therefore, we have $q^\pm \to \psi \vee \chi$ if and only

$$\exists r, s : (r \to \psi) \wedge (s \to \chi) \wedge (q^\pm \to r \vee s).$$

   Finally, the conclusion follows from the induction.

3. If $\varphi = \psi \wedge \chi$, then ψ and χ are also open. The case $q^\pm \to (\psi \wedge \chi)$ is trivial and we can focus on the second one. For $(\psi \wedge \chi) \to q^\pm$, since $(\psi \wedge \chi)$ is open, we know that it is an union of clopen subsets. Therefore, we have $(\psi \wedge \chi) \to q^\pm$ if and only if

$$\forall r : r \leq (\psi \wedge \chi) \Rightarrow (r \leq q^\pm),$$

   that is

$$\forall r : [(r \leq \psi) \wedge (r \leq \chi)] \Rightarrow (r \leq q^\pm).$$

   It is now enough to use the induction hypothesis on $r \leq \psi$ and $r \leq \chi$ to conclude the proof.

4. Suppose that $\varphi = \Box\psi$. Then $q^\pm \to \Box\psi$ is equivalent to $\blacklozenge q^\pm \to \psi$. Now, since $\blacklozenge q^\pm$ is closed and ψ if open, we know that $\blacklozenge q^\pm \to \psi$ if equivalent to

$$\exists r : (\blacklozenge q^\pm \leq r) \wedge (r \leq \psi).$$

   The conclusion once again follows from the induction on $r \leq \psi$. Moreover, for $\Box\psi \to q^\pm$, we can use the fact that $\Box\psi$ is open to have that the formula is equivalent to

$$\forall r : (r \leq \Box\psi) \Rightarrow r \leq q^\pm,$$

   and, since we just proved that $\Box\psi$ has a ucss, the proof is complete. Note that the remaining case $\varphi = \blacksquare\psi$ can be treated similarly.

□

We now use this proposition and the generalised Esakia Lemma 2.7.10 to conclude this section.

Definition 2.10.5. A bimodal formula φ is **g-closed** (resp. **g-open**) (where the letter g stands for generalised) if it is obtained from closed formulas (resp. open formulas) by applying $\wedge$, $\vee$, $\Box$ and $\blacksquare$ (resp. $\vee$, $\wedge$, $\Diamond$ and $\blacklozenge$). In particular, if φ is g-closed (resp. g-open) if and only if there is an open positive (resp. closed positive) formula $\chi(\underline{p})$ and closed (resp. open) formulas $\underline{\psi}$ such that $\varphi = \chi(\underline{\psi})$.

Theorem 2.10.6. *If φ_1 is g-open and φ_2 is g-closed, then $\varphi_1 \to \varphi_2$ has a css.*

Proof. By the definition, we know that $\varphi_1 = \chi_1(\underline{\psi}_1)$ where χ_1 is closed positive and $\underline{\psi}_1$ are open and that $\varphi_2 = \chi_2(\underline{\psi}_2)$, where χ_2 is open positive and $\underline{\psi}_2$ are closed. Now, by Proposition 2.7.10, we have that

$$\varphi_1 = \vee\{\chi_1(\underline{p}_1) \mid \underline{p}_1 \leq \underline{\psi}_1\} \text{ and } \varphi_2 = \wedge\{\chi_2(\underline{p}_2) \mid \underline{p}_2 \geq \underline{\psi}_2\}.$$

Hence, $\varphi_1 \to \varphi_2$ is equivalent to

$$\forall \underline{p}_1 \leq \underline{\psi}_1 \ \forall \underline{\psi}_2 \leq \underline{p}_2 : \chi_1(\underline{p}_1) \leq \chi_2(\underline{p}_2). \tag{2.15}$$

Since $\chi_1(\underline{p}_1)$ is closed and $\chi_2(\underline{p}_2)$ is open, it follows that (2.15) is equivalent to

$$\forall \underline{p}_1 \leq \underline{\psi}_1 \ \forall \underline{p}_2 \geq \underline{\psi}_2 \ \exists q : \chi_1(\underline{p}_1) \leq q \text{ and } q \leq \chi_2(\underline{p}_2).$$

We can hence conclude by Proposition 2.10.4. □

The g-closed and g-open fragments relate to the ones already mentioned in Section 2.7 as described in the next proposition.

Proposition 2.10.7. *Let φ be a bimodal formula.*

1. *If φ is s-positive, then φ is g-closed.*
2. *If $\varphi(\underline{p})$ is s-negative, then $\varphi^\partial(\neg\underline{p})$ is g-closed.*
3. *If $\varphi(\underline{p})$ is strongly positive, then $\varphi^\partial(\neg\underline{p})$ is g-closed.*
4. *If $\varphi(\underline{p})$ is s-untied, then $\varphi^\partial(\neg\underline{p})$ is g-closed.*
5. *If φ is an s-Sahlqvist formula, then φ is equivalent to a g-closed formula.*
6. *The formula φ is g-closed if and only if the formula φ^∂ (in the sense of Definition 2.7.3) is g-open.*

Proof. Items 1 and 6 follow immediately from the definitions while item 2 follows from item 1 and Lemma 2.7.4. Let us prove item 3. Since $\varphi(\underline{p})$ is strongly positive, it is in particular open. Therefore $\varphi^\partial(\neg\underline{p})$ is (g-)closed, as required. Let us prove item 4 by induction on the length of φ. The cases where φ is strongly positive or s-negative have already been considered in the previous item. If $\varphi(\underline{p}) = \psi(\underline{p}) \wedge \theta(\underline{p})$, then $\varphi^\partial(\neg\underline{p}) = \psi^\partial(\neg\underline{p}) \vee \theta^\partial(\neg\underline{p})$, and we can conclude by induction. If $\varphi(\underline{p}) = \Diamond\theta(\underline{p})$, then $\varphi^\partial(\underline{p}) = \Box\theta^\partial(\neg\underline{p})$, and we can again conclude by induction. Therefore, it remains to consider item 5. By definition, there exist an s-untied formula θ and an s-positive formula ψ such that $\varphi(\underline{p}) = \Box^\epsilon(\theta(\underline{p}) \to \psi(\underline{p}))$. In particular, for a subordination space $\mathfrak{X}$, we have

$$\mathfrak{X} \models \varphi \text{ if and only if } \mathfrak{X} \models \Box^\epsilon(\psi(\underline{p}) \vee \theta^\partial(\neg\underline{p})).$$

Now, $\psi(\underline{p}) \vee \theta^\partial(\neg\underline{p})$ is g-closed by the previous items and the conclusion follows from the definition of g-closed formulas.

□

In particular, we hence have the following corollary.

Corollary 2.10.8. *If φ is an s-Sahlqvist formula, then φ has a css.*

2.10.2 Subordination formulas

Definition 2.10.9 ([1]). The **subordination language** is constituted by:

- a set of Boolean variables $\mathrm{Var} = \{p, q, r, \dots\}$,
- the Boolean operations $\cup$ and $-$ and Boolean constants $0, 1$,
- the propositional connectives $\vee$, $\neg$ and the propositional constants $\top$ and $\bot$,
- the subordination connectives $\leq$ and $\prec$.

In addition to the usual shortcuts, we also have the following ones

$$a = b := (a \leq b) \vee (b \leq a) \mid a \neq b := \neg(a = b) \mid a \not\prec b := \neg(a \prec b).$$

From this language, we can define the **Boolean terms**, which are inductively constructed as follows:

$$a ::= p \mid 0 \mid 1 \mid a' \mid (a \cup b).$$

Then, we have the **subordination formulas** which are, for their part, inductively constructed as follows:

$$\varphi ::= a \leq b \mid a \prec b \mid \top \mid \bot \mid \neg\varphi \mid \varphi \vee \psi.$$

Note that in the subordination language, we indeed have two sets of connectives: one for the terms and one for the formulas.

Let us now turn on the definitions of validity.

Definition 2.10.10. As usual, a valuation v on a subordination space $\mathfrak{X} = (X, R)$ is a map $v : \mathrm{Var} \longrightarrow \mathrm{Clop}(X)$ which is inductively extend on the set of terms as follows: $v(1) = X$, $v(0) = \emptyset$, $v(a') = v(a)^c$ and $v(a \cup b) = v(a) \cup v(b)$. The validity of a subordination formula φ for this valuation v in $\mathfrak{X}$ is inductively defined as follows:

1. $\mathfrak{X} \models_v a \leq b$ if and only if $v(a) \subseteq v(b)$,
2. $\mathfrak{X} \models_v a \prec b$ if and only if for all $x \in v(a)$, $y\ R\ x$ implies $y \in v(b)$ (that is $R(-, v(a)) \subseteq v(b)$),
3. $\mathfrak{X} \not\models_v \bot$ and $\mathfrak{X} \models_v \top$,
4. $\mathfrak{X} \models_v \neg\varphi$ if and only if $\mathfrak{X} \not\models_v \varphi$,
5. $\mathfrak{X} \models_v \varphi \vee \psi$ if and only if $\mathfrak{X} \models_v \varphi$ or $\mathfrak{X} \models_v \psi$.

Let us now consider the following definition. A subordination formula φ has a first order correspondent if there exists a first order formula Φ in the language of the accessibility relation such that, for every subordination space $\mathfrak{X}$, we have

$$\mathfrak{X} \models \varphi \text{ if and only if } \mathfrak{X} \models \Phi. \tag{2.16}$$

Balbiani and Kikot established in [1] a family of subordination formulas, that we choose to call **Balbiani-Kikot formulas**[3] in this work, for which there exists a first order formula Φ satisfying (2.16). This fragment of formulas was further extended by Santoli in [66].

Since Balbiani-Kikot formulas behave like s-Sahlqvist formulas, one can wonder if it is possible to obtain the first family from the second, or vice-versa. However, this will be an impossible challenge. Indeed, consider the following examples from [74, Section 3].

[3] In [1, Section 2], Balbiani and Kikot used the name Sahlqvist formula. We changed it to prevent any possible confusion between Sahlqvist formulas for the standard tense language and Sahlqvist formulas for the subordination language.

1. The Balbiani-Kikot formula $\varphi : p \neq 0 \wedge q \neq 0 \rightarrow p \not\prec q'$ corresponds to the universality of R (that is $(\forall x, y)(x\ R\ y)$,) for which there is no correspondent bimodal formula.

2. The s-Sahlqvist formula $\varphi : \Box p \rightarrow \Box\Box p$ corresponds to the transitivity of R, for which there is no correspondent subordination formulas (recall that (S6) is not a subordination formula in the sense of Definition 2.10.9).

Therefore, we know that there can be no correspondence between s-Sahlqvist and Balbiani-Kikot formulas. However, Vakarelov gave in [74, Section 3] a translation from subordination formulas (or, more precisely pre-contact formulas, see Appendix C for further details) to formulas in a pluri-modal language, one of the modalities being the universal one. Since this discussion is more linked to the arbitrary signatures case than to the tense one, we will continue this discussion at the end of Chapter 4.

Chapter 3

Slanted duality

In this chapter, we first generalise the concept of subordination algebras to the concept of cs-lattices, that is distributive lattices endowed with a pre-**c**ontact relation $\mathcal{C}$ and a **s**ubordination relation $\prec$. Then, we give conditions on the relation $\mathcal{C}$ and $\prec$ to be generated by an unique relation on the Priestley dual.

We then go one step further by generalising the concept of cs-lattice to slanted lattices, that is distributive lattices endowed with maps that send elements of the said lattices to elements of their canonical extensions. Then, we introduce the slanted Priestley spaces as the topological part of the generalisation from subordination to slanted. In addition, we provide a suitable definition for the morphisms in these categories.

Some dualities with a similar sketch are to be considered. For instance, Sofronie-Stokkermans established in [67] to "clopen" correspondent of the duality. Moreover, Celani and Castro established in [12] the unary case. We hence prove that both these dualities are particular cases of the duality presented here.

We end the chapter with outlines for a universal algebraic study of slanted lattices.

3.1 Canonical extensions of maps

To complete the reminder about canonical extension theory started in Section 2.4, we will now recall how to extend a map $h : M \longrightarrow L$ between two lattices to a map $h : M^\delta \longrightarrow L^\delta$ between their canonical extensions.

Definition 3.1.1 ([34])**.** Let L and M be lattices and let $h : M \longrightarrow L$ be some map. The **σ-extension** h^σ of h is defined first for every $k \in \mathcal{K}(M^\delta)$ as

$$h^\sigma(k) = \wedge\{h(a) \mid k \leq a \in L\}$$

and then, for every $u \in L^\delta$,

$$h^\sigma(u) = \vee\{h^\sigma(k) \mid k \in \mathcal{K}(L^\delta) \text{ and } k \leq u\}.$$

The **π-extension** h^π of h is defined first for every $o \in \mathcal{O}(L^\delta)$ as

$$h^\pi(o) = \vee\{h(a) \mid a \in L \text{ and } a \leq o\}$$

and then, for every $u \in L^\delta$,

$$h^\pi(u) = \wedge\{h^\pi(o) \mid o \in \mathcal{O}(M^\delta) \text{ and } u \leq o\}.$$

Definition 3.1.2. We say that a map h between lattices is **smooth** if its σ-extension and its π-extension correspond. It this case, both extensions are denoted h^δ.

Lemma 3.1.3. *Let $h : L \longrightarrow M$ be a map between lattices,*

1. *If h^σ preserves non-empty joins, then h is smooth,*
2. *if h^π preserves non-empty meets, then h is smooth.*

Remark 3.1.4. Let L and M be lattices and $h : L^n \longrightarrow M$ a map. Then, h^σ and h^π should be map between $(L^\delta)^n$ and M^δ. But we have $(L^n)^\delta \cong (L^\delta)^n$ (see [34]). Therefore, h^σ and h^π can also be considered as n-ary maps. Moreover, if L^∂ denotes the order-dual lattice of L, one also has $(L^\partial)^\delta \cong (L^\delta)^\partial$.

Lemma 3.1.5. *Let $h : L^n \longrightarrow M$ be an order preserving map. If h preserves finite joins in its i^{th} coordinate, then h^σ preserves arbitrary non-empty joins in its i^{th} coordinate. Dually, if h preserves finite meets in its i^{th} coordinate, then h^σ preserves arbitrary non-empty meets in its i^{th} coordinate*

Corollary 3.1.6. *If h is a lattice morphism, then h^δ is a complete lattice morphism.*

Remark 3.1.7. It may seem that any application that respects finite meets or joins coordinatewise is smooth. While this is true for unary operators, it is not true for binary operators (see for instance [36, Example 6]). For non-smooth operators, we want of course to consider the extension that preserves the properties of the original map. For instance, if h preserves finite joins, we will consider its σ-extension.

3.2 Cs lattices

In Chapter 2, we saw that subordination relations on Boolean algebras are a concept dual to closed relations on Stone spaces. We also mentioned that subordination relations and pre-contact relations on Boolean algebras are interdefinable via the relation

$$a \,\mathcal{C}\, b \Leftrightarrow a \not\prec \neg b. \tag{3.1}$$

This situation mirrors the one existing in modal algebras, where modal operators are a concept dual to closed relations preserving clopen sets and where $\Diamond$ and $\Box$ operators are interdefinable via

$$\Box a = \neg\Diamond\neg a. \tag{3.2}$$

In short, in both cases, it is sufficient to have a relation on a Stone space to obtain two operations/relations on its dual algebra. Now, in a lattice environment, the interdefinitions (3.1) and (3.2) do not hold. Consequently, having a unique closed relation on a Priestley does not guaranteed the existence of two relations/operations on its dual.

In [58, Definition 4.1], Přenosil gave conditions for a modal lattice $\mathfrak{L} = (L, \Diamond, \Box)$ to have both modal operators $\Diamond$ and $\Box$ generated by a unique relation R on the Priestley dual of L. Namely, $\mathfrak{L}$ must satisfy the axioms

P. $\Diamond b \leq (\Box a \vee c)$ implies $\Diamond b \leq \Diamond(a \wedge b) \vee c$,

N. $\Diamond a \wedge c \leq \Box b$ implies $\Box(a \vee b) \wedge c \leq \Box b$.

This section is devoted to find conditions on a structure $\mathfrak{L} = (L, \mathcal{C}, \prec)$ such that $\mathcal{C}$ and $\prec$ are generated by a unique relation on the Priestley dual of L.

Definition 3.2.1.
- A **pre-contact lattice** is a structure $\mathfrak{L} = (L, \mathcal{C})$ where L is a bounded distributive lattice and $\mathcal{C}$ is a pre-contact relation on L, that is a binary relation such that

  C1. $1 \not\mathcal{C}\, 0$ and $0 \not\mathcal{C}\, 1$,

  C2. $a\, \mathcal{C}\, b \vee c$ implies $a\, \mathcal{C}\, b$ or $a\, \mathcal{C}\, c$,

  C2'. $a \vee b\, \mathcal{C}\, c$ implies $a\, \mathcal{C}\, c$ or $b\, \mathcal{C}\, c$,

  C3. $a \geq b\, \mathcal{C}\, c \leq d$ implies $a\, \mathcal{C}\, d$.

- A **subordination lattice** is a structure $\mathfrak{L} = (L, \prec)$ where L is a bounded distributive lattice and $\prec$ is a subordination relation in L, that is a binary relation such that

  S1. $0 \prec 0$ and $1 \prec 1$,

  S2. $a \prec b, c$ implies $a \prec b \wedge c$,

  S2'. $a, b \prec c$ implies $a \vee b \prec c$,

  S3. $a \leq b \prec c \leq d$ implies $a \prec d$.

- A **pre-contact subordination lattice** (in short **cs lattice**) is a structure $\mathfrak{L} = (L, \mathcal{C}, \prec)$ where $(L, \mathcal{C})$ is a pre-contact lattice, $(L, \prec)$ is a subordination lattice and where the relation $\mathcal{C}$ and $\prec$ are connected by the following axioms

  CS1. $a \prec (b \vee c)$ and $a \not\mathcal{C}\, b$ implies $a \prec c$,

  CS2. $a \not\mathcal{C}\, (b \wedge c)$ and $a \prec b$ implies $a \not\mathcal{C}\, c$.

Remark 3.2.2. As we already said in the introduction, we know that for Boolean algebras, pre-contact relations and subordination relations are bound by the equivalence (see Appendix C.1.5)

$$a \prec b \Leftrightarrow a \not\mathcal{C}\, \neg b. \tag{3.3}$$

However, this equivalence between $\prec$ and $\mathcal{C}$ does not hold anymore in a lattice context, obviously because of the possible non-existence of a complementary element $\neg b$.

Hence, axioms CS1 and CS2 of Definition 3.2.1 can be interpreted as a generalisation of (3.3) to lattices as the following properties are equivalent for a cs-lattice $\mathfrak{L} = (B, \mathcal{C}, \prec)$ where B is a Boolean algebra:

1. $(B, \mathcal{C}, \prec)$ satisfies (3.3),
2. $(B, \mathcal{C}, \prec)$ satisfies axioms CS1 and CS2.

Indeed, suppose that (3.3) is satisfied. Then, $a \prec (b \wedge c)$ and $a \not\mathcal{C}\, b$ implies $a \prec (b \vee c)$ and $a \prec \neg b$. Hence, by S2, we get

$$a \prec (b \vee c) \wedge \neg b = c \wedge \neg b \leq c,$$

and axiom CS1 then follows from S3. Moreover, if $a \not\mathcal{C}\, (b \wedge c)$ and $a \prec b$ then, again by S2,

$$a \prec (\neg b \vee \neg c) \wedge b = b \wedge \neg c \leq \neg c,$$

such that CS2 follows also from S3.

Now, suppose that CS1 and CS2 are satisfied and suppose first that $a \prec b$. By C3 and C1, we draw that $a \not\mathcal{C}\, 0 = (b \wedge \neg b)$ and therefore, by CS2, that $a \not\mathcal{C}\, \neg b$. Secondly, suppose that $a \not\mathcal{C}\, \neg b$. It follows from S1 and S4 that $a \prec 1 = (b \vee \neg b)$. Hence, we obtain $a \prec b$ from CS1.

Note that, quite obviously, if $(B, \mathcal{C}, \prec)$ is a cs lattice, then $(B, \prec)$ is a subordination algebra. Therefore, it has a dual subordination space (X, R). Now, we are able to construct both relations $\mathcal{C}$ and $\prec$ from R, since $a \prec b$ if and only if $R(-, \eta(a)) \subseteq \eta(b)$ and $a\, \mathcal{C}\, b$ if and only if the intersection $R(-, \eta(a)) \cap \eta(b)$ is not empty.

Now that the algebraic objects are settled, we can look for the topological ones. Of course, we begin first with a Priestley space endowed with two relations: one related to $\mathcal{C}$ and one to $\prec$. We will merge these two topological relations latter. For future convenience, we introduce the following notation.

Notation 3.2.3. Let X be any set and R, S be binary relations on X. We will denote by $R \circ S$ the binary relation defined by

$$x\ (R \circ S)\ y \Leftrightarrow \exists z \in X : x\ R\ z\ S\ y.$$

Definition 3.2.4. Let $\mathfrak{L} = (L, \mathcal{C}, \prec)$ be a cs-lattice, then $\mathfrak{X}_{\mathfrak{L}}$ denotes the tuple $(X, \leq, R_{\prec}, S_{\mathcal{C}})$ where $(X, \leq)$ is the Priestley dual of L and $R_{\prec}$ and $S_{\mathcal{C}}$ are binary relations on X defined by:

1. $x\ R_{\prec}\ y$ if $\prec(y, -) \subseteq x$.
2. $x\ S_{\mathcal{C}}\ y$ if $(y \times x) \subseteq \mathcal{C}$,

Of course the relation $R_{\prec}$ is defined in a similar way to the relation associated to a subordination relation in Chapter 2. Moreover, the relation $S_{\mathcal{C}}$ is symmetric to the usual accessibility relation used in pre-contact setting (see for instance [1] or [51]).

Definition 3.2.5. A **cs Priestley space** is a tuple $\mathfrak{X} = (X, \leq, R, S)$ where $(X, \leq)$ is a Priestley space and R and S are binary closed relations which satisfy the following axioms:

$T1$. $S = \leq \circ S \circ \geq$,

$T2$. $R = \geq \circ R \circ \geq$,

$TCS1$. $x\ R\ y$ implies that $z \leq x$ and $z\ (R \cap S)\ y$ for some $z \in X$,

$TCS2$. $x\ S\ y$ implies that $x \leq z$ and $z\ (R \cap S)\ y$ for some $z \in X$.

Proposition 3.2.6. *If $\mathfrak{L} = (L, \mathcal{C}, \prec)$ is a cs lattice, then $\mathfrak{X}_{\mathfrak{L}}$ is a cs Priestley space.*

Proof. $T1$. Consider $x, y \in X_L$ with $(x, y) \notin S_{\mathcal{C}}$. Then, there exists a pair $(a, b) \in (y \times x)$ such that $a \not{\mathcal{C}}\ b$. In particular, $(y, x) \in \eta(a) \times \eta(b)$ and $\eta(a) \times \eta(b) \subseteq S_{\mathcal{C}}^c$, it follows that $S_{\mathcal{C}}^c$ is open.

By reflexivity of the order, it is clear that $S_{\mathcal{C}} \subseteq \leq \circ S_{\mathcal{C}} \circ \geq$. Now, suppose that $x \leq \circ\ S_{\mathcal{C}} \circ \geq y$, that is, there exist $s, t \in X_L$ such that $x \leq s\ S_{\mathcal{C}}\ t \geq y$. In other words, we have

$$y \times x \subseteq t \times s \subseteq \mathcal{C},$$

whence, the conclusion.

$T2$. The proof is similar.

$TCS2$. We have to find a prime filter z such that $x \subseteq z$, $\prec(y, -) \subseteq z$ and $\not{\mathcal{C}}(y, -) \subseteq z$. Since the subset $\not{\mathcal{C}}(y, -)$ is an ideal, this can be done by proving that

$$\not{\mathcal{C}}(y, -) \cap \langle x \cup \prec(y, -)\rangle_{\text{filt}} = \emptyset.$$

If this is not the case, then there would be an element $b \in L$ such that there exists $a \in y$ with $a \not{\mathcal{C}}\ b$, and such that there exist an two elements $d \in x$ and $c \in L$ such that $b \geq c \wedge d$ and $a' \prec c$ for some $a' \in y$.

Using the axioms of $\mathcal{C}$ and $\prec$, we draw that $(a \wedge a') \not{\mathcal{C}}\ (c \wedge d)$ and $(a \wedge a') \prec c$. Therefore, by axiom CS2, it follows that $(a \wedge a') \not{\mathcal{C}}\ d$. But, we know that $d \in x$, $a \wedge a' \in y$ and $y \times x \subseteq \mathcal{C}$, hence this is absurd.

$TCS1$. With a similar argument, it follows from

$$\prec(y,-)\cap\langle x^c\cup \not{\mathcal{C}}(y,-)\rangle_{\mathrm{id}}=\emptyset$$

that $R_\prec$ and $S_\mathcal{C}$ satisfy TCS1.

□

Now, we have the dual versions of Definition 3.2.4 and Proposition 3.2.6.

Definition 3.2.7. If $\mathfrak{X}=(X,\leq,R,S)$ is a cs Priestley space, then we denote by $\mathfrak{L}_\mathfrak{X}$ the tuple $(L,\mathcal{C}_S,\prec_R)$ where $L=\uparrow\mathrm{Clop}(X)$ is the Priestley dual of $(X,\leq)$ and $\mathcal{C}_S$ and $\prec_R$ are the binary relations on L defined by

$$O\;\mathcal{C}_S\;U\Leftrightarrow S(U,-)\not\subseteq O^c \text{ and } O\prec_R U\Leftrightarrow R(-,O)\subseteq U.$$

Proposition 3.2.8. *If $\mathfrak{X}_\mathfrak{L}=(X,\leq,R,S)$ is a cs Priestley space, then $\mathfrak{L}_\mathfrak{X}$ is a cs lattice.*

Proof. It is easy to check that $\mathcal{C}_S$ and $R_\prec$ are respectively a pre-contact and a subordination relation. Hence, it remains to prove that they satisfy CS1 and CS2. We give the proof for CS1 and leave the other one to the reader. We have to show that for O,U and V three increasing clopen sets, we have

$$O\prec U\cup V \text{ and } O\not{\mathcal{C}}\,U\Rightarrow O\prec V,$$

or, equivalently,

$$O\not\prec V \text{ and } O\not{\mathcal{C}}\,V\Rightarrow O\not\prec U\cup V.$$

Hence, suppose that $R(-,O)\not\subseteq V$ and $S(U,-)\subseteq O^c$. It follows that there exists $x\in V^c$ and $y\in O$ such that $x\;R\;y$. By TCS1, there exists an element $z\in X$ such that $z\leq x$ and $z\;(S\cap R)\;y$. Hence, we have $z\in S(-,O)\subseteq U^c$ and, since V^c is decreasing, that $z\in V^c$. Consequently, z is an element of $R(-,O)\cap(U\cup V)^c$, and $O\not\prec U\cup V$. □

Now that we have determined how to construct a cs Priestley space from a cs lattice and vice-versa, it is time to look for the expected isomorphism theorem.

Theorem 3.2.9. *Let $\mathfrak{L}=(L,\mathcal{C},\prec)$ be a cs lattice, and $\mathfrak{X}(X,\leq,R,S)$ a cs Priestley space. Moreover, let us denote by η and ε the unit and the co-unit of the Priestley duality.*

1. *For every $a,b\in L$ we have*

$$a\;\mathcal{C}\;b \text{ iff } \eta(a)\;\mathcal{C}_{S_\mathcal{C}}\;\eta(b) \text{ and } a\prec b \text{ iff } \eta(a)\prec_{T_\prec}\eta(b).$$

2. *For every $x,y\in X$, we have*

$$x\;S\;y \text{ iff } \varepsilon(x)\;S_{\mathcal{C}_S}\;\varepsilon(y) \text{ and } x\;T\;y \text{ iff } \varepsilon(x)\;T_{\prec_T}\;\varepsilon(y).$$

Proof. As we have done before, we only give the proof for the pre-contact relation and its associated relation S.

1. First, let us suppose that $\eta(a)\;\mathcal{C}_{S_\mathcal{C}}\;\eta(b)$, hence the exists a filter x in $S_\mathcal{C}(\eta(b),-)\cap\eta(a)$. Therefore, there exists also a filter $y\in\eta(b)$ with $y\;S_\mathcal{C}\;x$. The conclusion follows immediately from

$$(a,b)\in x\times y\subseteq\mathcal{C}.$$

Now suppose that $a\;\mathcal{C}\;b$. We have to show that $S_\mathcal{C}(\eta(b),-)\cap\eta(a)\neq\emptyset$. It is easily checked that $\not{\mathcal{C}}(-,b)$ is an ideal of L, disjoint from $\uparrow a$. Hence, there exists a prime filter x such that $a\in x$ and $\not{\mathcal{C}}(-,b)\cap x=\emptyset$. From the latter, we get that $\not{\mathcal{C}}(x,-)\cap\uparrow b=\emptyset$. Again, it is not hard to prove that $\not{\mathcal{C}}(x,-)$ is an ideal, so there exists a prime filter y such that $b\in y$ and $\not{\mathcal{C}}(x,-)\cap y=\emptyset$, that is $x\times y\subseteq\mathcal{C}$, or $y\;S_\mathcal{C}\;x$ as required.

2. First, let us suppose that $x \; S \; y$. Then, for all increasing clopen sets $O \in \varepsilon(x)$ and $U \in \varepsilon(y)$, we have $S(O, -) \cap U \neq \emptyset$, i.e. $U \; \mathcal{C}_S \; O$. Therefore, we have $\varepsilon(y) \times \varepsilon(x) \subseteq \mathcal{C}_S$, which means, by definition, that $\varepsilon(x) \; S_{\mathcal{C}_S} \; \varepsilon(y)$.

   Now, suppose that $x \not S \; y$. Then, there exist two increasing clopen sets O and U such that $(O, U) \in \varepsilon(x) \times \varepsilon(y)$ and $O \times U \subseteq S^c$. Since the last inclusion is equivalent to $S(O, -) \subseteq U^c$, we have that $U \not\mathcal{C}_S \; O$ and therefore that $\varepsilon(y) \times \varepsilon(x) \not\subseteq \mathcal{C}_S$, as required.

□

Proposition 3.2.10. *Let $\mathfrak{X} = (X, \leq, R, S)$ be a cs-Priestley space. If T denotes the intersection $R \cap S$, then for all $y \in X$, we have*

$$\downarrow T(-, y) = S(-, y) \text{ and } \uparrow T(-, y) = R(-, y).$$

Proof. Suppose first that $x \in \downarrow T(-, y)$, then we have $x \leq z \; S \; y$ for some $z \in X$. Now, since S is a relation that satisfies T1, we have immediately that $x \; S \; y$. On the other hand, suppose that $x \in S(-, y)$. Then, by definition of cs Priestley spaces, there exists an element $z \in X$ such that

$$x \leq t \underbrace{(S \cap R)}_{=T} y,$$

as required. The remaining case is left to the reader. □

Definition 3.2.11. An **ucs Priestley space** is a triplet $\mathfrak{X} = (X, \leq, T)$ where $(X, \leq)$ is a Priestley space and T is a binary relation on X which is closed, convex in its left coordinate and decreasing in its right one.

It is clear that if $\mathfrak{X} = (X, \leq, R, S)$ is a cs Priestley space, then the triplet $\mathfrak{X}^\star := (X, \leq, R \cap S)$ is a ucs one. Moreover, in this case, the intersection $(R \cap S)$ determines exactly R and S, as we saw in Proposition 3.2.10. The next proposition will confirm that this association actually leads to a bijection.

Proposition 3.2.12. *Let $\mathfrak{X} = (X, \leq, T)$ be an ucs Priestley space. Then $\mathfrak{X}_\star := (X, \leq, R, S)$ where the relations R and S defined as $R = \geq \circ T$ and $S = \leq \circ S$ (recall Notation 3.2.3) is a cs Priestley space such that $T = R \cap S$.*

Proof. Let us start by noticing that, since T is decreasing in its second component, then $T = T \circ \geq$, so that

$$S = \leq \circ T = \leq \circ T \circ \geq = \leq \circ \leq \circ T \circ \geq = \leq \circ S \circ \geq,$$

which implies that S is indeed a relation satisfying T1. Similarly, we have that $R = \leq \circ R \circ \geq$, i.e. R satisfies T2. Finally, note that, as compositions of closed relations, R and S are closed relations.

To continue, let us prove that we have $T = R \cap S$. First, if $x \; T \; y$, we have $x \leq x \; T \; y$ and $x \geq x \; T \; y$. On the other hand, if $x \; R \cap S \; y$, then there exist s and t in X such that

$$x \leq s \; T \; y \text{ and } x \geq t \; T \; y. \tag{3.4}$$

Now, since T is convex in its left component, (3.4) clearly implies that $x \; T \; y$.

Finally, let us prove TCS1. Suppose that $x \; R \; y$. By definition, we have $x \geq z \; T \; y$ for some $z \in X$. Since T is equal to $R \cap S$, the conclusion is immediate. □

Remark 3.2.13. Of course, if $\mathfrak{X} = (X, \leq, T)$ is an ucs Priestley space, then it is possible to define directly a pre-contact and a subordination relation on $\uparrow\mathrm{Clop}(X)$, without using $\geq \circ T$ and $\leq \circ T$, by

$$O \prec_T U \Leftrightarrow T(-, O) \subseteq U \text{ and } O\, \mathcal{C}_T\, U \Leftrightarrow T(U, -) \not\subseteq O^c.$$

However, doing so, we will lose the isomorphism of Theorem 3.2.9.

Finally, we have the next immediate corollary as conclusion.

Corollary 3.2.14. *If $\mathfrak{L} = (L, \mathcal{C}, \prec)$ is a cs lattice, then there $\mathcal{C}$ and $\prec$ are generated by a unique binary relation T on* $\mathrm{Prim}(L)$ *such that* $(\mathrm{Prim}(L), \subseteq, T)$ *is an ucs Priestley space.*

Conversely, if $\mathfrak{X} = (X, \leq, T)$ is an ucs Priestley space, then $\mathfrak{L}_{X_\star}$ is a cs lattice.

3.3 Slanted lattices

To make the readability of the next sections easier, we introduce the following notations.

Notation 3.3.1. Let $\epsilon \in \{1, \partial\}^n$ be an order-type and L be a bounded lattice. We denote by L^ϵ the product $L^{\epsilon_1} \times \cdots \times L^{\epsilon_n}$ where $L^1 = L$ and L^∂ is the dual order lattice. We also denote by $\underset{\sim}{0}^\epsilon$ the bottom element of L^ϵ, that is $\underset{\sim}{0}^\epsilon$ is a vector of L^n such that $(\underset{\sim}{0}^\epsilon)_i = 0$ (resp. $= 1$) if $\epsilon_i = 1$ and $(\underset{\sim}{0}^\epsilon)_i = 1$ (resp. $\epsilon_i = \partial$). Similarly, $\underset{\sim}{1}^\epsilon$ will denote the top element of L^ϵ.

In the same vein, let $\epsilon \in \{1, \partial\}^n$ be an order-type, X be an arbitrary set and $y_1 \ldots, y_n$ be subsets of X. Then $\underset{\sim}{y}_\epsilon$ denotes the Cartesian product $y_1^{\epsilon_1} \times \ldots \times y_n^{\epsilon_n}$ where $y_i^{\epsilon_i} = y_i$ (resp. y_i^c) if $\epsilon_i = 1$ (resp. $\epsilon_i = \partial$).

Furthermore, for $a_1, \ldots, a_n \in L$ and for η the unit of the Priestley duality, i.e.

$$\eta : L \longrightarrow \uparrow\mathrm{Clop}(\mathrm{Prim}(L)) : a \longmapsto \{x \in \mathrm{Prim}(L) \mid a \in x\},$$

(see Appendix B.1.14) we will abuse notations and denote $\underset{\sim}{\eta(\underline{a})^\epsilon}$ by $\eta(\underset{\sim}{a}^\epsilon)$. It follows that, for $\underline{y}$ a vector of prime filters, we have the nice equivalence

$$\underline{y} \in \eta(\underset{\sim}{a}^\epsilon) \iff \underline{a} \in \underset{\sim}{y}^\epsilon.$$

To introduce one last notation, let us consider $(X, \leq)$ to be an ordered space. If $\underline{x}$ and $\underline{y}$ are elements of X^n, we write $\underline{x} \leq^\epsilon \underline{y}$ if and only if $\underline{x}_i \leq \underline{y}_i$ when $\epsilon_i = 1$ and $\underline{x}_i \geq \underline{y}_i$ when $\epsilon_i = \partial$ (substantially if $\underline{x}_i \leq^{\epsilon_i} \underline{y}_i$). Moreover, if A is a subset of X^n, we define

$$\uparrow^\epsilon A := \{\underline{y} \in X^n \mid \exists \underline{x} \in A : \underline{x} \leq^\epsilon \underline{y}\} \text{ and } \downarrow^\epsilon A := \{\underline{y} \in X^n \mid \exists \underline{x} \in A : \underline{y} \leq^\epsilon \underline{x}\}.$$

Finally, to further improve readability, we omit to write the superscript ϵ should the context cause no possible confusion.

Definition 3.3.2. Let L be a bounded distributive lattice, n be a natural number and $\epsilon \in \{1, \partial\}^n$ be an order-type. An n-ary **c-slanted operator** of order-type ϵ is a map $\triangle : L^n \longrightarrow L^\delta$ such that $\triangle\underline{a} \in \mathcal{K}(L^\delta)$ and

1. if $\epsilon_i = 1$, then we have
   (a) $\triangle(a_1, \ldots, 0, \ldots, a_n) = 0$;
   (b) $\triangle(a_1, \ldots, a_i \vee b_i, \ldots, a_n) = \triangle(a_1, \ldots, a_i, \ldots, a_n) \vee \triangle(a_1, \ldots, b_i, \ldots, a_n)$;
2. if $\epsilon_i = \partial$, then we have

(a) $\triangle(a_1, \ldots, 1, \ldots, a_n) = 0$;

(b) $\triangle(a_1, \ldots, a_i \wedge b_i, \ldots, a_n) = \triangle(a_1, \ldots, a_i, \ldots, a_n) \vee \triangle(a_1, \ldots, b_i, \ldots, a_n)$.

An n-ary **o-slanted operator** of order-type ϵ is a map $\triangledown : L^n \longrightarrow L^\delta$ such that $\triangledown \underline{a} \in \mathcal{O}(L^\delta)$ and

1. if $\epsilon_i = 1$, then we have
   (a) $\triangledown(a_1, \ldots, 1, \ldots, a_n) = 1$;
   (b) $\triangledown(a_1, \ldots, a_i \wedge b_i, \ldots, a_n) = \triangledown(a_1, \ldots, a_i, \ldots, a_n) \wedge \triangledown(a_1, \ldots, b_i, \ldots, a_n)$;
2. if $\epsilon_i = \partial$, then we have
   (a) $\triangledown(a_1, \ldots, 0, \ldots, a_n) = 1$;
   (b) $\triangledown(a_1, \ldots, a_i \vee b_i, \ldots, a_n) = \triangledown(a_1, \ldots, a_i, \ldots, a_n) \wedge \triangledown(a_1, \ldots, b_i, \ldots, a_n)$.

Now, a **slanted lattice** is a triplet $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ where L is a bounded distributive lattice, $\Gamma_1 = (\triangle_i \mid i \in I)$ is a family, possibly empty, of c-slanted operators on L and $\Gamma_2 = (\triangledown_j \mid j \in J)$ is a family, possibly empty, of o-slanted operators on L.

Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice with $\Gamma_1 = (\triangle_i \mid i \in I)$ and $\Gamma_2 = (\triangledown_j \mid j \in J)$. The family $(\epsilon_i \mid i \in I) \cup (\epsilon_j \mid j \in J)$, where ϵ_i is the arity of $\triangle_i$ and ϵ_j of $\triangledown_j$, is the **signature** of $\mathfrak{L}$.

The definition of morphisms for slanted lattices comes now quite naturally and was already suggested in [25].

Definition 3.3.3. Let $\mathfrak{M} = (M, \Gamma_1, \Gamma_2)$ and $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be slanted lattices whose signatures are identical. A map $h : M \longrightarrow L$ is a **slanted morphism** if h is a lattice morphism such that, for every $\triangle_i \in \Gamma_1$, $\triangledown_j \in \Gamma_2$ and every $\underline{a} \in M^n$, we have $h^\delta(\triangle_i \underline{a}) = \triangle_i h(\underline{a})$ and $h^\delta(\triangledown_j \underline{a}) = \triangledown_j h(\underline{a})$, namely the following diagrams are commutative:

$$\begin{tikzcd} M^\delta \arrow[r, "h^\delta"] & L^\delta \\ M^n \arrow[u, "\triangle_i"] \arrow[r, "h^n"'] & L^n \arrow[u, "\triangle_i"'] \end{tikzcd} \qquad \begin{tikzcd} M^\delta \arrow[r, "h^\delta"] & L^\delta \\ M^n \arrow[u, "\triangledown_j"] \arrow[r, "h^n"'] & L^n \arrow[u, "\triangledown_j"'] \end{tikzcd}$$

where h^δ is the canonical extension of h.

Proposition 3.3.4. *A lattice morphism $h : M \longrightarrow L$ is a slanted morphism between $\mathfrak{M} = (M, \Gamma_1, \Gamma_2)$ and $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ if and only if for all $\triangle \in \Gamma_1$ and all $\triangledown \in \Gamma_2$, one has:*

1. $\triangle \underline{a} \leq c$ *implies* $\triangle h(\underline{a}) \leq h(c)$;
2. $\triangledown \underline{a} \leq c$ *implies* $\triangledown h(\underline{a}) \leq h(c)$;
3. $\triangle h(\underline{a}) \leq b$ *implies* $\triangle \underline{a} \leq c$ *and* $h(c) \leq b$ *for some* $c \in M$;
4. $b \leq \triangledown h(\underline{a})$ *implies* $c \leq \triangledown \underline{a}$ *and* $b \leq h(c)$ *for some* $c \in M$.

Proof. Suppose first that h is a slanted morphism. Recall that, since $\triangle\underline{a}$ is closed for all $\underline{a} \in A^n$, we know that

$$h^\delta(\triangle\underline{a}) = \wedge\{h(c) : \triangle\underline{a} \leq c\} = \triangle(h(\underline{a})).$$

Therefore, if $\triangle\underline{a} \leq c$, it is clear that $\triangle(h(\underline{a})) \leq h(c)$. Moreover, if $\triangle(h(\underline{a})) \leq b$, then the existence of an element $c \in A$ such that $\triangle\underline{a} \leq c$ and $h(c) \leq b$ is assured by compactness.

On the other hand, suppose that h satisfies conditions 1. and 2. We have to prove that $\triangle(h(\underline{a})) = h^\delta(\triangle\underline{a})$. Now, recall that $\triangle(h(\underline{a}))$ and $\triangle\underline{a}$ are both closed, so that we have to prove that

$$\triangle(h(\underline{a})) = \wedge\{b \in B : \triangle(h(\underline{a})) \leq b\} = \wedge\{h(c) : \triangle\underline{a} \leq c \in A\} = h^\delta(\triangle\underline{a}).$$

First, we know that if $\triangle\underline{a} \leq c$, it comes by hypothesis that $\triangle(h(\underline{a})) \leq h(c)$. Therefore, we have immediately $\triangle(h(\underline{a})) \leq h^\delta(\triangle\underline{a})$. Secondly, if $\triangle(h(\underline{a})) \leq b$, then there exists $c \in A$ such that $\triangle\underline{a} \leq c$ and $h(c) \leq b$, that is $b \geq h^\delta(\triangle\underline{a})$, hence the conclusion.

The open case is proved dually and is left to the reader. □

Proposition 3.3.5. *If $h_1 : \mathfrak{M} \longrightarrow \mathfrak{L}$ and $h_2 : \mathfrak{L} \longrightarrow \mathfrak{N}$ are two slanted morphisms, then $h_2 \circ h_1$ is also a slanted morphism.*

Moreover, if h_1 is bijective, then $h_1^{-1} : \mathfrak{L} \longrightarrow \mathfrak{M}$ is also a slanted morphism.

Proof. To be convinced, one just has to recall that $(h_2 \circ h_1)^\delta = h_2^\delta \circ h_1^\delta$. (see [34, Lemma 4.5]). □

Definition 3.3.6. A **slanted isomorphism** is a bijective slanted morphism.

Finally, note that we have the following characterisation for isomorphisms between slanted lattices with identical signatures.

Proposition 3.3.7. *A map $h : M \longrightarrow L$ is a slanted isomorphism between $\mathfrak{M} = (M, \Gamma_1, \Gamma_2)$ and $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ if and only if h is a lattice isomorphism such that for all $\triangle \in \Gamma_1$ and all $\triangledown \in \Gamma_2$, we have*

$$\triangle\underline{a} \leq c \iff \triangle h(\underline{a}) \leq h(c) \text{ and } \triangledown\underline{a} \leq c \iff \triangledown h(\underline{a}) \leq h(c). \tag{3.5}$$

Proof. We suppose first that h is a slanted isomorphism. Then, there exists a slanted morphism $g : L \longrightarrow M$ such that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. Hence, it is clear that f (and g) is a lattice isomorphism. Since both the only if parts of (3.5) follow from Proposition 3.3.4, we can focus on the if parts. Suppose that $\triangle h(\underline{a}) \leq b$. Then, since g is a slanted morphism, we have

$$\triangle\underline{a} = \triangle g(h(\underline{a})) = g^\delta(\triangle(h\underline{a})) \leq g^\delta(h(b)) = g(h(b)) = b.$$

The open case is treated similarly.

Now, suppose that h is a lattice isomorphism which satisfies (3.5). Then, clearly, h^{-1} is also a lattice isomorphism which satisfies (3.5). Now, we prove that h satisfies the conditions of Proposition 3.3.4. We have immediately that items 1 and 2 are satisfied. Let $\underline{a}$ and b be such that $\triangle h(\underline{a}) \leq b$. Then, since h is bijective, we can suppose that $b = h(c)$ for some $c \in M$. Moreover, by (3.5), we obtain that $\triangle\underline{a} \leq c$ and, consequently, item 3 is proven. The remaining item is treated similarly. So, we just showed that h is a slanted morphism. Similarly, we can prove that h^{-1} is also a slanted morphism, which concludes the proof. □

Remark 3.3.8. Let L be a lattice. It is well known (see for instance [34, Lemma 3.3]) that there are lattice isomorphisms between the open elements of L^δ and the ideals of L as well as between the closed elements of L^δ and the filters of L. Hence, alternative definitions of c-slanted and o-slanted operators could be considered. For instance, an n-ary c-slanted operator of order-type $\epsilon = (\underline{1})$ is a map $\vartriangle : L^n \longrightarrow \mathcal{F}_L$ such that

$$\vartriangle(a_1, \dots, a_i \vee b_j, \dots, a_n) = \vartriangle(a_1, \dots, a_i, \dots, a_n) \cup \vartriangle(a_1, \dots, b_i, \dots, a_n)$$

and $\vartriangle(a_1, \dots, 0, \dots, a_n) = \emptyset$. With this definition, we retrieve the multi-operators of Raskin [59], the generalised implication of Castro, Celani and Jansana [13], the quasi-modal lattices and quasi-modal algebras of Celani and Castro [12] and [14].

Moreover, the notions of morphisms in the articles mentioned above correspond, with the required translation from closed and open elements to filters and ideals, to the notion of morphisms of Definition 3.3.3, as we show in the following examples.

Example 3.3.9. Of course, a first example of slanted lattices is given by the cs lattices of Section 3.2. Indeed, let $\mathfrak{L} = (L, \mathcal{C}, \prec)$. Its associated slanted lattice is $\mathfrak{L}^* = (L, \vartriangle_\prec, \triangledown_\mathcal{C})$ where $\vartriangle_\prec$ is the map defined as $a \longmapsto \wedge\{b \in L \mid a \prec b\}$ and $\triangledown_\mathcal{C}$ is the map defined as $a \longmapsto \vee\{b \in L \mid a \not\mathcal{C}\, b\}$. Note that $\mathfrak{L}^*$ is a slanted lattice which satisfies the following axioms:

CS1′. $\vartriangle_\prec a \leq b \vee c$ and $b \leq \triangledown_\mathcal{C} a$ implies $\vartriangle_\prec a \leq c$;

CS2′. $b \wedge c \leq \triangledown_\mathcal{C} a$ and $\vartriangle_\prec a \leq b$ implies $c \leq \triangledown_\mathcal{C} a$.

Conversely, if $\mathfrak{L} = (L, \vartriangle, \triangledown)$ is a slanted lattice which satisfies CS1' and CS2', then it associated cs lattice is $\mathfrak{L}_* = (L, \mathcal{C}_\triangledown, \prec_\vartriangle)$ where

$$a \prec_\vartriangle b \text{ iff } \vartriangle a \leq b \text{ and } a\, \mathcal{C}_\triangledown\, b \text{ iff } b \nleq \triangledown a.$$

Moreover, if $\mathfrak{L}$ is a cs lattice, then $\mathfrak{L} = (\mathfrak{L}^*)_*$ and if $\mathfrak{L}$ is a slanted lattice which satisfies CS1' and CS2', then $\mathfrak{L} = (\mathfrak{L}_*)^*$.

In particular, we note that a morphism between cs lattices should be a lattice morphism h which satisfies the following conditions:

1. $a \prec b$ implies $h(a) \prec h(b)$,
2. $a \not\mathcal{C}\, b$ implies $h(a) \not\mathcal{C}\, h(b)$,
3. $h(a) \prec b$ implies that $a \prec c$ and $h(c) \leq b$ for some c,
4. $b \not\mathcal{C}\, h(a)$ implies that $b \leq h(c)$ and $c \not\mathcal{C}\, a$ for some c.

Let us finally go back to the Boolean case where $a \prec b$ if and only if $a \not\mathcal{C} \neg b$. Hence, $\prec$ and $\mathcal{C}$ are linked together like the modal operators $\Diamond$ and $\Box$. However, the slanted operator associated to $\mathcal{C}$ is not the natural box associated to $\Diamond_\prec$. Indeed, the pre-contact relation $\mathcal{C}$ is associated to $\neg\Diamond_\prec$, or $\Box_\prec\neg$, as we observe with the following equalities:

$$\neg\Diamond_\prec a = \neg(\wedge\{b \mid a \prec b\}) = \vee\{\neg b \mid a \prec b\} = \vee\{b \mid a \not\mathcal{C}\, b\} = \triangledown_\mathcal{C}(a)$$

Example 3.3.10. A **generalized implication lattice** (or more simply a **gi-lattice**) [13] is a pair $\mathfrak{L} = (L, \Rightarrow)$ where L is a bounded distributive lattice and $\Rightarrow$ is a map from L^2 to the set of its ideals which satisfies for every a, b and $c \in L$ the following conditions:

1. $\Rightarrow(a, b) \cap \Rightarrow(a, c) = \Rightarrow(a, b \wedge c)$,
2. $\Rightarrow(a, b) \cap \Rightarrow(b, c) = \Rightarrow(a \vee b, c)$,
3. $\Rightarrow(a, b) \cap \Rightarrow(b, c) \subseteq \Rightarrow(a, c)$,

4. $\Rightarrow(a,a) = L$.

For each gi-lattice $\mathfrak{L}$, its associated slanted lattice is $\mathfrak{L}^* := (L, \triangledown_{\Rightarrow})$ where $\triangledown_{\Rightarrow} : L^2 \to L^\delta$ is defined as the operator that maps (a,b) to the open element $\vee\{c \in L \mid c \in \Rightarrow(a,b)\}$. In particular, $\triangledown_{\Rightarrow}$ is a binary o-slanted operator which order-type is $(\partial, 1)$ and which satisfies $\triangledown_{\Rightarrow}(a,a) = 1$ and $\triangledown_{\Rightarrow}(a,b) \wedge \triangledown_{\Rightarrow}(b,c) \leq \triangledown_{\Rightarrow}(a,c)$ for every $a,b,c \in L$. On the other hand, if $\triangledown$ is a binary o-slanted operator satisfying the previous properties, then $\mathfrak{L}_* = (L, \Rightarrow_{\triangledown})$ where $\Rightarrow_{\triangledown}(a,b) := \{c \in L \mid c \leq \triangledown(a,b)\}$, is a gi-lattice. Moreover, one has $(\mathfrak{L}_*)^* = \mathfrak{L}$ and $(\mathfrak{L}^*)_* = \mathfrak{L}$.

Now, for the morphisms, we recall a map $h(L, \Rightarrow) \longmapsto (M, \Rightarrow)$ between gi-lattices is a **gi-morphism** [13] if it is a lattice morphism such that, for any $a, b \in L$, we have

$$\langle h(\Rightarrow(a,b))\rangle_{\text{ id}} = \Rightarrow(h(a), h(b)). \tag{3.6}$$

Then, we have that $h : (L, \Rightarrow_{\triangledown}) \longrightarrow (M, \Rightarrow_{\triangledown})$ is a gi-moprhism if and only if $h : (L, \triangledown) \longrightarrow (M, \triangledown)$ is a slanted morphism.

Indeed, let us suppose first that h is a slanted morphism. We prove that h satisfies (3.6) by showing that the two inclusions hold. Let c be an element of $\Rightarrow_{\triangledown}(h(a), h(b))$, that is $c \leq \triangledown(h(a), h(b))$. Since h is a slanted morphism, we have that $c \leq h(d)$ and $d \leq \triangledown(a,b)$ for some d. Hence, we have $c \in \langle h(\Rightarrow(a,b))\rangle_{\text{ id}}$, as required. Now, let c be an element of $\langle h(\Rightarrow(a,b))\rangle_{\text{ id}}$. Then, we have $c \leq h(d)$ for some $d \leq \triangledown(a,b)$. It follows that

$$c \leq h(d) \leq \triangledown(h(a), h(b)),$$

which concludes the if part.

On the other hand, suppose that h is a gi-morphism and let a, b, c be such that $c \leq \triangledown(a,b)$. Then, we have

$$h(c) \leq h^\pi(\triangledown(a,b)) = \vee\{h(d) \mid d \leq \triangledown(a,b)\}.$$

Hence, by compactness, we have $h(c) \leq h(d)$ for some $d \leq \triangledown(a,b)$. In other words, we just obtained the equality

$$h(c) \in \langle h(\Rightarrow(a,b))\rangle_{\text{ id}} = \Rightarrow_{\triangledown}(h(a), h(b)),$$

which implies $h(c) \leq \triangledown(h(a), h(b))$. Finally, let a, b and c be such that $c \leq \triangledown(h(a), h(b))$. Then, we have

$$c \in \Rightarrow_{\triangledown}(h(a), h(b)) = \langle h(\Rightarrow(a,b))\rangle_{\text{ id}} ,$$

and, it follows that there exist $d \leq \triangledown(a,b)$ such that $c \leq h(c)$ as required.

Example 3.3.11. Of course, (classical) Boolean algebras with operators (see [48] and [49]) are slanted algebras. Moreover, clopen versions of slanted algebras have been studied. Indeed, consider the following definition given in [67, Definition 1]. Let L be a bounded distributive lattice. A **meet hemiantimorphism** on L is a map $f : L^n \longrightarrow L$ such that, for all $1 \leq i \leq n$, we have

1. $f(a_1, ..., a_{i_1}, 0, a_{i+1}, ..., a_n) = 1$,
2. $f(a_1, \ldots, a_i \vee b_i, \ldots, a_n) = f(a_1, \ldots, a_i, \ldots, a_n) \wedge f(a_1, \ldots, b_i, \ldots, a_n)$.

It clear that a meet hemiantimorphism is an o-slanted operator of order-type $\epsilon = \underline{\partial}$. Indeed, the elements of L are exactly the clopen elements of L^δ, so that f maps elements of L^n to elements of $\mathcal{O}(L^\delta)$. Moreover, a slanted morphism between lattices endowed with meet hemiantimorphisms are exactly the lattice morphisms that respect f in the classical sense. This observation follows easily from the fact that, for a lattice morphism $h : L \longrightarrow M$, h^δ restricted to L is actually h himself.

We also find in [67, Definition 1] the concepts of join hemimorphism, meet hemimorphism and join hemiantimorphism which correspond respectively to c-slanted operators of order-type $\underline{1}$, o-slanted operators of order-type $\underline{1}$ and c-slanted operators of order-type $\underline{\partial}$ which all map elements of L to clopen elements of L^δ.

Example 3.3.12. Linked to the previous one, the penultimate example is given by the residuals of standard operators on lattices. Let us consider an operator $f : L^n \longrightarrow L$ of order type $\underline{1}$. Then, f can be extended as a complete operator $f^\delta : (L^\delta)^n \to L$. Then, the residual $f_1^\sharp : (L^\delta) \to L^\delta$ of f^δ restricted to L is an o-slanted operator (see for instance [20, Section10]).

Example 3.3.13. If $(X, \leq)$ is a Priestley space, then $\mathrm{Clop}(X)$ is a Boolean algebra. We can define on it the operators

$$[\leq] : \mathrm{Clop}(X) \longrightarrow \mathcal{P}(X) : O \longmapsto (\downarrow O^c)^c \text{ and } <\leq>: \mathrm{Clop}(X) \longrightarrow \mathcal{P}(X) : O \longmapsto \downarrow O$$

which are respectively o-slanted and c-slanted. Note that Esakia spaces are the Priestley spaces such that these operator are clopen. Moreover, note that these operators are linked to the Gödel-McKinsey-Tarski companion of intuitionist logics (see for instance [21]).

3.4 Slanted Priestley spaces

Definition 3.4.1. Let R be an n-ary relation on $(X, \leq)$ an ordered set and $\epsilon \in \{1, \partial\}^n$ be an order-type. We say that R is a **relation of order-type** ϵ if for all $i \in \{1, \ldots, n\}$ we have that $(x_1, \ldots, x_i, \ldots, x_n) \in R$ and $x_i \leq^{\epsilon_i} y$ implies $(x_1, \ldots, y, \ldots, x_n) \in R$. That is, using Notation 3.3.1, R is a subset of X^n such that $R = \uparrow^\epsilon R$.

Notation 3.4.2. Using Notation 3.2.3, we have that a binary relation R on an ordered set is:

1. of order-type $(1, 1)$ if and only if $R = \geq \circ R \circ \leq$,
2. of order-type $(1, \partial)$ if and only if $R = \geq \circ R \circ \geq$,
3. of order-type $(\partial, 1)$ if and only if $R = \leq \circ R \circ \leq$,
4. of order-type (∂, ∂) if and only if $R = \leq \circ R \circ \geq$.

This kind of notations can of course easily be adapted to any pair of "compatible" n-ary relations.

Lemma 3.4.3. *Let $(X, \leq)$ be a Priestley space and R an n-ary relation of order-type $\epsilon \in \{1, \partial\}^n$. Then, for all $\underline{x} \notin R$ there exists $\underline{O}^{\epsilon^\partial} \in \uparrow \mathrm{Clop}(X)$, or equivalently $\underline{O}^{\epsilon} \in \downarrow \mathrm{Clop}(X)$, such that $\underline{x} \in \underline{O} \subseteq R^c$.*

Proof. Since $\underline{x} \notin R$ and since R is of order-type ϵ, then for all $\underline{y} \in R$, there exists $i \in \{1, \ldots, n\}$ such that $\underline{y}_i \not\leq^{\epsilon_i} \underline{x}_i$. Therefore, since $(X, \leq)$ is a Priestley space, there exists an increasing clopen set $O^i_{\underline{y}}$ such that $\underline{y}_i \in (O^i_{\underline{y}})^{\epsilon_i}$ and $\underline{x}_i \in (O^i_{\underline{y}})^{\epsilon_i^\partial}$. Set now $\underline{U}_{\underline{y}}$ as follows

$$\begin{cases} (\underline{U}_{\underline{y}})_i = (O^i_{\underline{y}})^{\epsilon_i^\partial} \\ (\underline{U}_{\underline{y}})_j = X \text{ if } j \neq i \end{cases}.$$

Now, R is compact and it can be proved quite easily that $\{(\underline{U}_{\underline{y}})^c \mid \underline{y} \in R\}$ is an open cover of R. Therefore, we have

$$R \subseteq (\underline{U}_{\underline{y}_1})^c \cup \ldots \cup (\underline{U}_{\underline{y}_m})^c$$

for some $\underline{y}_1, \ldots, \underline{y}_m \in R$. Finally, we have

$$\underline{x} \in \underbrace{\underline{U}_{\underline{y}_1} \cap \ldots \cap \underline{U}_{\underline{y}_m}}_{=\underline{O} \text{ with } \underline{O}_i = \bigcap_{j=1}^{m} (\underline{U}_{\underline{y}_j})_i} \subseteq R^c,$$

and the proof is completed. $\square$

Proposition 3.4.4. *Let L be a bounded distributive lattice and $(X, \leq)$ be a Priestley space which are dual from one another. Then*

1. *if $\triangle : L^n \longrightarrow L^\delta$ is an n-ary c-slanted operator of order-type ϵ, its associated relation $R_\triangle$ on X^{n+1}, defined by*

$$x\ R_\triangle\ \underline{y} \Leftrightarrow \{b \in L \mid \exists \underline{a} \in \underset{\sim}{y}^\epsilon : \triangle \underline{a} \leq b\} \subseteq x,$$

   is a closed $(n+1)$-ary relation of order-type $\epsilon' = (1, \epsilon^\partial)$;

2. *if R is a closed relation on X^{n+1} whose order-type is $\epsilon' = (1, \epsilon)$, its associated operator, $\triangle_R(\underline{O}) := R(-, \underset{\sim}{Q}^{\epsilon^\partial})$, is an n-ary c-slanted operator of order-type ϵ^∂ on L;*

3. *if $\triangledown : L^n \longrightarrow L^\delta$ is an n-ary o-slanted operator of order-type ϵ, its associated relation $S_\triangledown$ on X^{n+1}, defined by*

$$x\ S_\triangledown\ \underline{y} \Leftrightarrow \{b \in L \mid \exists \underline{a} \in \underset{\sim}{y}^{\epsilon^\partial} : b \leq \triangledown \underline{a}\} \subseteq x^c,$$

   is a closed $(n+1)$-ary relation of order-type $\epsilon' = (\partial, \epsilon)$;

4. *if S is a closed relation on X^{n+1} whose order-type is $\epsilon' = (\partial, \epsilon)$, it associated operator, $\triangledown_S(\underline{O}) := S(-, \underset{\sim}{Q}^{\epsilon^\partial})^c$, is an n-ary o-slanted operator of order-type ϵ on L.*

Proof. We give the proofs of the c-slanted case (Items 1 and 2) and leave the open one (Items 3 and 4) to the reader.

1. Let us show first that $R_\triangle$ is closed. Indeed, suppose that $(x, \underline{y}) \notin R_\triangle$. Then, by definition, there exists $\underline{a} \in L^n$ and $b \in L$ such that $\underline{a} \in \underset{\sim}{y}^\epsilon$, $b \notin x$ and $\triangle \underline{a} \leq b$. We then have

$$(x, \underline{y}) \in \eta(b)^c \times \eta(\underline{a}) \subseteq R^c,$$

   which leads to the conclusion.

   Now, we clearly have that $(x, \underline{y}) \in R_\triangle$ and $x \leq z$ implies that $(z, \underline{y}) \in R_\triangle$, so that $\epsilon'_1 = 1$. Finally, let $i \in \{1, \ldots, n\}$, $(x, \underline{y}) \in R$ and $y_i \leq^{\epsilon^\partial} z$, that is $z \leq^\epsilon y_i$. Then we have that

$$y_1^{\epsilon_1} \times \ldots \times z^{\epsilon_i} \times \ldots \times y_n^{\epsilon_n} \subseteq y_1^{\epsilon_1} \times \ldots \times y_i^{\epsilon_i} \times \ldots \times y_n^{\epsilon_n}$$

   and the conclusion is straightforward.

2. Since R is a closed relation and since Q^{ϵ^∂} is closed, it is clear that $\triangle_R(\underline{O})$ is a closed subset of X. Moreover, since the order-type of R is $(1, \epsilon)$, we have that $\triangle_R(\underline{O})$ is indeed an increasing set.

   Furthermore, if $\epsilon_i = 1$, that is $\epsilon_i^\partial = \partial$, then

$$\triangle_R(\ldots, X, \ldots) = R(-, \ldots, X^{\epsilon_i^\partial}, \ldots) = R(-, \ldots, \emptyset, \ldots) = \emptyset$$

and

$$\begin{aligned}\triangle_R(\ldots, O \cap U, \ldots) &= R(-, \ldots, (O \cap U)^{\epsilon_i^\partial}, \ldots) \\ &= R(-, \ldots, O^c \cup U^c, \ldots) \\ &= R(-, \ldots, O^c, \ldots) \cup R(-, \ldots, U^c, \ldots) \\ &= \triangle_R(\ldots, O, \ldots) \cup \triangle_R(\ldots, U, \ldots),\end{aligned}$$

as required. The remaining case $\epsilon_i = \partial$ is left to the reader.

□

Lemma 3.4.5. *Let L be a bounded distributive lattice, $\triangle : L^n \longrightarrow L^\delta$ and $\triangledown : L^n \longrightarrow L^\delta$ respectively a c-slanted operator and an o-slanted operator of order-type ϵ and, finally, let R and S be their respective associated closed relations, then for all $\underline{a} \in L^n$, we have*

$$\cap \{\eta(b) \mid \triangle\underline{a} \leq b \in L\} = R(-, \eta(\underset{\sim}{a}^\epsilon)) \tag{3.7}$$

and

$$\cap \{\eta(b)^c \mid L \ni b \leq \triangledown\underline{a}\} = S(-, \eta(\underset{\sim}{a}^{\epsilon^\partial})) \tag{3.8}$$

Proof. We prove (3.7) by double inclusion. First, suppose that $x \in R(-, \eta(\underset{\sim}{a}^\epsilon))$. Then $x\ R\ \underline{y}$, namely

$$\{b \in L \mid \exists \underline{c} \in \underset{\sim}{y}^\epsilon : \triangle\underline{c} \leq b\} \subseteq x,$$

for some $\underline{y} \in \eta(\underset{\sim}{a}^\epsilon)$. Therefore, it is clear that $x \in \eta(b)$ for all $b \in L$ such that $\triangle\underline{a} \leq b$.

On the other hand, suppose that $x \notin R(-, \eta(\underset{\sim}{a}^\epsilon))$. The procedure will be analogue to the one used in [42, Lemma 1.4] for the classical (i.e. non-slanted) case. We will build for all $\underline{y} \in \eta(\underline{a})$ a sequence $\underline{c_0}(\underline{y}), \ldots, \underline{c_n}(\underline{y})$ such that, for all $j \in \{0, \ldots, n\}$, we have

1. $\underline{y} \in \eta(\underset{\sim}{c_j}(\underline{y})^\epsilon)$,
2. $\{b \in L \mid \triangle\underline{c_j}(\underline{y}) \leq b\} \cap x^c \neq \emptyset$,
3. for all $k \in \{1, \ldots, j\}$, $\underline{a}_k \leq^{\epsilon_k} \underline{c_j}(\underline{y})_k$.

Once the sequences are built, we will able to conclude the proof. Indeed, for $j = n$, it will follow that, for all $\underline{y} \in \eta(\underline{a})$, there is $\underline{c_n}(\underline{y}) \in L^n$ such that

$$\emptyset \neq \{b \in L \mid \triangle\underline{c_n}(\underline{y}) \leq b\} \cap x^c \subseteq \{b \in L \mid \triangle\underline{a} \leq b\} \cap x^c.$$

Hence, we have $x \notin \cap\{\eta(b) \mid \triangle\underline{a} \leq b\}$, as required.

Therefore, let us build the sequences $\underline{c_0}(\underline{y}), \ldots, \underline{c_n}(\underline{y})$. We will proceed by induction on j. For $j = 0$, since $x \notin R(-, \eta(\underline{a}))$, for all $\underline{y} \in \eta(\underline{a})$, there exists $\underline{c} \in \underset{\sim}{y}$ such that $\{b \in L \mid \triangle\underline{c} \leq b\} \not\subseteq x$. Therefore, we see that $\underline{c_0}(\underline{y}) = \underline{c}$ satisfies the requirements.

Suppose now that we built $\underline{c_0}(\underline{y}), \ldots, \underline{c_j}(\underline{y})$ for some $j < k$ and for all $\underline{y} \in \eta(\underline{a})$. For an arbitrary $\underline{y} \in \eta(\underline{a})$, we let L_{j+1} denote the following set

$$L_{j+1}(\underline{y}) := \{\underline{z} \in \eta(\underline{a}) \mid \underline{y}_k = \underline{z}_k \text{ if } k \neq j+1\}.$$

We then claim that $\{\eta(\underline{c_j}(\underline{z})_{j+1})^{\epsilon_{j+1}} \mid \underline{z} \in L_{j+1}(\underline{y})\}$ is an open cover of $\eta(a_{j+1})^{\epsilon_{j+1}}$. Indeed, suppose that $z \in \eta(a_{j+1})^{\epsilon_{j+1}}$ and let $\underline{z}$ be the element of $L_{j+1}(\underline{y})$ such that $\underline{z}_{j+1} = z$. It follows, by the first point of the induction on j, that $z \in \eta(\underline{c_j}(\underline{z})_{j+1})^{\epsilon_{j+1}}$ and the claim is proved.

Now, since $\eta(a_{j+1})^{\epsilon_{j+1}}$ is compact, there are elements $\underline{z}^1, \ldots, \underline{z}^m \in L_{j+1}(\underline{y})$ such that

$$\begin{aligned} \eta(a_{j+1})^{\epsilon_{j+1}} &\subseteq \eta(\underline{c_j}(\underline{z}^1)_{j+1})^{\epsilon_{j+1}} \cup \ldots \cup \eta(\underline{c_j}(\underline{z}^m)_{j+1})^{\epsilon_{j+1}} \\ &= \eta((\bigvee_{i=1}^m)^{\epsilon_{j+1}} \underline{c_j}(\underline{z}^i)_{j+1})^{\epsilon_{j+1}}. \end{aligned} \tag{3.9}$$

Set now

$$\underline{c_{j+1}}(\underline{y})_{j+1} = (\bigvee_{i=1}^{m})^{\epsilon_{j+1}} \underline{c_j}(\underline{z}^i)_{j+1} \text{ and}$$
$$\underline{c_{j+1}}(\underline{y})_k = (\bigwedge_{i=1}^{m})^{\epsilon_k} \underline{c_j}(\underline{z}^i)_k \text{ if } k \neq j+1.$$

It remains to show that $\underline{c_{j+1}}(\underline{y})$ satisfies the conditions 1, 2 and 3.

1. First, we have
$$\underline{y}_{j+1} \in \eta(\underline{a}_{j+1})^{\epsilon_{j+1}} \subseteq \eta(\underline{c_{j+1}}(\underline{y})_{j+1})^{\epsilon_{j+1}}$$
and, if $k \neq j+1$, we have that $\underline{y}_k = \underline{z}^i_k$ for all $i \in \{1, \ldots, m\}$. Hence,
$$\underline{y}_k = \underline{z}^i_k \in \eta(\underline{c_j}(\underline{z}^i)_k)^{\epsilon_{j+1}}$$
by induction on j.

2. By induction on j, we know that
$$\{b \in L \mid \triangle \underline{c_j}(\underline{z}^i) \leq b\} \cap x^c \neq \emptyset$$
for all $i \in \{1, \ldots, m\}$. Moreover, we have
$$\begin{aligned} \triangle \underline{c_{j+1}}(\underline{y}) &= \triangle(\underline{c_{j+1}}(\underline{y})_1, \ldots, \underline{c_{j+1}}(\underline{y})_{j+1}, \ldots, \underline{c_{j+1}}(\underline{y})_n) \\ &= \triangle(\underline{c_{j+1}}(\underline{y})_1, \ldots, (\bigvee_{i=1}^{m})^{\epsilon_{j+1}} \underline{c_j}(\underline{z}^i)_{j+1}, \ldots, \underline{c_{j+1}}(\underline{y})_n) \\ &= \bigvee_{i=1}^{m} \triangle(\underline{c_{j+1}}(\underline{y})_1, \ldots, \underline{c_j}(\underline{z}^i)_{j+1}, \ldots, \underline{c_{j+1}}(\underline{y})_n) \\ &\leq \bigvee_{i=1}^{m} \triangle(\underline{c_j}(\underline{z}^i)). \end{aligned}$$
The conclusion now follows immediately from the fact that x^c is an ideal.

3. If $k \leq j$, we know, by induction, that $\underline{a}_k \leq^{\epsilon_k} \underline{c_j}(\underline{z}^i)_k$. Hence, we have
$$\underline{a}_k \leq^{\epsilon_k} (\bigwedge_{i=1}^{m})^{\epsilon_k} \underline{c_j}(\underline{z}^i)_k = \underline{c_{j+1}}(\underline{y})_k.$$
Finally, if $k = j+1$, then the conclusion follows from (3.9).

Hence, the equality (3.7) is proved. Obviously, one can prove (3.8) similarly. Note that, this time, the sequence $\underline{c_0}(\underline{y}), \ldots, \underline{c_n}(\underline{y})$ must satisfy the following conditions

1. $\underline{y} \in \eta(\underset{\sim}{c_j}(\underline{y})^{\epsilon^\partial})$,

2. $\{b \in L \mid b \leq \nabla \underline{c_j}(\underline{y})\} \cap x \neq \emptyset$,

3. for all $k \in \{1, \ldots, j\}$, $\underline{c_j}(\underline{y})_k \leq^{\epsilon_k} \underline{a}_k$.

$\square$

Definition 3.4.6. A **slanted Priestley space** is a triplet $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$ where $(X, \leq)$ is a Priestley space, $\Lambda_1 = (R_i \mid i \in I)$ is a family, possibly empty, such that for all $i \in I$, R_i is a closed $(n_i + 1)$-ary relation on X of order-type $(1, \epsilon_i)$ for some natural n_i and order-type $\epsilon_i \in \{1, \partial\}^{n_i}$ and $\Lambda_2 = (S_j \mid j \in J)$ is a family, possibly empty, such that for all $j \in J$, S_j is a closed $(n_j + 1)$-ary relation on X of order-type (∂, ϵ_j) for some natural n_j and some order-type $\epsilon_j \in \{1, \partial\}^{n_j}$. The family $(\epsilon_i \mid i \in I \cup J)$ is called the **signature** of $\mathfrak{X}$.

Examples of slanted Priestley spaces have already been studied in the literature, with slightly different axiomatisations. In [13], we have the Priestley duals of gi-lattices mentioned in Example 3.3.10. We now give additional examples.

Example 3.4.7. In [12, Definitions 3 and 12], the authors defined a **descriptive quasi-modal space** to be a tuple $(X, \leq, R_1, R_2)$ such that

1. $(X, \leq)$ is a Priestley space,

2. $\geq \circ R_1 \subseteq R_1$ and $\geq \circ R_2 \subseteq R_2$,

and such that for all $O \in {\uparrow}\mathrm{Clop}(X)$ and all $x \in X$, we have that

4. $\triangledown_{R_1}(O) = \{x \in X : R_1(x, -) \subseteq O\}$ is open and decreasing,

5. $\vartriangle_{R_2}(O) = \{x \in X : R_2(x, -) \cap O \neq \emptyset\}$ is closed and increasing,

6. $R_1(x, -)$ is closed and increasing,

7. $R_2(x, -)$ is closed and decreasing.

By routine calculations, one can show that $\triangledown_{R_1}(O) = R_1(-, O^c)^c$ and $\vartriangle_{R_2}(O) = R_2(-, O)$. Using this fact, it can easily be checked that $(X, \leq, R_1, R_2)$ is a descriptive quasi-modal space if and only if $(X, \leq, \{R_2\}, \{R_1\})$ is a slanted Priestley space whose signature is given by $((1, \partial), (\partial, 1))$.

Finally, note that descriptive quasi-modal spaces are defined to be Priestley duals of slanted lattice $\mathfrak{L} = (L, \vartriangle, \triangledown)$ where $\vartriangle$ is an unary c-slanted operator and $\triangledown$ an o-slanted one, both of order-type 1.

Example 3.4.8. In [67, Section 2.1], the author defined a **Ma relation** on a Priestley space $\mathfrak{X} = (X, R)$ to be a subset $Q \subseteq X^{n+1}$ for some $n \in \mathbb{N}$ such that:

1. for all $\underline{x} \in X^n$ and every $y, z \in X$, if $\underline{x}\ Q\ y$ and $y \geq z$, then $\underline{x}\ Q\ z$,

2. for every $y \in X$, $Q(-, y)$ is closed on $(X, \tau^{\uparrow})^n$, where $\tau^{\uparrow}$ denotes the topology of increasing open sets of X,

3. for all $O_1, \ldots, O_n \in \mathrm{Clop}(X)$, the set

$$h_Q(\underline{O}) := \{y \mid \forall \underline{x} \in X^n : \underline{x}\ Q\ y \Rightarrow \exists x_i \notin O_i\}$$

is clopen.

Moreover, we have that Ma relations on a Priestley spaces are in bijective correspondence with meet hemiantimorphisms of its Priestley dual (recall Example 3.3.11).

Consider now the relation $Q^{\sim}$ defined by

$$x\, Q^{\sim}\, \underline{y} \Leftrightarrow \underline{y}\, Q\, x.$$

Then, items 1 and 2 in the definition of Ma relation guarantee that $Q^{\sim}$ is a relation of order-type $\underline{\partial}$ (recall that closed sets of the topology $\tau^{\uparrow}$ are, by definition, decreasing sets). Moreover, items 2 and 3 guarantee that $Q^{\sim}$ is closed in X^{n+1}. Indeed, suppose that $(x, \underline{y}) \notin Q^{\sim}$, then it follows that $\underline{y}$ is an element of $Q^{\sim}(x, -)$, which is an open set by item 2 (and necessarily increasing by item 1). Therefore, there exist increasing clopen sets $O_1, \dots, O_n$ such that

$$\underline{y} \in O_1 \times \cdots \times O_n \subseteq Q^{\sim}(x, -)^c.$$

This implies in particular that $x \in h_Q(\underline{O})$. Hence, by item 3, we know that there exists a clopen O such that $x \in O \subseteq h_Q(\underline{O})$. It follows that

$$(x, \underline{y}) \in O \times \underline{O} \subseteq (Q^{\sim})^c,$$

and the proof is concluded.

Hence, using our notations, if Q is a Ma relation on a Priestley space $(X, \leq)$, then $(X, \leq, \emptyset, \{Q^{\sim}\})$ is a Slanted Priestley space whose signature is given by $(\underline{\partial})$ and such that $\triangledown_{Q^{\sim}}(\underline{O})$ is a clopen increasing set for all tuples $\underline{O}$ of increasing sets. This last property comes from the fact that meet hemiantimorphisms map clopen sets precisely to clopen sets.

To continue the description of slanted Priestley spaces, it is now natural to look for a definition of morphisms in the slanted topological category, duals to the algebraic slanted morphisms of Definition 3.3.3. Such topological morphisms have already been studied in the particular categories of [12, Definition 16] and [67, Definition 4] which we just discussed the objects.

Definition 3.4.9. Let $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$ and $\mathfrak{Y} = (Y, \leq, \Lambda_1, \Lambda_2)$ be two slanted Priestley spaces with identical signatures. We look for the properties that an increasing continuous function between X and Y should satisfy to be a "slanted function". By considering the Boolean case broached earlier, it seems natural to require that $f : X \longrightarrow Y$ should satisfy

$$f(R(x, -)) = R(f(x), -) \text{ and } (f(S(x, -)) = S(f(x), -)$$

for all $R \in \Lambda_1$ and all $S \in \Lambda_2$. Recall however that R and S are respectively of order-type $(1, \epsilon_1)$ and (∂, ϵ_2), so that we should actually have

$$\uparrow^{\epsilon_1} f(R(x, -)) = R(f(x), -) \text{ and } \uparrow^{\epsilon_2} f(S(x, -)) = S(f(x), -). \tag{3.10}$$

We therefore define $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ to be an **order slanted function** if f is an increasing continuous function satisfying (3.10) for all $x \in X$, all $R \in \Lambda_1$ and all $S \in \Lambda_2$.

Proposition 3.4.10. *If $f_1 : \mathfrak{X} \longrightarrow \mathfrak{Y}$ and $f_2 : \mathfrak{Y} \longrightarrow \mathfrak{Z}$ are two order slanted functions, then $f_2 \circ f_1 : \mathfrak{X} \longrightarrow \mathfrak{Z}$ is also an order slanted function.*

Proof. It is routine calculation. □

Definition 3.4.11. An order slanted function $f : (X, \Lambda_1, \Lambda_2) \longrightarrow (Y, \Lambda_1, \Lambda_2)$ is an **order slanted homeomorphism** if f is a bijective map that satisfies:

1. $x \leq y$ if and only if $f(x) \leq f(y)$ for all $x, y \in X$,

2. $x\ R\ \underline{y}$ if and only if $f(x)\ R\ f(\underline{y})$ for all $x \in X$, all $\underline{y} \in X^n$ and all $R \in \Gamma_1$,

3. $x\ S\ y$ if and only if $f(x)\ S\ f(\underline{y})$ for all $x \in X$, all $\underline{y} \in X^n$ and all $S \in \Gamma_2$,

Proposition 3.4.12. *A map $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an order slanted homeomorphism if and only if f and f^{-1} are order slanted functions.*

Proof. For the if part, by Priestley duality, it suffices to prove that

$$x\ R\ \underline{y} \iff f(x)\ R\ f(\underline{y}) \text{ and } x\ S\ \underline{y} \iff f(x)\ S\ f(\underline{y}).$$

Hence, suppose first that $x\ R\ \underline{y}$. Then, we have $f(\underline{y}) \in f(R(x,-))$. Moreover, since f is slanted, we also have $f(R(x,-)) \subseteq R(f(x),-)$, so that $f(x)\ R\ f(\underline{y})$. By a similar prove, we have that $f(x)\ R\ f(\underline{y})$ implies $x\ R\ \underline{y}$, using this time the fact that f^{-1} is slanted.

For the only if part, once again by Priestley duality, it suffices to prove that f and f^{-1} both satisfy (3.10). Hence, let $\underline{y}$ be an element of $R(f(x),-)$. Since f is bijective, we know that $\underline{y} = f(\underline{z})$ for some $\underline{z}$. Then, we have $f(x)\ R\ f(\underline{z})$ and, consequently, $x\ R\ \underline{z}$. In other words, we have $\underline{y} \in f(R(x,-)) \subseteq {\uparrow^{\epsilon}}(f(R(x,-))$. On the other hand, if $\underline{y} \in {\uparrow^{\epsilon}}(f(R(x,-))$, then we have $\underline{y} \geq^{\epsilon} f(\underline{z})$ and $x\ R\ \underline{z}$ for some $\underline{z}$. Since, once again, f is bijective, we know that $\underline{y} = f(\underline{t})$ for some $\underline{t}$. It follows that we have

$$x\ R\ \underline{z} \leq^{\epsilon} \underline{t}.$$

Now, the conclusion follows from the fact that R is of order-type $(1,\epsilon)$. The case of f^{-1} is treated similarly. □

The characterisation of isomorphisms for slanted Priestley spaces can be simplified one step further with the following proposition, whose proof is left to the reader.

Proposition 3.4.13. *A map $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an order slanted homeomorphism if and only if f is a bijective order slanted function which satisfies $x \leq y$ if and only if $f(x) \leq f(y)$.*

Notation 3.4.14. Let $\mathfrak{L} = (B, \Gamma_1, \Gamma_2)$ be a slanted lattice with $\Gamma_1 = (\triangle_i \mid i \in I)$ and $\Gamma_2 = (\triangledown_j \mid j \in J)$. We denote by $\mathfrak{X}_{\mathfrak{L}} = (X_L, \leq, \Lambda_{\Gamma_1}, \Lambda_{\Gamma_2})$ the slanted Priestley space whose underlying Priestley space is $(X_L, \leq) = (\mathrm{Prim}(L), \subseteq)$ and whose families of closed relations are $\Lambda_{\Gamma_1} = (R_i \mid i \in I)$ where R_i is the relation $R_{\triangle_i}$ for all $i \in I$ and $\Lambda_{\Gamma_2} = (S_j \mid j \in J)$ where S_j is the relation $S_{\triangledown_j}$ for all $j \in J$.

On the other hand, let $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$ be a slanted Priestley space with $\Lambda_1 = (R_i \mid i \in I)$ and $\Lambda_2 = (S_j \mid j \in J)$. We denote by $\mathfrak{L}_{\mathfrak{X}} = (L_X, \Gamma_{\Lambda_1}, \Gamma_{\Lambda_2})$ the slanted lattice whose underlying bounded distributive lattice is $L_X = {\uparrow}\mathrm{Clop}(X)$ and whose families of slanted operators are $\Gamma_{\Lambda_1} = (\triangle_i \mid i \in I)$ where $\triangle_i$ is the operator $\triangle_{R_i}$ for all $i \in I$ and $\Gamma_{\Lambda_2} = (\triangledown_j \mid j \in J)$ where $\triangledown_i$ is the operator $\triangledown_{S_j}$ for all $j \in J$.

Proposition 3.4.15. *1. Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ and $\mathfrak{L} = (M, \Gamma_1, \Gamma_2)$ be slanted lattices with identical signatures and $h : \mathfrak{M} \longrightarrow \mathfrak{L}$ be a slanted morphism. Then*

$$h^{-1} : \mathfrak{X}_{\mathfrak{L}} \longrightarrow \mathfrak{X}_{\mathfrak{M}} : x \longmapsto h^{-1}(x)$$

is an order slanted function.

2. let $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$ and $\mathfrak{Y} = (Y, \leq, \Lambda_1, \Lambda_2)$ be slanted Priestley spaces with identical signatures and $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be an order slanted function. Then,

$$f^{-1} : \mathfrak{L}_{\mathfrak{Y}} \longrightarrow \mathfrak{L}_{\mathfrak{X}} : O \longmapsto f^{-1}(O)$$

is a slanted morphism.

Proof. As usual, only the c-slanted case will be proved.

1. Let X denote Prim(L) and Y denote Prim(M). Moreover, let $\triangle_i \in \Gamma_1$ whose order-type is ϵ. We need to show that for every $x \in X$ and every $\underline{y} \in Y^{n_i}$, we have

$$\underline{y} \in R_i(h^{-1}(x), \underline{\ }) \iff \exists t \in h^{-1}(R_i(x, \underline{\ })) : \underline{y} \geq^{\epsilon^\partial} \underset{\sim}{z}. \tag{3.11}$$

Before we start to prove (3.11) itself, recall that $\underline{t} \in h^{-1}(R_i(x, \underline{\ }))$ if and only if there is $\underline{z} \in X^{n_i}$ such that

$$\underline{t} = h^{-1}(\underline{z}) \text{ and } \{b \in L \mid \exists \underline{a} \in \underset{\sim}{z} : \triangle_i \underline{a} \leq b\} \subseteq x.$$

Recall also that $\underline{y} \in R_i(h^{-1}(x), \underline{\ })$ if and only if

$$\{b \in M \mid \exists \underline{a} \in \underset{\sim}{y} : \triangle_i \underline{a} \leq b\} \subseteq h^{-1}(x).$$

We can now turn to the actual proof of (3.11).

$\Leftarrow$ Let b be an element of M such that $\triangle_i \underline{a} \leq b$ for some $\underline{a} \in \underset{\sim}{y}$. Hence, since h is a slanted morphism, we have

$$\triangle_i h(\underline{a}) = h^\delta(\triangle_i \underline{a}) \leq h^\delta(b) = h(b).$$

Moreover, Since $\underline{y} \geq^{\epsilon^\partial} h^{-1}(\underline{z})$ for some $\underline{z} \in X^{n_i}$ such that $x\ R_i\ \underline{z}$, we have that $\underline{a} \in \underset{\sim}{y} \subseteq h^{-1}(\underset{\sim}{z})$, that is $h(\underline{a}) \in \underset{\sim}{z}$, and therefore, $h(b) \in x$ or, in other words, $b \in h^{-1}(x)$.

$\Rightarrow$ Suppose now that $\underline{y} \notin \uparrow^{\epsilon^\partial} h^{-1}(R_i(x, \underline{\ }))$. Since h^{-1} is continuous and $R_i(x, \underline{\ })$ compact, we have that $h^{-1}(R_i(x, \underline{\ }))$ is compact and, therefore, that $\uparrow^{\epsilon^\partial} h^{-1}(R_i(x, \underline{\ }))$ is closed (cf. [38, Proposition VI-1.6.]) Therefore, by Lemma 3.4.3, there is $\underline{a} \in L^{n_i}$ such that $\underline{y} \in \eta(\underline{a})$ and

$$\eta(\underline{a}) \cap \uparrow^{\epsilon^\partial} h^{-1}(R_i(x, \underline{\ })) = \emptyset. \tag{3.12}$$

Recall that the proof is concluded if there exists $b \in M$ such that $\triangle_i \underline{a} \leq b$ and $b \notin h^{-1}(x)$. Let us proceed by contradiction and suppose that $h(b) \in x$ for each $b \in M$ that satisfies $\triangle_i \underline{a} \leq b$. As a direct consequence, we have that $c \in x$ for all $c \in L$ such that $\triangle_i h(\underline{a}) \leq c$. Indeed, if $\triangle_i(h(\underline{a})) \leq c$ then, by Proposition 3.3.4, there is $b \in M$ such that $\triangle_i \underline{a} \leq b$ and $h(b) \leq c$. Therefore, since x is a filter and $h(b) \in x$, it is clear that $c \in x$ for all $c \geq \triangle_i h(\underline{a})$. Consequently, we have

$$x \in \cap\{\eta(c) \mid \triangle_i h(\underline{a}) \leq c\} = R_i(-, \eta(h(\underline{a}))).$$

Henceforth, there is $\underline{t} \in X^{n_i}$ such that $x\ R_i\ \underline{t}$ and $\underline{t} \in \eta(h(\underline{a}))$ or, in other words, such that $h(\underline{a}) \in \underset{\sim}{t}$ or even $h^{-1}(\underline{t}) \in \eta(\underline{a})$. It follows that

$$h^{-1}(\underline{t}) \in \eta(\underline{a}) \cap h^{-1}(R_i(x, \underline{\ })),$$

which clearly contradicts (3.12).

2. Let R_i be a relation of Λ_1 whose order-type is $(1, \epsilon)$. We need to show that the following diagram is commutative

$$\begin{tikzcd}
\uparrow\mathcal{P}(Y) \arrow[r, "(f^{-1})^{\delta}"] & \uparrow\mathcal{P}(X) \\
\uparrow\mathrm{Clop}(Y)^n \arrow[u, "\Delta_i"] \arrow[r, "(f^{-1})^n"'] & \uparrow\mathrm{Clop}(X)^n \arrow[u, "\Delta_i"']
\end{tikzcd}$$

First of all note that $(f^{-1})^{\delta}$ is just f^{-1}, so that we need to show that for every $\underline{O} \in (\uparrow\mathrm{Clop}(Y))^n$, we have

$$f^{-1}(R_i(-, \underset{\sim}{Q}^{\epsilon^{\partial}})) = R(-, f^{-1}(\underset{\sim}{Q}^{\epsilon^{\partial}})).$$

Therefore, suppose on the one hand that $x \in f^{-1}(R_i(-, \underset{\sim}{Q}^{\epsilon^{\partial}}))$, that is there exists $\underset{\sim}{z} \in \underset{\sim}{Q}^{\epsilon^{\partial}}$ such that $f(x)\ R_i\ \underset{\sim}{z}$. It follows that

$$\underset{\sim}{z} \in R_i(f(x), \underline{-}) = \uparrow^{\epsilon} f(R_i(x, \underline{-})).$$

Therefore, we have $\underset{\sim}{z} \leq^{\epsilon^{\partial}} f(\underline{t})$ for some $\underline{t} \in R_i(x, -)$. But $\underset{\sim}{Q}^{\epsilon^{\partial}}$ is an element of $(\uparrow\mathrm{Clop}(X))^{\epsilon^{\partial}}$ and, hence, we obtain $f(\underline{t}) \in \underset{\sim}{Q}^{\epsilon^{\partial}}$ and also $x \in R_i(-, f^{-1}(\underset{\sim}{Q}^{\epsilon^{\partial}}))$.

Suppose now that $x\ R_i\ \underset{\sim}{z}$ for some $\underset{\sim}{z} \in f^{-1}(\underset{\sim}{Q}^{\epsilon^{\partial}})$. In particular, we have $\underset{\sim}{z} \in R_i(x, \underline{-})$ and therefore

$$f(\underset{\sim}{z}) \in f(R_i(x, \underline{-})) \subseteq R_i(f(x), \underline{-}).$$

Now, we have $f(x) \in R(-, \underset{\sim}{Q}^{\epsilon^{\partial}})$, that is $x \in f^{-1}(R(-, \underset{\sim}{Q}^{\epsilon^{\partial}}))$, as required.

□

We now have the duality theorem between slanted Priestley spaces and slanted lattices, which is a generalisation of the duality theorem in [12, Section 3.2] but also of the duality of Chapter 2.

Theorem 3.4.16. *Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice and $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$ be a slanted Priestley space. Then $\eta : \mathfrak{L} \longrightarrow \mathfrak{X}_{\mathfrak{L}_{\mathfrak{X}}}$ is a slanted isomorphism and $\varepsilon : \mathfrak{X} \longrightarrow \mathfrak{X}_{\mathfrak{L}_{\mathfrak{X}}}$ is an order slanted homeomorphism.*

Proof. It is clear by Lemma 3.4.5 that η is a slanted morphism. Indeed, we have, for a c-slanted operator $\Delta_i \in \Gamma_1$ of order-type ϵ,

$$\eta^{\delta}(\Delta_i \underline{a}) = \cap\{\eta(b) \mid \Delta_i \underline{a} \leq b\} = R_i(-, \eta(\underline{a}^{\epsilon})) = \Delta_{R_i}(\eta(\underline{a}))$$

(recall that R_i is of order-type $(1, \epsilon^{\partial})$) and for an o-slanted operator $\nabla_j \in \Gamma_2$ of order-type ϵ

$$\eta^{\delta}(\nabla_j \underline{a}) = \cup\{\eta(b) \mid b \mid b \leq \nabla_j \underline{a}\} = S_j(-, \eta(\underline{a}^{\epsilon^{\partial}})) = \nabla_{S_j}(\eta(\underline{a})).$$

Finally, since by Priestley duality the map η is a bijection, we have the conclusion.

Now, for the topological part, by Priestley duality, it is sufficient to prove that $x\ R_i\ \underline{y}$ if and only if $\varepsilon(x)\ R_{\Delta_i}\ \varepsilon(\underline{y})$. Hence, suppose first that $x\ R_i\ \underline{y}$. It follows that

$$\{O \in \uparrow\mathrm{Clop}(X) \mid \exists \underline{U} \in \uparrow\mathrm{Clop}(X) : \underset{\sim}{y} \in \underset{\sim}{U} \text{ and } R_i(-, \underset{\sim}{U}) \subseteq O\} \subseteq \varepsilon(x). \tag{3.13}$$

Now, since we have $\underset{\sim}{y} \in \underset{\sim}{U}$ if and only if $\underline{U} \in \varepsilon(\underset{\sim}{y})$, it is clear that (3.13) is equivalent to $\varepsilon(x)\ R_{\Delta_i}\ \varepsilon(\underline{y})$.

Finally, suppose that $x \not R_i\, y$. By Lemma 3.4.3, there exists $O \in {\downarrow}\mathrm{Clop}(X)$ and $\underset{\sim}{U}^{\epsilon^\partial} \in {\uparrow}\mathrm{Clop}(X)$ such that $(x, \underline{y}) \in O \times \underline{U} \subseteq {R_i}^c$. Henceforth, we have $\underline{U} \in \varepsilon(\underset{\sim}{y})$, $R_i(-, \underset{\sim}{U}) \subseteq O^c$ and $O^c \notin \varepsilon(x)$, which implies that $\varepsilon(x) \not R_{\triangle_i}\, \varepsilon(\underline{y})$, as required.

The last condition to satisfy, that is $x\ S_j\ \underline{y}$ if and only if $\varepsilon(x)\ S_{\nabla_j}\ \varepsilon(\underline{y})$ for all $S_j \in \Lambda_2$, is left to the reader. ☐

3.5 Canonical extensions of slanted lattices

As we recalled in Section 3.1, for L and M lattices, a map $h : L^n \longrightarrow M$ can be extended to a map $h^\delta : (L^\delta)^n \longrightarrow M^\delta$. In our situation, we have maps $\triangle : L^n \longrightarrow L^\delta$ and $\nabla : L^n \longrightarrow L^\delta$ that therefore should be extended to maps $\triangle^\delta : (L^\delta)^n \longrightarrow (L^\delta)^\delta$ and $\nabla^\delta : (L^\delta)^n \longrightarrow (L^\delta)^\delta$. However, in this particular case, these maps will be extended as follows $\triangle^\delta : (L^\delta)^n \longrightarrow L^\delta$ and $\nabla^\delta : (L^\delta)^n \longrightarrow L^\delta$ (as it is done in [35]). We know that such maps should exists thanks to the universal properties of canonical extension.

As a consequence of the choice we made for the extension, the canonical extensions of slanted lattices will be complete lattices with (clopen) operators. This choice is natural when we think of slanted lattices as models for logics. Let us recall indeed that we built canonical extensions of subordination algebras to obtain complete atomic modal algebras. Namely, we wanted to obtain a semantic where bimodal formulas could be valuated. The scope here is identical.

Definition 3.5.1. Let L be a bounded distributive lattice, n be a natural number and $\epsilon \in \{1, \partial\}^n$ be an order-type. An n-ary **operator** of order-type ϵ is a map $\triangle : L^n \longrightarrow L$ such that

1. if $\epsilon_i = 1$, then we have
   (a) $\triangle(a_1, \ldots, 0, \ldots, a_n) = 0$;
   (b) $\triangle(a_1, \ldots, a_i \vee b_i, \ldots, a_n) = \triangle(a_1, \ldots, a_i, \ldots, a_n) \vee \triangle(a_1, \ldots, b_i, \ldots, a_n)$;
2. if $\epsilon_i = \partial$, then we have
   (a) $\triangle(a_1, \ldots, 1, \ldots, a_n) = 0$;
   (b) $\triangle(a_1, \ldots, a_i \wedge b_i, \ldots, a_n) = \triangle(a_1, \ldots, a_i, \ldots, a_n) \vee \triangle(a_1, \ldots, b_i, \ldots, a_n)$.

An n-ary **dual operator** of order-type ϵ is a map $\nabla : L^n \longrightarrow L$ such that

1. if $\epsilon_i = 1$, then we have
   (a) $\nabla(a_1, \ldots, 1, \ldots, a_n) = 1$;
   (b) $\nabla(a_1, \ldots, a_i \wedge b_i, \ldots, a_n) = \nabla(a_1, \ldots, a_i, \ldots, a_n) \wedge \nabla(a_1, \ldots, b_i, \ldots, a_n)$;
2. if $\epsilon_i = \partial$, then we have
   (a) $\nabla(a_1, \ldots, 0, \ldots, a_n) = 1$;
   (b) $\nabla(a_1, \ldots, a_i \vee b_i, \ldots, a_n) = \nabla(a_1, \ldots, a_i, \ldots, a_n) \wedge \nabla(a_1, \ldots, b_i, \ldots, a_n)$.

A **distributive lattice expansion** (abbreviated in **DLE**) is a triplet $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ where L is a bounded distributive lattice, $\Gamma_1 = (\triangle_i \mid i \in I)$ is a family, possibly empty, of operators on L and $\Gamma_2 = (\nabla_j \mid j \in J)$ is a family, possibly empty, of dual operators on L.

Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice with $\Gamma_1 = (\triangle_i \mid i \in I)$ and $\Gamma_2 = (\nabla_j \mid j \in J)$. The family $(\epsilon_i \mid i \in I) \cup (\epsilon_j \mid j \in J)$, where ϵ_i is the arity of $\triangle_i$ and ϵ_j of ∇_j, is the **signature** of $\mathfrak{L}$.

Remark 3.5.2. As modal algebras were particular subordination algebras, it is clear that DLOs are particular slanted lattices. Indeed, the elements of L are clopen elements of L^δ such that every operator is a c-slanted one and every dual operator is an o-slanted one.

Definition 3.5.3. A **perfect DLE** is a DLE $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ such that

1. L is a perfect lattice , that is L is both completely join-generated by the set $J^\infty(L)$ of the completely join-irreducible elements of L, and completely meet-generated by the set $M^\infty(L)$ of the completely meet-irreducible elements of L.
2. the operators of Γ_1 and the dual operators of Γ_2 respect the infinitary versions of the distribution/reversion laws of Definition 3.5.1, for instance if $\triangle \in \Gamma_1$ is of order-type $\epsilon = \partial$, then
$$\triangle(\wedge S) = \vee\{\triangle s \mid s \in S\}$$
for all $S \subseteq L$.

Definition 3.5.4. Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice whose dual is $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$, then for every $\triangle_i \in \Gamma_1$ of order-type ϵ_i define
$$\triangle_i^\delta : (L^\delta)^n \longrightarrow L^\delta : \underline{E} \longmapsto R_i(-, \underset{\sim}{E}^{\epsilon_i})$$
and for every $\triangledown_j \in \Gamma_2$ of order-type ϵ_j define
$$\triangledown_j^\delta : (L^\delta)^n \longrightarrow L^\delta : \underline{E} \longmapsto S_j(-, \underset{\sim}{E}^{\epsilon_j^\partial})^c.$$
Finally, we denote $\Gamma_1^\delta = (\triangle_i^\delta \mid \triangle_i \in \Gamma_1)$ and $\Gamma_2^\delta = (\triangledown_j^\delta \mid \triangle_j \in \Gamma_2)$

Theorem 3.5.5. *If $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ is a slanted lattice, then $\mathfrak{L}^\delta := (L^\delta, \Gamma_1^\delta, \Gamma_2^\delta)$ is a perfect DLE such that for every $\triangle_i \in \Gamma_1$ and every $\triangledown_j \in \Gamma_2$ we have, for $\underline{a} \in L^n$*
$$\triangle_i^\delta(\underline{a}) = \triangle_i \underline{a} \text{ and } \triangledown_j^\delta(\underline{a}) = \triangledown_j \underline{a}$$

Proof. This is a direct consequence of Section 3.4. □

3.6 A universal algebra approach

In this short section, we present the beginning of a universal algebra approach of slanted lattices. Unary results in the Boolean setting were already given by Celani in [14].

We start the section with a little description of the behaviour of the canonical extension in respect to the usual constructions of universal algebra, namely subobject, quotient and finite product. Recall that we already broached the product in Section 3.1.

3.6.1 Canonical extensions and universal algebra

Sublattices

Let M be a sublattice of a lattice L. Then, the identity
$$i : M \longrightarrow L : a \longmapsto a$$
is an one-to-one morphism and, as such can, be extended to a one-to-one morphism ([34, Lemma 4.9])
$$i^\delta : M^\delta \longrightarrow L^\delta : u \longmapsto i^\delta(u).$$
It follows that M^δ is isomorphic to a sublattice of L^δ, which is clearly the complete sublattice of L^δ generated by M.

Quotients

Let θ be a lattice congruence on a lattice L. Then, the canonical projection

$$\pi : L \longrightarrow L/\theta : a \longmapsto a^\theta$$

is an onto morphism and, as such, can be extended to an onto morphism ([34, Lemma 4.9])

$$\pi^\delta : L^\delta \longrightarrow (L/\theta)^\delta : u \longmapsto \pi^\delta(u).$$

Now, by the first isomorphism theorem, we have

$$L^\delta / \ker(\pi^\delta) \cong (L/\theta)^\delta,$$

such that we can consider the following definition.

Definition 3.6.1. Let θ be a lattice congruence on a lattice L. The binary relation θ^δ on L^δ is defined as

$$u \; \theta^\delta \; v \iff \pi^\delta(u) = \pi^\delta(v).$$

In particular, for any $a, b \in L$, we have the following property

$$a \; \theta \; b \iff a \; \theta^\delta \; b.$$

Proposition 3.6.2. *Let L be a lattice and θ a congruence on L. Then, an element $u^{\theta^\delta} \in L^\delta/\theta^\delta$ is closed (resp. open) if and only if there exists $k \in \mathcal{K}(L^\delta)$ (resp. $o \in \mathcal{O}(L^\delta)$) such that $u \; \theta^\delta \; k$ (resp. $u \; \theta^\delta \; o$).*

Proof. Suppose first that $k \in \mathcal{K}(L^\delta)$, then we have

$$k^{\theta^\delta} = \pi^\delta(k) = \pi^\sigma(k) = \wedge\{\pi(a) \mid a \leq k\} = \wedge\{a^\theta \mid a \leq k\}.$$

Hence, k^δ is closed.

Suppose now that u^{θ^δ} is closed in L^δ/θ^δ. Then, we have

$$u^{\theta^\delta} = \wedge\{a^\theta \mid a^\theta \geq u^{\theta^\delta}\} = \wedge\{\pi(a) \mid a^\theta \leq u^{\theta^\delta}\} = \pi^\delta(\wedge\{a \mid a \leq u\}).$$

Hence, since $\wedge\{a \mid a \leq u\}$ is a closed element of L, the proof is concluded. □

Proposition 3.6.3. *If $h : L \longrightarrow M$ is a lattice morphism, then*

$$\ker(h)^\delta = \ker(h^\delta).$$

Proof. Now for $k_1, k_2 \in \mathcal{K}(L^\delta)$, we have

$$\begin{aligned}
& k_1 \ker(h)^\delta \; k_2 \\
\iff & (\forall a_2 \geq k_2)(\exists a_1 \geq k_1 : a_1 \ker(h) \; a_2) \text{ and } (\forall a_1 \geq k_1)(\exists a_2 \geq k_2 : a_1 \ker(h) \; a_2) \\
\iff & (\forall a_2 \geq k_2)(\exists a_1 \geq k_1 : h(a_1) = h(a_2)) \text{ and } (\forall a_1 \geq k_1)(\exists a_2 \geq k_2 : h(a_1) = h(a_2)) \\
\iff & h^\delta(k_1) = h^\delta(k_2) \\
\iff & k_1 \ker(h^\delta) \; k_2.
\end{aligned}$$

We can similarly prove that for $o_1, o_2 \in \mathcal{K}(L^\delta)$, we have

$$o_1 \ker(h)^\delta \; o_2 \Leftrightarrow o_1 \ker(h^\delta) \; o_2.$$

Finally, we have for $u, v \in L^\delta$

$$\begin{aligned}
& u \ker(h)^\delta v \\
\iff & (\forall o \geq v \text{ and } \forall k \leq u : (k \vee o) \ker(h)^\delta\, o) \text{ and } (\forall o \geq u \text{ and } \forall k \leq v : (k \vee o) \ker(h)^\delta\, o) \\
\iff & (\forall o \geq v \text{ and } \forall k \leq u : (k \vee o) \ker(h^\delta)\, o) \text{ and } (\forall o \geq u \text{ and } \forall k \leq v : (k \vee o) \ker(h^\delta)\, o) \\
\iff & (\forall o \geq v \text{ and } \forall k \leq u : h^\delta(k \vee o) = h^\delta(o)) \text{ and } (\forall o \geq u \text{ and } \forall k \leq v : h^\delta(k \vee o) = h^\delta(o)) \\
\iff & h^\delta(u) = h^\delta(v) \\
\iff & u \ker(h^\delta)\, v.
\end{aligned}$$

□

3.6.2 Subobject, quotient and finite product

Definition 3.6.4. Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ and $\mathfrak{M} = (M, \Gamma_1, \Gamma_2)$ be two slanted lattices with same signature. Then $\mathfrak{M}$ is a **sub-slanted lattice** of $\mathfrak{L}$ if M is a sublattice of L, each slanted operator $\circ \in \Gamma_1 \cup \Gamma_2$ on M is the restriction of $\circ$ on L and, for i the inclusion map, the following diagram is commutative

$$\begin{tikzcd}
M^\delta \arrow[r, "i^\delta"] & L^\delta \\
M^n \arrow[u, "\circ"] \arrow[r, "i^n"'] & L^n \arrow[u, "\circ"']
\end{tikzcd}$$

In other words, M is a sublattice of L such that the inclusion map is a slanted morphism.

Equivalently, M is a sublattice if i satisfies the following conditions for $\triangle \in \Gamma_1$ and $\triangledown \in \Gamma_2$:

1. $\triangle \underline{a} \leq c$ implies $\triangle i(\underline{a}) \leq i(c)$;
2. $\triangledown \underline{a} \leq c$ implies $\triangledown i(\underline{a}) \leq i(c)$;
3. $\triangle i(\underline{a}) \leq b$ implies $\triangle \underline{a} \leq c$ and $i(c) \leq b$ for some $c \in h(L)$;
4. $b \leq \triangledown i(\underline{a})$ implies $c \leq \triangledown \underline{a}$ and $b \leq i(c)$ for some $c \in h(L)$.

Conditions 1 and 2 are always trivially satisfied by definition and conditions 3 and 4 may be summarised in

3'. If $\triangle \underline{a} \leq b$ for $\underline{a} \in M^n$ and $b \in L$, then we have $\triangle \underline{a} \leq c \leq b$ for some $c \in M$,

4'. If $b \leq \triangledown \underline{a}$ for $\underline{a} \in M^n$ and $b \in B$, then we have $b \leq c \leq \triangledown \underline{a}$ for some $c \in M$.

Proposition 3.6.5. *Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ and $\mathfrak{M} = (M, \Gamma_1, \Gamma_2)$ be slanted lattices and $h : \mathfrak{L} \longrightarrow \mathfrak{M}$ be a slanted morphism. Then, $\mathfrak{M}^* = (h(L), \Gamma_1, \Gamma_2)$ is a sub-slanted lattice of $\mathfrak{M}$.*

Proof. We need to prove that $h(L)$ satisfies the conditions 3' and 4' of Definition 3.6.4. Let us prove 3'. Let $\underline{a} \in h(L)^n$ and $b \in M$ such that $\triangle \underline{a} \leq b$. By definition, there exists $\underline{a}' \in L$ such that $h(\underline{a}') = \underline{a}$ and, since h is an slanted morphism, $\triangle \underline{a} \leq b$ implies that there exists $c \in L$ such that $\triangle \underline{a}' \leq c$ and $h(c) \leq b$. It is sufficient to use the other property of slanted morphisms to obtain

$$\triangle \underline{a} = \triangle h(\underline{a}') \leq h(c) \leq b$$

and conclude the proof. □

Definition 3.6.6. Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice. A lattice congruence θ is a **slanted congruence** if and only if for every $\triangle \in \Gamma_1$ and $\triangledown \in \Gamma_2$, we have that $\underline{a}\ \theta\ \underline{b}$ implies $\triangle\underline{a}\ \theta^\delta\ \triangle\underline{b}$ and $\triangledown\underline{a}\ \theta^\delta\ \triangledown\underline{b}$.

Proposition 3.6.7. *Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice and let θ be a slanted congruence. Then, L/θ equipped with the operators*

$$\triangle^\theta(\underline{a}^\theta) = (\triangle(\underline{a}))^{\theta^\delta} \text{ and } \triangledown^\theta(\underline{a}^\theta) = (\triangledown(\underline{a}))^{\theta^\delta}$$

is a slanted lattice such that $\pi : a \longmapsto a^\theta$ is a slanted morphism.

Proof. As usual, we only give the proof for $\triangle$. The first thing to note is that $\triangle^\theta$ is well defined thanks to the definition of slanted congruence. Now, we have to check that $\triangle^\theta$ is a c-slanted operator which share a common order-type ϵ with $\triangle$. Consider the case where $\epsilon_i = 1$. Then, we have

$$\begin{aligned}
&\triangle^\theta(\ldots, a^\theta \vee b^\theta, \ldots)\\
=&\triangle^\theta(\ldots, (a \vee b)^\theta, \ldots)\\
=&(\triangle(\ldots, a \vee b, \ldots))^{\theta^\delta}\\
=&(\triangle(\ldots, a, \ldots) \vee \triangle(\ldots, b, \ldots))^{\theta^\delta}\\
=&(\triangle(\ldots, a, \ldots))^{\theta^\delta} \vee (\triangle(\ldots, b, \ldots))^{\theta^\delta}\\
=&\triangle^\theta(\ldots, a^\theta, \ldots) \vee \triangle^\theta(\ldots, b^\theta, \ldots)
\end{aligned}$$

and

$$\begin{aligned}
&\triangle^\theta(\cdots, 0^\theta, \cdots)\\
=&(\triangle(\ldots, 0, \ldots))^{\theta^\delta}\\
=&0^{\theta^\delta} = 0^\theta,
\end{aligned}$$

as required, the case $\epsilon_i = \partial$ is of course treated identically. Moreover, since $\triangle(\underline{a})$ is closed in L^δ, we have that $(\triangle(\underline{a}))^{\theta^\delta}$ is closed in L^δ/θ^∂.

Now, we have to check that π is indeed an a slanted morphism, that is, the following diagram is commutative.

$$\begin{array}{ccc}
L^\delta & \xrightarrow{\pi^\delta} & L^\delta/\theta^\delta\\
\triangle \uparrow & & \uparrow \triangle^\theta\\
L^n & \xrightarrow[\pi^n]{} & (L/\theta)^n
\end{array}$$

But this follows immediately from the definition of $\triangle^\theta$. $\square$

Proposition 3.6.8. *Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice and θ be a lattice congruence. Then, θ is a slanted congruence if and only if θ satisfied the following conditions:*

1. *$\underline{a}\ \theta\ \underline{b}$ and $\triangle\underline{b} \leq c$ implies $\triangle\underline{a} \leq d$ for some $d\ \theta\ c$,*

2. *$\underline{a}\ \theta\ \underline{b}$ and $c \leq \triangledown\underline{b}$ implies $d \leq \triangledown\underline{a}$ for some $d\ \theta\ c$.*

Proof. Suppose first that θ is a slanted congruence and consider that $\underline{a}\ \theta\ \underline{b}$ and $\triangle\underline{b} \leq c$. It follows that $\triangle\underline{a}\ \theta^\delta\ \triangle\underline{b}$, that is

$$\pi^\sigma(\triangle\underline{a}) = \pi^\delta(\triangle\underline{a}) = \pi^\delta(\triangle\underline{b}) = \pi^\sigma(\triangle\underline{b}) \leq \pi(c).$$

Now, we have

$$\pi^\sigma(\triangle\underline{a}) = \wedge\{\pi(d) \mid \triangle\underline{a} \leq d\} \leq \pi(c),$$

such that, by compactness, we can find an element d such that $\triangle a \leq d$ and $\pi(d) \leq \pi(c)$. It follows that $\triangle a \leq d \vee c$ and $\pi(d \vee c) = \pi(d) \vee \pi(c) = \pi(c)$, that is $(d \vee c)\ \theta\ c$, as required.

Suppose now that θ satisfies condition 1, we have to prove that $\underline{a}\ \theta\ \underline{b}$ implies $\triangle\underline{a}\ \theta^\delta\ \triangle\underline{b}$, that is

$$\wedge\{\pi(c) \mid \triangle\underline{a} \leq c\} = \wedge\{\pi(d) \mid \triangle\underline{b} \leq d\}.$$

It is sufficient to show that $\{\pi(c) \mid \triangle\underline{a} \leq c\} \subseteq \{\pi(d) \mid \triangle\underline{b} \leq d\}$. Indeed, by symmetry of the reasoning, we will also have the other inclusion. Let c such that $\triangle\underline{a} \leq c$. Then, by condition 1, since $\underline{a}\ \theta\ \underline{b}$, we have $\triangle\underline{b} \leq d$ for some $d\ \theta\ c$, that is $\pi(c) = \pi(d)$ and the conclusion is immediate. $\square$

Proposition 3.6.9. *If $h : L \longrightarrow M$ is a slanted morphism, then* $\ker(h)$ *is a slanted congruence.*

Proof. We already know that $\ker(h)$ is a lattice congruence. Therefore, we only have to prove that $\ker(h)$ satisfies

1. $\underline{a} \ker(h)\ \underline{b}$ and $\triangle\underline{b} \leq c$ implies $\triangle\underline{a} \leq d$ for some $d \ker(h)\ c$,

2. $\underline{a} \ker(h)\ \underline{b}$ and $c \leq \nabla\underline{b}$ implies $d \leq \nabla\underline{a}$ for some $d \ker(h)\ c$.

We prove 1. Suppose that $h(\underline{a}) = h(\underline{b})$ and $\triangle\underline{b} \leq c$. Since h is a slanted morphism, it follows that

$$\triangle h(\underline{a}) = \triangle h(\underline{b}) \leq h(c).$$

Moreover, there is an element d such that $\triangle\underline{a} \leq d$ and $h(d) \leq h(c)$. Finally, we have

$$\triangle\underline{a} \leq d \vee c \text{ and } h(d \vee c) = h(c),$$

as required.

$\square$

Definition 3.6.10. Let $\triangle^{L_1}$ and $\triangle^{L_2}$ be c-slanted operators with identical order type respectively on L_1 and L_2, we define on $L_1 \times L_2$ the operator

$$\triangle^{L_1 \times L_2} : (L_1 \times L_2)^n \longrightarrow L_1^\delta \times L_2^\delta : \underline{(a, b)} \longmapsto (\triangle^{L_1}\underline{a}, \triangle^{L_2}\underline{b}).$$

We define similarly the operator $\nabla^{L_1 \times L_2}$ for o-slanted operators.

Proposition 3.6.11. *Let $\mathfrak{L}_1 = (L_1, \Gamma_1^{L_1}, \Gamma_2^{L_1})$ and $\mathfrak{L}_2 = (L_2, \Gamma_1^{L_2}, \Gamma_2^{L_2})$ be slanted lattices with identical signature. Then $L_1 \times L_2$ endowed with the operators of Definition 3.6.10 is a slanted lattice, which will be denoted by $\mathfrak{L}_1 \times \mathfrak{L}_2$, which have the same signature as $\mathfrak{L}_1$ and $\mathfrak{L}_2$.*

Proof. First of all, remember that $(L_1^\delta, L_2^\delta) \cong (L_1 \times L_2)^\delta$ and that $\mathcal{K}((L_1 \times L_2)^\delta)$ correspond exactly to $\mathcal{K}(L_1^\delta) \times \mathcal{K}(L_2^\delta)$. Therefore, we just have to check that $\triangle^{L_1 \times L_2}$ have the same order-type ϵ as $\triangle^{L_1}$ and $\triangle^{L_2}$. We prove the case $\epsilon_i = 1$. We have

$$\begin{aligned}
&\triangle^{L_1\times L_2}(\ldots,(a,b)\vee(c,d),\ldots)\\
=&\triangle^{L_1\times L_2}(\ldots,(a\vee c, b\vee d),\ldots)\\
=&(\triangle^{L_1}(\ldots,a\vee c,\ldots),\triangle^{L_2}(\ldots,b\vee d,\ldots))\\
=&(\triangle^{L_1}(\ldots,a,\ldots)\vee\triangle^{L_1}(\ldots,c,\ldots),\triangle^{L_2}(\ldots,b,\ldots)\vee\triangle^{L_2}(\ldots,d,\ldots))\\
=&(\triangle^{L_1}(\ldots,a,\ldots),\triangle^{L_2}(\ldots,b,\ldots))\vee(\triangle^{L_1}(\ldots,c,\ldots),\triangle^{L_2}(\ldots,d,\ldots))\\
=&\triangle^{L_1\times L_2}(\ldots,(a,b),\ldots)\vee\triangle^{L_1\times L_2}(\ldots,(c,d),\ldots)
\end{aligned}$$

and

$$\begin{aligned}
&\triangle^{L_1\times L_2}(\ldots,(1,1),\ldots)\\
=&(\triangle^{L_1}(\ldots,1,\ldots),\triangle^{L_2}(\ldots,1,\ldots))\\
=&(1,1).
\end{aligned}$$

□

Proposition 3.6.12. *The projection maps $p_i : (a_1, a_2) \longmapsto a_i$ are slanted morphisms.*

Proof. It follows directly from the definitions. □

Chapter 4

Slanted canonicity

At the end of Chapter 2, we established a class of formulas, namely the s-Sahlqvist formulas, which were canonical for subordination algebras in the sense of Definition 2.4.5 and that admit a first order translation. In this section, we extend this result from the category of subordination algebras to the more general one of slanted lattices and we prove that s-Sahlqvist formulas are very specific examples of a more general class of canonical formulas: the class of analytic inductive formulas. This class was introduced in [41] in the context of the theory of analytic calculi in structural proof theory, to characterize the logics which can be presented by means of proper display calculi. This

The precautions used in Chapter 2, due to the inherent natures of subordination algebras, are still to be considered here. Indeed, the usual models of $\mathcal{L}_{\mathsf{DLE}}$-languages are "standard" distributive lattices expansions. Now, as subordination algebras were, in general, not standard modal algebras, the slanted lattices of Chapter 3 are not standard lattices expansions, since their associated operators are slanted. Hence, once again, the valuations of formulas may fail to be elements of the lattices, but are rather elements of their canonical extensions.

The method used to achieve canonicity in this chapter is based on the algorihtm ALBA (Ackermann Lemmas Based Algorithm) developed first for distributive logic in [19] and then for non-distributive logic in [20]. The results presented here concern the slanted (distributive) lattices with topological methods but can be proved for the more general case of non-distributive slanted lattices in purely constructive ways (that is without axiom of choice) as it is done in [25]. Once the process to prove canonicity is established, we comment in Section 4.7 the similitudes and differences with Chapter 2.

We end the chapter with a comparison between the Balbiani-Kikot formulas (already mentioned in Section 2.10) and the analytic inductive formulas of an extended language in the subordination setting.

4.1 The $\mathcal{L}_{\mathsf{DLE}}$-language

We start with the usual definitions of language, satisfaction and validity.

Definition 4.1.1. 1. The **language** $\mathcal{L}_{\mathrm{DLE}}(\Gamma_1, \Gamma_2)$, from now shortened as $\mathcal{L}_{\mathsf{DLE}}$ if the context causes no confusion, is constituted by:

- a denumerable set $\mathrm{Var} = \{p, q, r, \ldots\}$ of propositional variables,
- the classical lattices connectives $\wedge$ and $\vee$,

- the classical lattices constants $\top$ and $\bot$,
- disjoint sets of connectives Γ_1 and Γ_2. Each connective $\circ \in \Gamma_1 \cup \Gamma_2$ has an associated arity $n_\circ$ and an associated order-type $\epsilon_\circ$.

2. The **formulas** of $\mathcal{L}_{\mathsf{DLE}}$ are defined recursively as follow

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \circ(\underline{\varphi})$$

where $p \in \mathrm{Var}$ and $\circ \in \Gamma_1 \cup \Gamma_2$.

3. The **inequalities** of $\mathcal{L}_{\mathsf{DLE}}$ are expressions of the form $\varphi \leq \psi$ where φ and ψ are $\mathcal{L}_{\mathsf{DLE}}$ formulas.

Example 4.1.2. In Chapter 2, the language was given by $\Gamma_1 = \{\Diamond, \blacklozenge, \neg\}$ and $\Gamma_2 = \{\Box, \blacksquare, \rightarrow, \neg\}$ with the following order-types: $\varepsilon_\Diamond = \varepsilon_\blacklozenge = \varepsilon_\Box = \varepsilon_\blacksquare = 1$, $\varepsilon_\neg = \partial$ and $\varepsilon_\rightarrow = (\partial, 1)$.

The interpretation of the language $\mathcal{L}_{\mathsf{DLE}}$ in a slanted lattice $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ is of course constructed in a way similar to the one used for subordination algebras, namely, through canonical extension. Indeed, once again, the valuation of a formula may fail to be an element of the original lattice. Moreover, since $\mathfrak{L}^\delta$ is a $\mathcal{L}_{\mathsf{DLE}}$ lattice in the usual sense, we can use the usual notion of validity available on $\mathfrak{L}^\partial$.

Definition 4.1.3. Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice. A **valuation** on $\mathfrak{L}$ is a map $v : \mathrm{Var} \longrightarrow L$.

Definition 4.1.4. Let $\varphi \leq \psi$ be an $\mathcal{L}_{\mathsf{DLE}}$ inequality, $\mathfrak{L}$ be a slanted lattice and η be the canonical embedding from $\mathfrak{L}$ to $\mathfrak{L}^\delta$. We say that $\varphi \leq \psi$ is **valid** in $\mathfrak{L}$ for a valuation v, which is denoted by $\mathfrak{L} \models_v \varphi \leq \psi$, if $\mathfrak{L}^\delta \models_{\eta \circ v} \varphi \leq \psi$ in the usual sense, that is $(\eta \circ v)(\varphi) \leq (\eta \circ v)(\psi)$. We say that $\varphi \leq \psi$ is **satisfied** in $\mathfrak{L}$ if $\varphi \leq \psi$ is valid for all valuations.

On the topological side, the interpretation of $\mathcal{L}_{\mathsf{DLE}}$ in a slanted Priestley space $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$ is defined quite naturally, by extending a valuation on variables to one on formulas, using the different accessibility relations of Λ_1 and Λ_2.

Definition 4.1.5. Let $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$ be a slanted Priestley space. A **valuation** on $\mathfrak{X}$ is a map $v : \mathrm{Var} \longrightarrow \uparrow\!Of(X)$.

A valuation v is extended to the set of all $\mathcal{L}_{\mathsf{DLE}}$ formulas according to the following inductive rules:

- $v(\top) = X$, $v(\bot) = \emptyset$
- $v(\psi \vee \chi) = v(\psi) \cup v(\chi)$, $v(\psi \wedge \chi) = v(\psi) \cap v(\chi)$,
- if $\triangle_i \in \Gamma_1$ is of order-type ϵ_i, then $v(\triangle_i(\underline{\psi})) = R_i(-, v(\underset{\sim}{\psi}^{\epsilon_i}))$,
- if $\nabla_j \in \Gamma_2$ is of order-type ϵ_j, then $v(\nabla_j(\underline{\psi})) = S_i(-, v(\underset{\sim}{\psi}^{\epsilon_j^\partial}))^c$,

Definition 4.1.6. Let $\varphi \leq \psi$ be an $\mathcal{L}_{\mathsf{DLE}}$-inequality, $\mathfrak{X} = (X, \leq, \Gamma_1, \Gamma_2)$ a slanted Priestley space and $x \in X$.

1. We say that $\varphi \leq \psi$ is **valid** in $\mathfrak{X}$ for a valuation v, which is denoted by $\mathfrak{X} \models_v \varphi \leq \psi$, if $v(\varphi) \subseteq v(\psi)$.
2. We say that $\varphi \leq \psi$ is **satisfied**, which is denoted by $\mathfrak{X} \models \varphi \leq \psi$, if $\mathfrak{X} \models_v \varphi \leq \psi$ for all valuations v.

Finally, we can observe that the definition of satisfaction in slanted Priestley spaces coincides with the definition of satisfaction in slanted lattices, in the sense of the following theorem.

Theorem 4.1.7. *Let $\mathfrak{L}$ be a slanted lattice whose dual is the slanted Priestley space $\mathfrak{X}$. Then, for any $\mathcal{L}_{\mathsf{DLE}}$-inequality $\varphi \leq \psi$, we have*

$$\mathfrak{L} \models \varphi \leq \psi \textit{ iff } \mathfrak{X} \models \varphi \leq \psi.$$

Finally, we arrive at the notion of slanted canonicity, which is defined quite naturally.

Definition 4.1.8. An $\mathcal{L}_{\mathsf{DLE}}$-inequality $\varphi \leq \psi$ is **slanted canonical** if $\mathfrak{L} \models \varphi \leq \psi$ implies $\mathfrak{L}^\delta \models \varphi \leq \psi$ for any slanted lattice $\mathfrak{L}$.

On the topological side, if $\mathfrak{X}^\delta$ denotes the discrete version of a slanted Priestley space, then the definition of slanted canonicity corresponds to

$$\mathfrak{X} \models \varphi \leq \psi \text{ iff } \mathfrak{X}^\delta \models \varphi \leq \psi.$$

Now that the fundamentals have been established, we can look at the analytic inductive inequalities in the language $\mathcal{L}_{\mathsf{DLE}}$, which will be a fragment of canonical inequalities in the slanted setting.

4.2 Analytic and inductive inequalities

4.2.1 Generations trees and formulas

In this section, we introduce the terminology relative to the topic and illustrate the given definitions with short examples. We redirect the reader to [18], [19], [20] and [41] for more information and details.

Definition 4.2.1. For the next definitions, we will consider $\varphi(\underline{p})$ to be an $\mathcal{L}_{\mathsf{DLE}}$-formula and ϵ to be an **order-type** on $\underline{p} = (p_1, \ldots, p_n)$, that is a n-uple in $\{1, \partial\}^n$ for some natural n.

1. The **positive** (resp. **negative**) **generation tree** of φ is defined by labelling the root node of the generation tree of φ with $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows:
   (a) For any node labelled with $\wedge$, $\vee$, we assign the same sign to its children nodes.
   (b) For any node labelled with $\circ \in \Gamma_1 \cup \Gamma_2$ of arity $n_\circ \geq 1$, and for any $1 \leq i \leq n_\circ$, assign the same (resp. the opposite) sign to its i^{th} child node if $\varepsilon_\circ(i) = 1$ (resp. if $\varepsilon_\circ(i) = \partial$).
2. A node in a signed generation tree is said to be **positive** (resp. **negative**) if signed $+$ (resp. $-$).
3. For a formula $\varphi(p_1, \ldots, p_n)$, an order-type ϵ over n and $i \in \{1, \ldots, n\}$, a **ϵ-critical node** in a signed generation tree of φ is a leaf node $+p_i$ if $\epsilon_i = 1$ or $-p_i$ if $\epsilon_i = \partial$.
4. A **ϵ-critical branch** in a signed generation tree is a branch whose leaf is a ϵ-critical node.
5. Let $\varphi \leq \psi$ be a bimodal inequality. Its **generation tree** is the combination of the positive generation tree of φ and the negative generation tree of ψ.
6. An $\mathcal{L}_{\mathsf{DLE}}$-inequality $\varphi \leq \psi$ is said to be **uniform** in a given variable p if all occurrences of p in its generation tree are labelled with the same sign.

7. Let ϵ be an order-type. An $\mathcal{L}_{\mathsf{DLE}}$-inequality $\varphi \leq \psi$ is **ϵ-uniform** in a (sub)array $p_1, \ldots, p_n$ of its variables if $\varphi \leq \psi$ is uniform in p_i, for the sign dictated by ϵ_i, for all $i \in \{1, \ldots, n\}$.

Example 4.2.2. The positive signed generation tree of the the bimodal formula $(\blacksquare p \to p) \wedge (q \vee \Diamond r)$ is given by:

$$
\begin{tikzcd}
 & & \overset{+}{\wedge} \arrow[dl] \arrow[dr] & & \\
 & \overset{+}{\rightarrow} \arrow[dl] \arrow[d] & & \overset{+}{\vee} \arrow[d] \arrow[dr] & \\
\overset{-}{\blacksquare} \arrow[d] & \overset{+}{p} & & \overset{+}{q} & \overset{+}{\Diamond} \arrow[d] \\
\overset{-}{p} & & & & \overset{+}{r}
\end{tikzcd}
$$

We have that the node $\overset{-}{p}$ is ϵ_1-critical for $\epsilon_1 = (\partial, 1, 1)$, the node $\overset{+}{q}$ is ϵ_2-critical for $\epsilon_2 = (1, 1, \partial)$. Hence, for instance, the branch

$$
\begin{tikzcd}
\overset{+}{\wedge} \arrow[d] \\
\overset{+}{\rightarrow} \arrow[d] \\
\overset{-}{\blacksquare} \arrow[d] \\
\overset{-}{p}
\end{tikzcd}
$$

which ends with a ϵ_1-critical node is ϵ_1-critical.

Definition 4.2.3. For an $\mathcal{L}_{\mathsf{DLE}}$-formula $\varphi(p_1, ..., p_n)$ and an order-type ϵ, we say that the generation tree $+\varphi$ (resp. $-\varphi$) **agrees with** ϵ, and write $\epsilon(+\varphi)$ (resp. $\epsilon(-\varphi)$) if every leaf in the positive (resp. negative) generation tree of φ is ϵ-critical.

We will also write $+\varphi' \propto *\varphi$ (resp. $-\varphi' \propto *\varphi$), with $* \in \{-, +\}$, to indicate that the subformula φ' inherits the positive (resp. negative) signs from the signed generation tree.

Finally, we write $\epsilon(\varphi') \propto *\varphi$ (resp. $\epsilon^{\partial}(\varphi') \propto *\varphi$) to indicate that the signed subformula φ', with the signs inherited from $*\varphi$, agrees with ϵ (resp. ϵ^{∂}).

Example 4.2.4. Let us consider the signed generation tree given in Example 4.2.2. It cannot agree with any order-type since the variable p appears positively and negatively.

Now consider the negative generation tree of the formula $\psi(p, q) = \Box p \wedge \neg\Diamond q$. It is given by:

$$
\begin{tikzcd}
& \overset{-}{\wedge} \arrow[dl] \arrow[dr] & \\
\overset{-}{\Box} \arrow[d] & & \overset{-}{\neg} \arrow[d] \\
\overset{-}{p} & & \overset{+}{\Diamond} \arrow[d] \\
& & \overset{+}{q}
\end{tikzcd}
$$

Since p is always signed negatively and q always positively, we have that $-\psi$ agrees with the order-type $\epsilon = (\partial, 1)$. Moreover, consider the subformulas $\psi'(p, q) = \Box p$ and $\psi"(p, q) = \Diamond q$ of ψ and the order-type $\epsilon_1 = (\partial, \partial)$. Taking some time to write it properly, it should be clear that we have $-\psi' \propto -\psi$, $+\psi" \propto -\psi$, $\epsilon_1(\psi') \propto -\psi$ and $\epsilon_1^{\partial}(\psi") \propto -\psi"$.

Definition 4.2.5. We continue the presentation of the terminology of [20].

1. Nodes in signed generation trees will be called **Δ-adjoints**, **syntactically left residuals** (**SLR**), **syntactically right adjoint (SLR)** (**SRA**) or **syntactically right residual** (**SRR**) and will be split into **Skeleton** nodes and **PIA** nodes according to Table 4.1.

Skeleton	PIA
Δ-adjoints	Syntactically Right Adjoint (SRA)
$+ \quad \vee$	$+ \quad \wedge \quad \triangledown \quad$ with $n_{\triangledown} = 1$
$- \quad \wedge$	$- \quad \vee \quad \vartriangle \quad$ with $n_{\vartriangle} = 1$
Syntactically Left Residual (SLR)	Syntactically Right Residual (SRR)
$+ \quad \wedge \quad \vartriangle \quad$ with $n_{\vartriangle} \geq 1$	$+ \quad \vee \quad \triangledown \quad$ with $n_{\triangledown} \geq 2$
$- \quad \vee \quad \triangledown \quad$ with $n_{\triangledown} \geq 1$	$- \quad \wedge \quad \vartriangle \quad$ with $n_{\vartriangle} \geq 2$

Table 4.1: Skeleton and PIA nodes for $\mathcal{L}_{\mathsf{DLE}}$.

For the reader's information, the table of PIA and Skeleton nodes in the bimodal language of Chapter 2 are given in Table 4.2.

Skeleton	PIA
Δ-adjoints	Syntactically Right Adjoint (SRA)
$+ \quad \vee$	$+ \quad \wedge \ \Box \ \blacksquare \ \neg$
$- \quad \wedge$	$- \quad \vee \ \Diamond \ \blacklozenge \ \neg$
——	——
Syntactically Left Residual (SLR)	Syntactically Right Residual (SRR)
$+ \quad \wedge \ \Diamond \ \blacklozenge \ \neg$	$+ \quad \vee \ \rightarrow$
$- \quad \vee \ \Box \ \blacksquare \ \neg \ \rightarrow$	$- \quad \wedge$

Table 4.2: Skeleton and PIA nodes

2. A branch in a signed generation tree $*\varphi$ with $* \in \{+, -\}$ is a **good branch** if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA nodes and P_2 consists (apart from variables nodes) only of Skeleton nodes.

3. A branch is **excellent** if it is good and the path P_1 contains only SRA nodes.

4. If φ is a formula such that only PIA nodes occur in $+\varphi$ (resp. $-\varphi$), we will say that φ is a **positive** (resp. **negative**) **PIA formula**.

5. If φ is a formula such that only Skeleton nodes occur in $+\varphi$ (resp. $-\varphi$), we will say that φ is a **positive** (resp. **negative**) **Skeleton formula**.

4.2.2 Inductive inequalities

Definition 4.2.6. The inductive inequalities of $\mathcal{L}_{\mathsf{DLE}}$ are defined by the following construction.

1. For any order-type ϵ and any strict partial order Ω on $\{p_1, ..., p_n\}$, the signed generation tree $*\varphi$, where $* \in \{+, -\}$, of a formula $\varphi(p_1, ..., p_n)$ is (Ω, ϵ)**-inductive** if

   (a) for all $1 \leq i \leq n$, every ϵ-critical branch with leaf p_i is good,

   (b) when p_i is a critical variable, every SRR node occurring in the critical branch is of the form $\circ(\gamma_1, \ldots, \gamma_{j-1}, \beta, \gamma_{j+1}, \ldots, \gamma_n)$ where, for every $h \in \{1, \ldots, j-1, j+1, \ldots, n\}$, we have
   - $\epsilon^\partial(\gamma_h) \propto *\varphi$,
   - $p \leq_\Omega p_i$ for every p occurring in γ_h.

   We will refer to Ω as the **dependency order** on the variables.

2. An inequality $\varphi \leq \psi$ is (Ω, ϵ)**-inductive** if the signed generation trees $+\varphi$ and $-\psi$ are both (Ω, ϵ)-inductive.

3. An inequality $\varphi \leq \psi$ is **inductive** if it is (Ω, ϵ)-inductive for some Ω and ϵ.

The condition $\epsilon^\partial(\gamma_h) \propto *\varphi$ insures that there is no ϵ-critical variables except for p_i below the SRR node. As a particular case of the inductive inequalities, we have the Sahlqvist inequalities, which do not contain SRR node in their good branches.

Definition 4.2.7. 1. For an order-type ϵ, the signed generation tree $*\varphi$ of an $\mathcal{L}_{\mathsf{DLE}}$- formula $\varphi(p_1, ..., p_n)$ is ϵ**-Sahlqvist** if every ϵ-critical branch is excellent.

2. An inequality $\varphi \leq \psi$ is ϵ**-Sahlqvist** if the signed trees $+\varphi$ and $-\psi$ are both ϵ-Sahlqvist.

3. An inequality $\varphi \leq \psi$ is **Sahlqvist** if it is ϵ-Sahlqvist for some order-type ϵ.

Remark 4.2.8. If $*\varphi$ is a ϵ-Sahlqvist signed generation tree for some order-type ϵ, then it is a (Ω, ϵ)-inductive generation tree for every strict order Ω. Indeed, the strict order Ω impacts only SRR nodes, which do not occur in excellent branches.

Conversely, if the signed generation tree $*\varphi$ is (Ω, ϵ)-inductive for the strict order $\Omega = \emptyset$, then it is ϵ-Sahlqvist. Indeed, since $\Omega = \emptyset$, it is impossible to satisfy $p \leq_\Omega p_i$, thus no ϵ-critical branch can contain SRR nodes.

In [19], it is proved that every inductive inequality is canonical in a lattice expansion, that is a slanted lattice whose operators are clopen. However, we already observed in Chapter 2 that the passage from tense algebras to subordination algebras leads to restrictions in the computable canonical formulas, namely from Sahlqvist to s-Sahlqvist formulas. Such a restriction is still required in the formalisation of ALBA. We need to limit the inductive inequalities to the analytic inductive inequalities. The name analytic is not innocent since it was proved in [41] that they were inequalities in correspondence with analytic structural rules in display calculi. Their appearance in this apparently disconnected theory is still to be understood.

Definition 4.2.9. For every order-type ϵ and every strict partial order Ω on $\{p_1, ..., p_n\}$, the signed generation tree $*\varphi$ of a formula $\varphi(p_1, ..., p_n)$ is **analytic** (Ω, ϵ)**-inductive** (resp. **analytic** ϵ**-Sahlqvist**) if

1. $*\varphi$ is (Ω, ϵ)-inductive (resp. ϵ-Sahlqvist),
2. every branch (i.e. even non-critical ones) of $*\varphi$ is good.

The necessity to have good branches also in the non-critical branches will become clear in Section 4.4. But the reason is roughly the same as in Chapter 2. Indeed, during the execution of ALBA, we will try to eliminate all propositional variables of a given inequalities by generating a "minimal valuation". The critical occurrences of a variables will be the ones to generate this valuation while the non-critical ones will the occurrences receiving it. The good shapes of the non-critical branches is there to allow us to pull this minimal valuation up (think of the intersection lemma).

Notation 4.2.10. Following the notation of [17], we will sometimes represent analytic inductive inequalities as follows:

$$(\varphi \leq \psi)[\underline{\alpha}\,/!\underline{x}, \underline{\beta}\,/!\underline{y}, \underline{\gamma}\,/!\underline{z}, \underline{\delta}\,/!\underline{t}],$$

where $(\varphi \leq \psi)[!\underline{x}, !\underline{y}, !\underline{z}, !\underline{t}]$ is the Skeleton part of the given inequality, $\underline{\alpha}$ (resp. $\underline{\beta}$) denotes the positive (resp. negative) maximal PIA-subformulas, i.e. each α in $\underline{\alpha}$ and β in $\underline{\beta}$ contains at least one ϵ-critical occurrence of some propositional variable and, moreover:

1. for each $\alpha \in \underline{\alpha}$, either $+\alpha \propto +\varphi$ or $+\alpha \propto -\psi$,
2. for each $\beta \in \underline{\beta}$, either $-\beta \propto +\varphi$ or $-\beta \propto -\psi$,

and $\underline{\gamma}$ (resp. $\underline{\delta}$) denotes the positive (resp. negative) maximal ϵ^∂-uniform PIA-subformulas, i.e.:

1. for each $\gamma \in \underline{\gamma}$, either $+\gamma \propto +\varphi$ or $+\gamma \propto -\psi$,
2. for each $\delta \in \underline{\delta}$, either $-\delta \propto +\varphi$ or $-\delta \propto -\psi$.

Example 4.2.11 ([18]). 1. The signed formula $+\varphi(p, q) = +(\Box p \wedge \Diamond q)$ is ϵ-Sahlqvist for the order-type $\epsilon = (1, 1)$. The tree is given by

$$\begin{tikzcd} & \overset{+}{\wedge} \arrow[dl] \arrow[dr] & \\ \overset{+}{\Box} \arrow[d] & & \overset{+}{\Diamond} \arrow[d] \\ \overset{+}{p} & & \overset{+}{q} \end{tikzcd}$$

Hence both leaf nodes $+p$ and $+q$ are ϵ-critical, so that we have to show that every branch is excellent (since they are all critical).

First, the branch

$$\begin{tikzcd} \overset{+}{\wedge} \arrow[d] \\ \overset{+}{\Box} \arrow[d] \\ \overset{+}{p} \end{tikzcd}$$

is good as the node $+\wedge$ is Skeleton and the node $+\Box$ is PIA. Moreover, since $+\Box$ is in particular a SRA node, the branch is excellent.

Secondly, the branch

$$\begin{tikzcd} \overset{+}{\wedge} \arrow[d] \\ \overset{+}{\Diamond} \arrow[d] \\ \overset{+}{q} \end{tikzcd}$$

contains only Skeleton nodes and is therefore excellent.

Note that, in particular, $+\varphi(p,q)$ is also $(1,\partial)$-Sahlqvist, $(\partial,1)$-Sahlqvist and (∂,∂)-Sahlqvist.

2. The signed formula $-\psi(p,q) = -\Diamond(p\wedge q)$ is $(1,1)$-Sahlqvist but not $(1,\partial)$-Sahlqvist. Indeed, the signed generation tree is

$$\begin{tikzcd} & \overset{-}{\Diamond} \arrow[d] & \\ & \overset{-}{\wedge} \arrow[dl] \arrow[dr] & \\ \overset{-}{p} & & \overset{-}{q} \end{tikzcd}$$

and, hence, it does not contain any $(1,1)$-critical node.

On the other hand, the tree does not contain any excellent branch either, as the node $-\Diamond$ is PIA and the node $-\wedge$ is SRR.

3. With the first and the second item, we have that the inequality $(\Box p \wedge \Diamond q) \leq \Diamond(p\wedge q)$ is $(1,1)$-Sahlqvist but not $(1,\partial)$-Sahlqvist. Let us also use this example with the order-type (∂,∂) to illustrate Notation 4.2.10. The Skeleton part of $\Box p \wedge \Diamond q \leq \Diamond(p\wedge q)$ is given by

$$x \wedge \Diamond y \leq z.$$

The critical variables are concentrated in $-\beta(p,q) := -(\Diamond p \wedge q)$ which is negative PIA. Then, we have $\gamma_1 = \Box p$ and $\gamma_2 = q$ which are both ϵ^∂-uniform. Hence, we can rewrite $\Box p \wedge \Diamond q \leq \Diamond(p\wedge q)$ as $\gamma_1 \wedge \gamma_2 \leq \beta$.

4. We know that $(\Box p \wedge \Diamond q) \leq \Diamond(p\wedge q)$ is not a ϵ-Sahlqvist inequality for $\epsilon = (1,\partial)$, however it is (Ω,ϵ)-inductive for the strict order $p \leq_\Omega q$. Since we already proved that $+(\Box p \wedge \Diamond q)$ was ϵ-Sahlqvist and hence (Ω,ϵ)-inductive, we only have to focus on the tree

$$\begin{tikzcd} & \overset{-}{\Diamond} \arrow[d] & \\ & \overset{-}{\wedge} \arrow[dl] \arrow[dr] & \\ \overset{-}{p} & & \overset{-}{q} \end{tikzcd}$$

whose only $(1, \partial)$-critical branch is:

$$
\begin{array}{c}
\overset{-}{\Diamond} \\
\downarrow \\
\overset{-}{\wedge} \\
\downarrow \\
\overset{-}{q}
\end{array}
$$

Since $-\Diamond$ is an SRA node and $-\wedge$ is an SRR one, it is a good but not an excellent branch. Now, the only SRR node is $\wedge(p, q)$ and its critical variable is $-q$. Let us check that it satisfies the requirements of Definition 4.2.6. The node is of the form $\beta \wedge \gamma$ where $\beta = p$ and $\gamma = q$ and we have

- $\epsilon^{\partial}(p) \propto -\Diamond(p \wedge q)$ and
- $p \leq_{\Omega} q$.

Example 4.2.12. The inequality $\blacksquare(q \vee p) \leq \Diamond(r \wedge q)$ is not Sahlqvist. Consider indeed its generation tree:

$$
\begin{array}{ccccccc}
 & \overset{+}{\blacksquare} & & & & \overset{-}{\Diamond} & \\
 & \downarrow & & & & \downarrow & \\
 & \overset{+}{\vee} & & & & \overset{-}{\wedge} & \\
\swarrow & & \searrow & & \swarrow & & \searrow \\
\overset{+}{q} & & \overset{+}{p} & \leq & \overset{-}{q} & & \overset{-}{r}
\end{array}
$$

If $\epsilon(q) = 1$, then the critical branch ending with $+q$ is not excellent (but may be good) since the node $+\vee$ is either Skeleton or SRR. On the other hand, if $\epsilon(q) = \partial$, then the critical branch ending with $-q$ is not excellent (but may be good) either since the node $-\wedge$ is either Skeleton or SRR. Then, it is impossible for $\blacksquare(q \vee p) \leq \Diamond(r \wedge q)$ to be Sahlqvist.

Let us show that the inequality is (Ω, ϵ)-inductive for $\epsilon = (\partial, 1, 1)$ and the strict order $p \leq_{\Omega} q$. We only have one critical branch to examine: the one ending with $+q$. As we already noticed previously, this branch is good since it contains one SRA ($+\blacksquare$) and one SRR node ($+\vee$). Hence, to have an inductive inequality, we only have to check the conditions of Definition 4.2.6. Using the notations given there, the node is $\vee(\beta, \gamma)$ with $\beta = q$ and $\gamma = p$. Note that p is positive in $+\blacksquare(q \vee p)$, and hence that $\epsilon^{\partial}(p) \propto +\blacksquare(q \vee p)$. Finally, since $p \leq_{\Omega} q$, the inequality satisfies all the required conditions to be inductive. Moreover, since all its nodes are PIA nodes, it is clear that it is also analytic.

Example 4.2.13. The inequality $\top \leq \blacksquare(q \to \Box p) \to (\neg q \wedge p)$ is analytic $(\partial, 1)$-Sahlqvist, but not $(1, 1)$-Sahlqvist. Indeed, by observing the negative generation tree of $\blacksquare(q \to \Box p) \to (\neg q \wedge p)$, one remarks that the critical branch ending in $+p$ is not excellent, and so, the inequality cannot be $(1, 1)$-Sahlqvist.

Since we already defined a notion of Sahlqvist formulas in Chapter 2, we know illustrate how this previous concept interacts with the the terminology of the current chapter.

Lemma 4.2.14. *Let φ denote a bimodal formula and ϵ the order-type $\underline{1}$.*

1. *If φ is closed (resp. open), then $-\varphi$ is PIA (resp. Skeleton).*
2. *If φ is positive (resp. negative), then $-\varphi$ (resp. $+\varphi$) does not contain any ϵ-critical node.*
3. *If φ is strongly positive, then $+\varphi$ is PIA and any one of its branches is excellent.*
4. *If φ is s-negative (resp. s-positive), then $+\varphi$ (resp $-\varphi$) does not contain any ϵ-critical (resp. ϵ^{∂}-critical) node and any one of its branches is good.*
5. *If φ is s-untied, then $+\varphi$ is analytic ϵ-Sahlqvist.*
6. *If φ is s-Sahlqvist, then $-\varphi$ is analytic ϵ-Sahlqvist and so is $\top \leq \varphi$.*

Proof.

1. This follows immediately from the definitions.
2. By construction of φ and the way in which the sign propagates through the tree, it is clear that every leaf node in $-\varphi$ (resp. $+\varphi$) will inherit the negative sign, and therefore will not be ϵ-critical.
3. By construction, φ contains only nodes of the form $+\wedge$, $+\Box$ or $+\blacksquare$, which are all SRA nodes.
4. Since an s-negative formula is in particular negative, it is clear from item 2 that $+\varphi$ does not contain any ϵ-critical node. Let us now show that every branch of $+\varphi$ is good. Now, by the definition of an s-negative formula, we know that every node of $+\varphi$ (apart from variable nodes) is positive and that there is no positive (black or white) diamond under the scope of a positive (black or white) box. Hence, every branch in $+\varphi$ is good.

   The s-positive case is proved dually and left to the reader.
5. Since φ is built from s-negative and strongly positive formulas using only $\wedge$, $\Diamond$ and $\blacklozenge$, which are all Skeleton nodes when positively signed, it follows immediately from the previous items.
6. We have that the negative generation tree of φ is of the form

$$\begin{tikzcd}
 & \overset{-}{\Box^{\epsilon}} \arrow[d] & \\
 & \overset{-}{\rightarrow} \arrow[dl] \arrow[dr] & \\
\overset{+}{\varphi_1} & & \overset{-}{\varphi_2}
\end{tikzcd}$$

   with φ_1 s-untied and φ_2 s-positive. Since both nodes $-\Box^{\epsilon}$ and $- \rightarrow$ are Skeleton, the conclusion follows immediately from the previous items.

$\Box$

To consider the other direction, we will use the following notation: if p is a variable, $p^1 := p$ and $p^{\partial} := \neg p$. To extend this notation to vector of variables, if ϵ is an order-type and $\underline{p} = (p_1, \ldots, p_n)$ is a vector of variables then $\underline{p}^{\epsilon}$ denotes the vector $(p_1^{\epsilon_1}, \ldots, p_n^{\epsilon_n})$.

Since we are in a Boolean setting, some modifications can be operated on the formulas thanks to the existence of the connective $\neg$. Indeed, first, it is possible to consider a node $\rightarrow$ as the contraction of $\neg \cdot \vee \cdot$. Then, in a formula φ, we can always move a node $\neg$ down to the leaf nodes via the equivalence between $\neg\Diamond\neg$ and $\Box$, between $\neg(\cdot \wedge \cdot)$ and $\neg\cdot \vee \neg\cdot$, etc.

Lemma 4.2.15. *Let $\varphi(\underline{p})$ and $\psi(\underline{p})$ be bimodal formulas, $\tilde{\varphi}$ and $\tilde{\psi}$ be the bimodal formulas obtained with the modification we just described and $\mathfrak{X} = (X, R)$ be a subordination space.*

1. *For every valuation $v : \mathrm{Var} \longrightarrow \mathrm{Clop}(X)$, we have $v(\varphi) = v(\tilde{\varphi})$.*
2. *We have $\mathfrak{X} \models \varphi \leq \psi$ if and only if $\mathfrak{X} \models \tilde{\varphi} \leq \tilde{\psi}$.*
3. *For a variable p of φ, an order-type ϵ and $* \in \{-, +\}$, an occurrence of p is a ϵ-critical node in $*\varphi$ if and only if it is a ϵ-critical node in $*\tilde{\varphi}$.*
4. *For a variable p of φ and $* \in \{-, +\}$, an occurrence of p is the leaf of a good branch in $*\varphi$ if and only if it is the leaf of a good branch in $*\tilde{\varphi}$.*
5. *For an order-type ϵ, Ω a strict partial order and $* \in \{-, +\}$, $*\varphi$ is (analytic) (Ω, ϵ)-inductive (resp. ϵ-Sahlqvist) if and only if $*\tilde{\varphi}$ is (analytic) (Ω, ϵ)-inductive (resp. ϵ-Sahlqvist).*
6. *The inequality $\varphi \leq \psi$ is (Ω, ϵ) is (analytic) (Ω, ϵ)-inductive (resp. ϵ-Sahlqvist) if and only if $\tilde{\varphi} \leq \tilde{\psi}$ is (analytic) (Ω, ϵ)-inductive (resp. ϵ-Sahlqvist).*

Finally, for the sake of readability in the next lemma, if ϵ is an order-type and φ is a bimodal formula whose variables are $\underline{p}$, we will write $\varphi(\underline{p}, \underline{p})$ to separate the ϵ-critical occurrences of $\underline{p}$ (in the first coordinate) from the non-ϵ-critical occurrences of $\underline{p}$ (in the second coordinate).

Lemma 4.2.16. *Let φ and ψ denote bimodal formulas.*

1. *If $-\psi(\underline{p})$ is analytic ϵ-Sahlqvist, then $\psi(\underline{p}^{\epsilon^\partial}, \underline{p}^{\epsilon})$ is equivalent to an s-positive formula.*
2. *If $+\varphi(\underline{p})$ is analytic ϵ-Sahlqvist, then $\varphi(\underline{p}^{\epsilon}, \underline{p}^{\epsilon^\partial})$ is equivalent to an s-negative formula.*
3. *If $(\varphi \leq \psi)(\underline{p})$ is an analytic ϵ-Sahlqvist inequality, then $(\varphi \rightarrow \psi)(\underline{p}^{\epsilon}, \underline{p}^{\epsilon^\partial})$ is equivalent to an s-Sahlqvist formula.*

Proof. We only prove item 1, as item 2 is proved similarly and item 3 is an immediate consequence of items 1 and 2.

As we saw in Lemma 4.2.15, $\psi(\underline{p})$ is equivalent to the formula $\tilde{\psi}(\underline{p})$, whose negative generation tree is also analytic ϵ-Sahlqvist. Now, suppose that $\epsilon_i = \partial$, then the critical occurrences of p_i in $\tilde{\psi}$ are not preceded by $\neg$ (if not they would be positively signed), but so do non-critical occurrences. And, on the contrary, if $\epsilon_i = \partial$, then the occurrences of p_i which are preceded by $\neg$ are the critical ones. Therefore, $\tilde{\psi}(\underline{p}^{\epsilon^\partial}, \underline{p}^{\epsilon})$ is equivalent to a positive formula $\chi(\underline{p})$. It remains to show that $\chi(p)$ is s-positive, but this a consequence of the analyticity of $\tilde{\psi}$. Indeed, by construction, $-\tilde{\psi}$ only contains negative nodes (in the exception of the variable ones). Moreover, since every branch of $-\tilde{\psi}$ is good, we know that $\tilde{\psi}$, and hence χ, does not contain any $-\Box$ in the scope of a $-\Diamond$, as required.

$\square$

4.3 The $\mathcal{L}^*_{\mathsf{DLE}}$-language

The expanded $\mathcal{L}_{\mathsf{DLE}}$-language, we are about to describe was introduced in [18, Chapter 36.2] in order to express minimal topological valuations in an algebraic setting. Let us return to the modal setting one moment for the sake of this introduction and consider for instance the formula $\varphi = \Box p$. For a valuation v on a modal space X and an element $x \in X$, one has

$$x \in v(\varphi) \Leftrightarrow R(\{x\}, -) \subseteq v(p). \tag{4.1}$$

This equivalence leads us to introduce a new kind of variables that will be interpreted as singletons instead of clopen sets, the soon to be defined nominals $\boldsymbol{i}$, and a new symbol $\blacklozenge$ to describe the map

$$E \in \mathcal{P}(X) \longmapsto R(E, -) \in \mathcal{P}(X),$$

so that the equivalence (4.1) can be equivalently presented as

$$\boldsymbol{i} \leq \Box p \Leftrightarrow \blacklozenge \boldsymbol{i} \leq p.$$

Let us remark that, in the subordination setting, there is no need to introduce the new symbol $\blacklozenge$ which is already part of the base language. Indeed, the accessibility relation of a subordination space (unlike the modal one but similarly to the tense one) is not asymmetrical in its properties and hence does not favour one side (see the definition of modal space in B.3.2). Therefore, in the subordination setting, to go from the basic language to the expanded one only requires to add a new kind of variables.

In the general slanted setting, we did not specify that the set of connectives Γ_1 and Γ_2 should be closed under residuation. This is why the "black" symbols should be added in the expansion $\mathcal{L}^*_{\mathsf{DLE}}$ of $\mathcal{L}_{\mathsf{DLE}}$. The next example illustrates how we should behave with residuations of slanted operators. They are a consequence of the following equivalence:

$$R(-, A_1, \dots, A_{j-1}, A_j, A_{j+1}, \dots, A_n) \subseteq A \iff A_j \subseteq R(A^c, A_1, \dots, A_{j-1}, -, A_{j+1}, \dots, A_n)^c.$$

Hence, for instance, consider the connective $\vartriangle \in \Gamma_1$ of order-type $(1, \partial)$ and its associated relation R, that is

$$\vartriangle : (O_1, O_2) \longmapsto R(-, O_1, O_2^c).$$

We have the residuals

1. $\vartriangle^\sharp_1(U, O_2) = R(U^c, -, O_2^c)^c$ is an o-slanted operator of order-type $(1, 1)$, such that we have $\vartriangle(O_1, O_2) \leq U$ if and only if $U \leq \vartriangle^\sharp_1(U, O_2)$,

2. $\vartriangle^\sharp_2(O_1, U) = R(U^c, O_1, -)$ is a c-slanted operator of order-type $(1, \partial)$, such that we have $\vartriangle(O_1, O_2) \leq U$ if and only if $\vartriangle^\sharp_2(O_1, U) \leq O_2$,

Note that $\vartriangle^\sharp_1$ and $\vartriangle^\sharp_2$ are well defined. Indeed, since $\vartriangle$ is of order-type $(1, \partial)$, we know that R is (as a relation) of order-type $(1, \partial, 1)$. Therefore, $\vartriangle^\sharp_1(U, O_2)$ and $\vartriangle^\sharp_2(O_1, U)$ are both increasing subsets of $\mathfrak{X}$.

On the other hand, for a connective $\triangledown \in \Gamma_2$ of order-type $(1, \partial)$, we have:

1. $\triangledown^\flat_1$ is a c-slanted operator of order-type $(1, 1)$ such that $U \leq \triangledown(O_1, O_2)$ if and only if $\triangledown^\flat_1(U, O_2) \leq O_1$,

2. $\triangledown^\flat_2$ is an o-slanted operator of order type $(1, \partial)$ such that $U \leq \triangledown(O_1, O_2)$ if and only if $O_2 \leq \triangledown^\flat_2(O_1, U)$.

From the previous example, we can extrapolate the generalisation given in the next theorem, which is given without proof.

Theorem 4.3.1. *Let $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$ be a slanted lattice, $\triangle \in \Gamma_1$ of order-type $\epsilon_\triangle$ and $\triangledown \in \Gamma_2$ of order-type $\epsilon_\triangledown$. We have*

1. *if $\epsilon_\triangle(i) = 1$, then there exists an o-slanted operator $\triangle_i^\sharp$ such that:*
   (a) *$\epsilon_{\triangle_i^\sharp}(i) = 1$ and $\epsilon_{\triangle_i^\sharp}(j) = (\epsilon_\triangle(j))$ for any $j \neq i$,*
   (b) *$\triangle(a_1, \ldots, a_i, \ldots a_{n_\triangle}) \leq b$ if and only if $a_i \leq \triangle_i^\sharp(a_1, \ldots, b, \ldots, a_{n_\triangle})$,*
2. *if $\epsilon_\triangle(i) = \partial$, then there exists a c-slanted operator $\triangle_i^\sharp$ such that:*
   (a) *$\epsilon_{\triangle_i^\sharp}(i) = \partial$ and $\epsilon_{\triangle_i^\sharp}(j) = (\epsilon_\triangle(j))^\partial$ for any $j \neq i$,*
   (b) *$\triangle(a_1, \ldots, a_i, \ldots a_{n_\triangle}) \leq b$ if and only if $\triangle_i^\sharp(a_1, \ldots, b, \ldots, a_{n_\triangle}) \leq a_i$,*
3. *if $\epsilon_\triangledown(i) = 1$, then there exists a c-slanted operator $\triangledown_i^\flat$ such that:*
   (a) *then $\epsilon_{\triangledown_i^\flat}(i) = 1$ and $\epsilon_{\triangledown_i^\flat}(j) = (\epsilon_\triangledown(j))^\partial$ for any $j \neq i$,*
   (b) *$b \leq \triangledown(a_1, \ldots, a_i, \ldots a_{n_\triangledown})$ if and only $\triangledown_i^\flat(a_1, \ldots, b, \ldots, a_{n_\triangledown}) \leq a_i$,*
4. *if $\epsilon_\triangledown(i) = \partial$, then there exists a c-slanted operator $\triangledown_i^\sharp$ such that:*
   (a) *$\epsilon_{\triangledown_i^\flat}(i) = \partial$ and $\epsilon_{\triangledown_i^\flat}(j) = \epsilon_\triangledown(j)$ for any $j \neq i$,*
   (b) *$b \leq \triangledown(a_1, \ldots, a_i, \ldots a_{n_\triangledown})$ iff $\triangledown_i^\flat(a_1, \ldots, b, \ldots, a_{n_\triangledown}) \geq a_i$.*

The additives connectives being defined, we can look at the new variables. They will be split in two disjoints sets: the first one is the set of **nominals** Nom $= \{\boldsymbol{i}, \boldsymbol{j}, \ldots\}$ and the second the one of the set **co-nominals** Co-Nom $= \{\boldsymbol{m}, \boldsymbol{n}, \ldots\}$.

For a slanted Priestley space $\mathfrak{X} = (X, \leq, \Lambda_1, \Lambda_2)$, the valuation of an element $\boldsymbol{i} \in$ Nom will be a subset of the form $\uparrow x$ for some $x \in X$. On the other hand, the valuation of an element $\boldsymbol{m} \in$ Co-Nom will be a subset of the form $X \setminus \downarrow x$ for some $x \in X$.

For a slanted lattice $\mathfrak{L} = (L, \Gamma_1, \Gamma_2)$, the valuation of an element $\boldsymbol{i} \in$ Nom will be a completely join irreducible element of L^∂ and the valuation of $\boldsymbol{j} \in$ Co-Nom will be a completely meet irreducible element of L^∂.

Now that we have all the ingredients of the new expanded language, we can define it properly.

Definition 4.3.2. The **expanded language** $\mathcal{L}^*_{\mathsf{DLE}}(\Gamma_1^*, \Gamma_2^*)$, from now denoted by $\mathcal{L}^*_{\mathsf{DLE}}$ is constituted by the original language $\mathcal{L}_{\mathsf{DLE}}$ (see Definition 4.1.1) augmented with the set Nom of nominals and Co-Nom of co-nominals and the sets of connectives Γ_1^* and Γ_2^* definied as follows:

1. $\Gamma_1^* = \Gamma_1 \cup \{\triangledown_i^\flat \mid \triangledown \in \Gamma_2 \text{ and } \epsilon_\triangledown(i) = 1\} \cup \{\triangle_i^\sharp \mid \triangle \in \Gamma_1 \text{ and } \epsilon_\triangle(i) = \partial\}$,
2. $\Gamma_2^* = \Gamma_2 \cup \{\triangle_i^\sharp \mid \triangle \in \Gamma_1 \text{ and } \epsilon_\triangle(i) = 1\} \cup \{\triangledown_i^\flat \mid \triangledown \in \Gamma_2 \text{ and } \epsilon_\triangledown(i) = \partial\}$.

The $\mathcal{L}^*_{\mathsf{DLE}}$**-formulas** are then defined via the following inductive rules:

$$\varphi ::= \boldsymbol{i} \mid \boldsymbol{j} \mid \chi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \triangle(\underline{\varphi}, \underline{\psi}) \mid \triangledown(\underline{\varphi}, \underline{\psi})$$

where $\boldsymbol{i} \in$ Nom, $\boldsymbol{j} \in$ Co-Nom, χ is an $\mathcal{L}_{\mathsf{DLE}}$-formula, $\triangle \in \Gamma_1^*$ and $\triangledown \in \Gamma_2^*$. Note that we write $\circ(\underline{\varphi}, \underline{\psi})$ to indicate that $\underline{\varphi}$ is used in the coordinates of $\circ$ whose order type is 1 and $\underline{\psi}$ in the coordinates whose order-type is ∂. In other words, we may suppose that the order-type of the connectives are always of the form $(\underline{1}, \underline{\partial})$.

As they will be the sinews of war, we define the **pure formulas** to be the $\mathcal{L}^*_{\mathsf{DLE}}$-formulas where no (propositional) variable occurs.

Of course, $\mathcal{L}^*_{\mathsf{DLE}}$-formulas can be assigned to subsets of a slanted Priestley space via a valuation, as $\mathcal{L}_{\mathsf{DLE}}$-formulas. The two procedures are quite similar and we will therefore often refer to Section 4.1 for precise definitions. For a slanted Priestley space $\mathfrak{X}$, let us denote by $\mathfrak{X}^\delta$ the order-Kripke structure associated to $\mathfrak{X}$ (that is, the image of $\mathfrak{X}$ via the forgetful functor). From the definitions of valuation we just established, we have almost directly the next lemmas.

Lemma 4.3.3. *Let φ be an $\mathcal{L}^*_{\mathsf{DLE}}$-formula, $\mathfrak{L}$ be a slanted lattice, $\mathfrak{L}^\delta$ its canonical extension and $\mathfrak{X}$ its dual slanted Priestley space.*

1. *We have $\mathfrak{L} \models \varphi \leq \psi$ if and only $\mathfrak{X} \models \varphi \leq \psi$ and $\mathfrak{L}^\delta \models \varphi \leq \psi$ if and only $\mathfrak{X}^\delta \models \varphi \leq \psi$.*
2. *If $\mathfrak{X}^\delta \models \varphi \leq \psi$ then $\mathfrak{X} \models \varphi \leq \psi$,*
3. *If φ is pure, then $\mathfrak{X} \models \varphi \leq \psi$ implies $\mathfrak{X}^\delta \models \varphi \leq \psi$.*

We observed in Chapter 2 that being able to determine whether clopen sets were mapped to an open or to a closed one was of major importance to use correctly the intersection lemma. This why we introduced the notion of closed and open formulas in Definition 2.7.1. Of course, we have here notions of closed and open formulas, that we introduce now.

Definition 4.3.4. The **strictly syntacticly closed formulas** (shortened as **ssc** formulas) and **strictly syntacticly open formulas** (shortened as **sso** formulas) of $\mathcal{L}^*_{\mathsf{DLE}}$ are simultaneously defined by the following induction:

$$\varphi ::= p \mid \boldsymbol{j} \mid \top \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \triangle^*(\underline{\varphi}, \underline{\psi}), \tag{Ssc}$$
$$\psi ::= p \mid \boldsymbol{m} \mid \top \mid \bot \mid \psi \vee \psi \mid \psi \wedge \psi \mid \nabla^*(\underline{\psi}, \underline{\varphi}) \tag{Sso}$$

with $p \in \mathrm{Var}$, $\boldsymbol{j} \in \mathrm{Nom}$, $\boldsymbol{m} \in \text{Co-Nom}$, $\triangle^* \in \Gamma_1^*$ and $\nabla \in \Gamma_2^*$.

We extend our notation $\triangle^*(\underline{\varphi}, \underline{\psi})$ to formulas, and directly obtain the next lemma from the definitions.

Lemma 4.3.5. *For all ssc formulas $\varphi(\underline{!x}, \underline{!y})$ and all sso formulas $\psi(\underline{!x}, \underline{!y})$ which are positive in any x in $\underline{!x}$ and negative in any y in $\underline{!y}$, and all tuples $\underline{\varphi'}$ and $\underline{\psi'}$ of scc formulas and sso formulas respectively,*

1. *$\varphi[\underline{\varphi'}/\underline{!x}, \underline{\psi'}/\underline{!y}]$ is ssc;*
2. *$\psi[\underline{\psi'}/\underline{!x}, \underline{\varphi'}/\underline{!y}]$ is sso.*

The denominations of closed/open formulas is clear in the sense that the valuations of ssc formulas φ on a slanted Priestley space $\mathfrak{X}$ are closed subsets of $\mathfrak{X}$ and the valuations of sso formulas ψ are open subsets. This can be observed by a simultaneous induction on the length of φ and ψ thanks to a property which is the n-ary version of Lemma 2.1.3.

Lemma 4.3.6. *Let $\mathfrak{X}$ be a slanted Priestley and $F_1, \ldots, F_n$ closed subset of $\mathfrak{X}$. Then, for every $R \in \Lambda_1$ and $S \in \Lambda_2$, we have that*

$$R(F_1, \ldots, F_{j_1}, -, F_{j+1}, \ldots, F_n) \text{ and } S(F_1, \ldots, F_{j_1}, -, F_{j+1}, \ldots, F_n)$$

are closed subsets of $\mathfrak{X}$.

As we will see in the section 4.5, the residuation of the connectives in a PIA formula will play an important role in the execution of the algorithm ALBA. For future convenience, we introduce the following notations and results from [25].

Definition 4.3.7. For every definite positive PIA $\mathcal{L}_{\mathsf{DLE}}$-formula $\varphi = \varphi(!x, \underline{z})$, and any definite negative PIA $\mathcal{L}_{\mathsf{DLE}}$-formula $\psi = \psi(!x, \underline{z})$ such that the variable x occurs in them exactly once, the $\mathcal{L}^*_{\mathsf{DLE}}$-formulas $\mathsf{LA}(\varphi)(u, \underline{z})$ and $\mathsf{RA}(\psi)(u, \underline{z})$ (for $u \in \mathrm{Var}\setminus(x \cup \underline{z})$) are defined by simultaneous recursion as follows:

$$
\begin{array}{rcl}
\mathsf{LA}(x) & = & u; \\
\mathsf{LA}(\nabla(\underline{\varphi_{-j}(\underline{z})}, \varphi_j(x, \underline{z}), \underline{\psi(\underline{z})})) & = & \mathsf{LA}(\varphi_j)(\nabla^\flat_j(\underline{\varphi_{-j}(\underline{z})}, u, \underline{\psi(\underline{z})}), \underline{z}); \\
\mathsf{LA}(\nabla(\underline{\varphi(\underline{z})}, \underline{\psi_{-j}(\underline{z})}, \psi_j(x, \underline{z}))) & = & \mathsf{RA}(\psi_j)(\nabla^\flat_j(\underline{\varphi(\underline{z})}, \underline{\psi_{-j}(\underline{z})}, u), \underline{z}); \\
\\
\mathsf{RA}(x) & = & u; \\
\mathsf{RA}(\Delta(\underline{\psi_{-j}(\underline{z})}, \psi_j(x, \underline{z}), \underline{\varphi(\underline{z})})) & = & \mathsf{RA}(\psi_j)(\Delta^\sharp_j(\underline{\psi_{-j}(\underline{z})}, u, \underline{\varphi(\underline{z})}), \underline{z}); \\
\mathsf{RA}(\Delta(\underline{\psi(\underline{z})}, \underline{\varphi_{-j}(\underline{z})}, \varphi_j(x, \underline{z}))) & = & \mathsf{LA}(\varphi_j)(\Delta^\sharp_j(\underline{\psi(\underline{z})}, \underline{\varphi_{-j}(\underline{z})}, u), \underline{z}).
\end{array}
$$

Above, $\underline{\varphi_{-j}}$ denotes the vector obtained by removing the jth coordinate of $\underline{\varphi}$.

Lemma 4.3.8. *For every definite positive PIA $\mathcal{L}_{\mathsf{DLE}}$-formula $\varphi = \varphi(!x, \underline{z})$, and any definite negative PIA $\mathcal{L}_{\mathsf{DLE}}$-formula $\psi = \psi(!x, \underline{z})$ such that x occurs in them exactly once,*

1. *if $+x \prec +\varphi$ then $\mathsf{LA}(\varphi)(u, \underline{z})$ is monotone in u and for each z in $\underline{z}$, $\mathsf{LA}(\varphi)(u, \underline{z})$ has the opposite polarity to the polarity of φ in z;*
2. *if $-x \prec +\varphi$ then $\mathsf{LA}(\varphi)(u, \underline{z})$ is antitone in u and for each z in $\underline{z}$, $\mathsf{LA}(\varphi)(u, \underline{z})$ has the same polarity as φ in z;*
3. *if $+x \prec +\psi$ then $\mathsf{RA}(\psi)(u, \underline{z})$ is monotone in u and for each z in $\underline{z}$, $\mathsf{RA}(\psi)(u, \underline{z})$ has the opposite polarity to the polarity of ψ in z;*
4. *if $-x \prec +\psi$ then $\mathsf{RA}(\psi)(u, \underline{z})$ is antitone in u and for each z in $\underline{z}$, $\mathsf{RA}(\psi)(u, \underline{z})$ has the same polarity as ψ in z.*

Proof. By simultaneous induction on φ and ψ. If $\varphi = \psi = x$, then the assumptions of item 1 and 3 are satisfied; then $\mathsf{RA}(\psi) = \mathsf{LA}(\varphi) = u$ is clearly monotone in u and the second part of the statement is vacuously satisfied. As to the inductive step, if $\varphi(!x, \underline{z}) = g(\underline{\varphi'_{-j}(\underline{z})}, \varphi'_j(x, \underline{z}), \underline{\psi'(\underline{z})})$, with each φ' in $\underline{\varphi'}$ being positive PIA and each ψ' in $\underline{\psi'}$ being negative PIA, then $\nabla^\flat_j \in \Gamma^*_1$ is monotone in its j^{th} coordinate and has the opposite polarity of ϵ_∇ in all the other coordinates. Hence, $\nabla^\flat_j(\underline{\varphi'_{-j}(\underline{z})}, u, \underline{\psi'(\underline{z})})$ has the opposite polarity of $\varphi(!x, \underline{z})$ in each z in $\underline{z}$. Two cases can occur: (a) if $+x \prec +\varphi_j$, then by induction hypothesis, $\mathsf{LA}(\varphi_j)(u', \underline{z})$ is monotone in u', and has the opposite polarity of φ_j in every z in $\underline{z}$. Hence,

$$\mathsf{LA}(\varphi) = \mathsf{LA}(\varphi_j)(\nabla^\flat_j(\underline{\varphi'_{-j}(\underline{z})}, u, \underline{\psi'(\underline{z})})/u', \underline{z})$$

is monotone in u and has the opposite polarity to the polarity of φ in each z in $\underline{z}$. (b) if $-x \prec +\varphi_j$, then by induction hypothesis, $\mathsf{LA}(\varphi_j)(u', \underline{z})$ is antitone in u', and has the same polarity as φ_j in every z in $\underline{z}$. Hence,

$$\mathsf{LA}(\varphi) = \mathsf{LA}(\varphi_j)(\nabla^\flat_j(\underline{\varphi'_{-j}(\underline{z})}, u, \underline{\psi'(\underline{z})})/u', \underline{z})$$

is antitone in u and has the same polarity as φ in each z in $\underline{z}$. The remaining cases are $\varphi := \nabla(\underline{\varphi'(\underline{z})}, \underline{\psi'_{-h}(\underline{z})}, \psi_h(x, \underline{z}))$, $\psi := \Delta(\underline{\varphi'_{-j}(\underline{z})}, \varphi'_j(x, \underline{z}), \underline{\psi'(\underline{z})})$, and $\psi := \Delta(\underline{\varphi'(\underline{z})}, \underline{\psi'_{-h}(\underline{z})}, \psi'_h(x, \underline{z}))$ and are shown in a similar way. $\square$

Lemma 4.3.9. *If $\alpha(!x)$ is a definite positive PIA $\mathcal{L}_{\mathrm{LE}}$-formula and $\beta(!x)$ is a definite negative PIA $\mathcal{L}_{\mathrm{LE}}$-formula, then*

1. *α is sso and β is ssc.*
2. *If $+x \propto +\alpha$ and $+x \propto +\beta$, then $\mathsf{LA}(\alpha)[\boldsymbol{j}/!u]$ is ssc and $\mathsf{RA}(\beta)[\boldsymbol{m}/!u]$ is sso.*
3. *If $-x \propto +\alpha$ and $-x \propto +\beta$, then $\mathsf{LA}(\alpha)[\mathbf{j}/!u]$ is sso and $\mathsf{RA}(\beta)[\mathbf{m}/!u]$ is ssc.*

Proof. 1. Straightforward by simultaneous induction on α and β.

2. and 3. We proceed by simultaneous induction on α and β.

If $\alpha = \beta = x$, then the assumptions of item 2 are satisfied; then $\mathsf{LA}(\alpha)[\mathbf{j}/!u] = \mathbf{j}/u$ is clearly ssc and $\mathsf{RA}(\beta)[\mathbf{m}/!u] = \mathbf{m}/u$ is clearly sso.

As to the inductive step, if $\alpha = \nabla(\underline{\varphi}, \underline{\psi})$, with each φ in $\underline{\varphi}$ positive PIA (hence, by item 1, sso) and each ψ in $\underline{\psi}$ negative PIA (hence, by item 1, ssc), and the only occurrence of x is in φ_h, then φ_h is positive PIA, and moreover, $\Delta_h^\flat \in \Gamma_1^*$ is positive in its hth coordinate and has the opposite polarity of ϵ_∇ in all the other coordinates. Hence, $\nabla_h^\flat(\underline{\varphi_{-h}}, \mathbf{j}/!u, \underline{\psi})$ is ssc. Two cases can occur: (a) if $+x \propto +\alpha$, then $+x \propto +\varphi_h$, hence by induction hypothesis, $\mathsf{LA}(\varphi_h)[\mathbf{i}/!u']$ is ssc, and moreover, $+u' \propto \mathsf{LA}(\varphi_h)(u')$ (cf. Lemma 4.3.8). Hence,

$$\mathsf{LA}(\alpha)[\mathbf{j}/!u] = \mathsf{LA}(\varphi_h)[\nabla_h^\flat(\underline{\varphi_{-h}}, \mathbf{j}/!u, \underline{\psi})/!u']$$

is ssc (cf. Lemma 4.3.5). (b) if $-x \propto +\alpha$, then $-x \propto +\varphi_h$, hence by induction hypothesis, $\mathsf{LA}(\varphi_h)[\mathbf{i}/!u']$ is sso, and moreover, $-u' \propto \mathsf{LA}(\varphi_h)(u')$ (cf. Lemma 4.3.8). Hence,

$$\mathsf{LA}(\alpha)[\mathbf{j}/!u] = \mathsf{LA}(\varphi_h)[\nabla_h^\flat(\underline{\varphi_{-h}}, \mathbf{j}/!u, \underline{\psi})/!u']$$

is sso (cf. Lemma 4.3.5). The remaining cases are $\alpha = \nabla(\underline{\varphi}, \underline{\psi})$ such that the only occurrence of x is in ψ_h, $\beta = \Delta(\underline{\varphi}, \underline{\psi})$ with x occurring in φ_h or ψ_h, and are shown in a similar way. $\square$

Example 4.3.10. Let us illustrate the inequality obtained by applying the process given in Lemma 4.3.9 to the inequality $\Diamond q \wedge \blacklozenge(\neg p \wedge r) \leq \boldsymbol{m}$ with $\epsilon(p) = 1$. We have the following succession of equivalences

$$\begin{aligned}
& \Diamond q \wedge \blacklozenge(\neg p \wedge r) \leq \boldsymbol{m} \\
\iff & \blacklozenge(\neg p \wedge r) \leq \boldsymbol{m} \vee \neg \Diamond q \\
\iff & \neg p \wedge r \leq \Box(\boldsymbol{m} \vee \neg \Diamond q) \\
\iff & \neg p \leq \Box(\boldsymbol{m} \vee \neg \Diamond q) \vee \neg r \\
\iff & r \wedge \neg \Box(\boldsymbol{m} \vee \neg \Diamond q) \leq p.
\end{aligned}$$

Here, the formula $\beta_p(p, q, r)$ is $\Diamond q \wedge \blacklozenge(\neg p \wedge r)$ and the formula $\mathsf{RA}(\beta_p)(\boldsymbol{m}, q, r)$ is $r \wedge \neg \Box(\boldsymbol{m} \vee \neg \Diamond q)$ or, equivalently, $r \wedge \Diamond(\neg \boldsymbol{m} \wedge \Diamond q)$. Notice that q and r occur negatively in β_q and in $\mathsf{RA}(\beta_p)$, that $\boldsymbol{m}$ occurs positively in $\mathsf{RA}(\beta_p)$ and that $\mathsf{RA}(\beta_p)$ is indeed ssc.

4.4 Preliminaries for ALBA

In this section, we introduce some results that will be of great use in the execution of the algorithm ALBA. They are topological versions of lemmas used in [19] and [20] by Conradie and Palmigiano.

Definition 4.4.1. 1. An $\mathcal{L}^*_{\mathsf{DLE}}$-formula φ is **positive** (resp. **negative**) in a variable p if all occurrences p are positive (resp. negative) in φ. An $\mathcal{L}^*_{\mathsf{DLE}}$-formula φ is **monotone** in a variable p if φ is positive or negative in p.

2. An $\mathcal{L}^*_{\mathsf{DLE}}$-inequality $\varphi \leq \psi$ is **positive** (resp. **negative**) in a variable p if φ is positive (resp. negative) in p and ψ is negative (resp. positive) in p. An $\mathcal{L}^*_{\mathsf{DLE}}$-inequality $\varphi \leq \psi$ is **monotone** in a variable p if $\varphi \leq \psi$ is positive or negative in p.

Proposition 4.4.2. *If $\varphi \leq \psi$ is an $\mathcal{L}^*_{\mathsf{DLE}}$-inequality positive in a variable p, then for any slanted Priestley space $\mathfrak{X}$ we have*

$$\mathfrak{X} \models \varphi(p) \leq \psi(p) \iff \mathfrak{X} \models \varphi(\top) \leq \psi(\top).$$

Dually, if $\varphi \leq \psi$ is an $\mathcal{L}^*_{\mathsf{DLE}}$-inequality negative in a variable p, then for any subordination space $\mathfrak{X} = (X, R)$ we have

$$\mathfrak{X} \models \varphi(p) \leq \psi(p) \iff \mathfrak{X} \models \varphi(\bot) \leq \psi(\bot).$$

Proof. We prove the positive case and leave the negative one to the reader.

First, we have that $\mathfrak{X} \models \varphi(p) \leq \psi(p)$ immediately implies $\mathfrak{X} \models \varphi(\top) \leq \psi(\top)$ as the latter is equivalent to $\mathfrak{X} \models_v \varphi(p) \leq \psi(p)$ for the valuation $v(p) = X$. Suppose now that $X \models \varphi(\top) \leq \psi(\top)$, that is suppose that $\varphi(X) \subseteq \psi(X)$. Since φ is positive in p and ψ is negative in p, for any increasing clopen set O, we have

$$\varphi(O) \subseteq \varphi(X) \subseteq \psi(X) \subseteq \psi(O).$$

In other words, for all valuations $v : \mathrm{Var} \longrightarrow {\uparrow}\mathrm{Clop}(X)$, we have $\varphi(v(p)) \subseteq \psi(v(p))$, as required. □

Lemma 4.4.3 (Distribution lemma)**.** *If $\varphi(!x), \psi(!x), \xi(!x), \chi(!x)$ are $\mathcal{L}_{\mathsf{DLE}}$ formulas, $\mathfrak{X}$ an slanted Priestley space and $(S_j \mid j \in I)$ a family of increasing subsets of $\mathfrak{X}$ then:*

1. *$\varphi(\bigcup_{j\in I} S_j) = \bigcup\{\varphi(S_j) \mid j \in I\}$, when $+x \propto +\varphi(!x)$ and in $+\varphi(!x)$ the branch ending in $+x$ is SLR;*

2. *$\psi(\bigcap_{j\in I} S_j) = \bigcup\{\psi(S_j) \mid j \in I\}$, when $-x \propto +\psi(!x)$ and in $+\psi(!x)$ the branch ending in $-x$ is SLR;*

3. *$\xi(\bigcap_{j\in I} S_j) = \bigcap\{\xi(S_j) \mid j \in I\}$, when $-x \propto -\xi(!x)$ and in $-\xi(!x)$ the branch ending in $-x$ is SLR;*

4. *$\chi(\bigcup_{j\in I} S_j) = \bigcap\{\chi(S_j) \mid j \in I\}$, when $+x \propto -\chi(!x)$ and in $-\chi(!x)$ the branch ending in $+x$ is SLR.*

Proof. The proof is by simultaneous induction on φ, ψ, ξ and χ. The base cases for $\bot$, $\top$, and x, when applicable, are trivial. We check the inductive cases for φ, and note that all the other cases follow in a similar way.

φ of the form $\triangle(\varphi_1, \ldots, \varphi_i(!x), \ldots, \varphi_{n_\triangle})$ with $\triangle \in \Gamma_1$ and $\epsilon_\triangle(i) = 1$: By the assumption of a unique occurrence of x in φ, the variable x occurs in φ_i for exactly one index $1 \leq i \leq n_\triangle$. The assumption that $\epsilon_\triangle(i) = 1$ implies that $+x \propto +\varphi_i$. Then, we have

$$\begin{array}{rcl}\varphi(\bigcup_{j\in I} S_j) & = & \triangle(\varphi_1,\ldots,\varphi_i(\bigcup_{j\in I} a_j)\ldots,\varphi_{n_\triangle})\\ & = & \triangle(\varphi_1,\ldots,\bigcup_{j\in I}\varphi_i(S_j)\ldots,\varphi_{n_\triangle})\\ & = & R(-,\ldots,\bigcup_{j\in I}\varphi_i(S_j),\ldots)\\ & = & \bigcup_{j\in I} R(-,\ldots,\varphi_i(S_j),\ldots)=\bigcup_{j\in I}\varphi(S_j),\end{array}$$

where R is the $(1,\epsilon^\partial_\triangle)$-relation associated to $\triangle$ and where the second equality holds by the inductive hypothesis, since the branch of $+\varphi$ ending in $+x$ is SLR, and it traverses $+\varphi_i$.

φ **of the form** $\triangle(\varphi_1,\ldots,\chi_i(!x),\ldots,\varphi_{n_\triangle})$ **with** $\triangle\in\Gamma_1$ **and** $\epsilon_\triangle(i)=\partial$**:** By the assumption of a unique occurrence of x in φ, the variable x occurs in ψ_i for exactly one index $1\le i\le n_\triangle$. The assumption that $\epsilon_\triangle(i)=\partial$ implies that $+x\propto-\chi_i$. Then

$$\begin{array}{rcl}\varphi(\bigcup_{j\in I} S_j) & = & \triangle(\varphi_1,\ldots,\chi_i(\bigcup_{j\in I} a_j)\ldots,\varphi_{n_\triangle})\\ & = & \triangle(\varphi_1,\ldots,\bigcap_{j\in I}\chi_i(a_j)\ldots,\varphi_{n_\triangle})\\ & = & R(-,\ldots,(\bigcap_{j\in I}\chi_i(S_j))^c,\ldots)\\ & = & R(-,\ldots,\bigcup_{j\in I}\chi_i(S_j)^c,\ldots)\\ & = & \bigcup_{j\in I} R(-,\ldots,\chi_i(S_j)^c,\ldots)=\bigcup_{j\in I}\varphi(S_j),\end{array}$$

where R is the $(1,\epsilon^\partial_\triangle)$-relation associated to $\triangle$ and the second equality holds by the inductive hypothesis, since the branch of $+\varphi$ ending in $+x$ is SLR, and it traverses $-\chi_i$.

ψ **of the form** $\triangle(\psi_1,\ldots,\psi_i(!x),\ldots,\psi_{n_\triangle})$ with $\triangle\in\Gamma_1$ and $\epsilon_\triangle(i)=1$ or $f(\psi_1,\ldots,\xi_i(!x),\ldots,\varphi_{n_\triangle})$ with $\triangle\in\Gamma_1$ and $\epsilon_\triangle(i)=\partial$.

ξ **of the form** $\triangledown(\xi_1,\ldots,\xi_i(!x),\ldots,\xi_{n_\triangledown})$ with $\triangledown\in\Gamma_2$ and $\epsilon_\triangledown(i)=1$ or $g(\xi_1,\ldots,\psi_i(!x),\ldots,\xi_{n_\triangledown})$ with $\triangledown\in\Gamma_2$ and $\epsilon_\triangledown(i)=\partial$.

χ **of the form** $\triangledown(\chi_1,\ldots,\chi_i(!x),\ldots,\chi_{n_\triangledown})$ with $\triangledown\in\Gamma_2$ and $\epsilon_\triangledown(i)=1$ or $\triangledown(\chi_1,\ldots,\varphi_i(!x),\ldots,\xi_{n_\triangledown})$ with $\triangledown\in\Gamma_2$ and $\epsilon_\triangledown(i)=\partial$.

□

The next two lemmas will be the n-ary versions of Esakia's and intersection lemma (see 2.1.4 and 2.7.11).

Lemma 4.4.4. *Let $(X,\le)$ be a Priestley space and R an n-ary closed relation on X and $\mathcal{F}_1,\ldots,\mathcal{F}_n$ be filtered families of closed subsets of X, then for every $j\in\{1,\ldots,n\}$ we have*

$$R(\cap\mathcal{F}_1,\ldots,\cap\mathcal{F}_{j-1},-,\cap\mathcal{F}_{j+1},\ldots,\cap\mathcal{F}_n)=\bigcap\{R(F_1,\ldots,F_{j-1},-,F_{j+1},\ldots,F_n)\mid F_i\in\mathcal{F}_i\}.$$

Proof. Let us do the proof for the ternary case. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filtered families of closed subsets. For every $F_2\in\mathcal{F}_2$, let us define the binary relation R_{F_2} to be $R(-,-,F_2)$. We know that R_{F_2} is a closed relation and so, we can apply Esakia's lemma to obtain

$$\begin{array}{l}\bigcap\{R(-,F_1,F_2)\mid F_1\in\mathcal{F}_1,F_2\in\mathcal{F}_2\}\\ =\bigcap\{\bigcap\{R_{F_2}(-,F_1)\mid F_1\in\mathcal{F}_1\}\mid F_2\in\mathcal{F}_2\}\\ =\bigcap\{R_{F_2}(-,\cap\mathcal{F}_1)\mid F_2\in\mathcal{F}_2\}\\ =\bigcap\{R(-,\cap\mathcal{F}_1,F_2)\mid F_2\in\mathcal{F}_2\}.\end{array}$$

We can use Esakia's lemma once again for the binary relation $R_{\mathcal{F}}$ defined by $R(-, \cap\mathcal{F}, -)$, which is closed, to obtain

$$\begin{aligned}&\bigcap\{R(-, \cap\mathcal{F}_1, F_2) \mid F_2 \in \mathcal{F}_2\}\\ =&\bigcap\{R_{\mathcal{F}_1}(-, F_2) \mid F_2 \in \mathcal{F}_2\}\\ =&R_{\mathcal{F}_1}(-, \cap\mathcal{F}_2)\\ =&R(-, \cap\mathcal{F}_1, \cap\mathcal{F}_2),\end{aligned}$$

as required. □

Let us now introduce a notation for the next theorem. If φ is a formula, p is a variable occurring in φ, v a valuation on a slanted Priestley space $\mathfrak{X}$ and $S \subseteq \mathfrak{X}$, then $\varphi(v(p)/S)$ will denote the subset of $\mathfrak{X}$ which is obtained via the usual method for calculating the valuation v of φ (see Definition 4.1.5), but where, instead of mapping p to $v(p)$, we map it to S. If the context causes no confusion, we will simply write $\varphi(S)$.

Lemma 4.4.5. *Let φ and ψ be respectively ssc and sso, $\mathfrak{X}$ a slanted Priestley space, $\mathcal{F}$ a filtered family of closed subsets, $\mathcal{O}$ a directed family of open subsets and $\underline{O}$ a vector of clopen subsets of $\mathfrak{X}$. Then, for any variable p of φ or ψ and for any valuation v in $\mathfrak{X}$:*

1. *(a) If φ is positive in p, then $\varphi(v(p)/\cap\mathcal{F}) = \bigcap\{\varphi(v(p)/F) \mid F \in \mathcal{F}\}$,*
   (b) If ψ is negative in p, then $\varphi(v(p)/\cap\mathcal{F}) = \bigcup\{\varphi(v(p)/F) \mid F \in \mathcal{F}\}$,
2. *(c) If φ is negative in p, then $\varphi(v(p)/\cup\mathcal{O}) = \bigcap\{\varphi(v(p)/O) \mid O \in \mathcal{O}\}$,*
   (d) If ψ is positive in p, then $\varphi(v(p)/\cup\mathcal{O}) = \bigcup\{\varphi(v(p)/O) \mid O \in \mathcal{O}\}$,

Proof. 1. We proceed by simultaneous induction on the lengths of φ and ψ. The cases $\varphi = \varphi_1 \wedge \varphi_2$, $\varphi = \varphi_1 \vee \varphi_2$ (and respective for ψ) are treated exactly as in the unary setting (see Lemma 2.7.10). Now suppose that $\varphi = \triangle^*(\underline{\varphi}, \underline{\psi})$, then p occurs positively in all φ_j of $\underline{\varphi}$ and negatively in all ψ_k or $\underline{\psi}$. Hence, by induction, we have

$$\varphi(\cap\mathcal{F}) = \triangle^*(\underline{\varphi}(\cap\mathcal{F}), \underline{\psi}(\cap\mathcal{F})) = \triangle^*(\cap\,\underline{\varphi}(F), \cup\,\underline{\psi}(F)).$$

Since $\triangle^*$ is a c-slanted operator of order-type $(\underline{1}, \underline{\partial})$, if we denote by R^* its associated closed relation, we have

$$\begin{aligned}\varphi(\cap\mathcal{F}) =&\triangle^*(\cap\,\underline{\varphi}(F), \cup\,\underline{\psi}(F))\\ =&R^*(-, \cap\,\underline{\varphi}(F), (\cup\,\underline{\psi}(F))^c)\\ =&R^*(-, \cap\,\underline{\varphi}(F), \cap\,\underline{\psi}(F)^c)\\ =&R^*(-, \bigcap_{F_1\in\mathcal{F}} \varphi_1(F_1), ..., \bigcap_{F_n\in\mathcal{F}} \varphi_n(F_n), \bigcap_{F_{n+1}\in\mathcal{F}} \psi_{n+1}(F_{n+1})^c, ..., \bigcap_{F_k\in\mathcal{F}} \psi_k(F_k)^c)\end{aligned}$$

Now, since every φ_i is positive in p and every ψ_j is negative in p, we have that the families $\mathcal{F}_i = \{(\varphi_i(F_i)) \mid F_i \in \mathcal{F}\}$ and $\mathcal{F}_j = \{(\psi_j(F_j))^c \mid F_j \in \mathcal{F}\}$ are all filtered. Hence, by Lemma 4.4.4, we have

$$\varphi(\cap\mathcal{F}) = \bigcap\{R^*(-, \varphi_1(F_1), \ldots, \psi_k(F_k)^c) \mid F_1, \ldots, F_k \in \mathcal{F}\}.$$

To conclude the proof, we finally have to show that

$$\bigcap\{R^*(-, \varphi_1(F_1), \ldots, \psi_k(F_k)^c) \mid F_1, \ldots, F_k \in \mathcal{F}\} = \bigcap\{R^*(-, \varphi_1(F), \ldots, \psi_k(F)^c) \mid F \in \mathcal{F}\}.$$

The inclusion $\subseteq$ is clear. For the inclusion $\supseteq$, it is sufficient to note that, since $\mathcal{F}$ is filtered, for all $F_1, ..., F_k \in \mathcal{F}$, there is an element $F \in \mathcal{F}$ such that $F \subseteq F_1 \cap \cdots \cap F_k$ and, therefore, such that $\varphi_i(F) \subseteq \varphi_i(F_i)$ and $\psi_j(F_j) \subseteq \psi_j(F)$, which leads to the conclusion. □

Corollary 4.4.6 (Righthanded Ackermann lemma). *Let φ be an ssc formula and p a variable not occurring in φ. Let $\delta_1(p), \ldots, \delta_n(p)$ be ssc formulas where p occurs positively and $\gamma_1(p), \ldots, \gamma_n(p)$ sso formulas where p occurs negatively. For any slanted Priestley space $\mathfrak{X}$ and for any valuation v on $\mathfrak{X}$, we have*

$$\mathfrak{X} \models_v \delta_i(\varphi) \leq \gamma_i(\varphi) \text{ for all } i \leq n$$

if and only if there is a valuation v' on $\mathfrak{X}$ that may only differ from v in p such that

$$\mathfrak{X} \models_{v'} \varphi \leq p \text{ and } \mathfrak{X} \models_{v'} \delta_i(p) \leq \gamma_i(p) \text{ for all } i \leq n.$$

Proof. For the if part , note that since p does not occur in φ, we have trivially $v(\varphi) = v'(\varphi)$. Hence, we only have to use the monotonicity of δ_i and the antitonicity of γ_i to obtain

$$v(\delta_i(\varphi)) = \delta_i(v(\varphi)) = \delta_i(v'(\varphi)) \leq \delta_i(v'(p)) \leq \gamma_i(v'(p)) \leq \gamma_i(v'(\varphi)) = v(\gamma_i(\varphi)).$$

We now prove the only if part. First, note that since φ is an ssc formula, its valuation $v(\varphi)$ is a closed increasing set. Consequently, we have

$$v(\varphi) = \bigcap\{O \in {\uparrow}\mathrm{Clop}(X) \mid v(\varphi) \subseteq O\}.$$

Hence, since by hypothesis $\mathfrak{X} \models_v \delta_i(\varphi) \leq \gamma_i(\varphi)$, we have

$$\delta_i(\bigcap\{O \in {\uparrow}\mathrm{Clop}(X) \mid v(\varphi) \subseteq O\}) \subseteq \gamma_i(\cap\{\{O \in {\uparrow}\mathrm{Clop}(X) \mid v(\varphi) \subseteq O\}),$$

which is equivalent to

$$\bigcap\{\delta_i(O) \in {\uparrow}\mathrm{Clop}(X) \mid v(\varphi) \subseteq O\} \subseteq \bigcup\{\gamma_i(O) \in {\uparrow}\mathrm{Clop}(X) \mid v(\varphi) \subseteq O\}$$

by Lemma 4.4.5. Then, $\delta_i(O)$ is closed and $\gamma_i(O)$ is open for every increasing clopen set O. Therefore, by compactness, there exist $O_1, \ldots, O_m$ and $U_1, \ldots, U_m$ clopen sets containing $v(\varphi)$ such that

$$\delta_i(O_1) \cap \cdots \cap \delta_i(O_m) \subseteq \gamma_i(U_1) \cap \cdots \cap \gamma_i(U_m).$$

Set V_i as $O_1 \cap \cdots \cap O_m \cap U_1 \cap \cdots \cap U_m$, then V_i is an increasing clopen set containing $v(\varphi)$ and satisfying $\delta_i(V_i) \subseteq \gamma_i(V_i)$, by monotonicity of δ_i and antitonicity of γ_i. Finally, set O as $V_1 \cap \cdots \cap V_n$. We have that O is an increasing clopen set containing $v(\varphi)$ and such that $\delta_i(O) \subseteq \gamma_i(O)$ for all $i \leq n$. We just have to let v' be the valuation defined as $v'(p) = O$ and $v'(q) = v(q)$ for all $q \neq p$ to conclude. □

Corollary 4.4.7 (Lefthanded Ackermann lemma). *Let ψ be an sso formula and p a variable not occurring in ψ. Let $\delta_1(p), \ldots, \delta_n(p)$ be ssc formulas where p occurs negatively and $\gamma_1(p), \ldots, \gamma_n(p)$ sso formula where p occurs positively. For any slanted Priestley space $\mathfrak{X}$ and for any valuation v on $\mathfrak{X}$, we have*

$$\mathfrak{X} \models_v \delta_i(\psi) \leq \gamma_i(\psi) \text{ for all } i \leq n$$

if and only if there is a valuation v' on $\mathfrak{X}$ that may only differ from v in p such that

$$X \models_{v'} p \leq \psi \text{ and } X \models_{v'} \delta_i(p) \leq \gamma_i(p) \text{ for all } i \leq n.$$

Proof. The proof is similar to the one of Corollary 4.4.6 with the required adaptations. □

4.5 ALBA on analytic inductive inequalities

The aim of the algorithm ALBA is to determine whether a $\mathcal{L}_{\mathsf{DLE}}$-inequality $\varphi \leq \psi$ can be transformed into a set of pure inequalities $\mathsf{ALBA}(\varphi \leq \psi) := \{\varphi_i \leq \psi_i \mid i \in \{1, \ldots, n\}\}$ such that, for a slanted lattice $\mathfrak{L}$, we have

$$\mathfrak{L} \models \varphi \leq \psi \iff \mathfrak{L} \models \mathsf{ALBA}(\varphi \leq \psi). \tag{4.2}$$

Note that if this equivalence is achieved, we will immediately have that $\varphi \leq \psi$ admits a first order translation. Concerning canonicity, we should also prove that the equivalence (4.2) remains valid for perfects distributive lattices (i.e. discrete ones), which is proved for instance in [20]. Hence, we have for every canonical extension $\mathfrak{L}^\delta$ of a slanted lattice

$$\mathfrak{L}^\delta \models \varphi \leq \psi \iff \mathfrak{L}^\delta \models \mathsf{ALBA}(\varphi \leq \psi). \tag{4.3}$$

Finally, using Lemma 4.3.3 and the fact that $\mathsf{ALBA}(\varphi \leq \psi)$ is a set of pure inequalities, we have a bridge between (4.2) and (4.3), to an obtain the following diagram:

$$\begin{array}{ccc} \mathfrak{L} \models \varphi \leq \psi & & \mathfrak{L}^\delta \models \varphi \leq \psi \\ \Updownarrow & & \Updownarrow \\ \mathfrak{L} \models \mathsf{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathfrak{L}^\delta \models \mathsf{ALBA}(\varphi \leq \psi) \end{array}$$

Alongside the description of the algorithm, we prove that every one of its stages transforms its input in an equivalent (for slanted lattices) output. We also highlight where the different properties of analytic inductive inequalities are required. Finally, at the end of the algorithm, we will state the one theorem that generalises Theorems 2.7.15 and 2.7.16.

Stage 1 : Preprocessing and initialization

Stage 1.1

ALBA receives an analytic (Ω, ϵ)-inductive inequality $\varphi \leq \psi$ and applies the following *rules for elimination of monotone variables* (whose soundness is guaranteed by Proposition 4.4.2) exhaustively

$$\frac{\varphi(p) \leq \psi(p)}{\varphi(\top) \leq \psi(\top)} \qquad \frac{\chi(p) \leq \xi(p)}{\chi(\bot) \leq \xi(\bot)}$$

for $\varphi(p) \leq \psi(p)$ positive and $\chi(p) \leq \xi(p)$ negative in p, respectively.

Let us note that a non-critical variable is always monotone. Indeed, otherwise at least one of its occurrence should be critical. Therefore, after Stage 1.1 we can consider, without loss of generality, that **the inequality contains only critical variables.**

Stage 1.2

ALBA exhaustively distributes nodes $\vartriangle \in \Gamma_1$ over $\vee$ and nodes $\triangledown \in \Gamma_2$ over $\wedge$, according to their respective order-type, so as to bring occurrences of $\wedge$ and $\vee$ to the surface whenever this is possible and then eliminate them via exhaustive applications of *splitting rules*

$$\frac{\varphi \leq \psi \wedge \xi}{\varphi \leq \psi \quad \varphi \leq \xi} \qquad \frac{\varphi \vee \chi \leq \psi}{\varphi \leq \psi \quad \chi \leq \psi}$$

Definition 4.5.1. An inductive inequality is **definite** if the Skeleton nodes occurring in its critical branches are SLR nodes.

When Stage 1.2 is completed on an analytic inequality, we obtain a set of definite inequalities $\{\varphi'_i \leq \psi'_i \mid 1 \leq i \leq n\}$. Indeed, suppose there is a $+\vee$ node in ψ'_i or φ'_i. Then, it is necessarily under the scope of a $+\triangledown$ or a $-\vartriangle$, since otherwise it would have disappeared with the splitting rules. Now $+\triangledown$ and $-\vartriangle$ are PIA nodes (see Table 4.1) and every branch of ψ'_i and φ'_i is good, hence $+\vee$ must also be a PIA node. Of course, the argument is similar in the case of $-\wedge$ nodes.

As a consequence, we will from now on assume to work with definite analytic inductive inequalities only.

Stage 1.3

Let $(\varphi \leq \psi)[\underline{\alpha}/!\underline{x}, \underline{\beta}/!\underline{y}, \underline{\gamma}/!\underline{z}, \underline{\delta}/!\underline{t}]$ (recall Notation 4.2.10) denote one of the inequalities resulting from Stage 1.2. The algorithm ALBA transforms it into the following *initial quasi-inequality*

$$\frac{(\varphi \leq \psi)[\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}]}{\forall \underline{\boldsymbol{j}}\ \forall \underline{\boldsymbol{m}}\ \forall \underline{\boldsymbol{i}}\ \forall \underline{\boldsymbol{n}}((\underline{\boldsymbol{j}} \leq \underline{\alpha}\ \&\ \underline{\beta} \leq \underline{\boldsymbol{m}}\ \&\ \underline{\boldsymbol{i}} \leq \underline{\gamma}\ \&\ \underline{\delta} \leq \underline{\boldsymbol{n}}) \Rightarrow (\varphi \leq \psi)[!\underline{\boldsymbol{j}}, !\underline{\boldsymbol{m}}, !\underline{\boldsymbol{i}}, !\underline{\boldsymbol{n}}])} \tag{4.4}$$

In the quasi-inequality above, symbols such as $\underline{\boldsymbol{j}} \leq \underline{\alpha}$ denote the conjunction of inequalities of the form $\boldsymbol{j}_k \leq \alpha_k$ for each $\boldsymbol{j}_k \in \underline{\boldsymbol{j}}$ and each $\alpha_k \in \underline{\alpha}$. The soundness of this transition follows almost immediately from Proposition 4.4.3. For instance, consider the case $\varphi(\alpha/x) \leq \psi$ with $+\alpha \propto +\varphi$ (remember Notation 4.2.10). First, we have, almost by definition, that $\alpha = \cup\{\boldsymbol{j} \in J^\infty \mid \boldsymbol{j} \leq \alpha\}$. Then, since we assumed to work only with definite inequalities, we know that the branch of φ ending in α is an SLR one. It follows that $\varphi(\alpha) = \cup\{\varphi(\boldsymbol{j}) \mid \alpha \geq \boldsymbol{j} \in J^\infty\}$. Hence, we have that $\varphi(\alpha) \leq \psi$ is equivalent to $\boldsymbol{j} \leq \alpha \Rightarrow \varphi(\boldsymbol{j}) \leq \psi$, as required.

Stage 1.4

Before moving each quasi-inequality separately to Stage 2 (described below), by exhaustively applying splitting rules to the top-most nodes of the formulas in $\underline{\alpha}$ and $\underline{\beta}$, it is possible to transform the resulting inequality of (4.4) into one of similar shape in which each α_k in $\underline{\alpha}$ and β_k in $\underline{\beta}$ contains exactly one ϵ-critical occurrence.

Indeed, suppose for instance that $\beta \leq \boldsymbol{m}$ and β contains two ϵ-critical occurrences. Necessarily, β must contain an SRR node (recall that β is a definite PIA formula). Then, β cannot satisfy the condition (b) in Definition 4.2.6 of inductive formulas, which is absurd.

Consequently, the quasi-inequality will be represented as follows:

$$\begin{aligned} \forall \underline{\boldsymbol{j}}\ \forall \underline{\boldsymbol{m}}\ \forall \underline{\boldsymbol{i}}\ \forall \underline{\boldsymbol{n}}((\underline{\boldsymbol{j}} \leq \underline{\alpha}_p\ \&\ \underline{\boldsymbol{j}} \leq \underline{\alpha}_q\ \&\ \underline{\beta}_p \leq \underline{\boldsymbol{m}}\ \&\ \underline{\beta}_q \leq \underline{\boldsymbol{m}}\ \&\ \underline{\boldsymbol{i}} \leq \underline{\gamma}\ \&\ \underline{\delta} \leq \underline{\boldsymbol{n}}) \\ \Rightarrow (\varphi \leq \psi)[!\underline{\boldsymbol{j}}/!\underline{x}, !\underline{\boldsymbol{m}}/!\underline{y}, !\underline{\boldsymbol{i}}/!\underline{z}, !\underline{\boldsymbol{n}}/!\underline{t}]), \end{aligned} \tag{4.5}$$

where $\underline{p}$ (resp. $\underline{q}$) is the vector of the variables in $\varphi \leq \psi$ such that $\epsilon(p) = 1$ (resp. $\epsilon(q) = \partial$) and the subscript in each PIA-formula in $\underline{\alpha}$ and $\underline{\beta}$ indicates the unique ϵ-critical variable occurrence contained in that formula.

Stage 2: Reduction and elimination

Stage 2.1

From now on, we will work on the of inequalities in the antecedent of an (initial) quasi-inequality $(\underline{\boldsymbol{j}} \leq \underline{\alpha}, ...)$. The objective is to remove all propositional variables from it. The execution will be separated in two. In the first, we merely modify the inequalities to obtain inequalities of the form required to use Ackermann lemmas, namely $\xi \leq p$ or $q \leq \xi$. It will be done by using the splitting rules already introduced in Stage 1.2 and the following *residuation rules* (whose soundness is guaranteed by Theorem 4.3.1): For every $\triangle \in \Gamma_1$ and $\nabla \in \Gamma_2$, and any $1 \leq i \leq n_\triangle$ and $1 \leq j \leq n_\nabla$,

$$\frac{\triangle(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_\nabla}) \leq \psi}{\varphi_i \leq \nabla_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_\nabla})}\ \epsilon_\nabla(i) = 1 \qquad \frac{f(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_f}) \leq \psi}{f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f}) \leq \varphi_i}\ \epsilon_f(i) = \partial$$

$$\frac{\psi \leq g(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_\nabla})}{g_i^\flat(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_\nabla}) \leq \varphi_i}\ \epsilon_\nabla(i) = 1 \qquad \frac{\psi \leq g(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_\nabla})}{\varphi_i \leq g_i^\flat(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_g})}\ \epsilon_\nabla(i) = \partial$$

Using the notations of Definition 4.3.7, the antecedent of (4.5) can be rewritten as

$$\begin{aligned} &\underline{\mathsf{LA}(\alpha_p)(\boldsymbol{j}, \underline{p}, \underline{q})} \leq \underline{p}\ \&\ \underline{\mathsf{RA}(\beta_p)(\boldsymbol{m}, \underline{p}, \underline{q})} \leq \underline{p}\ \&\ \underline{q} \leq \underline{\mathsf{LA}(\alpha_q)(\boldsymbol{j}, \underline{p}, \underline{q})} \\ &\quad \&\ \underline{q} \leq \underline{\mathsf{RA}(\beta_q)(\boldsymbol{m}, \underline{p}, \underline{q})}\ \&\ \underline{\boldsymbol{i}} \leq \underline{\gamma}\ \&\ \underline{\delta} \leq \neg\underline{\boldsymbol{n}}. \end{aligned} \tag{4.6}$$

Note that the non-critical variables in $\underline{p}$ and $\underline{q}$ actually occurring in each formula $\mathsf{LA}(\alpha_p)(\boldsymbol{j}, \underline{p}, \underline{q})$, $\mathsf{RA}(\beta_p)(\boldsymbol{m}, \underline{p}, \underline{q})$, $\mathsf{LA}(\alpha_q)(\boldsymbol{j}, \underline{p}, \underline{q})$ and $\mathsf{RA}(\beta_q)(\boldsymbol{m}, \underline{p}, \underline{q})$ are those that are strictly Ω-smaller than the critical variable indicated in the subscript of the given PIA-formula.

We are now ready to actually remove all variables from the antecedent. This will be achieved thanks to the *Ackermann rules*:

1. *Right Ackermann Rule*

$$\frac{\&\{\chi_i \leq p \mid 1 \leq i \leq n\}\ \&\{\delta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\}\ \&\{\theta_k \leq \mu_k \mid 1 \leq k \leq l\}}{\&\{\delta_j(\bigvee_{i=1}^n \chi_i) \leq \gamma_j(\bigvee_{i=1}^n \chi_i) \mid 1 \leq j \leq m\}\ \&\{\theta_k \leq \mu_k \mid 1 \leq k \leq l\}}\ (RAR)$$

where:

- p does not occur in $\chi_1, \ldots, \chi_n$ or in $\theta_1 \leq \mu_1, \ldots, \theta_l \leq \mu_l$,
- $\delta_1(p), \ldots, \delta_m(p)$ are positive in p, and
- $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p.

2. *Left Ackermann Rule*

$$\frac{\&\{p \leq \chi_i \mid 1 \leq i \leq n\}\ \&\{\delta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\}\ \&\{\theta_k \leq \mu_k \mid 1 \leq k \leq l\}}{\&\{\delta_j(\bigwedge_{i=1}^n \chi_i) \leq \gamma_j(\bigwedge_{i=1}^n \chi_i) \mid 1 \leq j \leq m\}\ \&\{\theta_k \leq \mu_k \mid 1 \leq k \leq l\}}\ (LAR)$$

where:

- p does not occur in $\chi_1, \ldots, \chi_n$ or in $\theta_1 \leq \mu_1, \ldots, \theta_l \leq \mu_l$,
- $\delta_1(p), \ldots, \delta_m(p)$ are negative in p, and
- $\gamma_1(p), \ldots, \gamma_m(p)$ are positive in p.

The soundness of theses rules is guaranteed by Corollaries 4.4.6 and 4.4.7. Now, it should be checked that (4.6) has the shape required to apply Ackermann rules. This is where we see the importance of the analyticity of the original inequality. Indeed, since every branch of $\varphi \leq \psi$ is good, it is clear that γ is a positive PIA formula (recall Notation 4.2.10), while δ is a negative PIA formula and that both are ϵ^∂-uniform. Recall also that $\epsilon(p) = 1$ for every $p \in \underline{p}$ and that $\epsilon(q) = \partial$ for every $q \in \underline{q}$, so that we have $-p \propto +\gamma$, $-p \propto -\delta$,$+q \propto +\gamma$ and $+ \propto -\delta$. Moreover, we know firstly that $\boldsymbol{i}$ and $\boldsymbol{n}$ are respectively closed and open by definition and secondly that every γ is sso and that every δ is ssc by Lemma 4.3.9. Again by Lemma 4.3.9, we know that $\mathsf{LA}(\alpha_p)$ and $\mathsf{RA}(\beta_p)$ are ssc and that $\mathsf{LA}(\alpha_q)$ and $\mathsf{RA}(\beta_q)$ are sso. Hence, we may apply the right Ackermann rule to every $p \in \underline{p}$ and the left one to every $q \in \underline{q}$.

Definition 4.5.2. For every $p \in \underline{p}$ and $q \in \underline{q}$, we define the sets $\mathrm{Mv}(p)$ and $\mathrm{Mv}(q)$ by recursion on Ω as follows:

1. $\mathrm{Mv}(p) := \{\mathsf{LA}(\alpha_p)(\boldsymbol{j}_k, \underline{\mathrm{mv}(p)}/\underline{p}, \underline{\mathrm{mv}(q)}/\underline{q}), \mathsf{RA}(\beta_p)(\boldsymbol{m}_h, \underline{\mathrm{mv}(p)}/\underline{p}, \underline{\mathrm{mv}(q)}/\underline{q} \mid 1 \leq k \leq n_{i_1}, 1 \leq h \leq n_{i_2}, \underline{\mathrm{mv}(p)} \in \underline{\mathrm{Mv}(p)}, \underline{\mathrm{mv}(q)} \in \underline{\mathrm{Mv}(q)})\}$
2. $\mathrm{Mv}(q) := \{\mathsf{LA}(\alpha_q)(\boldsymbol{j}_h, \underline{\mathrm{mv}(p)}/\underline{p}, \underline{\mathrm{mv}(q)}/\underline{q}), \mathsf{RA}(\beta_q)[\boldsymbol{m}_k, \underline{\mathrm{mv}(p)}/\underline{q}, \underline{\mathrm{mv}(q)}/\underline{q} \mid 1 \leq h \leq m_{j_1}, 1 \leq k \leq m_{j_2}, \underline{\mathrm{mv}(p)} \in \underline{\mathrm{Mv}(p)}, \underline{\mathrm{mv}(q)} \in \underline{\mathrm{Mv}(q)}\}$

where n_{i_1} (resp. n_{i_2}) is the number of occurrences of p in α formulas (resp. in β formulas) for every $p \in \underline{p}$ and m_{j_1} (resp. m_{j_2}) is the number of occurrences of q in α formulas (resp. in β formulas) for every $q \in \underline{q}$.

Lemma 4.5.3. *The elements of $Mv(p)$ (resp. $Mv(q)$) are closed (resp. open) pure formulas whose nominal occurrences are respectively $\underline{\boldsymbol{j}}$ and $\underline{\boldsymbol{m}}$.*

Proof. This is done by induction on the strict order Ω. We know by Lemma 4.3.9 that the formulas $\mathsf{LA}(\alpha_p)(\boldsymbol{j}, \underline{p}, \underline{q})$ and $\mathsf{RA}(\beta_q)(\boldsymbol{m}, \underline{p}, \underline{q})$ are ssc formulas while both the formulas $\mathsf{LA}(\alpha_q)(\boldsymbol{j}, \underline{p}, \underline{q})$ and $\mathsf{RA}(\beta_p)(\boldsymbol{m}, \underline{p}, \underline{q})$ are sso. It is then sufficient to use induction and Lemma 4.3.5 to conclude the proof. ☐

By induction on Ω, we can apply the Ackermann rules exhaustively so as to eliminate all variables $\underline{p}$ and $\underline{q}$. Then the antecedent of the resulting purified quasi-inequality has the following form:

$$\underline{\boldsymbol{i}} \leq \underline{\gamma}(\underline{\vee \mathrm{Mv}(p)}/\underline{p}, \underline{\wedge \mathrm{Mv}(q)}/\underline{q}) \ \& \ \underline{\delta}(\underline{\wedge \mathrm{Mv}(p)}/\underline{p}, \underline{\wedge \mathrm{Mv}(q)}/\underline{q}) \leq \underline{\boldsymbol{n}}. \tag{4.7}$$

Lemma 4.5.4. *The inequalities of* (4.7) *are such that their left-hand sides are closed and their right-hand sides are open.*

Proof. Since we know that $\boldsymbol{i}$ is closed and that $\boldsymbol{n}$ is open by definition, we can focus on the formulas $\gamma(\underline{\vee \mathrm{Mv}(p)}/\underline{p}, \underline{\wedge \mathrm{Mv}(q)}/\underline{q})$ and $\delta(\underline{\wedge \mathrm{Mv}(p)}/\underline{p}, \underline{\wedge \mathrm{Mv}(q)}/\underline{q})$. First, let us show that $\gamma(\underline{\mathrm{mv}(p)}/\underline{p}, \underline{\mathrm{mv}(q)}/\underline{q})$ is closed for every $\underline{\mathrm{mv}(p)} \in \underline{\mathrm{Mv}(p)}$ and $\underline{\mathrm{mv}(q)} \in \underline{\mathrm{Mv}(q)}$.

Recall, from Notation 4.2.10, that $\gamma(\underline{p}, \underline{q})$ is a definite positive formula which is ϵ^∂-uniform as subformula of the original inequality. Recall also that $\epsilon(p) = 1$ for every $p \in \underline{p}$ and that $\epsilon(q) = \partial$ for every $q \in \underline{q}$. Therefore, we have $-p \propto +\gamma$ and $+q \propto +\gamma$ for every $p \in \underline{p}$ and every $q \in \underline{q}$. Now comes the importance of the analytic status of the original inequality. Indeed, since every branch of $\varphi \leq \psi$ is good, it is clear that γ is a PIA formula (again from Notation 4.2.10). Hence, it is a positive PIA formula. Furthermore, recall that, by Lemma 4.3.9, we know that γ is an open formula and that by Lemma 4.5.3 we have that $\mathrm{mv}(p)$ is ssc and $\mathrm{mv}(q)$ is sso. Therefore, the conclusion is an immediate conclusion of Lemma 4.3.5.

To finally complete the proof, we will once again use the fact that γ is a positive PIA formula (that is it is practically a box) and therefore, for every variable p with $-p \propto +\gamma$, we have

$$\gamma(\vee_{i=1}^{n}\chi_i/p) = \wedge_{i=1}^{n}\gamma(\chi_i/p)$$

and for every variable q with $+q \propto +\gamma$, we have

$$\gamma(\wedge_{i=1}^{n}\chi_i/q) = \wedge_{i=1}^{n}\gamma(\chi_i/q).$$

It follows that $\gamma(\underline{\vee\mathrm{Mv}(p)}/\underline{p}, \underline{\wedge\mathrm{Mv}(q)}/\underline{q})$ is a closed formula as conjunction of smaller closed formulas. □

For the sake of aesthetic, let us shorten the formula $\underline{\gamma}(\underline{\vee\mathrm{Mv}(p)}/\underline{p}, \underline{\wedge\mathrm{Mv}(q)}/\underline{q})$ into $\underline{\gamma}(\underline{\boldsymbol{j}}, \underline{\boldsymbol{m}})$ and the formula $\underline{\delta}(\underline{\wedge\mathrm{Mv}(p)}/\underline{p}, \underline{\wedge\mathrm{Mv}(q)}/\underline{q})$ into $\underline{\delta}(\underline{\boldsymbol{j}}, \underline{\boldsymbol{m}})$. The quasi-inequality 4.5 is hence equivalently rewritten as

$$\forall \boldsymbol{j} \forall \boldsymbol{m} \forall \boldsymbol{i} \forall \boldsymbol{n} (\underline{\boldsymbol{i}} \leq \underline{\gamma}(\underline{\boldsymbol{j}}, \underline{\boldsymbol{m}}) \ \& \ \underline{\delta}(\underline{\boldsymbol{j}}, \underline{\boldsymbol{m}}) \leq \underline{\boldsymbol{n}} \Rightarrow (\varphi \leq \psi)(\underline{\boldsymbol{j}}, \underline{\boldsymbol{m}}, \underline{\boldsymbol{i}}, \underline{\boldsymbol{n}})). \tag{4.8}$$

And we finally arrive at the final stage of ALBA.

Stage 2.2

The quasi-inequality (4.8) is equivalent to the pure inequality

$$(\varphi \leq \psi)(\underline{\boldsymbol{j}}, \underline{\boldsymbol{m}}, \gamma(\underline{\boldsymbol{j}}, \underline{\boldsymbol{m}}), \underline{\delta}(\underline{\boldsymbol{j}}, \underline{\boldsymbol{m}})) \tag{4.9}$$

The direction (4.9) ⇒ (4.8) is immediate. To prove the direction (4.8) ⇒ (4.9), we have to remember Notation 4.2.10 where we saw that φ was closed and ψ was open. It is then sufficient to apply Proposition 4.4.3 to conclude.

Conclusion

In every stage of ALBA, we transform the input in a semanticly equivalent output, starting from an analytic inductive inequality $\varphi \leq \psi$ to finally obtain a set of pure inequalities, set which is denoted by $\mathsf{ALBA}(\varphi \leq \psi)$. Therefore, for every slanted lattice $\mathfrak{L}$, we have

$$\mathfrak{L} \models \varphi \leq \psi \text{ iff } \mathfrak{L} \models \mathsf{ALBA}(\varphi \leq \psi).$$

It follows that the analytic inductive inequality $\varphi \leq \psi$ admits a first order translation.

Moreover, since every inequality in $\mathsf{ALBA}(\varphi \leq \psi)$ is pure, we have also

$$\mathfrak{L} \models \mathsf{ALBA}(\varphi \leq \psi) \text{ iff } \mathfrak{L}^\delta \models \mathsf{ALBA}(\varphi \leq \psi).$$

Now, it is proved in [20, Theorem 6.1] that the algorithm ALBA we discussed here is reversible not only for clopen valuations but also for arbitrary valuations. Consequently, we have

$$\mathfrak{L}^\delta \models \mathsf{ALBA}(\varphi \leq \psi) \text{ iff } \mathfrak{L}^\delta \models \varphi \leq \psi.$$

In other words, the inductive inequality $\varphi \leq \psi$ is canonical (in the sense of Definition 4.1.8). We summarise this conclusion in the next theorem.

Theorem 4.5.5. *Let $\varphi \leq \psi$ be an analytic inductive inequality. Then*

1. *there exists a first order formula Φ in the language of the accessibility relations, effectively computable from $\varphi \leq \psi$, such that for any slanted $\mathfrak{L}$ with dual $\mathfrak{X}$*

$$\mathfrak{L} \models \varphi \leq \psi \textit{ iff } \mathfrak{X} \models \Phi,$$

2. *for any slanted lattice $\mathfrak{L}$, we have*

$$\mathfrak{L} \models \varphi \leq \psi \textit{ iff } \mathfrak{L}^\delta \models \varphi \leq \psi.$$

4.6 Examples

Consider the inequality $(\Box p \wedge \Diamond q) \leq \Diamond(p \wedge q)$. We saw in Example 4.2.11 that it was both $(1,1)$-Sahlqvist and $(\Omega,(1,\partial))$-inductive for the strict partial order $p \leq_\Omega q$. It is quite easy to check that this inequality is also analytic. Hence, we can let ALBA run on the inequality with two different set-ups. Remember that the generation trees of the inequality are given by

$$
\begin{array}{ccccccc}
 & \overset{+}{\wedge} & & & & \overset{-}{\Diamond} & \\
\swarrow & & \searrow & & & \downarrow & \\
\overset{+}{\Box} & & \overset{+}{\Diamond} & \leq & & \overset{-}{\wedge} & \\
\downarrow & & \downarrow & & \swarrow & & \searrow \\
\overset{+}{p} & & \overset{+}{q} & & \overset{-}{p} & & \overset{-}{q}
\end{array}
$$

Example 4.6.1 (Sahlqvist set-up)**.** The critical nodes are $+p$ and $+q$. Using Notation 4.2.10, the inequality $(\Box p \wedge \Diamond q) \leq \Diamond(p \wedge q)$ becomes $\alpha_p \wedge \Diamond \alpha_q \leq \gamma$ with

- $x \wedge \Diamond y \leq t$, the Skeleton part of the inequality.
- $\alpha_p = \Box p$, a positive PIA formula (and hence open),
- $\alpha_q = q$, a positive PIA formula (and hence open),
- $\delta = \Diamond(p \wedge q)$, a negative $(1,1)^\partial$-uniform and closed formula.

Stage 1.1 and Stage 1.2 do not apply in this example, so that we can directly move to Stage 1.3 and obtain

$$\forall \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{m}(\boldsymbol{i} \leq \alpha_p \ \&\ \boldsymbol{j} \leq \alpha_q \ \&\ \delta \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \boldsymbol{m}).$$

Now, by applying residuation rules of Stage 2, we obtain

$$\forall \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{m}(\blacklozenge \boldsymbol{i} \leq p \ \&\ \boldsymbol{j} \leq q \ \&\ \Diamond(p \wedge q) \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \boldsymbol{m}).$$

Notice now that $\blacklozenge \boldsymbol{i}$ is closed and p does not occur in it, $\Diamond(p \wedge q)$ is closed and positive in p and $\neg \boldsymbol{m}$ is open (and negative in p). Hence, we can apply the Right handed Ackermann rule to obtain

$$\forall \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{m}(\boldsymbol{j} \leq q \ \&\ \Diamond(\blacklozenge \boldsymbol{i} \wedge q) \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \boldsymbol{m}).$$

Using a second time the Right handed Ackermann rule for q, we obtain

$$\forall \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{m}(\Diamond(\blacklozenge \boldsymbol{i} \wedge \boldsymbol{j}) \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \boldsymbol{m}),$$

which is equivalent to

$$\boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \Diamond(\blacklozenge \boldsymbol{i} \wedge \boldsymbol{j}), \tag{4.10}$$

a pure expanded bimodal formula. The first order translation of (4.10) is

$$\{x\} \cap R(-, y) \subseteq R(-, R(x, -) \cap \{y\})$$

which can easily be seen to be equivalent to

$$x \, R \, y \Rightarrow x \, R \, y.$$

Example 4.6.2 (Inductive set-up)**.** The critical nodes are $+p$ and $-q$. With Notation 4.2.10, the inequality $(\Box p \wedge \Diamond q) \leq \Diamond(p \wedge q)$ becomes $\alpha_p \wedge \Diamond\gamma \leq \beta_q$ with

1. $x \wedge \Diamond y \leq t$, the Skeleton part of the inequality.
2. $\alpha_p = \Box p$, a positive PIA formula (and hence open),
3. $\beta_q = \Diamond(p \wedge q)$, a negative PIA formula (and hence closed),
4. $\gamma = q$, a $(1, \partial)^\partial$-uniform and open formula.

Once again, we can move directly to Stage 1.3 to obtain

$$\forall \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{m}(\boldsymbol{i} \leq \alpha_p \ \&\ \beta_q \leq \boldsymbol{m} \ \&\ \boldsymbol{j} \leq \gamma \Rightarrow \boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \boldsymbol{m}).$$

Now, by applying residuation rules of Stage 2, we obtain

$$\forall \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{m}(\blacklozenge \boldsymbol{i} \leq p \ \&\ p \wedge q \leq \blacksquare \boldsymbol{m} \ \&\ \boldsymbol{j} \leq q \Rightarrow \boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \boldsymbol{m})$$

which is equivalent to (recall Lemma 4.3.8)

$$\forall \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{m}(\blacklozenge \boldsymbol{i} \leq p \ \&\ q \leq \blacksquare \boldsymbol{m} \vee \neg p \ \&\ \boldsymbol{j} \leq q \Rightarrow \boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \boldsymbol{m}).$$

We have in particular $\mathsf{LA}(\alpha_p)(\boldsymbol{i}) = \blacklozenge \boldsymbol{i}$ and $\mathsf{RA}(\beta_q)(\boldsymbol{m}, p) = \blacksquare \boldsymbol{m} \vee \neg p$.

It is now time to define the sets $\mathrm{Mv}(p)$ and $\mathrm{Mv}(q)$. Since the strict partial order is given by $p \leq_\Omega q$, we start with $\mathrm{Mv}(p)$, which is given by $\{\blacklozenge \boldsymbol{i}\}$. Now, $\mathrm{Mv}(q)$ is given by

$$\mathrm{Mv}(q) = \{\mathsf{RA}(\beta_q)(\boldsymbol{m}, \mathrm{mv}(p)) \mid \mathrm{mv}(p) \in \mathrm{Mv}(p)\} = \{\blacksquare \boldsymbol{m} \vee \neg \blacklozenge \boldsymbol{i}\}.$$

Hence, we get

$$\forall \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{m}(\boldsymbol{j} \leq \blacksquare(\boldsymbol{m} \vee \neg \blacklozenge) \boldsymbol{i} \Rightarrow \boldsymbol{i} \wedge \Diamond \boldsymbol{j} \leq \boldsymbol{m}),$$

which is equivalent to

$$(\blacksquare \boldsymbol{m} \vee \neg \blacklozenge \boldsymbol{i}) \leq \blacksquare(\boldsymbol{m} \vee \neg \boldsymbol{i}), \tag{4.11}$$

a pure expanded bimodal formula. The first order translation of (4.11) is

$$R(-, y)^c \cup R(-, x)^c \subseteq R(-, \{x\} \cap \{y\})^c$$

which is equivalent to

$$[(x \neq y) \wedge (\forall z)(z \not R\, y \wedge z \not R\, x \Rightarrow z \neq z)] \vee [(x = y) \wedge (\forall z)(z \not R\, x \Rightarrow z \not R\, x)].$$

This last first order formula is shown to be equivalent, with some concentration, to

$$x\ R\ y \Rightarrow x\ R\ y.$$

It follows that the pure inequalities (4.10) and (4.11) obtained from $(\Box p \wedge \Diamond q) \leq \Diamond(p \wedge q)$ in two different settings are fortunately equivalent.

Example 4.6.3. Remember that in Example 2.8.2, we saw that the first order equivalent of the inequality $\Diamond\Box p \leq \Box\Diamond p$ was

$$(x\ R\ y \wedge x\ R\ z) \Rightarrow (\exists t)(y\ R\ t \wedge z\ R\ t). \tag{4.12}$$

Of course, we find this equivalence also in this formalism. First, we destructure our inequality as usual in $\Diamond\alpha_p \leq \Box\delta$ where

1. $\Diamond x \leq \Box y$, is the Skeleton part of the inequality,
2. $\alpha_p = \Box p$, is an open formula,
3. $\delta = \Diamond p$, is a closed formula.

We then have the following sequence of equivalences

$$\begin{aligned}
&\Diamond\Box p \leq \Box\Diamond p \\
\iff &\forall \boldsymbol{j}, \boldsymbol{n}(\boldsymbol{j} \leq \Box p \wedge \Diamond p \leq \boldsymbol{n} \Rightarrow \Diamond \boldsymbol{j} \leq \Box(\boldsymbol{n})) \\
\iff &\forall \boldsymbol{j}, \boldsymbol{n}(\blacklozenge \boldsymbol{j} \leq p \wedge \Diamond p \leq \boldsymbol{n} \Rightarrow \Diamond \boldsymbol{j} \leq \Box(\boldsymbol{n})) \\
\iff &\forall \boldsymbol{j}, \boldsymbol{n}(\Diamond\blacklozenge \boldsymbol{j} \leq \neg \boldsymbol{n} \Rightarrow \Diamond \boldsymbol{j} \leq \Box(\boldsymbol{n})) \\
\iff &\forall \boldsymbol{j}(\Diamond \boldsymbol{j} \leq \Box\Diamond\blacklozenge \boldsymbol{j}) \\
\iff &\forall \boldsymbol{j}(\blacklozenge\Diamond \boldsymbol{j} \leq \Diamond\blacklozenge \boldsymbol{j}).
\end{aligned}$$

Finally, the first order equivalent of the last expression is

$$R(R(-, x), -) \subseteq R(-, R(x, -))$$

which is indeed equivalent to (4.12).

4.7 Comparison with Chapter 2

We have seen two different processes (one in Section 2.7 and one in this current chapter) to determine whether a given formula/inequality is canonical/translatable or not. In this short section, we discuss the differences and similarities between them.

The main idea remains identical: identify correct shapes of formulas to remove every occurrences of propositional variables via suitable "minimal valuations". Here stands the major difference between the two chapters, which is actually the major difference between the Sahlqvist and the inductive inequalities; the existence of an order to create these minimal valuations. In Sahlqvist inequality, the minimal valuations can be determined somehow "simultaneously" while the order Ω of inductive inequalities determine in which order we have to calculate the minimal valuation associated the variables of the inequality (as it can be seen for instance in Example 4.6.2). But, since this inductive case was not present in Chapter 2, we will not overly linger over it and instead compare the resolution Sahlqvist inequalities. Consider for instance the following executions for the formula $\blacksquare p \to p$, the the inequality $\blacksquare p \leq p$ for $\epsilon = 1$.

	Chapter 2		Chapter 4
	$\blacksquare p \to p$		$\blacksquare p \leq p$
$\Leftrightarrow$	$\blacksquare O \subseteq O$	$\Leftrightarrow$	$\boldsymbol{i} \leq \blacksquare p$ and $p \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$
$\Leftrightarrow$	$x \in \blacksquare O \Rightarrow x \in O$	$\Leftrightarrow$	$\Diamond \boldsymbol{i} \leq p$ and $p \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$
$\Leftrightarrow$	$R(-, x) \subseteq O \Rightarrow x \in O$	$\Leftrightarrow$	$\Diamond \boldsymbol{i} \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m}$
$\Leftrightarrow$	$x \in \overline{R(-, x)} = R(-, x)$	$\Leftrightarrow$	$\boldsymbol{i} \leq \Diamond \boldsymbol{i}$

Table 4.3: Executions for $\blacksquare p \leq p$

In both cases, the minimal valuation is given by $R(-, x)$, that is $\Diamond \boldsymbol{i}$, and is generated by the first occurrence of p (the one under the scope of $\blacksquare$). We hence saw here that the Lemma 2.7.9 which guarantees the passage from $x \in \blacksquare O$ to $R(x, -) \subseteq O$, is a disguised use of the adjunction

$p \leq \blacksquare q$ if and only if $\Diamond p \leq q$. Alternatively, we could have solved the inequality $\blacksquare p \leq p$ as ∂-Sahlqvist. In this case, we have the following execution and the minimal valuation is generated by the second occurrence of p.

$$\begin{array}{ll} & \blacksquare p \leq p \\ \Leftrightarrow & \boldsymbol{i} \leq \blacksquare p \text{ and } p \leq \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m} \\ \Leftrightarrow & \boldsymbol{i} \leq \blacksquare \boldsymbol{m} \Rightarrow \boldsymbol{i} \leq \boldsymbol{m} \\ \Leftrightarrow & \blacksquare \boldsymbol{m} \leq \boldsymbol{m} \end{array}$$

Table 4.4: Execution for $\blacksquare p \leq p$

In particular, let us note that critical variable in the process of Chapter 2 can only occur in the strongly positive part of the formula. On the other hand, for the process of 4, the critical occurrence may appear everywhere in the inequality. Of course, the fact that Boolean algebras constitute the setting of Chapter 2 part off nullifies this difference ($\varphi \to \psi$ is equivalent to $\neg\psi \to \neg\varphi$).

The main ingredient of ALBA if of course the Ackermann lemmas. In Tables 4.3 and 4.4, the Ackermann lemmas are used to transform $\Diamond \boldsymbol{i} \leq p$ and $p \leq \boldsymbol{m}$ into $\Diamond \boldsymbol{i} \leq \boldsymbol{m}$ on one side and $\boldsymbol{i} \leq \blacksquare p$ and $p \leq \boldsymbol{m}$ into $\boldsymbol{i} \leq \blacksquare \boldsymbol{m}$ on the other side. In the process of Chapter 2, the Ackermann passage is somehow shortcut as we went from

$$\Diamond \boldsymbol{i} \leq p \text{ (that is } R(-,x) \subseteq O) \ \Rightarrow \boldsymbol{i} \leq p \ (x \in O)$$

to $\boldsymbol{i} \leq \Diamond \boldsymbol{i}$, using the intersection lemma (recall that the intersection lemma is a key element in the proof of Ackermann lemmas). The limitation of this option is that clopen sets O can only be considered as closed sets, since they approximate R-expressions (which are in particular closed sets) from above.

Now, let us consider an example where the comparison of the methods is less straightforward: the formula $\Box(\Diamond\Box p \to p)$, or, equivalently, the inequality $\top \leq \Box(\Diamond\Box p \to p)$. We have the following executions:

Chapter 2	Chapter 4
$\Box(\Diamond\Box p \to p)$	$\top \leq \Box(\Diamond\Box p \to p)$
$\Leftrightarrow\ x\,R\,y \Rightarrow (y \in \Diamond\Box O \Rightarrow y \in O)$	$\Leftrightarrow\ \boldsymbol{i} \leq \Box p \text{ and } p \leq \boldsymbol{m} \Rightarrow \top \leq \Box(\Diamond \boldsymbol{i} \to \boldsymbol{m})$
$\Leftrightarrow\ x\,R\,y \Rightarrow ((\exists t)(y\,R\,t \text{ and } R(t,-) \subseteq O) \Rightarrow y \in O)$	$\Leftrightarrow\ \blacklozenge \boldsymbol{i} \leq p \text{ and } p \leq \boldsymbol{m} \Rightarrow \top \leq \Box(\Diamond \boldsymbol{i} \to \boldsymbol{m})$
$\Leftrightarrow\ x\,R\,y \Rightarrow (y\,R\,t \Rightarrow (R(t,-) \subseteq O \Rightarrow y \in O))$	$\Leftrightarrow\ \blacklozenge \boldsymbol{i} \leq \boldsymbol{m} \Rightarrow \top \leq \Box(\Diamond \boldsymbol{i} \to \boldsymbol{m})$
$\Leftrightarrow\ x\,R\,y \Rightarrow (y\,R\,t \Rightarrow t\,R\,y)$.	$\Leftrightarrow\ \top \leq \Box(\Diamond \boldsymbol{i} \to \blacklozenge \boldsymbol{i})$
	$\Leftrightarrow\ \Box(\Diamond \boldsymbol{i} \to \blacklozenge \boldsymbol{i})$.

Let us recall that in Chapter 2, the creation of the minimal valuation for p (that is O) is not immediate (see the proof of Theorem 2.7.15). Indeed, we first have to "extract" the part of the formula $R(t,-) \subseteq O$ to find the minimal valuation of p (in this case $R(t,-)$ or, equivalently, $\blacklozenge \boldsymbol{i}$)and subsequently use the intersection lemma. Now, in Chapter 4, this "extraction" occurs when we separate the Skeleton part of the inequality (in this case $\top \leq \Box(\Diamond x \to y)$) from its PIA subtrees $\Box p$ and p.

4.8 Balbiani-Kikot formulas

It is now time to continue the discussion started in Section 2.10. We first need to establish the Balbiani-Kikot fragment of subordination formulas (in the sense of Definition 2.10.9).

Definition 4.8.1 ([1])**.** Let a be a subordination term. We say that a is **positive** if it is obtained from Boolean variables and the constant 1 by applying $\cup$ and $\cap$.

Definition 4.8.2 ([1])**.** Let φ be a subordination formula. We say that φ is

1. **negation-free** if it is obtained from the atomic formulas $a \mathcal{C} b$ (where $a \mathcal{C} b$ is a shortcut for $a \not\prec b'$), $a \neq 0$ and the constant $\top$ by applying $\vee$ and $\wedge$,
2. **positive** if it obtained from the atomic formulas $a \neq 0$, $\neg a = 0$, $a \mathcal{C} b$ and $\neg a \not\mathcal{C} \neg b$ (with a and b positive terms) and the constant $\top$ by applying $\wedge$ and $\vee$,
3. **Balbiani-Kikot** if it is of the form $\varphi_1 \to \varphi_2$ with φ_1 a negation-free formula and φ_2 a positive formula.

We already saw in Section 2.10 that there was no hope to obtain some kind of equivalence between Balbiani-Kikot formulas (in the subordination language) and s-Sahlqvist formulas (in the standard tense language). However, it is possible to translate subordination formulas into the standard tense language when this language is extended with an **universal** modality $\Diamond_u$. We recall that a modal operator $\Diamond_u$ is universal if $\Diamond_u a = 1$ for all $a \neq 0$. Equivalently, an universal modality corresponds to the relation $R = \nabla$. With this new modal operator, Vakarelov proposed in [74] the following translation τ:

$\tau(p) = p$	for all Boolean variables p
$\tau(0) = \bot$	
$\tau(a \cup b) = \tau(a) \vee \tau(b)$	for all Boolean terms a and b
$\tau(a') = \neg\tau(a)$	for all Boolean terms a
$\tau(a \mathcal{C} b) = \Diamond_u(\Diamond\tau(a) \wedge \tau(b))$	for all Boolean terms a and b
$\tau(a \leq b) = \Box_u(\tau(a) \to \tau(b))$	for all Boolean terms a and b
$\tau(\top) = \top$	
$\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi)$	for all subordination formulas ϕ and ψ
$\tau(\neg\phi) = \neg\tau(\phi)$	for all subordination formulas ϕ.

It means that, instead of working with the tense language, we will work with the language $\mathcal{L} = \{\wedge, \vee, \neg, \Diamond, \blacklozenge, \Diamond_u\}$.

Theorem 4.8.3 ([74])**.** *Let $\mathfrak{X}$ be a subordination space (or a Kripke structure) and φ be a subordination formula. We have*

$$\mathfrak{X} \models \varphi \text{ if and only if } \mathfrak{X} \models \tau(\varphi).$$

Hence, in particular, we also have the following immediate theorem.

Theorem 4.8.4. *Let φ and ψ be subordination formulas. If $\tau(\varphi) \leq \tau(\psi)$ is an analytic inductive inequality, then $\varphi \to \psi$ is canonical and there exists a first order formula Φ such that for every subordination space, we have*

$$\mathfrak{X} \models \varphi \to \psi \text{ if and only if } \mathfrak{X} \models \Phi.$$

Now, we want to determine how the translation $\tau(\varphi)$ of a Balbiani-Kikot formula φ slots into the (analytic) inductive formulas of the current chapter.

Theorem 4.8.5. *Let φ and ψ denote subordination formulas and a denote a Boolean term. Then*

1. *$+\tau(a)$ and $-\tau(a)$ are both Skeleton and PIA,*
2. *if φ is negation-free, then $+\tau(\varphi)$ is Skeleton,*
3. *if φ is positive, then all variable nodes of $-\tau(\varphi)$ are negative and all branches of $-\tau(\varphi)$ are good,*
4. *if $\varphi \to \psi$ is a Balbiani-Kikot formula then $\tau(\varphi) \leq \tau(\psi)$ is an analytic $\underline{1}$-Sahlqvist inequality.*

Proof. 1. It immediately follows from the fact that the nodes $-\neg, +\neg, +\vee, -\vee, +\wedge$ and $-\wedge$ are all Skeleton and PIA.

2. Since $+\wedge$ and $+\vee$ are Skeleton nodes, it is sufficient to prove that the positive trees $+\tau(a \neq 0)$ and $+\tau(a\, \mathcal{C}\, b)$ are Skeleton for all Boolean terms a and b.

   - The formula $a \neq 0$ is trivially equivalent to the formula $a \not\leq 0$. Hence, we can consider the positive generation tree of

     $$\tau(a \not\leq 0) = \neg\Box_u(\tau(a) \to \bot),$$

     which is given by

     $$\begin{tikzcd} & \overset{+}{\neg} \arrow[d] & \\ & \overset{-}{\Box_u} \arrow[d] & \\ & \overset{-}{\to} \arrow[dl] \arrow[dr] & \\ \overset{+}{\tau(a)} & & \overset{-}{\bot} \end{tikzcd}$$

     By Item 1, we have that every node in this tree is indeed Skeleton.

   - The positive generation tree of

     $$\tau(a\, \mathcal{C}\, b) = \Diamond_u(\Diamond\tau(a) \wedge \tau(b))$$

     is given by

     $$\begin{tikzcd} & \overset{+}{\Diamond_u} \arrow[d] & \\ & \overset{+}{\wedge} \arrow[dl] \arrow[dr] & \\ \overset{+}{\Diamond} \arrow[d] & & \overset{+}{\tau(b)} \\ \overset{+}{\tau(a)} & & \end{tikzcd}$$

     and the conclusion once again follows from Item 1.

3. Since $-\wedge$ and $-\vee$ are Skeleton nodes and their children nodes are negative, it is sufficient to prove that, in the negative generation trees of the formulas $\tau(a \neq 0)$, $\tau(-a = 0)$, $\tau(a \; \mathcal{C} \; b)$ and $\tau(-a \not\mathcal{C} -b)$, all variables nodes are negative and all branches are good when a and b are positive Boolean terms.

   We already have all the requisite trees from the previous item. Indeed, for the trees $-\tau(a \neq 0)$ and $-\tau(a' = 0)$, we know that we will obtain $-\tau(a)$ as child node of $+\rightarrow$ in the first case and as children node of $+\neg$ in the second case. Now, since a is a positive term, we have that every variable node of $+\tau(a)$ is negative, as required. Moreover, we have that $-\tau(a)$ is PIA from item 1. Hence, the conclusion follows from the fact that $-\neg$, $+\Box_u$ and $+\rightarrow$ are PIA nodes in the first case, and from the fact that $-\Box_u$, $-\rightarrow$ and $+\neg$ are Skeleton nodes in the second case. The proofs for the remaining cases ($a \; \mathcal{C} \; b$ and $-a \not\mathcal{C} -b$) are similar.

4. We have that every branch of $+\tau(\varphi)$ is good (since we only have Skeleton nodes) by item 2 and that every branch of $-\tau(\psi)$ is good by item 3. Moreover, since every variable node of $-\tau(\psi)$ is negative, we know that it contains no $\underline{1}$-critical occurrence of variable. Hence, every possible critical occurrence is contained in $+\tau(\varphi)$. Now, $+\tau(\varphi)$ only contains Skeleton nodes and, therefore, excellent branches.

$\square$

Remark 4.8.6. Recall that the Balbiani-Kikot fragment was extended in [66, Section 6] to what we could call Santoli formulas. Unfortunately, Theorem 4.8.5 does not extend to this greater fragment of formulas, this is due to the presence of non-separating formulas in the consequent of the implication. Consider for instance the subordination formula

$$\top \rightarrow (q \; \mathcal{C} \; p \vee q \; \mathcal{C} \; p') \wedge (q \; \mathcal{C} \; p \vee p \; \mathcal{C} \; q')$$

which is Santoli but such that its translation is not inductive analytic.

Remark 4.8.7. Looking carefully to the translation τ proposed by Vakarelov, we can notice that a valuation of $\tau(a \; \mathcal{C} \; b)$ and $\tau(a \leq b)$ can only be equal to 0 or 1. Therefore, there is another possible translation from subordination language to a modal language with two (binary) operators which maps elements of B to 0 or 1, and hence, not slanted (we choose to blend the language with its interpretation for the sake of readability). This is the option considered for instance in [4, Section 7]

4.9 An application: Canonicity via translation

Let us conclude this chapter with an application of the canonicity and correspondence theorem of Section 4.5. But, first, let us explain how to obtain a slanted algebra from a bounded distributive lattice L. By Priestley duality (see Appendix B.1), we know that the elements of L can be considered as the increasing clopen sets of a Priestley space $\mathfrak{X}_L = (X_L, \leq)$, which is, in particular, a Stone space. Therefore, we can endowed the Boolean algebra $\mathrm{Clop}(X_L)$ with the c-slanted operators

$$\lozenge_{\leq} : O \longmapsto {\downarrow} O \text{ and } \lozenge_{\geq} : O \longmapsto {\uparrow} O$$

to obtain a slanted algebra $\mathfrak{B} = (\mathrm{Clop}(X_L), \lozenge_{\leq}, \lozenge_{\geq})$ associated to the original lattice L.

Note that, in certain cases, the operators $\lozenge_{\leq}$ and $\lozenge_{\geq}$ are actually operators in the standard sense. For instance, if L is a Heyting algebra, that is if for every $a, b \in L$ there exists a greatest c such that $a \wedge c \leq b$, then, by Esakia duality [33], we know that the operator $\lozenge_{\leq}$ is standard.

Moreover, the operator $\Diamond_{\geq}$ is standard when L is a co-Heyting algebra and both are standard when L is bi-Heyting.

Moreover, thanks to [67], we know that every standard operator f (or g) of order-type $\epsilon \in 1, \partial^n$ on a bounded distributive lattice L correspond to an $(n+1)$-ary closed relation, with additional properties, on its dual. Therefore, since a closed relation is the only requirement to define slanted operators on a Stone space (there is no order), we also have a slanted operator f° (or g°) corresponding to the initial standard operator.

These constructions are considered in [21] where Gödel-McKinsey-Tarski type translations (GMT-type translations) are used to obtain Sahlqvist correspondence and canonicity as transfer results. The idea is that a GMT-type type translation τ_ϵ is defined for each variable p of a set of variables Var according to an order-type ϵ over Var. The dependence on the order-type is required to preserve the syntactic shape of (Ω, ϵ)-inductive inequalities when passing from an arbitrary distributive lattices expansions (DLE) language to its corresponding target Boolean algebras expansions (BAE) language, augmented with the modalities $\Diamond_{\leq}$ and $\Diamond_{\geq}$ we introduced earlier.

With these parametric translations, the correspondence via translations is obtained for inductive inequalities in an arbitrary DLE-language (see [21, Theorem 6.1]), however the canonicity via translations is only obtained for the particular case of bi-Heyting algebras (see [21, Theorem 7.1]). This result was not directly extended to the general case because the operators $\Diamond_{\leq}$ and $\Diamond_{\geq}$ are slanted and not standard and the results presented here and in [25] were not available. Before we cut to the chase, let us introduce the following notations for further uses.

Notation 4.9.1. Let $(X, \leq)$ be a Priestley space (resp. an ordered set), $\{R_i \mid i \in I\}$ a set of relations increasing in their first coordinate, $\{S_j \mid j \in J\}$ a set of relations decreasing in their first coordinate and $\{\epsilon_k \mid k \in I \cup J\}$ order-types. Then:

1. $\mathfrak{B}_X$ (resp. $\mathfrak{B}^\star_X$) denotes the Boolean algebra $\mathrm{Clop}(X)$ (resp. $\mathcal{P}(X)$) endowed with the operators:
   - $\triangle^\circ_i : (E_1, \ldots, E_{n_i}) \longmapsto R_i(-, E_1, \ldots, E_{n_i})$ for all $i \in I$,
   - $\triangledown^\circ_j : (E_1, \ldots, E_{n_j}) \longmapsto S_i(-, E^c_1, \ldots, E^c_{n_j})^c$ for all $j \in J$.
   - $\Diamond_{\leq} : O \longmapsto {\downarrow} O$ and $\Diamond_{\geq} : O \longmapsto {\uparrow} O$.
2. $\mathfrak{L}_X$ (resp. $\mathfrak{L}^\star_X$) denotes the distributive bounded lattice ${\uparrow}\mathrm{Clop}(X)$ (resp. ${\uparrow}\mathcal{P}(X)$) endowed with the operators:
   - $\triangle_i : (E_1, \ldots, E_{n_i}) \longmapsto R_i(-, E_1^{\epsilon_i(1)}, \ldots, E_{n_i}^{\epsilon_i(n_i)})$ for all $i \in I$,
   - $\triangle_i : (E_1, \ldots, E_{n_i}) \longmapsto S_j(-, E_1^{\epsilon_j(1)}, \ldots, E_{n_j}^{\epsilon_j(n_j)})^c$ for all $j \in J$.

Note that, as we mentioned earlier, $\mathfrak{B}_X$ is in general a slanted algebra, while $\mathfrak{L}_X$ is a standard DLE.

Definition 4.9.2. We freely use the notations and definitions of [21, Section 5.2.1]. Let $\mathcal{L}_{\mathsf{DLE}} = \mathcal{L}_{\mathsf{DLE}}(\mathcal{F}, \mathcal{G})$ be an arbitrary normal DLE-signature and let ϵ be an order-type over Var, the signature of the target language of the parametric GMT-type translations τ_ϵ is the normal BAE-signature $\mathcal{L}^\circ_{\mathrm{BAE}} = \mathcal{L}_{\mathrm{BAE}}(\mathcal{F}^\circ, \mathcal{G}^\circ)$ where $\mathcal{F}^\circ := \{\Diamond_{\geq}\} \cup \{f^\circ \mid f \in \mathcal{F}\}$, and $\mathcal{G}^\circ := \{\Box_{\leq}\} \cup \{g^\circ \mid g \in \mathcal{G}\}$, and for every $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$), the connective f° (resp. g°) is such that $n_{f^\circ} = n_f$ (resp. $n_{g^\circ} = n_g$) and $\epsilon_{f^\circ}(i) = 1$ for each $1 \leq i \leq n_f$ (resp. $\epsilon_{g^\circ}(i) = 1$ for each $1 \leq i \leq n_g$).

The target language for the parametrized GMT translations over Var is given by

$$\mathcal{L}^\circ_{\mathrm{BAE}} \ni \alpha ::= p \mid \bot \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \mid \neg\alpha \mid f^\circ(\overline{\alpha}) \mid g^\circ(\overline{\alpha}) \mid \Diamond_{\geq}\alpha \mid \Box_{\leq}\alpha.$$

Then, for any order-type ϵ on Var, the translation $\tau_\epsilon : \mathcal{L}_{\mathrm{DLE}} \to \mathcal{L}^{\circ}_{\mathrm{BAE}}$ is defined by the following recursion:

$$\tau_\epsilon(p) = \begin{cases} \Box_{\leq} p & \text{if } \epsilon(p) = 1 \\ \Diamond_{\geq} p & \text{if } \epsilon(p) = \partial, \end{cases} \qquad \begin{array}{rcl} \tau_\epsilon(\bot) & = & \bot \\ \tau_\epsilon(\top) & = & \top \\ \tau_\epsilon(\phi \wedge \psi) & = & \tau_\epsilon(\phi) \wedge \tau_\epsilon(\psi) \\ \tau_\epsilon(\phi \vee \psi) & = & \tau_\epsilon(\phi) \vee \tau_\epsilon(\psi) \\ \tau_\epsilon(f(\overline{\phi})) & = & \Diamond_{\geq} f^{\circ}(\overline{\tau_\epsilon(\phi)}^{\epsilon_f}) \\ \tau_\epsilon(g(\overline{\phi})) & = & \Box_{\leq} g^{\circ}(\overline{\tau_\epsilon(\phi)}^{\epsilon_g}) \end{array}$$

where for each order-type η on n and any n-tuple $\overline{\psi}$ of $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-formulas, $\overline{\psi}^{\eta}$ denotes the n-tuple $(\psi'_i)_{i=1}^{n}$, where $\psi'_i = \psi_i$ if $\eta(i) = 1$ and $\psi'_i = \neg\psi_i$ if $\eta(i) = \partial$.

We now have the following theorems from [21, Corollary 5.13 and Proposition 7.4].

Theorem 4.9.3. *Let $(X, \leq)$ be a Priestley space (resp. an ordered set) and $\mathfrak{L}_X$ and $\mathfrak{B}_X$ (resp. $\mathfrak{L}^{\star}_X$ and $\mathfrak{B}^{\star}_X$) be the algebras introduced in Notation 4.9.1. Then, for any order-type ϵ on* Var *and for any inequality $\varphi \leq \psi$ in $\mathcal{L}_{\mathsf{DLE}}$, we have*

$$\mathfrak{L}_X \models \varphi \leq \psi \iff \mathfrak{B}_X \models \tau_\epsilon(\varphi) \leq \tau_\epsilon(\psi)$$

and

$$\mathfrak{L}^{\star}_X \models \varphi \leq \psi \iff \mathfrak{B}^{\star}_X \models \tau_\epsilon(\varphi) \leq \tau_\epsilon(\psi).$$

As an immediate corollary of the previous theorem, we have that when the inequality $\tau_\epsilon(\varphi) \leq \tau_\epsilon(\psi)$ is canonical (in the sense of Theorem 4.5.5), then the original inequality $\varphi \leq \psi$ is also canonical (in the standard sense). In particular, we have therefore that $\varphi \leq \psi$ is canonical whenever $\tau_\epsilon(\varphi) \leq \tau_\epsilon(\psi)$ is analytic inductive. One may note that the translation τ_ϵ does not preserve the analytic properties since it is intended to preserve only the shape of critical branches: a $\Box_{\leq}$ connective is added in front of every occurrences of a variable p such that $\epsilon(p) = 1$ and a $\Diamond_{\geq}$ connective is added in front of every occurrences of a variable q such that $\epsilon(q) = \partial$. Therefore, in the generations tree of $\tau_\epsilon(\varphi) \leq \tau_\epsilon(\psi)$, we have PIA nodes right before the leaf nodes of ϵ-critical branches. Unfortunately, it also means that we have Skeleton nodes right before the leaf nodes of non-critical branches. Let us consider the following examples.

Example 4.9.4.

1. The inequality $\Diamond\Box p \leq \Box\Diamond p$ is analytic ϵ-inductive for $\epsilon(p) = 1$ and its τ_ϵ-translation, which is given by

   $$\Diamond_{\geq}\Diamond^{\circ}\Box_{\leq}\Box^{\circ}\Box_{\leq} p \leq \Box_{\leq}\Box^{\circ}\Diamond_{\geq}\Diamond^{\circ}\Box_{\geq} p,$$

   is not analytic since the branch associated with $\Box_{\leq}\Box^{\circ}\Diamond_{\geq}\Diamond^{\circ}\Box_{\geq} p$ is not good.

2. The inequality $\Diamond p \vee q \leq \Box p \wedge \Diamond q$ is analytic ϵ-inductive for $\epsilon(p, q) = (1, \partial)$ and its τ_ϵ-translation, which is given by

   $$\Diamond_{\geq}\Diamond^{\circ}\Box_{\leq} p \vee \Diamond_{\leq} q \leq \Box_{\leq}\Box^{\circ}\Box_{\leq} p \wedge \Diamond_{\geq}\Diamond^{\circ}\Diamond_{\geq} q,$$

   is analytic.

Hence, we can observe that an τ_ϵ-translation of an analytic inductive inequality remains analytic if and only if the PIA parts of the non-critical branches are either reduced to a variable node or variable frees. This observation leads to the following definition.

Definition 4.9.5. Let ϵ be an order type on Var, we say that an (Ω, ϵ)-analytic inductive inequality $(\varphi \leq \psi)[\overline{\alpha}/!\overline{x}, \overline{\beta}/!\overline{y}, \overline{\gamma}/!\overline{z}, \overline{\delta}/!\overline{t}]$ (as in Notation 4.2.10) is τ_ϵ-**transferable** if for every maximal positive (resp. negative) ϵ^∂-uniform PIA-subformula γ in $\overline{\gamma}$ (resp. δ in $\overline{\delta}$), either $\gamma = q$ (resp. $\delta = p$) for some $q \in \mathrm{Var}$ (resp. $p \in \mathrm{Var}$) such that $\epsilon(q) = \partial$ (resp. $\epsilon(p) = 1$), or γ (resp. δ) does not contain atomic propositions at all.

From the definition above, we immediately have the following proposition.

Proposition 4.9.6. *If $\varphi \leq \psi$ is a τ_ϵ-transferable (Ω, ϵ)-analytic inductive $\mathcal{L}_{\mathrm{DLE}}$-inequality ,the $\mathcal{L}_{\mathrm{BAE}}$-inequality $\tau_\epsilon(\phi) \leq \tau_\epsilon(\psi)$ is analytic inductive, and hence canonical (in the sense of Theorem 4.5.5).*

Hence, finally, we can extend [21, Theorem 7.2] as follows:

Theorem 4.9.7 (Canonicity via translation)**.** *For any order type ϵ and any strict order Ω on* Var*, the slanted canonicity theorem for analytic (Ω, ϵ)-inductive $\mathcal{L}^{\circ}_{\mathrm{BAE}}$-inequalities transfers to τ_ϵ-transferable analytic (Ω, ϵ)-inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities.*

Chapter 5

Gelfand duality for compact po-spaces

In this chapter, our main goal is to mimic what happened when we went from Stone duality to Priestley duality, more precisely when we went to Stone spaces to Priestley spaces, that is "ordered Stone spaces".

In [7], Bezhanishvili, Morandi and Olberding established a duality between the categories **ubal** and **KHaus** (see Appendix B.4 for details). This duality may be seen as a particular side of a square of dualities and equivalences.

$$
\begin{tikzcd}
\mathbf{KHaus} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbf{KRFrm} \arrow[d, dash] \\
\mathbf{ubal} & \mathbf{DeV}
\end{tikzcd}
$$

In this square, we already explored the duality **KHaus** - **DeV** in Chapter 1 and gave a sketch of the duality **KHaus** - **ubal** in Section B.4. For the duality between **KHaus** and the category of compact regular frames and frames homomorphisms **KRFrm**, we redirect the reader to [46, III.1.8 and III.1.10] for definitions and a proof of the duality.

Bezhanishvili and Harding extended three of these dualities and equivalences in their article [6].

$$
\begin{tikzcd}
\mathbf{StKSp} \arrow[rrr, dash] & & & \mathbf{StKFrm} \arrow[ddd, dash] \\
& \mathbf{KHaus} \arrow[ul] \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbf{KRFrm} \arrow[ur] \arrow[d, dash] & \\
& \mathbf{ubal} & \mathbf{DeV} \arrow[dr] & \\
& & & \mathbf{PrFrm}
\end{tikzcd}
$$

It appears that the fourth corner of the external square is missing. In order to complete it, we will make use of a category equivalent to **StKSp** (the category of stably compact spaces with proper continuous functions): the category **KPSp** of compact po-spaces and continuous increasing functions, which explains the parallel with the transition from Stone to Priestley we made earlier.

5.1 Stably compact and compact po-spaces

In this section, we will briefly describe the equivalence between **KPSp** and **StKSp**. The results and definitions are mostly from to [38, Section VI-6] and [53].

Definition 5.1.1. A **compact po-space** (for partially ordered space - also sometimes called a **Nachbin space**) is a pair $(X, \leq)$ where X is a compact space and $\leq$ is an order on X which is closed in X^2.

Remark 5.1.2. Recall that a topological space X, whose topology is denoted by τ, endowed with a closed order is necessarily Hausdorff and that $\tau^{\uparrow}$, the set of increasing open sets, is a topology on X. (see Proposition B.1.7 in the Appendix)

Definition 5.1.3. We denote by **KPSp** the category whose objects are compact po-spaces and whose morphisms are continuous increasing functions.

In order to characterise the category of stably compact spaces, some preliminary definitions will be required.

Definition 5.1.4. Let X be a topological space and $S \subseteq X$.

1. The **saturation** of S, denoted by S^s, is the intersection of the open sets of X containing S. Moreover, we say that S is **saturated** if it is equal to its saturation.
2. We say that S is **irreducible** if $S \subseteq F_1 \cup F_2$ for F_1 and F_2 closed sets of X implies that $S \subseteq F_i$ for at least one $i \in \{1, 2\}$.

We are now ready to give the definition of stably compact spaces.

Definition 5.1.5.
1. Let X be a topological space. We say that X is **sober** if every irreducible closed subset of X is the closure of a singleton. We say that X is **coherent** if its set of compact saturated sets is closed under finite intersection. We say that X is **locally compact** if every $x \in X$ admits a base of compact neighbourhoods. Finally, we say that X is **stably compact** if it is compact, locally compact, coherent and sober.
2. A map $f : X \longrightarrow Y$ between topological spaces is said to be **proper** if for every saturated compact subset K of Y, the set $f^{-1}(K)$ is compact.
3. We denote by **StKSp** the category whose objects are stably compact spaces and whose morphisms are proper continuous functions.

Theorem 5.1.6. *Let $(X, \leq)$ and $(Y, \leq)$ be compact po-spaces and let $f : X \longrightarrow Y$ be an increasing continuous function. Then, $\mathrm{Up}(X) = (X, \tau^{\uparrow})$ is a stably compact space and*

$$\mathrm{Up}(f) : \mathrm{Up}(X) \longrightarrow \mathrm{Up}(Y) : x \longmapsto f(x)$$

is a proper continuous function.

Proof. The proof strongly relies on two remarks. The first one is that an increasing subset K of $(X, \leq)$ is compact in the original topology τ if and only it is compact in the topology $\tau^{\uparrow}$. And the second is that a locally compact Hausdorff space is sober if and only if it is well-filtered. □

Now, we have to find a way to obtain a compact po-space from a stably compact space. According to Theorem 5.1.6, we should keep the same underlying set and define a new topology, as well as an order, on it to obtain an equivalence.

Definition 5.1.7. 1. Let (X, τ) be a stably compact space. The **canonical order** on X is defined as follows
$$x \leq_\tau y \Leftrightarrow x \in \overline{\{y\}}.$$

2. Let (X, τ) be a topological space. The **co-compact topology**, denoted by τ^k, on X is the topology whose closed sets are generated by the compact saturated sets of τ. The **patch topology**, denoted by π, is the smallest topology that contains τ and τ^k.

Theorem 5.1.8. *Let (X, τ) and (Y, τ) be stably compact spaces and let $f : X \longrightarrow Y$ be a proper continuous function. Then* $\mathrm{Ord}(X) = (X, \pi, \leq_\tau)$ *is a compact po-space and*
$$\mathrm{Ord}(f) : \mathrm{Ord}(X) \longrightarrow \mathrm{Ord}(Y) : x \longmapsto f(x)$$
is an increasing continuous function.

Proof. The compactness of (X, π) is again proved thanks to the fact that (X, τ) is well-filtered. The proof that the order is closed exploits the fact that $\downarrow_\tau x$ is a closed set of τ. □

It remains to prove that the functors Ord and Up establish an equivalence. This can be achieved by showing that $\mathrm{Ord} \circ \mathrm{Up}$ and $\mathrm{Up} \circ \mathrm{Ord}$ are the identity functor of their respective category. This proof heavily relies on the following proposition.

Proposition 5.1.9. *Let (X, τ) be a topological space. The following are equivalent:*

1. *(X, τ) is stably compact,*
2. *$(X, \pi, \leq_\tau)$ is a compact po-space such that $\pi^\uparrow = \tau$ and $\pi^\downarrow = \tau^k$,*
3. *there exists a topology σ on X such that $(X, \sigma, \leq_\tau)$ is a compact po-space and $\sigma^\uparrow = \tau$.*
4. *(X, τ) is locally compact and (X, π) is compact.*

We can now change the squares of dualities and equivalences with the following one.

$$\begin{tikzcd}
\mathbf{KPSp} \arrow[rrr, dash] & & & \mathbf{StKFrm} \arrow[ddd, dash] \\
& \mathbf{KHaus} \arrow[ul] \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbf{KRFrm} \arrow[ur] \arrow[d, dash] & \\
& \mathbf{ubal} & \mathbf{DeV} \arrow[dr] & \\
& & & \mathbf{PrFrm}
\end{tikzcd} \tag{5.1}$$

5.2 Semibals

Unlike the non-ordered case of compact Hausdorff space (see Appendix B.4), the ring of continuous functions $C(X, \mathbb{R})$ cannot be used to characterise a compact po-space X. A quick way to convince to see this is to consider a compact po-space $(X, \tau, \leq)$ with an order that is not the equality (examples of such spaces are given by Priestley duals of bounded distributive lattices). Then, as we already stated in Remark 5.1.2, (X, τ) is a compact Hausdorff space, and hence

$(X, \tau, =)$ is a compact po-space. Now, $(X, \tau, \leq)$ and $(X, \tau, =)$ obviously share the same ring of continuous functions and are anything but isomorphic in **KPSp**.

As we will prove later on, the role of $C(X, \mathbb{R})$ will be fulfilled by the set of positive increasing continuous functions, which will be denoted by $I(X, \mathbb{R}^+)$. But, as this set is clearly not closed under opposites, it is not a ring. This is why we turn to a category of semirings (see for instance [40]) to describe it.

Definition 5.2.1. 1. A **semiring** is an algebra $A = (A, +, \cdot, 0, 1)$ where the algebras $(A, +, 0)$ and $(A, \cdot, 1)$ are commutative monoids, the distributive law holds and multiplications by 0 annihilates A.

2. A **po-semiring** (for partially ordered semiring) is an ordered structure $A = (A, \leq)$ where A is a semiring and the following axioms hold:

   (a) $a \leq b$ if and only if $a + c \leq b + c$,

   (b) $a \leq b$ and $c \leq d$ implies $ad + bc \leq bd + ac$.

A trivial observation about a po-semiring A is that it is additively cancellable, i.e. for every $a, b, c \in A$, $a + c = b + c$ implies $a = b$. This follows immediately from the antisymmetry of the order $\leq$ and from point 2.(a) of Definition 5.2.1. Moreover, a direct consequence of point 2.(b) is that $a \leq b$ and $0 \leq c$ implies $ac \leq bc$.

Finally, if a, b, c and d are elements of A such that $a \leq b$ and $c \leq d$, then $a+c \leq b+d$. Indeed, by 2.(a), we have $a + c \leq b + c \leq b + d$. The conclusion then follows from the transitivity of $\leq$.

Definition 5.2.2. Let A be a po-semiring. We say that A is:

1. **positive** if every element of A is greater than 0,

2. **bounded** if for every element $a \in A$, there exists $n \in \mathbb{N}$ such that $a \leq n \cdot 1$,

3. **Archimedean** if for each $a, b, c, d \in A$, $n \cdot a + c \leq n \cdot b + d$ for every $n \in \mathbb{N}$ implies that $a \leq b$.

Recall that a po-ring B is said to be Archimedean if for each $a, b \in B$, $n \cdot a \leq b$ for each $n \in \mathbb{N}$ implies $a \leq 0$. Therefore, it may seem at first glance that the two definitions of being Archimedean differ. We will see later that this new definition of Archimedean is the one actually required. Anyway we can understand now why it is not fundamentally different from the "original definition". Indeed, suppose that a semiring admits opposites. In this case, we have

$$n \cdot a + c \leq n \cdot b + d \Leftrightarrow n \cdot (a - b) \leq d - c$$

and

$$a \leq b \Leftrightarrow a - b \leq 0.$$

Hence, we get back the usual definition of Archimedean.

Definition 5.2.3. A po-semiring A is:

1. an **ℓ-semiring** if $(A, \leq)$ is a lattice such that

   (a) $(a \vee b) + c = (a + c) \vee (b + c)$,

   (b) $(a \wedge b) + c = (a + c) \wedge (b + c)$.

2. a **(real) semialgebra** if it is endowed with a map

$$A \times \mathbb{R}^+ \longrightarrow A : (a, r) \longmapsto r \cdot a$$

such that:

(a) $a \leq b$ implies $r \cdot a \leq r \cdot b$,

(b) $r \cdot (a + b) = r \cdot a + r \cdot b$ and $(r + s) \cdot a = r \cdot a + s \cdot a$,

(c) $r \cdot (a \cdot b) = (r \cdot a)b$ and $r \cdot (s \cdot a) = (rs) \cdot a$,

(d) $1 \cdot a = a$.

3. a **semibal** if it is a semialgebra and a positive bounded Archimedan ℓ-semiring.

Proposition 5.2.4. *Let A be a semibal, $a, b \in A$ and $r \in \mathbb{R}^+$. Almost immediately from the definitions, we can derive the following equalities:*

$$0a = 0, \; r(a \vee b) = ra \vee rb, \; r(a \wedge b) = ra \wedge rb.$$

Proof. The first equality follows from the fact that A is cancellable and the equality

$$0a + 0 = 0a = (0 + 0)a = 0a + 0a.$$

The second equality is trivially true when $r = 0$. In the other cases, we have $ra, rb \leq r(a \vee b)$, whence $ra \vee rb \leq r(a \vee b)$. Moreover, if $ra, rb \leq c$, then $a \vee b \leq \frac{1}{r}c$, or in other terms, $r(a \vee b) \leq c$, which concludes the proof.

The third equality is proved in a similar way. □

Example 5.2.5. 1. As stated in the introduction, for X a compact po-space, we wanted a category to describe the set $I(X, \mathbb{R}^+)$ of all increasing continuous function from X to $\mathbb{R}^+$. It is then not a surprise that the main examples of semibals are given by these sets.

2. Consider B to be a bal (see Appendix B.4.1). Its positive cone

$$B^+ := \{b \in B \mid b \geq 0\}$$

with the operations inherited from B is a semibal. While this may seem trivial, it is important to note that it is a valid example because B satisfies $0 \leq 1$.

3. The set $\mathbb{R}^+$ with the usual operations is a semibal.

Definition 5.2.6. We denote by **sbal** the category whose objects are semibals and whose morphisms, named **semibal morphisms**, are defined in the natural way, that is they respect the operations of semibals.

Now that we have a category candidated to fill the missing corner of Diagram (5.1), we have to check if it respects the following conditions:

1. it admits **ubal** as a subcategory,

2. it is dually equivalent to **KPSp**,

3. the duality between **ubal** and **KHaus** is extended by the duality between **sbal** and **KPSp**.

5.3 The $\cdot^b$ functor

Just as **KHaus** is a (full) subcategory of **KPSp**, we have to determine if **bal** is a subcategory of **sbal**. While, in the topological case, the statement is quite obvious (just consider the equality as an order), it will require further work in the algebraic case.

In this section, we will describe a functor from **sbal** to the category **bal** and its adjoint functor from **bal** to **sbal**. Note that the former functor will enable us to transfer results obtained in **bal** to **sbal**. These functors will be equivalent respectively to the forgetful functor from **KPSp** to **KHaus** and to its adjoint, the inclusion functor.

Definition 5.3.1. Let A be an ℓ-semiring. We define the binary relation $\sim$ on A^2 as follows

$$(a,b) \sim (c,d) \Leftrightarrow a+d = b+c. \tag{5.2}$$

Proposition 5.3.2. *The relation defined in* (5.2) *is an equivalence relation.*

Proof. It is easy to prove that $\sim$ is symmetric and reflexive. Now, suppose that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. That is $a+d = b+c$ and $c+f = d+e$. Adding the two together, we obtain

$$a+f+(d+c) = e+b+(d+c).$$

What remains to do is use cancellability to get $a+f = e+b$, that is $(a,b) \sim (e,f)$. $\square$

Note that the relation $\sim$ is a, without any pun intended, natural way to construct opposites in an environment that lacks this property. Indeed, it is analogue, if not identical, to the relation used to construct $\mathbb{Z}$ from $\mathbb{N}$. Hence, the equivalence class $(a,b)^\sim$ may be considered as the difference $a-b$. Consequently, the set $A^2/_\sim$ may be considered as $A-A$, that is the set of differences of elements in A. It follows that the product $(a,b)^\sim \cdot (c,d)^\sim$, still to be defined, can be seen as

$$(a-b)\cdot(c-d) = (ac+bd)-(ad+bc) = (ac+bd, ad+bc)^\sim.$$

We will now prove that the natural definition of the operations on $A^2/_\sim$ does not depend on the chosen representative.

Lemma 5.3.3. *If A is an ℓ-semiring and if*

$$(a_1,b_1) \sim (a_2,b_2) \text{ and } (c_1,d_1) \sim (c_2,d_2),$$

then:

1. $(a_1+c_1, b_1+d_1) \sim (a_2+c_2, b_2+d_2)$,
2. $(a_1c_1+b_1d_1, a_1d_1+b_1c_1) \sim (a_2c_2+b_2d_2, a_2d_2+b_2c_2)$,
3. $a_1+d_1 \leq b_1+c_1$ *if and only* $a_2+d_2 \leq b_2+c_2$.

Proof. 1. Since we have $a_1+b_2 = a_2+b_1$ and $c_1+d_2 = c_2+d_1$, it is clear that

$$a_1+c_1+b_2+d_2 = b_1+d_1+a_2+c_2.$$

2. Still from $a_1+b_2 = a_2+b_1$ and $c_1+d_2 = c_2+d_1$, we obtain

- $c_1a_1+c_1b_2 = c_1b_1+c_1a_2$,
- $d_1a_2+d_1b_1 = d_1a_1+d_1b_2$,

- $a_2c_1 + a_2d_2 = a_2c_2 + a_2d_1$,
- $b_2c_2 + b_2d_1 = b_2c_1 + b_2d_2$.

It suffices now to add the four equalities together and then use the fact that A is cancellable to get the conclusion.

3. We have, by hypothesis and cancellability,

$$\begin{aligned} & a_1 + d_1 \leq b_1 + c_1 \\ \iff & \underbrace{a_1 + b_2}_{=b_1+a_2} + d_1 + d_2 \leq b_1 + \underbrace{c_1 + d_2}_{=c_2+d_1} + b_2 \\ \iff & a_2 + d_2 \leq b_2 + c_2. \end{aligned}$$

□

Using Lemma 5.3.3, we are now going to prove that $A^2/\sim$ is as bal. Of course the properties bounded, Archimedean and $\mathbb{R}$-algebra of an sbal A are intended to implies their respective correspondents in $A^2/\sim$, as we now see.

Lemma 5.3.4. *Let A be an ℓ-semiring and let A^b be defined as $A^2/_\sim$. The set A^b equipped with the operations $+$, $\cdot$ and the order $\leq$ defined as :*

1. $(a,b)^\sim + (c,d)^\sim = (a+c, b+d)^\sim$,
2. $(a,b)^\sim \cdot (c,d)^\sim = (ac+bd, ad+bc)^\sim$,
3. $(a,b)^\sim \leq (c,d)^\sim \Leftrightarrow a+d \leq b+c$,

is an ℓ-ring.

Proof. Thanks to Lemma 5.3.3, we know that the operations and the order are well-defined.

It is not hard to prove that A^b is a ring whose identity elements are $(0,0)^\sim$ for addition and $(1,0)^\sim$ for multiplication. What remains to prove is that A^b is a lattice and that

(c) $(a_1,b_1)^\sim \leq (a_2,b_2)^\sim$ implies $(a_1,b_1)^\sim + (c,d)^\sim \leq (a_2,b_2) + (c,d)^\sim$,

(d) $0 \leq (a,b)^\sim, (c,d)^\sim$ implies $0 \leq (a,b)^\sim \cdot (c,d)^\sim$.

Let us show that $(a,b)^\sim \vee (c,d)^\sim$ exists and is equal to $((a+d)\vee(b+c), b+d)^\sim$. Indeed, we have first $(a,b)^\sim \leq ((a+d)\vee(b+c), b+d)^\sim$ since

$$a+b+d \leq ((a+d)\vee(b+c)) + b.$$

Similarly, we have $(c,d)^\sim \leq ((a+d)\vee(b+c), b+d)^\sim$. Moreover, let $(e,f)^\sim$ be an equivalence class such that $(a,b)^\sim, (c,d)^\sim \leq (e,f)^\sim$. It follows that $c+f \leq d+e$ and $a+f \leq b+e$, hence $a+d+f \leq b+d+e$ and $b+c+f \leq b+d+e$. Now, it follows that

$$((a+d)\vee(b+c)) + f = (a+d+f)\vee(b+c+f) \leq b+d+e,$$

hence $((a+d)\vee(b+c), b+d)^\sim \leq (e,f)^\sim$.

In an analogue way, one can prove that $(a,b)^\sim \wedge (c,d)^\sim$ exists and is equal to $((a+d)\wedge(b+c), b+d)^\sim$.

One just needs to use the definitions to prove (c) and (d).

□

Lemma 5.3.5. *If A is a positive bounded ℓ-semiring, then A^b is a bounded ℓ-ring.*

Proof. Let $(a,b)^\sim$ be an element of A^b. We have to find a natural n such that $(a,b)^\sim \leq n\cdot(1,0)^\sim$, or in other terms, such that $a \leq n \cdot 1 + b$. But, since A is bounded, there exists $n' \in \mathbb{N}$ such that $a \leq n' \cdot 1$ and, since A is positive, $0 \leq b$. Hence, we get

$$a = a + 0 \leq n' \cdot 1 + b,$$

as required.

□

Lemma 5.3.6. *If A is an Archimedean ℓ-semiring, then A^b is an Archimedean ℓ-ring.*

Proof. Suppose that $(a,b)^\sim, (c,d)^\sim \in A^b$ are such that $n \cdot (a,b)^\sim \leq (c,d)^\sim$. That is, in other words, such that $n \cdot a + d \leq c + n \cdot b$, for all $n \in \mathbb{N}$. Since A is Archimedean, it implies that $a \leq b$ and, hence, that $(a,b)^\sim \leq (0,0)^\sim$ as required. □

Finally, we just have to focus on the $\mathbb{R}$-algebra structure of bals and semibals.

Lemma 5.3.7. *If A is a semialgebra, $(a,b) \sim (c,d)$ and $r, s \in \mathbb{R}^+$, then*

$$(ra + sb, rb + sa) \sim (rc + sd, rd + sc). \tag{5.3}$$

Proof. Since (5.3) is equivalent to

$$r(a+d) + s(b+c) = r(b+c) + s(a+d)$$

and since, by hypothesis, $a + d = b + c$, the conclusion is immediate. □

We now arrive at the theorem that conclude the construction of A^b.

Theorem 5.3.8. *If A is a semibal, then A^b equipped with the scalar multiplication defined by*

$$(r - s) \cdot (a,b)^\sim = (ra + sb, rb + sa)^\sim \ \textit{(with } r, s \in \mathbb{R}^+\textit{)}$$

is a bal.

Proof. Thanks to Lemma 5.3.7, we know that the scalar multiplication is well-defined. Moreover, since we already proved Lemmas 5.3.4, 5.3.5 and 5.3.6, we only have to to show that A^b is an $\mathbb{R}$-algebra such that $(0,0)^\sim \leq (a,b)^\sim$ and $r \in \mathbb{R}^+$ implies $0 \leq r(a,b)^\sim$.

It is straightforward to prove that A^b is an $\mathbb{R}$-algebra. We will just prove, as an example, that

$$((r-s)(a,b)^\sim)(c,d)^\sim = (r-s)((a,b)^\sim(c,d)^\sim).$$

We have indeed

$$\begin{aligned}
((r-s)(a,b)^\sim)(c,d)^\sim &= (ra + sb, sa + rb)^\sim(c,d)^\sim \\
&= (rac + sbc + sad + rbd, rad + sbd + sac + rbc)^\sim \\
&= (r(ac+bd) + s(bc+ad), r(ad+bc) + s(bd+ac))^\sim \\
&= (r-s)(ac+bd, bc+ad)^\sim \\
&= (r-s)((a,b)^\sim(c,d)^\sim).
\end{aligned}$$

We just need the definition of semialgebra to prove the last part. □

To conclude the construction of the passage from **sbal** to **bal**, we should now determine how to lift a morphism between sbals to a morphisms between their associated bals.

Proposition 5.3.9. *Let α be a morphism of* **sbal**(A, C)*. The application $\alpha^b : A^b \longrightarrow C^b$ defined by*

$$\alpha^b((a,b)^\sim) = (\alpha(a), \alpha(b))^\sim$$

is a morphism of **bal**(A^b, C^b)*.*

Proof. First, we have to prove that α^b is well-defined. But this is clear since, on one hand, $(a,b) \sim (c,d)$ means that $a + d = b + c$ and, on the other hand, α respects the addition. Hence, $\alpha(a) + \alpha(d) = \alpha(b) + \alpha(c)$ as required.

In addition, we have to prove that the map α^b respects every operation $\circ$ in $\{+, \cdot, \wedge, \vee, (r\cdot)_{(r\in\mathbb{R})}\}$. Let us show it for $\vee$, keeping in mind that the proofs of the remaining cases follow similar paths. We have

$$\begin{aligned}
\alpha^b((a,b)^\sim \vee (c,d)^\sim) &= \alpha^b(((a+d) \vee (b+c), b+d)^\sim) \\
&= (\alpha((a+d) \vee (b+c)), \alpha(b+d))^\sim \\
&= ((\alpha(a) + \alpha(d)) \vee (\alpha(b) + \alpha(c)), \alpha(b) + \alpha(d))^\sim \\
&= (\alpha(a), \alpha(b))^\sim \vee (\alpha(c), \alpha(d))^\sim \\
&= \alpha^b((a,b)^\sim) \vee \alpha^b((c,d)^\sim).
\end{aligned}$$

□

Definition 5.3.10. We denote by $\cdot^b$ the functor from **sbal** to **bal** which maps a semibal A to the bal A^b and a semibal morphism α to the bal morphism α^b.

Remark 5.3.11. In order to continue our presentation of the functor $\cdot^b$ and its soon to be defined adjoint, we will digress briefly giving a summary of some properties of ℓ-rings. An interested reader can find the wanted proofs in [8, Chapters XIII to XVII].

Property	Required object
$a \leq b \Rightarrow -b \leq -a$ $a \leq c, b \leq d \Rightarrow a+b \leq c+d$ $a \geq 0$ et $b \leq c \Rightarrow ab \leq ac$	partially ordered ring
$-(a \vee b) = (-a) \wedge (-b)$ $-(a \wedge b) = (-a) \vee (-b)$ $a + (b \wedge c) = (a+b) \wedge (a+c)$ $n.b \geq 0 \Rightarrow b \geq 0$ $a + (b \vee c) = (a+b) \vee (a+c)$ $a + b = (a \vee b) + (a \wedge b)$ $a, b \geq 0 \Rightarrow a + b \geq a \vee b$ $\lvert a+b \rvert \leq \lvert a \rvert + \lvert b \rvert$ $\lvert a \rvert = 0 \Rightarrow a = 0$ $a \geq 0 \Rightarrow ab \vee ac \leq a(b \vee c)$ $a \geq 0 \Rightarrow ab \wedge ac \geq a(b \wedge c)$ $a \wedge b = 0$ et $a \wedge c = 0 \Rightarrow a \wedge (b+c) = 0$ $a \vee b = 0$ et $a \vee c = 0 \Rightarrow a \vee (b+c) = 0$	ℓ-ring
distributive lattice	ℓ-ring
$a \wedge b = 0 \Rightarrow ab = 0$ $\lvert a \rvert \wedge 1 = 0 \Rightarrow a = 0$ $a \geq 0 \Rightarrow a(b \vee c) = ab \vee ac$ $a \geq 0 \Rightarrow a(b \wedge c) = ab \wedge ac$ $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$ $a^2 \geq 0$ $1 \geq 0$	f-ring
(totally) ordered	integer f-ring
f-ring	bounded ℓ-ring

For the sake of completeness, we define the concept of f**-ring**: it is an ℓ-ring satisfying the following axiom:

$$a \wedge b = 0 \text{ and } c \geq 0 \Rightarrow ca \wedge b = ac \wedge b = 0.$$

Definition 5.3.12. For an element b of an ℓ-ring B, define

$$b^+ := b \vee 0 \text{ and } b^- := (-b) \vee 0.$$

Using properties of ℓ-rings, it is not hard to prove that

$$b^+ - b^- = (b \vee 0) + (b \wedge 0) = b.$$

In particular, this will imply that an object of the category **bal** is completely determined by its positive cone, as we will see latter.

Now that we have the functor from **sbal** to **bal**, we can focus on the other direction. As Example 5.2.5 suggests, this functor should map a bal B to its positive cone B^+. The following proposition completely determines the behaviour of the functor from **bal** to **sbal**.

Proposition 5.3.13. *If α is a morphism of* $\mathbf{bal}(B, D)$*, then its restriction to B^+, denoted by α^+, is a morphism of* $\mathbf{sbal}(B^+, D^+)$*.*

Proof. First of all, if $b \in B^+$, then $\alpha(b) \geq \alpha(0) = 0$, so that we have $\alpha(b) \in D^+$ as required. In addition, α^+ obviously respects all the operations of semibals. □

Proposition 5.3.14. *For each bal B, the map*

$$\eta_B : B \longrightarrow (B^+)^b : b \longmapsto (b^+, b^-)^\sim$$

is an isomorphism in the category **bal** *such that, for every $\alpha \in$* **bal**(B, D)*, the following diagram is commutative.*

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "\alpha"'] & (B^+)^b \arrow[d, "(\alpha^+)^b"] \\ D \arrow[r, "\eta_D"'] & (D^+)^b \end{tikzcd}$$

Proof. 1. The map η_B is injective. Consider two elements $a, b \in B$ such that $(a^+, a^-)^\sim = (b^+, b^-)^\sim$. By definition, it means that $a^+ + b^- = b^+ + a^-$, or equivalently that

$$a = a^+ - a^- = b^+ - b^- = b,$$

as required.

2. The map η_B is onto. Consider $(a, c)^\sim \in (B^+)^b$ and let $b = a - c$. Then, we have

$$b = b^+ - b^- = a - c \Leftrightarrow b^+ + c = b^- + a,$$

that is $(b^+, b^-) \sim (a, c)$ as required.

3. The map η_B is a morphism in **bal**.

   (a) Using the properties of Remark 5.3.11, we have immediately that

   $$1^+ = 1 \vee 0 = 1 \text{ and } 1^- = (-1) \vee 0 = 0,$$

   such that $\eta_B(1) = (1, 0)^\sim$.

   (b) It follows from

   $$a^+ + b^+ - a^- - b^- = (a^+ - a^-) + (b^+ - b^-) = a + b = (a + b)^+ - (a + b)^-$$

   that

   $$(a^+ + b^+, a^- + b^-) \sim ((a + b)^+, (a + b)^-),$$

   that is $\eta_B(a) + \eta_B(b) = \eta_B(a + b)$.

   (c) To prove that $\eta_B(a) \cdot \eta_B(b) = \eta_B(a \cdot b)$, it suffices to check if $(ab)^+ = a^+b^+ + a^-b^-$ and $(ab)^- = a^+b^- + a^-b^+$. We prove the first equality:

   $$\begin{aligned} (ab)^+ &= (ab) \vee 0 \\ &= [(a^+ - a^-)(b^+ - b^-)] \vee 0 \\ &= [(a^+b^+ + a^-b^-) + (-a^-b^+ - a^+b^-)] \vee 0 \\ &= [(a^+b^+ + a^-b^-) \vee 0] + [(-a^-b^+ - a^+b^-) \vee 0]. \end{aligned}$$

   The conclusion now follows from the fact that $(a^+b^+ + a^-b^-) \geq 0$ and $(-a^-b^+ - a^+b^-) \leq 0$.

(d) We know that η_B is a bijection. Moreover, for every $a, b \in B$, we have that $a \leq b$ if and only if $\eta_B(a) \leq \eta_B(b)$ (Recall that the order of $(B^+)^b$ is given in 5.3.4). This is enough to prove that $\eta_B(a \wedge b) = \eta_B(a) \wedge \eta_B(b)$ and that $\eta_B(a \vee b) = \eta_B(a) \vee \eta_B(b)$.

4. The diagram is commutative. Indeed, for $b \in B$, we have

$$\alpha(b)^+ = \alpha(b) \vee 0 = \alpha(b) \vee \alpha(0) = \alpha(b \vee 0) = \alpha(b^+),$$

and similarly $\alpha(b)^- = \alpha(b^-)$. Hence, we have

$$\eta_D(\alpha(b)) = (\alpha(b)^+, \alpha(b)^-)^\sim = (\alpha^+(b^+), \alpha^+(b^-))^\sim = (\alpha^+)^b(\eta_B(b)).$$

□

Proposition 5.3.15. *For each sbal A, the map*

$$\varepsilon_A : A \longrightarrow (A^b)^+ : a \longmapsto (a, 0)^\sim$$

is an injective morphism in the category **sbal** *such that, for every morphism α of* **sbal**(A, C)*, the following diagram is commutative.*

$$\begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[d, "\alpha"'] & (A^b)^+ \arrow[d, "(\alpha^b)^+"] \\ C \arrow[r, "\varepsilon_C"'] & (C^b)^+ \end{tikzcd}$$

Proof. 1. The map ε_A is well-defined since, A being positive, we have $0 \leq a$ and therefore $(a, 0)^\sim \geq 0_{(A^b)} = (0, 0)^\sim$.

2. The fact that ε_A is one-to-one follows immediately from the definition of the equivalence relation $\sim$ in (5.2).

3. The fact that the map ε_A is a morphism in **sbal** follows also immediately from (5.2) and from the definition of the operations in A^b.

4. The commutativity of the diagram follows from these equalities

$$((a, b)^\sim)^+ = (a, 0)^\sim \text{ and } ((a, b)^\sim)^- = (b, 0)^\sim.$$

□

Example 5.3.16. Let us note that, in general, the co-unit ε_A of Proposition 5.3.15 is not an isomorphism. For instance, this is the case if we consider the semibal $A = I(X, \mathbb{R}^+)$ of Example 5.2.5 with $X = [0, 1]$. The bal generated by A, namely $I(X, \mathbb{R}^+)^b$, is not the whole set of continuous functions $C(X, \mathbb{R})$, but the set $BV(X, \mathbb{R})$ of functions of bounded variation, which is strictly included in $C(X, \mathbb{R})$ (see for instance [61] for definitions and proofs). We are now going to show that the semibals $I(X, \mathbb{R}^+)$ and $(I(X, \mathbb{R}^+)^b)^+$, namely $BV(X, \mathbb{R})^+$, are not isomorphic.

Consider the function $f : x \longmapsto x$ and $g : x \longmapsto x^2$ in $I(X, \mathbb{R}^+)$. These functions are such that $f \geq g$ and for every $h \in I(X, \mathbb{R}^+)$, $f \neq g + h$. Indeed, otherwise we would have that $f - g$ is an element of $I(X, \mathbb{R}^+)$, which is obviously false. Now, for every elements f', g' of $BV(X, \mathbb{R})^+$ such that $f' \geq g'$, the difference $h' = f' - g'$ is also an element of $BV(X, \mathbb{R})^+$. In particular, it follows that there exists a function h' such that $f' = g' + h'$. Since we showed that $I(X, \mathbb{R}^+)$ does not have this property, $\varepsilon_{I(X,\mathbb{R}^+)}$ cannot be an isomorphism.

Theorem 5.3.17. *The functor* $\cdot^b$: **sbal** $\longrightarrow$ **bal** *is a right adjoint of the functor* $\cdot^+$: **bal** $\longrightarrow$ **sbal**. *The unit is the isomorphism* η_B *while the co-unit is the embedding* ε_A.

Proof. The proof follows almost immediately from what we just proved and defined. $\square$

With the help of the functor $\cdot^b$, we can transfer information obtained in the category **bal** to the category **sbal**. Let us focus on topology for a moment.

Recall that A^b endowed with the uniform norm

$$||(a,b)^\sim|| = \inf\{\lambda \in \mathbb{R} \mid (a,b)^\sim \vee (b,a)^\sim \leq \lambda\},$$

is a topological algebra (see [7, p.444]), that is every operations in A is continuous for the topology defined by this norm, and bal morphisms are continuous.

Of course, $A \cong (A^b)^+$ is endowed with the norm and topology induced by A^b. It follows from the corresponding facts in A^b that A is a topological semialgebra (that is every operations on A is continuous) and that semibal morphisms α are continuous.

5.4 Congruences and ℓ-ideals

In this section, we introduce the counterpart of the maximal ℓ-ideals used in the ring setting: congruences. Thanks to these congruences, we will be able to prove a representation theorem for semibals, as a first step for the duality we are seeking to establish. Recall that a congruence for an algebra of language $\mathcal{L} = \{f_i \mid i \in I\}$ is an equivalence relation which respects all the operations f_i, for $i \in I$.

Also recall that an ℓ-ideal of a bal B is a ring ideal I such that for all $a, b \in B$, $|a| \leq |b|$ and $b \in I$ implies $a \in I$. While this definition is widely used, in this section it will be be more simple to work with an alternative definition given in Proposition 5.4.1.

Finally, for the sake of simplicity, through this section, we will denote a semibal by A and its generated bal by B and denote the equivalence class $(a,b)^\sim$ by $a - b$, morally $(a,0)^\sim - (b,0)^\sim$.

Proposition 5.4.1. *A ring ideal I of a bal B is an ℓ-ring if and only if it is closed under $\vee$ and convex, that is $a \leq b \leq c$ and $a, c \in I$ implies $b \in I$.*

Proof. Consider first that I is an ℓ-ideal. Note that since $||a|| = |a|$ for all $a \in B$, it is clear that I is closed under taking absolute values. The convexity of I then follows from this remark. Indeed, set $a, c \in I$ and $b \in B$ such that $a \leq b \leq c$. It follows that

$$a \wedge c \leq b \leq a \vee c. \tag{5.4}$$

Starting from (5.4), we obtain that

$$\begin{aligned} |b| = (-b) \vee b &\leq -(a \wedge c) \vee (a \vee c) \\ &= (-a \vee -c) \vee (a \vee c) \\ &= (-a \vee a) \vee (-c \vee c) \\ &= |a| \vee |c| \leq |a| + |c| \in I, \end{aligned}$$

where the last inequality stems from

$$|a| + |c| = (|a| \wedge |c|) + (|a| \vee |c|) \geq |a| \vee |c|.$$

Hence, we have $|b| \leq ||a| + |c||$, and so $b \in I$ as required. It remains to prove that I is closed under $\vee$ and this follows easily from

$$|a \vee b| \leq |a| \vee |b| \leq |a| + |b|.$$

On the other hand, suppose that I is a ring ideal convex and closed under $\vee$ and that $|a| \leq |b|$ for $b \in I$. Since $b \in I$ and since I is a ring ideal, we have that $-b \in I$. It follows that $|b| = b \vee (-b) \in I$ and that $-|b| \in I$. Using the convexity of I and the fact that $-|b| \leq |a| \leq |b|$, we obtain that $|a| \in I$. Finally, using again the convexity and the fact that

$$a \wedge -a = -|a| \leq a \leq |a|,$$

we have $a \in I$. □

Remark 5.4.2. Note that for a ring ideal I of a bal the following are equivalent:

- I is closed under $\vee$,
- I is closed under $\wedge$.

Indeed, suppose I closed under $\vee$ and let $a, b \in I$. Then $-a, -b \in I$ implies

$$-(a \wedge b) = (-a) \vee (-b) \in I$$

and, consequently, $a \wedge b \in I$.

Definition 5.4.3. Let A be a semibal. A congruence θ on A is said to be **strong** if

$$(a + c)\ \theta\ (b + c) \Rightarrow a\ \theta\ b. \tag{5.5}$$

We use the denomination $\mathrm{Con}_\ell(A)$ for the set of all strong congruences on A and $\mathrm{MaxCon_l}(A)$, or more simply X_A, for the set of all the ones maximal with respect to inclusion.

Proposition 5.4.4. *If θ is a strong congruence of A, then the subset I_θ of A^b defined by*

$$I_\theta := \{(a, b)^\sim \mid a\ \theta\ b\}$$

is an ℓ-ideal of B.

Proof. As θ is in particular a semiring congruence, it is easy to prove that I_θ is a ring ideal. Consequently, it suffices we prove that I_θ is closed under $\vee$ and convex.

1. Closure under $\vee$.
   Let $(a_i, b_i)^\sim \in I_\theta$ for $i = 1, 2$. We want to prove that

   $$(a_1, b_1)^\sim \vee (a_2, b_2)^\sim = ((a_1 + b_2) \vee (a_2 + b_1), b_1 + b_2)^\sim \in I_\theta,$$

   that is

   $$(a_1 + b_2) \vee (a_2 + b_1)\ \theta\ b_1 + b_2. \tag{5.6}$$

   But, $a_1\ \theta\ b_1$ implies $a_1 + b_2\ \theta\ b_1 + b_2$ and $a_2\ \theta\ b_2$ implies $a_2 + b_1\ \theta\ b_2 + b_1$. Hence, we obtain (5.6) as required.

2. Convexity.
   Let $(a_i, b_i)^\sim \in I_\theta$ for $i = 1, 2$ and

$$(a_1, b_1)^\sim \leq (c, d)^\sim \leq (a_2, b_2)^\sim.$$

   In other words, we have $a_i \;\theta\; b_i$, $a_1+d \leq c+b_1$ and $c+b_2 \leq a_2+d$. This can be reformulated as $(c + b_2) \wedge (a_2 + d) = a_2 + d$, and more generally, $(c + b_2) \wedge (a_2 + d) \;\theta\; a_2 + d$. Using $a_2 \;\theta\; b_2$, we get

$$a_2 + d \;\theta\; (c + a_2) \wedge (d + a_2) = (c \wedge d) + a_2.$$

   Finally, since θ is strong, this is equivalent to $d \;\theta\; (c \wedge d)$.

   Using $a_1 \;\theta\; b_1$ and with a similar proof, one gets $c \;\theta\; (c \wedge d)$ and, consequently, $c \;\theta\; d$, as required.

□

Proposition 5.4.5. *If I is an ℓ-ideal of B, then the binary relation θ_I on A defined by*

$$(a, b) \in \theta_I \Leftrightarrow (a, b)^\sim \in I$$

is a strong congruence of A.

Proof. From the fact that I is in particular a subgroup of $(B, +)$, it follows that θ_I is an equivalence relation.

Moreover, it is clear that I being an ring ideal implies that θ_I respects $+$ and $\cdot$. Therefore, it remains to prove that θ_I respects $\vee$ and $\wedge$ and that it is strong. Let us prove it for $\vee$, noticing that the proof for $\wedge$ is identical, thanks to Remarks 5.4.2. Suppose that $a_i \;\theta_I\; b_i$ for $i = 1, 2$. We have $(a_i, b_i)^\sim \in I$ and $(b_i, a_i)^\sim \in I$. Now, since I is closed under $\vee$, it follows that

$$(a_1, b_1)^\sim \vee (a_2, b_2)^\sim = ((a_1 + b_2) \vee (b_1 + a_2), b_1 + b_2)^\sim \in I$$

and

$$(b_1, a_1)^\sim \vee (b_2, a_2)^\sim = ((a_1 + b_2) \vee (b_1 + a_2), a_1 + a_2)^\sim \in I.$$

We are going to prove that

$$\begin{aligned}(a_1 + a_2, (a_1 + b_2) \vee (b_1 + a_2))^\sim \leq& (a_1 \vee a_2, b_1 \vee b_2)^\sim &(5.7)\\ &\leq ((a_1 + b_2) \vee (b_1 + a_2), b_1 + b_2)^\sim, &(5.8)\end{aligned}$$

the conclusion will then follow from the convexity of I.

For (5.7), we have

$$\begin{aligned}&(a_1 + a_2, (a_1 + b_2) \vee (b_1 + a_2))^\sim \leq (a_1 \vee a_2, b_1 \vee b_2)^\sim\\ \Leftrightarrow\;& a_1 + a_2 + (b_1 \vee b_2) \leq ((a_1 + b_2) \vee (b_1 + a_2)) + (a_1 \vee a_2)\\ \Leftrightarrow\;& (a_1 + a_2 + b_1) \vee (a_1 + a_2 + b_2) \leq (a_1 + b_2 + (a_1 \vee a_2)) \vee (b_1 + a_2 + (a_1 \vee a_2))\\ \Leftrightarrow\;& (a_1 + a_2 + b_1) \vee (a_1 + a_2 + b_2) \leq (a_1 + ((a_1 + b_2) \vee (a_2 + b_2))\\ &\qquad\qquad\qquad\qquad\qquad\qquad \vee (a_2 + ((a_1 + b_1) \vee (a_2 + b_1))\end{aligned}$$

And, since

$$(a_1 + a_2 + b_1) \leq (a_2 + ((a_1 + b_1) \vee (a_2 + b_1))$$

and

$$(a_1 + a_2 + b_2) \leq (a_1 + ((a_1 + b_2) \vee (a_2 + b_2)),$$

the inequality (5.7) is proved. Of course, the procedure to prove the inequality (5.8) is similar.

It remains to prove that θ_I is strong. But, for $a, b \in A$, we have

$$\begin{aligned} & a + c\,\theta_I\,b + c \\ \Leftrightarrow\ & (a+c, b+c)^\sim \in I \\ \Leftrightarrow\ & (a,b)^\sim + \underbrace{(c,c)^\sim}_{=0} \in I \\ \Leftrightarrow\ & (a,b)^\sim \in I \Leftrightarrow a\,\theta_I\,b, \end{aligned}$$

as required. □

We now have the following theorem, which is a direct consequence of Propositions 5.4.4 and 5.4.5, and its useful corollaries.

Theorem 5.4.6. *Let us denote by* $\mathrm{Id}_\ell(A^b)$ *the set of all* ℓ*-ideals of* A^b*. The maps*

$$I_A : \mathrm{Con}_\ell(A) \longrightarrow \mathrm{Id}_\ell(A^b) : \theta \longrightarrow I_\theta \text{ and } I_A^{-1} : \mathrm{Id}_\ell(A^b) \longrightarrow \mathrm{Con}_\ell(A) : I \longrightarrow \theta_I$$

are bijections inverse of each others and preserve inclusions. Hence, they are also bijections between $\mathrm{MaxId}_\ell(A^b)$ *and* $\mathrm{MaxCon}_{\mathrm{1}\ell}(A)$*.*

Corollary 5.4.7. *Let A be a semibal. The set X_A equipped with the topology generated by the sets*

$$\omega_A((a,c)) := \{\theta \in X_A \mid (a,c) \notin \theta\},$$

with $a, c \in A$ is a compact Hausdorff space.

Proof. We know by [44] that $\mathrm{MaxId}_\ell(A^b)$ equipped with the topology generated by the sets

$$\omega((a,b)^\sim) = \{I \in \mathrm{MaxId}_\ell(A^b) \mid (a,b)^\sim \notin I\},$$

with $(a,b)^\sim \in A^b$ is a compact Hausdorff space. Therefore, the result is trivial when the bijection of Theorem 5.4.6 is taken into account. □

Corollary 5.4.8. *For any semibal A and any $\theta \in X_A$, $A/_\theta \cong \mathbb{R}^+$.*

Proof. By Theorem 5.4.6, I_θ is a maximal ℓ-ideal, hence (see [7, p. 441]) there exist a unique isomorphism

$$\lambda : A^b/_\theta \longrightarrow \mathbb{R}\,.$$

If $a \in A$, then $a^{I_\theta} \geq 0$, therefore $\lambda(a^{I_\theta}) \geq 0$ and λ is an isomorphism between $A/_\theta$ and $\mathbb{R}^+$. □

Corollary 5.4.9. *Let A be a semibal. Then the intersection of all maximal strong ℓ-congruences of A collapses to the equality.*

Proof. By the results of Johnson in [45, Definition II.1.3 and Theorem II.2.11], we know that the intersection of all maximal ℓ-ideals of an object of **bal** is equal to the singleton $\{0\}$. Hence, if $a\,\theta\,b$ for all $\theta \in X_A$, we have that $(a,b) \sim (0,0)$ and therefore, that $a = b$. □

Remark 5.4.10. The unique morphism $\lambda : A^b/_{I_\theta} \longrightarrow \mathbb{R}$ of Corollary 5.4.8 is described in [44] as follows: $\lambda(a)$ is the unique real such that $a - \lambda(a) \in I_\theta$. In particular, this means that, for $a \in A$, we have that $\lambda(a)$ is the unique positive real such that $(a, \lambda(a)) \in \theta$ and that, for $r \in \mathbb{R}^+$, $\lambda(r) = r$.

Definition 5.4.11. If θ is a congruence of X_A, let us denote by $\lambda(-,\theta)$ the isomorphism $A/_\theta \longrightarrow \mathbb{R}^+$ given in Corollary 5.4.8. We define on X_A the binary relation $\leq$ as follow

$$\theta \leq \xi \Leftrightarrow \lambda(a,\theta) \leq \lambda(a,\xi)\ \forall a \in A. \tag{5.9}$$

Example 5.4.12. The case of maximal ℓ-congruences of $I(X,\mathbb{R}^+)$, for X a compact po-space, was studied by Hansoul in [43]. He proved that an ℓ-congruence θ is maximal if and only if there exists an element $x \in X$ such that

$$\theta = \theta_x := \{(f,g) \in I(X,\mathbb{R}^+) : f(x) = g(x)\}.$$

In this case, it is clear that $\lambda(f,\theta_x) = f(x)$ for all $f \in I(X,\mathbb{R}^+)$, so that we obtain

$$\theta_x \leq \theta_y \Leftrightarrow f(x) \leq f(y)\ \forall f \in I(X,\mathbb{R}^+).$$

Proposition 5.4.13. *Let $\theta \in X_A$. Then*

$$\alpha_\theta : A \longrightarrow \mathbb{R}^+ : a \longmapsto \lambda(a,\theta)$$

is a semibal morphism such that $\theta = \ker(\alpha_\theta)$.

Proof. It suffices to check that the map α_θ satisfies all the required properties and this follows immediately from θ being an ℓ-congruence and the uniqueness of $\lambda(a,\theta)$ for a $a \in A$. □

Proposition 5.4.14. *Let A be a semibal. Then $a \leq b$ in A if and only if $\lambda(a,\theta) \leq \lambda(b,\theta)$ for all $\theta \in X_A$.*

Proof. The only if part follows immediately from Proposition 5.4.13. Now, for the if part, suppose that $a \not\leq b$, hence that $a \wedge b \neq a$. By Corollary 5.4.9, $(a \wedge b, a) \notin \theta$ for some $\theta \in X_A$. Let us proceed by contradiction and suppose that

$$\lambda_1 = \lambda(a,\theta) \leq \lambda(b,\theta) = \lambda_2.$$

We have

$$a\ \theta\ \lambda_1\ \theta\ \lambda_1 \wedge \lambda_2\ \theta\ a \wedge b,$$

which is a contradiction. □

Theorem 5.4.15. *The relation $\leq$ defined in* (5.9) *is a closed order on X_A. Hence, $(X_A,\leq)$ is a compact po-space.*

Proof. It is clear that $\leq$ is reflexive and transitive, so that we just need to prove that $\leq$ is antisymmetric for it to be an order. Suppose that $\lambda(a,\theta) = \lambda(a,\xi)$ for all $a \in A$. We have to prove that $\theta = \xi$, but since they are maximal ℓ-congruences, we just have to show that $\theta \subseteq \xi$. Hence, suppose that $a\ \theta\ b$. Then, by Remark 5.4.10, we have that $\lambda(a,\theta) = \lambda(b,\theta)$. It follows now from our hypothesis that $\lambda(a,\xi) = \lambda(b,\xi)$. Therefore, we have

$$a\ \xi\ \lambda(a,\xi)\ \xi\ \lambda(b,\xi)\ \xi\ b,$$

that is $a\ \xi\ b$.

Now, to show that the order $\leq$ is closed, suppose that $\theta \not\leq \xi$. It means that there exists $a \in A$ such that $\lambda(a,\theta) > \lambda(a,\xi)$. Set O and U as follow:

$$O = \cup\{\omega(a, r\cdot 1) \mid r < \lambda(a,\theta)\} \text{ and } U = \cup\{\omega(a, r\cdot 1) \mid r \geq \lambda(a,\theta)\}.$$

It is clear that O is a neighbourhood of θ while U is a neighbourhood of ξ. To conclude, we need to prove that for all $\alpha \in O$ and for all $\beta \in U$, we have $\alpha \not\leq \beta$. But, if $\alpha \in O$, we know that $(a, r) \notin \alpha$ for all $r < \lambda(a, \theta)$, which in particular implies that $\lambda(a, \alpha) \geq \lambda(a, \theta)$. Similarly, from $\beta \in U$, we can deduce that $\lambda(a, \beta) < \lambda(a, \theta)$. Therefore, we have

$$\lambda(a, \beta) < \lambda(a, \theta) \leq \lambda(a, \alpha)$$

and $\alpha \not\leq \beta$. □

Theorem 5.4.16 (Representation theorem)**.** *Let A be a semibal. Then the map*

$$\Lambda_A : A \longrightarrow I(X_A, \mathbb{R}^+) : a \longmapsto: \lambda(a, -) : \theta \longmapsto \lambda(a, \theta)$$

is an embedding in **sbal** *such that $a \leq b$ if and only if $\Lambda_A(a) \leq \Lambda_A(b)$.*

Proof. First of all, let us prove that Λ_A is well-defined, i.e. that $\lambda(a, -)$ is indeed continuous and increasing for all $a \in A$. Given the definition of the order on X_A in (5.9), the map $\lambda(a, -)$ is trivially increasing. Therefore, let us focus on continuity. If $]r, s[$ is an open interval of $\mathbb{R}^+$, then

$$\begin{aligned}\lambda(a, -)^{-1}(]r, s[) &= \{\theta \in X_A : r < \lambda(a, \theta) < s\} \\ &= \cup\{\omega(a, r'.1) \mid r' \leq r\} \cap \cup\{\omega(a, s'.1) \mid s' \geq s\}.\end{aligned}$$

Hence, $\lambda(a, -)$ is continuous.

Moreover, by Corollary 5.4.9, it is clear that Λ_A is an embedding. Now, since the elements of X_A are congruences, we have, given Remark 5.4.10, that Λ_A is a morphism of **bal**.

Finally, let us prove that $\Lambda_A(a) \leq \Lambda_A(b)$ implies $a \leq b$ (the other direction being a direct consequence of Λ_A being a morphism of **sbal**). By contraposition, suppose that $a \not\leq b$, hence that $a \wedge b \neq a$. By Corollary 5.4.9, this implies that there is a congruence $\theta \in X_A$ such that $((a \wedge b), a) \notin \theta$. In other words, θ is a congruence such that

$$\lambda(a, \theta) \wedge \lambda(b, \theta) = \lambda(a \wedge b, \theta) \neq \lambda(a, \theta).$$

Therefore, we have $\lambda(a, \theta) \not\leq \lambda(b, \theta)$ and, consequently, $\Lambda_A(a) \not\leq \Lambda_A(b)$. □

5.5 Quotients

In this short section, which is complementary to Section 5.4, we will prove that the quotient of an object in **sbal** by a maximal ℓ-congruence is still an object of **sbal**.

While it is clear, by universal algebra theory, that $A/_\theta$ is a semiring, an $\mathbb{R}^+$-semialgebra and a lattice with the operations defined as usual in quotients, some work has to be done for the other properties.

Let us note, for future convenience, that the order on $A/_\theta$ is given by

$$a^\theta \leq b^\theta \Leftrightarrow (a \vee b)\ \theta\ b \Leftrightarrow (a \wedge b)\ \theta\ a, \tag{5.10}$$

while the order on $A^b/_{I_\theta}$ is given by

$$(a, b)^\sim + I_\theta \leq (c, d)^\sim + I_\theta \Leftrightarrow \exists i \in I_\theta : (a, b)^\sim \leq (c, d)^\sim + i. \tag{5.11}$$

(see for instance [68]).

Proposition 5.5.1. *Let A be a semibal and $\theta \in \mathrm{Con}_\ell(A)$. Then, the application*

$$\varphi : (A/_\theta)^b \longrightarrow A^b/_{I_\theta} : (a^\theta, b^\theta)^\sim \longmapsto (a,b)^\sim + I_\theta,$$

is a ring, an $\mathbb{R}$-algebra and a lattice isomorphism.

Proof. It is clear that φ is onto. Now, let us prove that

$$(a^\theta, b^\theta)^\sim \leq (c^\theta, d^\theta)^\sim \Leftrightarrow \varphi((a,b)^\sim) \leq \varphi((c,d)^\sim). \tag{5.12}$$

On one hand, we have

$$\begin{aligned}(a^\theta, b^\theta)^\sim \leq (c^\theta, d^\theta)^\sim &\Leftrightarrow\ a^\theta + d^\theta \leq b^\theta + c^\theta \\ &\Leftrightarrow\ (a+d)^\theta \leq (b+c)^\theta \\ &\Leftrightarrow (a+d) \vee (b+c)\ \theta\ (b+c) \\ &\Leftrightarrow\ \underbrace{((a+d) \vee (b+c), (b+c))^\sim}_{=(a+d,b+c)^\sim \vee (0,0)^\sim} \in I_\theta.\end{aligned}$$

From $(a+d, b+c)^\sim \vee (0,0)^\sim \in I_\theta$, it follows that there exists an element $i \in I_\theta$ such that

$$(a,b)^\sim + (d,c)^\sim = (a+d, b+c)^\sim \leq (a+d, b+c)^\sim \vee (0,0)^\sim = i.$$

Therefore, we proved that

$$(a^\theta, b^\theta)^\sim \leq (c^\theta, d^\theta)^\sim \Rightarrow \exists i \in I_\theta : (a,b)^\sim \leq (c,d)^\sim + i,$$

On the other hand, suppose that there exists $i \in I_\theta$ such that $(a,b)^\sim \leq (c,d)^\sim + i$. Since I_θ is an ℓ-ideal, we have by definition that $|i| \in I_\theta$. Consequently, we obtain

$$I_\theta \ni 0 \leq ((a,b)^\sim + (d,c)^\sim) \vee 0 \leq |i| \in I_\theta.$$

Now, as I_θ is convex, this implies that $((a,b)^\sim + (d,c)^\sim) \vee 0 \in I_\theta$, which we just proved to be equivalent to $(a^\theta, b^\theta)^\sim \leq (c^\theta, d^\theta)^\sim$. Hence, we have (5.12) which, conveniently, implies that φ is one-to-one and a lattice morphism.

What remains to prove is that φ is a ring and an $\mathbb{R}$-algebra morphism, but this follows immediately from the definitions of the operations on A^b and from the fact that θ is a congruence. □

We will now use this proposition and the fact that the quotient of a bal by a maximal ℓ-ideal is a bal (see [7], p. 440) to obtain the result we were looking for.

Theorem 5.5.2. *If A is a semibal and $\theta \in X_A$, then $A/_\theta$ is a semibal.*

Proof. From $\theta \in X_A$ comes $I_\theta \in \mathrm{MaxId}_\ell(A^b)$ and, consequently, that $A^b/_{I_\theta}$ is a bal.

1. $A/_\theta$ is a po-semiring.

   (a) We have

$$\begin{aligned}a^\theta \leq b^\theta &\Leftrightarrow\ (a \vee b)\ \theta\ b \\ &\Leftrightarrow\ (a \vee b) + c\ \theta\ b + c \\ &\Leftrightarrow\ (a+c) \vee (b+c)\ \theta\ b + c \\ &\Leftrightarrow\ (a+b)^\theta \leq (b+c)^\theta \\ &\Leftrightarrow\ a^\theta + c^\theta \leq b^\theta + c^\theta,\end{aligned}$$

   where we used that θ is strong and that A is an ℓ-semiring.

(b) Suppose that $a^\theta \leq b^\theta$ and $c^\theta \leq d^\theta$. This implies that

$$(b^\theta, a^\theta)^\sim, (d^\theta, c^\theta)^\sim \geq (0^\theta, 0^\theta)^\sim.$$

Now, as this implies, by Proposition 5.5.1, that

$$(b,a)^\sim + I_\theta, (d,c)^\sim + I_\theta \geq I_\theta,$$

and since $A^b/_{I_\theta}$ is a bal, we have

$$\underbrace{((b,a)^\sim \cdot (d,c)^\sim) + I_\theta}_{=(bd+ac,bc+ad)^\sim + I_\theta} \geq I_\theta.$$

Again by Proposition 5.5.1, this implies that

$$((bd+ac)^\theta, (bc+ad)^\theta)^\sim \geq (0,0)^\sim,$$

that is $(bd+ac)^\theta \geq (bc+ad)^\theta$ as required.

2. $A/_\theta$ is positive. We have $0^\theta \leq a^\theta$ if and only if $(0 \vee a)\ \theta\ a$. But, as A is positive, $(0 \vee a) = a$ and the conclusion is immediate.

3. $A/_\theta$ is bounded. There exists $n \in \mathbb{N}$ such that $a^\theta \leq n \cdot 1^\theta$ if and only if there exists $n \in \mathbb{N}$ such that $a \vee n \cdot 1\ \theta\ n \cdot 1$. This follows immediately from A being bounded.

4. $A/_\theta$ is Archimedean. Suppose that, for each $n \in \mathbb{N}$

$$n \cdot a^\theta + b^\theta \leq n \cdot c^\theta + d^\theta.$$

This implies that

$$(n \cdot a^\theta, n \cdot c^\theta)^\sim \leq (d^\theta, b^\theta)^\sim,$$

and, by Proposition 5.5.1, that

$$n \cdot (a,c)^\sim + I_\theta \leq (d,b)^\sim + I_\theta.$$

As $A^b/_{I_\theta}$ is Archimedean, this implies that $(a,c)^\sim + I_\theta \leq I_\theta$ and, consequently, that $a^\theta \leq c^\theta$.

5. We already know that $A/_\theta$ is a lattice and , since θ is an ℓ-congruence, it is clear that $A/_\theta$ is an ℓ-semiring.

Finally, since we also already know that $A/_\theta$ is a $\mathbb{R}^+$-semialgebra, the proof is concluded. □

Remark 5.5.3. In the proof of Theorem 5.5.2, we used several times the fact that $A^b/_{I_\theta}$ is a bal. Actually, if I_θ is not a maximal ℓ-ideal, $A^b/_{I_\theta}$ conserves virtually all the properties of A^b but one: A^b may fail to be Archimedean.

Of course, the semibal morphism Λ_A given in Theorem 5.4.16 is intended to be the unit of our duality between a subcategory of **sbal** and **KPSp**. However, being a unit in a duality requires being an isomorphism, which is currently not the case. In this section, we will develop additional properties of Λ_A and in Section 5.7, we will find conditions under which it is indeed an isomorphism.

Definition 5.5.4. The object mapping $A \longrightarrow X_A$ is extended to a functor $\chi : \mathbf{sbal} \longrightarrow \mathbf{KPSp}$ by defining for a morphism $\alpha \in \mathbf{sbal}(A, C)$ the dual map $\alpha^\star \in \mathbf{KPSp}(X_C, X_A)$ such that, for $\theta \in X_C$,

$$(a, b) \in \alpha^\star(\theta) \text{ if and only if } (\alpha(a), \alpha(b)) \in \theta,$$

(in other words, $\alpha^\star(\theta) = (\alpha \times \alpha)^{-1}(\theta)$). Let us now check that this functor is well-defined.

Lemma 5.5.5. *Let A and C be semibals and α a morphism from A to C. Then, the map $\alpha^\star$ is equal to the composition $(I_A^{-1} \circ \mathrm{MaxId}_\ell(\alpha^b) \circ I_C)$, where $\mathrm{MaxId}_\ell(\alpha^b)$ is defined as*

$$\mathrm{MaxId}_\ell(\alpha^b) : \mathrm{MaxId}_\ell(C) \longrightarrow \mathrm{MaxId}_\ell(A) : I \longmapsto (\alpha^b)^{-1}(I)$$

(see Appendix B.4.5). In other words, for every $\theta \in \mathrm{MaxCon}_\mathrm{l}(C)$, we have

$$I_{\alpha^\star(\theta)} = (\alpha^b)^{-1}(I_\theta).$$

Proof. For θ, a maximal strong ℓ-congruence of C, we have that:

$$\begin{aligned} &(a, b)^\sim \in I_{\alpha^\star(\theta)} \\ \Leftrightarrow\ &(a, b) \in \alpha^\star(\theta) \\ \Leftrightarrow\ &(\alpha(a), \alpha(b)) \in \theta \\ \Leftrightarrow\ &(\alpha(a), \alpha(b))^\sim \in I_\theta \Leftrightarrow \alpha^b((a, b)^\sim)) \in I_\theta. \end{aligned}$$

□

Corollary 5.5.6. *Let A and C be semibals, α a morphism from A to C and θ be a strong ℓ-congruence on A, then $\alpha^\star(\theta)$ is a maximal strong ℓ-congruence.*

Proof. This a direct consequence of Lemma 5.5.5 and Theorem 5.4.6. □

Lemma 5.5.7. *Let α be a morphism of* $\mathbf{bal}(A, C)$*. Then, for any $\theta \in X_C$ and $a \in A$,*

$$\lambda(a, \alpha^\star(\theta)) = \lambda(\alpha(a), \theta).$$

Proof. Let us denote $\lambda(a, \alpha^\star(\theta))$ by r. We have $r\ \alpha^\star(\theta)\ a$, which is, by the definition of $\alpha^\star(\theta)$, equivalent to $\alpha(r)\ \theta\ \alpha(a)$. Now, since $r = \alpha(r)$, we have $r\ \theta\ \alpha(a)$. Hence, by uniqueness of r, we have immediately that $r = \lambda(\alpha(a), \theta)$. □

Proposition 5.5.8. *If α is a morphism of* $\mathbf{sbal}(A, C)$*, then $\alpha^\star$ is a morphism of* $\mathbf{KPSp}(X_C, X_A)$*.*

Proof. Namely, we need to prove that $\alpha^\star$ is continuous and increasing. We know by Gelfand duality that $\mathrm{MaxId}_\ell(\alpha^b)$ is continuous. Moreover, we know that the maps I_A and I_C are homeomorphisms. Therefore, by Lemma 5.5.5, $\alpha^\star$ is continuous.

Now, let $\theta, \xi \in X_C$ be such that for all $c \in C$, $\lambda(c, \theta) \leq \lambda(c, \xi)$. We want to prove that, for all a

$$\lambda_1 = \lambda(a, \alpha^\star(\theta)) \leq \lambda(a, \alpha^\star(\xi)) = \lambda_2.$$

But, by Lemma 5.5.7, we have $\lambda_1 = \lambda(\alpha(a), \theta)$ and $\lambda_2 = \lambda(\alpha(a), \xi)$ and, hence, $\lambda_1 \leq \lambda_2$, as required. □

Finally, to actually have a functor, we need to have the following proposition, whose proof is obvious.

Proposition 5.5.9. *1. If $\alpha : A \longrightarrow C$ and $\beta : C \longrightarrow D$ are semibal morphisms, then*

$$(\beta \circ \alpha)^\star = \alpha^\star \circ \beta^\star.$$

2. If 1_A is the identity morphism of A, then $(1_A)^\star$ is the identity morphism of X_A.

Now that the functor from **sbal** to **KPSp** is defined, we can focus on the functor in the other direction which obviously maps a compact po-space X to its semiring of increasing positive continuous functions $I(X, \mathbb{R}^+)$.

Definition 5.5.10. For a morphism $f \in \mathbf{KPSp}(X, Y)$, its dual map $f_\star \in \mathbf{sbal}(I(Y, \mathbb{R}^+), I(X, \mathbb{R}^+))$ is defined by

$$f_\star : g \longmapsto g \circ f.$$

Again, we have this trivial proposition.

Proposition 5.5.11. *1. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are increasing continuous functions, then*

$$(g \circ f)_\star = f_\star \circ g_\star.$$

2. If 1_X is the identity morphism of X, then $(1_X)_\star$ is the identity morphism of $I(X, \mathbb{R}^+)$.

Let us now focus on a sequence of propositions whose proofs follow the line of the bal case, established by Bezhanishvili, Morandi and Olberding in [7, Lemma 2.9]. Nevertheless, we need to introduce new concepts which will take the roles of epimorphism and embedding.

Definition 5.5.12. 1. A semibal morphism $\alpha : A \longrightarrow C$ is an **order epimorphism** if for every semibal morphisms β_1 and $\beta_2 : C \longrightarrow D$, $(\beta_1 \circ \alpha) \leq (\beta_2 \circ \alpha)$ implies $\beta_1 \leq \beta_2$ (with the inequalities defined pointwise).

2. A function $f : X \longrightarrow Y$ between compact po-spaces is an **order embedding** if $f(x) \leq f(y)$ implies $x \leq y$.

Of course the ordered properties imply respectively the non-ordered properties which they are associated to.

Proposition 5.5.13. *A semibal morphism $\alpha : A \longrightarrow C$ is an order epimorphism if and only if $\alpha^\star$ is an order embedding.*

Proof. $\Rightarrow$ Let $\theta_1 \not\leq \theta_2$ in X_C, that is, there exists an element $c \in C$ such that $\lambda(c, \theta_1) > \lambda(c, \theta_2)$. By Proposition 5.4.13, there are semibal morphisms $\beta_i : C \longrightarrow \mathbb{R}^+ : a \longrightarrow \lambda(a, \theta_i)$ such that $\theta_i = \ker(\beta_i)$ for $i = 1, 2$. We have that $\beta_1 \not\leq \beta_2$ as, otherwise we would have for c

$$\lambda(c, \theta_1) = \beta_1(\lambda(c, \theta_1)) = \beta_1(c) \leq \beta_2(c) = \lambda(c, \theta_2),$$

which is absurd.

Hence, since α is order epic, it follows that $\beta_1 \circ \alpha \not\leq \beta_2 \circ \alpha$, and, more specifically, that there exists an element $a \in A$ such that

$$\lambda(\alpha(a), \theta_1) = \beta_1(\alpha(a)) \not\leq \beta_2(\alpha(a)) = \lambda(\alpha(a), \theta_2).$$

In other words, since $\lambda(\alpha(a), \theta_i) = \lambda(a, \alpha^\star(\theta_i))$, there exists $a \in A$ such that

$$\lambda(a, \alpha^\star(\theta_1)) \not\leq \lambda(a, \alpha^\star(\theta_2)),$$

that is $\alpha^\star(\theta_1) \not\leq \alpha^\star(\theta_2)$, as required.

$\Leftarrow$ Suppose that α is not an order epimorphism. There exist semibal morphisms $\beta_1, \beta_2 : C \longrightarrow D$ such that $\beta_1 \not\leq \beta_2$ and $\beta_1 \circ \alpha \leq \beta_2 \circ \alpha$. This implies that, for some $c \in C$, we have $\beta_1(c) \not\leq \beta_2(c)$, hence

$$\beta_1(c) \wedge \beta_2(c) \neq \beta_1(c).$$

Then, by Corollary 5.4.9, there exists $\theta \in X_D$ such that

$$(\beta_1(c), \beta_2(c) \wedge \beta_1(c)) \notin \theta. \tag{5.13}$$

We have

$$\lambda_1 = \lambda(\beta_1(c), \theta) > \lambda(\beta_2(c), \theta) = \lambda_2. \tag{5.14}$$

Indeed, if it was not the case, we would have $\lambda_1 \leq \lambda_2$ and

$$\beta_1(c) \; \theta \; \lambda_1 \; \theta \; (\lambda_1 \wedge \lambda_2) \; \theta \; (\beta_1(c) \wedge \beta_2(c)),$$

which contradicts (5.13).

Now, as $\lambda(\beta_i(c), \theta) = \lambda(c, \beta_i^\star(c))$, it follows from the inequality (5.14) that $\beta_1^\star(\theta) \not\leq \beta_2^\star(\theta)$. Consequently, since $\alpha^\star$ is an order embedding, we have $\alpha^\star(\beta_1^\star(\theta)) \not\leq \alpha^\star(\beta_2^\star(\theta))$. Therefore, there exists $a \in A$ with

$$\lambda'_1 = \lambda(a, \alpha^\star(\beta_1^\star(\theta))) > \lambda(a, \alpha^\star(\beta_2^\star(\theta))) = \lambda'_2. \tag{5.15}$$

Also, we have that $\lambda'_i = \lambda(\beta_i(\alpha(a)), \theta)$ for $i =, 1, 2$. Moreover, we have by the assumptions we made on β_1 and β_2 that $\beta_1(\alpha(a)) \leq \beta_2(\alpha(a))$. It follows, via Proposition 5.4.13, that $\lambda'_1 \leq \lambda'_2$, contradicting (5.15).

□

Two remarks have to be done about the proof Proposition 5.5.13. The first one is that, with a similar proof, one can demonstrate easily the non-ordered case, stated below for the sake of completeness.

The second one is that, with a closer look at the proof, we can notice that the morphisms β_1 and β_2 used in the if part of the proof are quite specific: they map elements in $\mathbb{R}^+$. It may lead us to consider another definition of order epimorphism, equivalent to the first one.

A semibal morphism $\alpha : A \longrightarrow C$ is a **weak order epimorphism** if for every semibal morphisms $\beta_1, \beta_2 : C \longrightarrow \mathbb{R}^+$, $\beta_1 \circ \alpha \leq \beta_2 \circ \alpha$ implies $\beta_1 \leq \beta_2$. Surprisingly at first glance, the notions of order epimorphism and weak order epimorphism are equivalent. Indeed, first, it is clear that the weak property is vacuously a consequence of the other one. On the other hand, we just said that if α was a weak order epimorphism, then $\alpha^\star$ must be an order embedding and, consequently, α must be an order epimorphism.

Let us summarise both these remarks in the following proposition.

Proposition 5.5.14. *1. A semibal morphism is an order epimorphism if and only if it is a weak order epimorphism.*

2. A semibal morphism α is an epimorphism if and only if $\alpha^\star$ is an embedding.

Lemma 5.5.15. *Let $\alpha : A \longrightarrow C$ be a one-to-one semibal morphism and θ be a proper ℓ-congruence of C. Then the ℓ-congruence generated by*

$$(\alpha \times \alpha)(\theta) := \{(\alpha(a), \alpha(b)) : a \; \theta \; b\}$$

is a proper ℓ-congruence of C.

Proof. Notice that an ℓ-congruence θ is proper if and only its associated ideal I_θ is proper, which is equivalent to ask that $(1,0)^\sim \notin I_\theta$, that is $(1,0) \notin \theta$. Hence, we have to prove that $(1,0)$ is not in the ℓ-congruence generated by $(\alpha \times \alpha)(\theta)$ and this is just routine. $\square$

Proposition 5.5.16. *A semibal morphism $\alpha : A \longrightarrow C$ is one-to-one if and only if $\alpha^\star$ is onto.*

Proof. $\Rightarrow$ Let $\theta \in X_A$. Since α is one-to-one, by Lemma 5.5.15, we know that there exists a maximal ℓ-congruence ξ such that $(\alpha \times \alpha)(\theta) \subseteq \xi$.

It is sufficient to show that $\theta \subseteq \alpha^\star(\xi)$ since, by maximality of θ, it will imply that $\theta = \alpha^\star(\xi)$. We have

$$(a,b) \in \theta \Rightarrow (\alpha(a), \alpha(b)) \in (\alpha \times \alpha)(\theta) \subseteq \xi,$$

which, consequently, implies $(a,b) \in \alpha^\star(\xi)$, as required.

$\Leftarrow$ Let us suppose that α is not one-to-one, that is there exist distinct elements $a, b \in A$ such that $\alpha(a) = \alpha(b)$. Recall that, by Corollary 5.4.9, we know that there exist $\theta \in X_A$ such that $(a,b) \notin \theta$.

Now, as $\alpha^\star$ is onto, there exists $\xi \in X_C$ such that $\alpha^\star(\xi) = \theta$. Furthermore, since they are equal, we trivially have $\alpha(a)\ \xi\ \alpha(b)$, implying that $a\ \theta\ b$, which is absurd.

$\square$

Theorem 5.5.17. *A semibal morphism $\alpha : A \longrightarrow C$ is a one-to-one order epimorphism if and only if $\alpha^\star$ is an isomorphism in* **KPSp**.

Proof. It is clear that $\alpha^\star$ is an isomorphism of **KPSp** if and only if it is an onto order embedding. Hence, the proof follows immediately from the previous proposition. $\square$

As a corollary of what we developed in this section, we have the following improvement of Theorem 5.4.16.

Theorem 5.5.18. *Let A be a semibal. Then Λ_A is a one-to-one order epimorphism such that, for every semibal morphism $\alpha : a \longrightarrow C$ the following diagram is commutative.*

$$\begin{tikzcd}
A \arrow[r, "\Lambda_A"] \arrow[d, "\alpha"'] & I(X_A, \mathbb{R}^+) \arrow[d, "(\alpha^\star)_\star"] \\
C \arrow[r, "\Lambda_C"'] & I(X_C, \mathbb{R}^+)
\end{tikzcd}$$

Proof. In [43], Hansoul proved that the map

$$\zeta_X : X \longrightarrow X_{I(X,\mathbb{R}^+)} : x \longmapsto \theta_x := \{(f,g) : f(x) = g(x)\}$$

was an isomorphism in **KPSp**. Letting $X = X_A$, one gets, for $\theta \in X_A$

$$(a,c) \in \Lambda_A^\star(\zeta_{X_A}(\theta)) \Leftrightarrow \lambda(a,-)\ \zeta_{X_A}(\theta)\ \lambda(c,-) \Leftrightarrow \lambda(a,\theta) = \lambda(c,\theta) \Leftrightarrow (a,c) \in \theta.$$

Hence, $\Lambda_A^\star \circ \zeta_{X_A}$ is the identity and $\Lambda_A^\star$ is an isomorphism.

Moreover, for all $a \in A$ and $\theta \in X_C$, we have the following equality:

$$\Lambda_C(\alpha(a))(\theta) = \lambda(\alpha(a), \theta) = \lambda(a, \alpha^\star(\theta)) = \Lambda_A(a)((\alpha^\star)(\theta)) = (\alpha^\star)_\star(\Lambda_A(a))(\theta).$$

So, the functions $\Lambda_C(\alpha(a))$ and $(\alpha^\star)_\star(\Lambda_A(a))$ are identical, as required. $\square$

5.6 A weak ordered Stone-Weierstrass theorem

It must no go unnoticed that, in Theorem 5.5.18, we established that $\Lambda_A^\star$ was an isomorphism. Hence, a semibal A and its associated semiring of functions $I(X_A, \mathbb{R}^+)$ have isomorphic po-spaces of maximal ℓ-congruences. We have to find conditions on A to lift this isomorphism to A and $I(X_A, \mathbb{R}^+)$.

What is expected, as in the bal case, is to encounter some (uniform) completeness condition. But we have to introduce another fundamental concept distinguishing A from $I(X_A, \mathbb{R}^+)$: if $f \in I(X_A, \mathbb{R}^+)$ and $g \leq f$ is a constant function of $I(X_A, \mathbb{R}^+)$, then the difference $f - g$ is still an element of $I(X_A, \mathbb{R}^+)$. Hence, $I(X_A, \mathbb{R}^+)$ has a property that we will define as admitting difference with constants

Definition 5.6.1. A positive semiring A **admits difference with constants** (**admits dc** for short) if for all $a \in A$ an $r \in \mathbb{R}^+$

$$a \geq r \cdot 1 \Rightarrow (\exists b)(a = b + r \cdot 1). \tag{5.16}$$

Proposition 5.6.2. *Let A be a semibal that admits dc and θ and ξ elements of X_A, then*

$$\theta \leq \xi \Leftrightarrow 0^\xi \subseteq 0^\theta.$$

Proof. For the if part, we know that $a \mathrel{\xi} \lambda(a, \xi)$. Hence, it follows that $(a \vee \lambda(a, \xi)) \mathrel{\xi} \lambda(a, \xi)$. Recall that $\lambda(a, \xi)$ is a positive real number. Therefore, by (5.16), there exists $b \in A$ satisfying $a \vee \lambda(a, \xi) = b + \lambda(a, \xi)$. Thus, we have $(b + \lambda(a, \xi)) \mathrel{\xi} \lambda(a, \xi)$ and, by strongness of ξ, $b \mathrel{\xi} 0$. By the assumptions we made, it follows that $b \mathrel{\theta} 0$, so that

$$(a \vee \lambda(a, \xi)) = (b + \lambda(a, \xi)) \mathrel{\theta} \lambda(a, \xi).$$

Consequently, we have $(\lambda(a, \theta) \vee \lambda(a, \xi)) \mathrel{\theta} \lambda(a, \xi)$, whence $(\lambda(a, \theta) \vee \lambda(a, \xi)) = \lambda(a, \xi)$.

For the only if part, for $a \in 0^\xi$, we have

$$0 \leq \lambda(a, \theta) \leq \lambda(a, \xi) = 0.$$

It implies that $\lambda(a, \theta) = 0$ and hence that $a \in 0^\theta$. □

In particular, this proposition shows that the order on $X_{I(X, \mathbb{R}^+)}$ defined in [43] coincides with the order defined in (5.9). We need this property to obtain an ordered version of the Stone-Weierstrass theorem.

Definition 5.6.3. We say that a subset A' of a semibal A **order-separates the point of** X_A if it satisfies for all $\theta, \xi \in X_A$

$$\theta \not\leq \xi \Rightarrow (\exists a \in A')(\lambda(a, \theta) > \lambda(a, \xi)).$$

In particular, a subset A' of $I(X, \mathbb{R}^+)$ order-separates the points of X if it satisfies, for all $x, y \in X$

$$x \not\leq y \Rightarrow (\exists f \in A')(f(x) > f(y)).$$

Lemma 5.6.4. *Let X be a compact po-space and let A be an ℓ-subalgebra of $I(X, \mathbb{R}^+)$ that order-separates the points of X and admits difference with constants. Then A satisfies the following stronger separation property: $x \not\leq y$ in X and $r > s \in \mathbb{R}^+$ implies the existence of a function $f \in A$ such that $f(x) = r$ and $f(y) = s$.*

Proof. Suppose x, y, r, s as in the statement. By Definition 5.6.3, there is $f \in A$ such that $f(x) > f(y)$. Set m as

$$m = \frac{r-s}{f(x)-f(y)} > 0.$$

Then, we have $(mf + r) \vee mf(x) \geq mf(x)$. So that, since A admits dc, there is $b \in A$ such that

$$(mf + r) \vee mf(x) = b + mf(x).$$

Let us prove that $b(x) = r$ and $b(y) = s$.

First, we have

$$b(x) + mf(x) = (mf(x) + r) \vee mf(x) = mf(x) + r,$$

so that $b(x) = r$. On the other hand, we have

$$b(y) + mf(x) = (mf(y) + r) \vee mf(x).$$

Now, since

$$mf(y) + r \geq mf(x) \Leftrightarrow r \geq m(f(x) - f(y)) = r - s,$$

we can conclude that $b(y) + mf(x) = mf(y) + r$, that is $b(y) = s$. □

Theorem 5.6.5 (Weak ordered Stone-Weierstrass theorem)**.** *If X is a compact po-space and A an ℓ-subalgebra of $I(X, \mathbb{R}^+)$ that order-separates points of X and admits dc, then A is uniformly dense in $I(X, \mathbb{R}^+)$ (that is dense for the topology of the uniform norm defined on $C(X, \mathbb{R})$, see Definition B.4.7).*

Proof. We follow the lines of the proof of the unordered Stone-Weierstrass theorem (see for instance [62]). Let $\varepsilon > 0$ and $f \in I(X, \mathbb{R}^+)$. We have to prove that there exists a function $g \in A$ with $||f - g|| \leq \varepsilon$.

For a pair $(x, y) \in X^2$, we know that there exists a function $f_{xy} \in A$ such that $f_{xy}(x) = f(x)$ and $f_{xy}(y) = f(y)$. Let

$$O_{xy} = \{z \in X \mid f(z) \leq f_{xy}(z) + \varepsilon\},$$
$$U_{xy} = \{z \in X \mid f_{xy}(z) \leq f(z) + \varepsilon\}.$$

For a fixed $x_0 \in X$,

$$\rho_{x_0} := \{O_{x_0 y} \mid y \in X\}$$

is an open cover of X, and since X is compact, there exists $F \subseteq X$ finite such that

$$X = \cup\{O_{x_0 y} \mid y \in F\}.$$

Let $f_{x_0} = \vee\{f_{xy} \mid y \in F\}$, which is an element of A. Then, for all $z \in X$, there exists $y \in F$ with $z \in O_{x_0 y}$, which implies that

$$f(z) \leq f_{x_0 y}(z) + \varepsilon \leq f_{x_0}(z) + \varepsilon.$$

Now, if $N_{x_0} = \cap\{U_{x_0 y} \mid y \in F\}$, N_{x_0} is an open neighbourhood of x_0, such that, for all $z \in N_{x_0}$,

$$f_{x_0}(z) \leq f(z) + \varepsilon.$$

Moreover,

$$\rho := \{N_{x_0} \mid x_0 \in X\}$$

is an open cover of X. Once again, there exists $G \subseteq X$ finite such that $\{N_{x_0} \mid x_0 \in G\}$ is a cover of X. Let $g = \wedge\{f_{x_0} \mid x_0 \in G\}$., which is also an element of A.

For each $z \in X$, there exists $x_0 \in G$ with $z \in N_{x_0}$, so

$$g(z) \leq f_{x_0}(z) = \vee\{f_{x_0 y}(z) \mid y \in F\} \leq f(z) + \varepsilon.$$

On the other hand, since $f(z) \leq f_{x_0}(z) + \varepsilon$ for all $z \in X$ and all $x_0 \in G$, we have that

$$f(z) \leq g(z) + \varepsilon.$$

In a nutshell, we have for all $z \in X$, $g(z) \leq f(z) + \varepsilon$ and $f(z) \leq g(z) + \varepsilon$, hence the conclusion. □

5.7 Stone semirings

We now have all the required tools to finish this chapter and establish the duality between **KPSp** and the category whose objects are what we shall call Stone semirings. Moreover, we have to prove that this duality extends the duality between the categories **ubal** and **KHaus** (see AppendixB.4).

Lemma 5.7.1. *If $\alpha : A \longrightarrow C$ is a semibal morphism, the following conditions are equivalent:*

1. *α is a one-to-one order epimorphism,*
2. *$\alpha^\star$ is an isomorphism in* **KPSp***,*
3. *there exists a one-to-one order epimorphism $\beta : C \longrightarrow I(X_A, \mathbb{R}^+)$ such that $\beta \circ \alpha = \Lambda_A$.*

Proof. (1. ⇔ 2.) This is exactly Theorem 5.5.17.

(2. ⇒ 3.) If $\alpha^\star$ is an isomorphism in **KPSp**, then

$$(\alpha^\star)_\star : I(X_A, \mathbb{R}^+) \longrightarrow I(X_B, \mathbb{R}^+) : f \longmapsto f \circ \alpha^\star$$

and $(\alpha^\star)_\star^{-1}$ are isomorphisms in **sbal**.

Let β be the following semibal morphism

$$\beta : C \longrightarrow I(X_A, \mathbb{R}^+) : c \longmapsto (\alpha^\star)_\star^{-1}(\Lambda_C(c)).$$

Since $(\alpha^\star)_\star^{-1}$ and Λ_C are one-to-one order epimorphisms, the former because it is an isomorphism and the latter by Theorem 5.5.18, β is also a one-to-one order epimorphism. So, it remains to prove that $\beta \circ \alpha = \Lambda_A$, which is, by definition of β, equivalent to prove that $\Lambda_C \circ \alpha = (\alpha_\star)^\star \circ \Lambda_A$. Since the latter equation was proved to be true in Theorem 5.5.18, the proof is concluded.

(3. ⇒ 2.) Since $\beta \circ \alpha = \Lambda_A$, $\alpha^\star \circ \beta^\star = \Lambda_A^\star$. Hence, as $\Lambda^\star$ and $\beta^\star$ are isomorphisms, so is $\alpha^\star$. □

Lemma 5.7.2. *If A is a semibal that admits dc and $\alpha : A \longrightarrow C$ is a semibal morphism, then the following conditions are equivalent:*

1. *α is a one-to-one order epimorphism,*
2. *$\alpha^\star$ is an isomorphism in* **KPSp***,*

4. *α is one-to-one and $\alpha(A)$ order separates the points of X_C,*

5. *$\alpha(A)$ is dense in C.*

Proof. (2. $\Rightarrow$ 4.) Let $\theta \not\lesssim \xi$ in X_C. As $\alpha^\star$ is an isomorphism, it follows that $\alpha^\star(\theta) \not\lesssim \alpha^\star(\xi)$. Hence, there exists $a \in A$ with $\lambda(a, \alpha^\star(\theta)) > \lambda(a, \alpha^\star(\xi)$, that is $\lambda(\alpha(a), \theta) > \lambda(\alpha(a), \xi)$, as required.

(4. $\Rightarrow$ 5.) If $\alpha(A)$ admits dc, then this will be a direct corollary of Theorem 5.6.5. Therefore, suppose that $\alpha(a) \geq r = \alpha(r)$. We have $\alpha(a) = \alpha(a \vee r)$ and, by injectivity, $a = a \vee r$. Hence, there exists $b \in A$ such that $a = r + b$, and, moreover, such that $\alpha(a) = r + \alpha(b)$.

(5. $\Rightarrow$ 1.) Let $\beta_1, \beta_2 : C \longrightarrow D$ be semibal morphisms such that $\beta_1 \circ \alpha \leq \beta_2 \circ \alpha$. Since $\alpha(A)$ is dense in B, for all $c \in C$ and for all $n \in \mathbb{N}$, there exists $a_n \in A$ such that $||\alpha(a_n) - c||$[1] $\leq \frac{1}{n}$. Hence, it is clear that $(\alpha(a_n))_{n \in \mathbb{N}}$ is a sequence that converges to c.

Now, for all congruences θ in X_D, the map $\lambda(\beta_i(\cdot), \theta)$ is continuous. Consequently, the sequence $(\lambda(\beta_i(\alpha(a_n)), \theta)_{n \in \mathbb{N}}$ converges to $\lambda(\beta_i(c), \theta)$. We thus have

$$\lambda(\beta_1(c), \theta) \leftarrow \lambda(\beta_1(\alpha(a_n)), \theta) \leq \lambda(\beta_2(\alpha(a_n)), \theta) \rightarrow \lambda(\beta_2(c), \theta).$$

Also, as the order on $\mathbb{R}^+$ is closed, it follows that $\lambda(\beta_1(c), \theta) \leq \lambda(\beta_2(c), \theta)$ for all $\theta \in X_D$. Therefore, by Proposition 5.4.14, we have $\beta_1(c) \leq \beta_2(c)$ as required.

□

Theorem 5.7.3. *Let A be a semibal. The following conditions are equivalent:*

1. *A is uniformly complete and admits dc,*

2. *every one-to-one order epimorphism $\alpha : A \longrightarrow C$ is an isomorphism,*

3. *Λ_A is an isomorphism.*

Proof. (1. $\Rightarrow$ 2.) We just have to prove that α is onto. Let c be an element of C, we need to prove that there exists an element $a \in A$ such that $\alpha(a) = c$. By Lemma 5.7.2, $\alpha(A)$ is dense in C. So, for all $n \in \mathbb{N}$, there exists an element $a_n \in A$ such that $||\alpha(a_n) - c|| \leq \frac{1}{n}$. Moreover, for all $n, m \in \mathbb{N}$, we have[2]

$$||a_n - a_m|| \leq ||\alpha^b(a_n - a_m)||. \tag{5.17}$$

Now, the sequence $(\alpha(a_n)_{n \in \mathbb{N}})$ converges to c by construction. Therefore, it is a Cauchy sequence. Hence, (5.17) implies that $(a_n)_{n \in \mathbb{N}}$ is also a Cauchy sequence and, as A is complete, it converges to a for some $a \in A$.

Now, let $\theta \in X_C$. As α and $\lambda(\cdot, \theta)$ are continuous, we have

$$\underbrace{\lambda(c, \theta) \leftarrow \lambda(\alpha(a_n), \theta) \rightarrow \lambda(\alpha(a), \theta).}_{\text{convergences in } \mathbb{R}}$$

It follows that $\lambda(c, \theta) = \lambda(\alpha(a), \theta)$, hence $\alpha(a)\ \theta\ c$. Since this is true for all $\theta \in X_C$, the equality $\alpha(a) = c$ follows from Corollary 5.4.9.

(2. $\Rightarrow$ 3.) By Theorem 5.5.18.

[1] Here it is important to remember that we work actually in A^b.

[2] Idem.

(3. $\Rightarrow$ 1.) Since $I(X_A, \mathbb{R}^+)$ admits dc and is complete, the conclusion is immediate. □

Definition 5.7.4. Let A be a semibal. We say that A is a **Stone semiring** if it is uniformly complete and admits dc. Moreover, we denote by **usbal** the full subcategory of **sbal** whose objects are Stone semirings.

Notation 5.7.5. Let us denote by χ the functor **sbal** $\longrightarrow$ **KPSp** which maps A to X_A and $\alpha : A \longrightarrow C$ to $\alpha^\star : X_C \longrightarrow X_A$. Also, let us denote by ι the functor **KPSp** $\longrightarrow$ **sbal** which maps X to $I(X, \mathbb{R}^+)$ and $f : X \longrightarrow Y$ to $f_\star : I(Y, \mathbb{R}^+) \longrightarrow I(X, \mathbb{R}^+)$.

Theorem 5.7.6 (Gelfand duality for compact po-spaces)**.** *Let X be an object of* **KPSp** *and A one of* **usbal**. *The applications Λ_A and ζ_X are natural isomorphisms for their respective categories. It follows that the functor χ and ι establish a dual equivalence between* **KPSp** *and* **usbal**.

Proof. By Theorems 5.7.3 and 5.5.18, Λ_A is a natural isomorphism.

On the other hand, by [43], ζ_X is an isomorphism and we have, for all $x \in X$ and $f \in \mathbf{KPSp}(X, Y)$ that

$$\begin{aligned}
(g, h) \in (f_\star)^\star(\zeta_X(x)) &\Leftrightarrow (f_\star(g), f_\star(h)) \in \zeta_X(x) \\
&\Leftrightarrow (g(f), h(f)) \in \theta_X \\
&\Leftrightarrow g(f(x)) = h(f(x)) \\
&\Leftrightarrow (g, h) \in \theta_{f(x)} = \zeta_Y(f(x)).
\end{aligned}$$

□

Theorem 5.7.7. *The duality between* **ubal** *and* **KHaus** *is a restriction of the duality establish in the previous Theorem.*

Proof. The conclusion follows easily from two facts. Firstly, it is not hard to prove that if B is bal, then B^+ is a Stone semiring. And, by Theorem 5.4.6, we have $\mathrm{MaxId}_\ell(B) \cong X_{B^+}$. Secondly, for a compact Hausdorff space X ordered by the equality, $I(X, \mathbb{R}^+) = C(X, \mathbb{R})^+$ and $(C(X, \mathbb{R})^+)^b \cong C(X, \mathbb{R})$ by Proposition 5.3.14. □

So, we have completed the outer square of dualities and equivalences, initiated by Bezhanishvili and Harding in [6]:

$$\begin{tikzcd}
\mathbf{KPSp} \arrow[rrr, no head] \arrow[ddd, dashed, no head] & & & \mathbf{StKFrm} \arrow[ddd, no head] \\
& \mathbf{KHaus} \arrow[ul] \arrow[r, no head] \arrow[d, no head] \arrow[dr, no head] & \mathbf{KRFrm} \arrow[ur] \arrow[d, no head] & \\
& \mathbf{ubal} \arrow[dl, dashed] & \mathbf{DeV} \arrow[dr] & \\
\mathbf{usbal} & & & \mathbf{PrFrm}
\end{tikzcd}$$

Conclusion and future work

Conclusion

In this thesis, we explored how to define subordination algebras, which are a generalisation of de Vries algebras [26], as models for tense logics. We proceeded via a suitable definition of their canonical extensions. To obtain these extensions, we established a strong topological duality between subordination algebras and subordination spaces and a strong discrete duality between complete atomic subordination algebras and Kripke structures. It should be noted that the topological duality encompasses several previous dualities: namely the modal one, the weak one [5] and the black one [14].

Once the dualities were settled, the construction of the canonical extension was obtained as it is classically done in modal setting: namely, concatenating the topological and the discrete duality with the forgetful functor such as to have a functor from subordination algebras to complete atomic subordination algebras. Finally, we pointed out that complete atomic subordination algebras and complete atomic tense algebras are equivalent categories, hence concluding the first step.

As soon as subordination algebras were recognised as model for tense logics, we studied completeness, correspondence and canonicity results. The family of canonical, and translatable, formulas presented here, in the subordination setting, were a refinement of the canonical formulas for tense algebras. Namely, we moved from inductive formulas [19] to analytic inductive formulas (introduced in [41] in a totally different context). The main reason behind this refinement was the restrictions we conceded to guarantee the generalised Esakia lemma (Proposition 2.7.10). We also had an opportunity in Sections 2.10 and 4.8 to discuss the relation between the subordination and the subordination and the tense language. In particular, we established a fragment of subordination statements which admit equivalent tense formulas.

The subordination algebras were then presented as a particular case of slanted lattices. Therefore, it seemed natural to extend (as it is done algebraicly in [25]) the results previously obtained to this wider class of algebras. We gave in this thesis the required topological tools and describe the outlines of the canonicity result in Chapter 3. Then, we gave a duality result between slanted lattices and slanted Priestley spaces and the resulting topological construction of canonical extensions of slanted lattices.

We also took advantage of the opportunity given by the slanted setting to determine conditions for a pre-contact relation [28] and a subordination relation to be generated by a unique closed relation on the dual.

Finally, we turned to another generalisation of de Vries algebras: the category **PrFrm** of Proximity frames. They are part of an "outside" triangle (see [6]) of equivalences and dualities between the categories **KPSp**, **StKFrm** and **PrFrm**. This outside triangle was an incomplete generalisation of an inside square of dualities and equivalences whose corners were **DeV**, **KHaus**, **KRFrm** and the category **ubal**, introduced in [7]. Hence, there were a missing corner in the outside square. We filled the gap with the category **usbal** which we proved to be dual to **KPSp**. Roughly speaking, the duality was obtained by an adequate axiomatisation of semi-rings of positive increasing continuous functions associated to compact po-spaces. To actually conclude the generalisation, we also established an adjunction between the categories **usbal** and **ubal** that mirrored the forgetful and the inclusion functors between **KPSp** and **KHaus**.

Future works

We present now some questions and ideas for future research.

Untranslatable formulas in the subordination setting

In Chapter 2, we saw that there were standard modal formulas which were canonical for standard modal algebras but not for subordination ones (with our usual counterexample formula $p \to \Diamond\Box p$). What we did not see however is the existence of translatable formulas in the modal setting but not in the subordination one. An obvious candidate should be $p \to \Diamond\Box p$ which is already known to not admit the "modal" translation (see Example 2.9.1).

Complete the universal algebra approach of slanted lattices

Universal algebra already turned out to be of great use in the characterisation of canonical formulas for subordination algebras. See for instance Corollary 2.6.9 and Theorem 2.7.16. The main remaining problem in this area is certainly to determine wether a categorical product exists or not. Indeed, since the canonical extension of an arbitrary Cartesian product is not the Cartesian product of the canonical extensions, we cannot play the usual game. Moreover, it was already remarked in [24] that the Cartesian product was not the categorical one for subordination algebras.

A de Vries-like duality for slanted lattices

While the category **SubAlg** is clearly a generalisation of the category **DeV** at the objects level, it is not the case anymore at the morphisms one. The other blatant difference between **SubAlg** and **DeV** lies in their respective dual category **SubSp** and **KHaus**: the latter contains all the required information in is topological part and, hence, does not require the presence of a binary relation relation as the former. To find an actual generalisation of de Vries duality for subordination algebras, we should turn to the work of Düntsh and Winter in [31]. Note that this generalisation does not consider the question of morphisms and is restricted to the Boolean setting.

Three languages in slanted lattices

In the introduction of Section 2.7, we stated that subordination algebras were the right environment to explore the relations between three languages: the modal/tense one, the language of the subordination relation $\prec$ and the language of the accessibility relation R. Such interconnections are also naturally available in slanted lattices since, for instance, from an n-ary c-slanted operator $\triangle$ comes the subordination-like relation $(\underline{a}, b) \in \prec_\triangle$ if and only if $\triangle(\underline{a}) \leq b$.

Topology in display calculi

We saw in Chapter 3 that the analytic inductive inequalities where a fragment of the canonical inequalities on slanted lattices. Recall that the analytic inductive inequalities were introduced in [41] in the context of analytic calculi in structural proof theory, to characterize the logics which can be presented by means of proper display calculi. The reasons behind this unexpected result are still to be explored.

Close the squares of Chapters 1 and 5

The corner of the squares of equivalences and dualities in Chapters 1 and 5, namely

$$\begin{tikzcd}
\mathbf{KPSp} \arrow[rrr, no head] \arrow[ddd, dashed, no head] & & & \mathbf{StKFrm} \arrow[ddd, no head] \\
& \mathbf{KHaus} \arrow[ul] \arrow[r, no head] \arrow[d, no head] \arrow[dr, no head] & \mathbf{KRFrm} \arrow[ur] \arrow[d, no head] & \\
& \mathbf{ubal} \arrow[dl, dashed] & \mathbf{DeV} \arrow[dr] & \\
\mathbf{usbal} & & & \mathbf{PrFrm}
\end{tikzcd}$$

are well established. However, several direct descriptions of the functors that link them together are missing. In Chapter 1, we gave the functor from **ubal** to **DeV** but the converse functor is still missing. Moreover, for instance, the functors between **usbal** and **PrFrm** also lack of a direct description.

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Appendix A

Notions of category theory

A.1 Introductory example

In this appendix, we describe the duality between Boolean algebras and Stone spaces due to Stone (see [70]) and use it as an introductory example for the terminology of category theory.

We assume the reader is familiar with notions and techniques in universal algebras, Boolean algebras and topological spaces. Indeed, we will limit ourself to give the required basic definitions and results. But we redirect the interested reader to our standard references, which are respectively [11], [39] and [50]. We also redirect the reader to [52] for a larger insight into category theory.

A.1.1 Boolean algebras and Stone spaces

Definition A.1.1. 1. A **lattice** is an algebra $L = (L, \wedge, \vee)$ where $\wedge$ and $\vee$ are binary operations such that for every $a, b \in L$

   B1. $\vee$ and $\wedge$ are idempotent,

   B2. $\vee$ and $\wedge$ are commutative,

   B3. $\vee$ and $\wedge$ are associative,

   B4. $a \vee (a \wedge b) = a = a \wedge (a \vee b)$.

2. A lattice L is **distributive** if for every $a, b, c \in L$, we have

   B5. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

3. A lattice L is **bounded** if there exist elements $0, 1 \in L$ such that for all $a \in L$

   B6 $a \wedge 1 = a$ and $a \vee 0 = a$.

4. A lattice L is **complete** if for every subset $S \subseteq L$ there exist an element $a \in L$ such that $a \leq s$ for every $s \in S$ and such that

$$(c \leq s \ \forall s \in S) \Rightarrow c \leq a.$$

   It is usual to denote this element a by $\wedge S$.

5. A **Boolean algebra** is an algebra $B = (B, \wedge, \vee, \neg)$ such that $(B, \wedge, \vee)$ is a distributive lattice and $\neg$ is an unary operation on B such that for every $a \in L$

B7 $a \wedge \neg a = 0$ and $a \vee \neg a = 1$,

B8 $\neg(\neg a) = a$,

B9 $\neg(a \wedge b) = \neg a \wedge \neg b$ and $\neg(a \vee b) = \neg a \vee \neg b$.

The element $\neg a$ is called the **complement** of a.

6. Let B, C be Boolean algebras. A map $h : B \longrightarrow C$ is a **Boolean morphism** if h respects $\vee$, $\wedge$, 0 and 1.

Lemma A.1.2. *1. In presence of B1 - B4, the axiom B5 is equivalent to the axiom*

B5' $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

2. If B is a lattice, then the binary relation $\leq$ on B defined by

$$a \leq b \Leftrightarrow a = a \wedge b$$

is an order.

3. If B is a Boolean algebra, then for $a, b \in B$, $a \leq b$ is equivalent to $a \wedge \neg b = 0$.

Definition A.1.3. Let B be a lattice and F a non-empty subset of B. We said that F is a **filter** if F is:

1. an increasing subset, that is $a \in F$ and $a \leq b$ implies $b \in F$ and
2. closed under $\wedge$, that is, $a, b \in F$ implies $a \wedge b \in F$.

Dually, a subset I of B is an **ideal** if I is a decreasing subset which is closed under $\vee$.

A filter F of B is an **ultrafilter** if it is maximal among the proper filters of B with respect to the inclusion. Usually, we will denote an ultrafilter by x. Also, we say that an ideal I of B is a **maximal ideal** if it is maximal among the proper ideals of B with respect to the inclusion.

A filter F of B is a **prime filter** if it satisfies for every $a, b \in B$

$$a \vee b \in F \Rightarrow a \in F \text{ or } b \in F.$$

Dually, an ideal I of B is a **prime ideal** if it satisfies for every $a, b \in B$

$$a \wedge b \in I \Rightarrow a \in I \text{ or } b \in I.$$

Notation A.1.4. Let B be a Boolean algebra and S be a subset of B. We denote

$$\uparrow S = \{a \in B \mid \exists s \in S : s \leq a\} \text{ and } \downarrow S = \{a \in B \mid \exists s \in S : s \geq a\}.$$

Of course, if $S = \{a\}$, we will write $\uparrow a$ and $\downarrow a$ instead of $\uparrow \{a\}$ and $\downarrow \{a\}$.

With this notation in mind, F is a filter if and only if $F = \uparrow F$ and $F = F \wedge F$.

In a Boolean algebra, the notions of ultrafilter and prime filter coincide, as shown in the next theorem. We also give other characterisations of ultrafilters.

Theorem A.1.5. *Let F be a filter of a Boolean algebra B.*

1. The filter F is an ultrafilter if and only if it is a prime filter.

2. There is an ultrafilter x such that $F \subseteq x$.

3. *The filter F is an ultrafilter if and only F^c is a maximal ideal.*
4. *The filter F is prime if and only F^c is a prime ideal.*

Theorem A.1.6 (Stone Theorem)**.** *Let B be a Boolean algebra, F a filter of B and I an ideal of B. There exists an ultrafilter x such that $F \subseteq x$ and $x \cap I = \emptyset$.*

We can now take an interest in the topological side of Stone duality.

Definition A.1.7.

1. A **topological space** is a pair (X, τ) where X is a set and τ is a collection of subsets of X such that:
   (a) $\emptyset \in \tau$ and $X \in \tau$,
   (b) τ is closed under finite intersections,
   (c) τ is closed under arbitrary unions.

   The elements of τ are called **open sets** and τ itself is called a **topology**. Furthermore, a **closed set** is the complementary of an open set and a **clopen set** is a set which is both open and closed.

   Let us note that we will often, if not always, abuse notation and write X instead of (X, τ) if the context is clear.

2. Let X, Y be two topological spaces. A map $f : X \longrightarrow Y$ is a **continuous function** if for every open set O of Y, $f^{-1}(O)$ is an open set of X.

Definition A.1.8. Let (X, τ) be a topological space. We say that (X, τ) is:

1. **Hausdorff** if for every pair x, y of distinct elements of X, there exist two disjoints elements O_x and O_y in τ such that $x \in O_x$ and $y \in O_y$,
2. **compact** if for every family $(O_i \mid i \in I)$ of τ such that $X = \cup\{O_i \mid i \in I\}$, there exists a finite subset $I' \subseteq I$ such that $X = \cup\{O_i \mid i \in I'\}$.
3. **zero dimensional** if for every $O \in \tau$, there is a family $(U_i \mid i \in I)$ of clopen sets such that $O = \cup_i U_i$,
4. a **Stone space** if it is Hausdorff, compact and zero dimensional.

Theorem A.1.9. *Let $f : X \longrightarrow Y$ be a continuous bijection between X a compact space and Y a Hausdorff space. Then f^{-1} is also a continuous function.*

A.1.2 Stone duality

We now prove step by step the duality between Boolean algebras and Stone spaces, beginning with the construction of a Stone space X_B associated to a Boolean algebra B. We denote by $X_B = \mathrm{Ult}(B)$ the set of ultrafilters on B. We can endow $\mathrm{Ult}(B)$ with a topology where the open sets are unions of sets of the form

$$\eta(a) := \{x \in \mathrm{Ult}(B) \mid x \ni a\} \tag{A.1}$$

with $a \in B$ (we say that the topology is **generated** by $\{\eta(a) \mid a \in B\}$). We now have to prove that $\mathrm{Ult}(B)$ is a Stone space. To help us with the proof, we can consider first this preliminary Lemma.

Lemma A.1.10. *If B is a Boolean algebra and $a, b \in B$, then*

1. $\eta(a)^c = \eta(\neg a)$,
2. $\eta(a \wedge b) = \eta(a) \cap \eta(b)$,
3. $\eta(a \vee b) = \eta(a) \cup \eta(b)$,
4. $\eta(0) = \emptyset$ *and* $\eta(1) = \mathrm{Ult}(B)$.

Proof. The proof follows immediately from the definition of ultrafilter and from Theorem A.1.5. $\square$

Theorem A.1.11. *If B is a Boolean algebra, then* $\mathrm{Ult}(B)$ *is a Stone space.*

Proof. By Lemma A.1.10, it is clear that $\mathrm{Ult}(B)$ is a topological space and that $\eta(a)$ is a clopen set for every $a \in B$. Consequently, the space $\mathrm{Ult}(B)$ is zero dimensional.

We prove now that $\mathrm{Ult}(B)$ is a Hausdorff space. Let x and y be two distinct elements of $\mathrm{Ult}(B)$. Then, there is an element $a \in B$ such that $\eta(a) \ni x$ and $\eta(\neg a) \ni y$. Since $\eta(a) \cap \eta(\neg a) = \eta(0) = \emptyset$, the conclusion is immediate.

Finally, we have to show that $\mathrm{Ult}(B)$ is a compact space. Suppose that

$$\mathrm{Ult}(B) \subseteq \cup\{\eta(s) \mid s \in S \subseteq B\}.$$

To conclude the proof, it is sufficient to find elements $s_1, \ldots, s_n \in S$ that satisfy $s_1 \vee \ldots \vee s_n = 1$, as we would have

$$\mathrm{Ult}(B) = \eta(1) = \eta(s_1) \cup \ldots \cup \eta(s_n).$$

Suppose by contradiction that it is impossible to find such $s_1, ..., s_n$. Then S is contained in some maximal ideal $\mathfrak{I}$. It follows that $x = \mathfrak{I}^c$ is an ultrafilter such that $x \notin \eta(s)$ for every $s \in S$, which is impossible. $\square$

Let us now consider the other direction, that is, we want to associate a Boolean algebra to a given Stone space.

Theorem A.1.12. *If X is a Stone space, then*

$$\mathrm{Clop}(X) := \{O \subseteq X \mid O \textit{ is a clopen set}\}$$

is Boolean algebra with operations $\cap$, $\cup$, $\cdot^c$ and constants $\emptyset$ and X.

Proof. It is purely routine. $\square$

A question that arises naturally now is what happens when we combine both constructions. What we would like to obtain is that B and $\mathrm{Clop}(\mathrm{Ult}(B))$ and X and $\mathrm{Ult}(\mathrm{Clop}(X))$ are "analogue", at least up to isomorphism. But, of course, we have first to define what isomorphic means in both situations.

Definition A.1.13.
1. We say that two Boolean algebras B and C are **isomorphic**, and denote it by $B \cong C$, if there exists a bijective Boolean morphism $h : B \longrightarrow C$.
2. We say that two topological spaces X and Y are **homeomorphic**, and also denote it by $X \cong Y$, if there exists a continuous bijective function $f : X \longrightarrow Y$ such that $f^{-1} : Y \longrightarrow X$ is also continuous.

We can remark almost immediately that there is a substantial difference between the algebraic and the topological case. It is actually due to the fact that if h is a bijective Boolean morphism, then h^{-1} is still a Boolean morphism, whereas it is not the case for f a continuous function. Nevertheless, for Stone spaces, thanks to Theorem A.1.9, we know that we can restrict our definition to a more "algebraic one".

With these considerations in mind, it remains to prove that B is isomorphic to Clop(Ult(B)) and that X is homeomorphic to Ult(Clop(X)). Using (A.1) and Lemma A.1.10, we already know that the map

$$\eta : B \longrightarrow \mathrm{Clop}(\mathrm{Ult}(B)) : a \longmapsto \eta(a)$$

is well-defined and is a Boolean morphism. Hence, it remains to prove that is one-to-one and onto.

Proposition A.1.14. *The map η defined in* (A.1) *is bijective.*

Proof. To show that η is one-to-one, suppose $a, b \in B$ such that $a \not\leq b$. It follows that $(a \wedge \neg b) \neq 0$ and, consequently, there exists $x \in \mathrm{Ult}(B)$ such that $(a \wedge \neg b) \in x$. Therefore, we have $a \in x$ and $b \notin x$ or, in other terms, $x \in \eta(a)$ and $x \notin \eta(b)$.

Now we want to prove that, for a clopen set O of Ult(B), there exists $a \in B$ such that $O = \eta(a)$. Since O is clopen, it is in particular open. Therefore, we have

$$O = \cup\{\eta(s) \mid s \in S\}$$

for a subset S of B. Moreover, since O is closed, it is also compact. Hence, we have

$$O = \eta(s_1) \cup ... \cup \eta(s_n) = \eta(s_1 \vee ... \vee s_n),$$

for some $s_1, ..., s_n \in S$, which concludes the proof. $\square$

On the other direction, let us define ε as the following assignment

$$\varepsilon : X \longrightarrow \mathrm{Ult}(\mathrm{Clop}(X)) : x \longmapsto \{O \in \mathrm{Clop}(X) \mid O \ni x\} \tag{A.2}$$

and let us prove it is a suitable function.

Proposition A.1.15. *Let X be a Stone space. The map ε defined in* (A.2) *is a homeomorphism.*

Proof. We have first to prove that ε is well-defined. It is clear that $\varepsilon(x)$ is a filter for every $x \in X$. Moreover, since

$$O \notin \varepsilon(x) \Leftrightarrow x \notin O \Leftrightarrow x \in O^c \Leftrightarrow O^c \in \varepsilon(x),$$

$\varepsilon(x)$ is in particular an ultrafilter.

We show now that ε is continuous. Let O be a clopen set of X, we have

$$\begin{aligned} \varepsilon^{-1}(\eta(O)) &= \{x \mid \varepsilon(x) \in \eta(O)\} \\ &= \{x \mid O \in \varepsilon(x)\} \\ &= \{x \mid x \in O\} = O, \end{aligned}$$

which is sufficient.

Since it follows easily from the fact that X a Hausdorff space that ε is one-to-one, it remains to prove that ε is onto. It is enough to prove that if $\mathfrak{F}$ is an ultrafilter on Clop(X), then there exists an $x \in X$ such that

$$x \in \cap\{O \mid O \in \mathfrak{F}\}.$$

Indeed, we will then have $\mathfrak{F} \subseteq \varepsilon(x)$ and therefore, by maximality of $\mathfrak{F}$ that $\mathfrak{F} = \varepsilon(x)$. But, the intersection $\cap\{O \mid O \in \mathfrak{F}\}$ is not empty since every finite intersection of elements of $\mathfrak{F}$ is not empty and X is compact. $\square$

We can now summarise the previous propositions by the following theorem.

Theorem A.1.16. *If B is a Boolean algebra and X is a Stone space, then $B \cong \mathrm{Clop}(\mathrm{Ult}(B))$ and $X \cong \mathrm{Ult}(\mathrm{Clop}(X))$.*

To continue our exploration of Stone duality, we need to extend Theorem A.1.16, established between Boolean algebras and Stone spaces, to morphisms and continuous functions. We proceed as follows.

Proposition A.1.17.

1. *Let B and C be two Boolean algebras and $h : B \longrightarrow C$ be a Boolean morphism, then*
$$h^\star : \mathrm{Ult}(C) \longrightarrow \mathrm{Ult}(B) : x \longmapsto h^{-1}(x)$$
is a continuous function.

2. *Let X and Y be two Stone spaces and $f : X \longrightarrow Y$ be a continuous function, then*
$$f_\star : \mathrm{Clop}(Y) \longrightarrow \mathrm{Clop}(X) : O \longmapsto f^{-1}(O)$$
is a Boolean morphism.

Proof.

1. First of all, we prove that $h^\star$ is well-defined. Indeed, for $x \in \mathrm{Ult}(C)$, $h^{-1}(x)$ is clearly a filter and since
$$a \in h^{-1}(x) \Leftrightarrow h(a) \in x \Leftarrow \neg h(a) \notin x \Leftrightarrow h(\neg a) \in x \Leftrightarrow \neg a \notin h^{-1}(x),$$
it is also a maximal one. Then, we prove that $h^\star$ is continuous. Let b be an element of B, we have
$$x \in (h^\star)^{-1}(\eta(b)) \Leftrightarrow h^\star(x) \in \eta(b) \Leftrightarrow b \in h^\star(x) \Leftrightarrow h(b) \in x \Leftrightarrow x \in \eta(h(b)).$$

2. It follows immediately from f continuous that $f_\star$ is well-defined and from the definitions that this is a Boolean morphism.

□

Theorem A.1.18.

1. *Let B and C be two Boolean algebras and $h, g : B \longrightarrow C$ be Boolean morphisms, then $h^\star = g^\star$ implies $h = g$. Moreover, we have that $(h^\star)_\star = \eta \circ h \circ \eta^{-1}$.*

2. *Let X and Y be two Stone spaces and $f, g : X \longrightarrow Y$ be continuous functions, then $f_\star = g_\star$ implies $f = g$. Moreover, we have that $(f_\star)^\star = \varepsilon \circ h \circ \varepsilon^{-1}$.*

Proof.

1. Suppose that there is $a \in B$ such that $h(a) \neq g(a)$. Then there exists an ultrafilter x such that $g(a) \in x$ and $h(a) \notin x$. This latter statement is clearly absurd since $h^\star = g^\star$ implies
$$\eta(h(a)) = h^\star(\eta(a)) = g^\star(\eta(a)) = \eta(g(a)).$$
Now, it is not hard to prove that for every $a \in B$, $\eta(h(a)) = (h^\star)_\star(\eta(a))$. Indeed,
$$\begin{aligned}(h^\star)_\star(\eta(a)) &= (h^\star)^{-1}(\eta(a)) \\ &= \{x \in \mathrm{Ult}(C) \mid h^\star(x) \in \eta(a)\} \\ &= \{x \in \mathrm{Ult}(C) \mid a \in h^\star(x)\} \\ &= \{x \in \mathrm{Ult}(c) \mid h(a) \in x\} = \eta(h(a))\end{aligned}$$

2. The proof follows the same process as the one of item 1.

□

What happened here between Boolean algebras and Stone spaces is just one example among many of two families of mathematical objects which may seem different at first glance but actually contain analogue information. Such relations are formalised in the mathematical field called category theory and we will now introduce it shortly.

A.2 First definitions in category theory

First of all, let us note that we will not use all the available powerful machinery in category theory as we will mostly applied it in specific cases. As we said previously, an interested reader should be directed to [52] for more on this subject.

A.2.1 Categories

Definition A.2.1. A **category** $\mathbf{C}$ is a triple $(\mathrm{Ob}(\mathbf{C}), \mathrm{Hom}(\mathbf{C}), \circ)$ where

- $\mathrm{Ob}(\mathbf{C})$ is the class containing the **objects** of $\mathbf{C}$, note that we will often say, through misuse of language, that an object c is in $\mathbf{C}$ instead of $\mathrm{Ob}(\mathbf{C})$,
- $\mathrm{Hom}(\mathbf{C})$ is the class containing the **morphisms**, or **arrows**, between two objects of $\mathbf{C}$. To every morphism f is associated a **domain** object c and an **image** object d. Often, we will say that "f is a morphism from c to d" and write it $f : c \longrightarrow d$. We denote then by $\mathrm{Hom}_{\mathbf{C}}(c, d)$, or more simply by $\mathbf{C}(c, d)$, the class of morphisms from c to d.
- $\circ$ is a binary operation, called **morphisms composition**, such that for every c, d, e objects of $\mathbf{C}$ and for every f in $\mathbf{C}(c, d)$ and every g in $\mathbf{C}(d, e)$, the element $g \circ f$ is a morphism of $\mathbf{C}(c, e)$. Moreover, $\circ$ satisfies the following properties.
  1. For every object c, there exists a morphism in $\mathbf{C}(c, c)$, denoted 1_c and called the **identity morphism**, such that for every morphism $f \in \mathbf{C}(c, d)$ and $g \in \mathbf{C}(d, c)$, we have $1_c \circ g = g$ and $f \circ 1_c = f$.
  2. The operation is associative.

Example A.2.2. There are legions of examples of categories. Of course, what we have in mind after the introductory example are the category denoted **Bool** whose objects are Boolean algebras, morphisms are Boolean morphisms and morphisms composition is simply the habitual composition and the category denoted **Stone** whose objects are Stone spaces, morphisms are continuous functions and morphisms composition is again the habitual composition.

Here are some other examples. The first one is given by a Boolean algebra B itself. Indeed B is a category whose objects are the elements of B and where there is a morphism between two elements $a, b \in B$ when $a \leq b$. Since $a \leq a$, there is an identity morphism and the other requirements of composition follow simply from transitivity.

Secondly, we could consider the category $\mathbf{Stone}^\star$ whose objects are Stone spaces and morphisms are arbitrary functions, instead of continuous one.

These two examples lead to some remarks:

1. The morphisms of a category are not necessarily maps between objects. Furthermore, the morphisms composition is not necessarily the habitual composition, even when the morphisms are maps. We encountered such a situation with the category **DeV** of de Vries algebras in Chapter 1.

2. A given class of objects may be associated to several different classes of morphisms. An important example of this situation appeared in Chapter 2 where four different categories, namely **Sub**, $\Diamond$**Sub**, $\blacklozenge$**Sub** and **sSub**, share a common class of objects, subordination algebras, but have different classes of morphisms.

Let us now define some special sorts of morphisms that will come in handy in our approach of the category **sbal** of semi-bounded Archimedean ℓ-algebras

Definition A.2.3. Let $\mathbf{C}$ be a category and c, d be two objects of C. A morphism f in $\mathbf{C}(c, d)$ is said to be:

1. an **isomorphism** if there exists a morphism $g \in \mathbf{C}(d, c)$ such that $f \circ g$ and $g \circ f$ are identities morphisms,
2. an **epimorphism** if for every pair $g_1, g_2 \in \mathbf{C}(d, e)$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$,
3. a **monomorphism** if for every pair $g_1, g_2 \in \mathbf{C}(e, c)$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$,
4. a **bimorphism** if it is both an epimorphism and a monomorphism.

A.2.2 Functors

Definition A.2.4. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A **(covariant) functor** T from $\mathbf{C}$ to $\mathbf{D}$ is a map that assigns to each object c in $\mathbf{C}$ an object $T(c)$ in $\mathbf{D}$ and to each morphism f in $\mathbf{C}(c_1, c_2)$ a morphism $T(f)$ in $\mathbf{D}(T(c_1), T(c_2))$ such that for every c in $\mathbf{C}$, we have $T(1_c) = 1_{T(c)}$ and such that for every pair f, g of morphisms in $\mathbf{C}$, we have $T(f \circ g) = T(f) \circ T(g)$.

A mindful reader might have noticed that the definition of functor we encountered, while quite natural, does not correspond to the situation between Boolean algebras and Stone spaces exposed earlier in Proposition A.1.17. For this reason, we need to introduce another kind of functors.

Definition A.2.5. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A **contravariant functor** T from $\mathbf{C}$ to $\mathbf{D}$ is a map that assigns to each object c in $\mathbf{C}$ an object $T(c)$ in $\mathbf{D}$ and to each morphism f in $\mathbf{C}(c_1, c_2)$ a morphism $T(f)$ in $\mathbf{D}(T(c_2), T(c_1))$ such that for every c in $\mathbf{C}$, we have $T(1_c) = 1_{T(c)}$ and such that for every pair f, g of morphisms in $\mathbf{C}$, we have $T(f \circ g) = T(g) \circ T(f)$.

Remark A.2.6. It is worth noticing that we can compose functors. For instance, if T is a functor from $\mathbf{C}$ to $\mathbf{D}$ and if S is a functor from $\mathbf{D}$ to $\mathbf{E}$, then $T \circ S$ is the functor from $\mathbf{C}$ to $\mathbf{E}$ which sends an object c to the object $S(T(c))$ and a morphisms f to the morphism $S(T(f))$. The composition of two covariant or contravariant functors gives a covariant functor, while the composition of a covariant and a contravariant functor gives a contravariant one.

One-to-one and onto concepts can be generalised to the functors situation as we will see with the next definition.

Definition A.2.7. Let T be a functor from the category $\mathbf{C}$ to the category $\mathbf{D}$. We say that T is :

- **faithful** if for any morphism g in $\mathbf{D}(T(c), T(c'))$, there exists a morphism f in $\mathbf{C}(c, c')$ such that $g = T(f)$.
- **full** if for every pair c, c' of objects in $\mathbf{C}$ and for every pair of morphisms f, g in $\mathbf{C}(c, c')$, $T(f) = T(g)$ implies $f = g$.

- an **isomorphism** if T is a bijection both for objects and for morphisms.

Example A.2.8. We will give a first theoretical but very important example of functor. Let $\mathbf{C}$ be a category such for every pair (c,d) of objects in $\mathbf{C}$, the class $\mathbf{C}(c,d)$ is a set. Let c be a fixed object in $\mathbf{C}$ et let us consider the covariant functor $\mathrm{Hom}(c,\cdot)$ from $\mathbf{C}$ to $\mathbf{Set}$ which sends an object d in $\mathbf{C}$ to $\mathbf{C}(c,d)$ and a morphism $f : d \longrightarrow d'$ to the map

$$\mathrm{Hom}(c,f) : \mathbf{C}(c,d) \longrightarrow \mathbf{C}(c,d') : g \longmapsto f \circ g.$$

This last map will be denoted $f_\star$(not to be confused with $f_\star$ in A.1.17) and will be called **left composition with** f.

Still considering the category $\mathbf{C}$ and the fixed object c, let us define the contravariant functor $\mathrm{Hom}(\cdot,c)$ which sends an object d to the set $\mathbf{C}(d,c)$ and a morphism $f : d \longrightarrow d'$ to the map

$$\mathrm{Hom}(f,c) : \mathrm{Hom}(d',c) \longrightarrow \mathrm{Hom}(d,c) : g \longrightarrow g \circ f.$$

This map is often denoted $f^\star$ (still not to be confused with $h^\star$ in Proposition A.1.17) and called **right composition with** f.

Before giving other examples of functors, let us define the notion of subcategory.

Definition A.2.9. Let $\mathbf{C}$ be a category. A **subcategory** $\mathbf{S}$ of $\mathbf{C}$ is a collection $\mathrm{Ob}(\mathbf{S})$ of some of the objects of $\mathbf{C}$ and a collection $\mathrm{Hom}(\mathbf{S})$ of some of the arrows of $\mathbf{C}$ such that for every object s in $\mathrm{Ob}(\mathbf{S})$, the idendity arrow 1_s is in $\mathrm{Hom}(\mathbf{S})$, for every arrow f in $\mathrm{Hom}(\mathbf{S})$, the domain and the image of f are in $\mathrm{Ob}(\mathbf{S})$ and for every pair (f,f') of composable arrows in $\mathrm{Hom}(\mathbf{S})$, their composition $f \circ f'$ is also in $\mathrm{Hom}(\mathbf{S})$.

It is not hard to convince itself that a subcategory is a category and that the map which sends every object every arrows in $\mathbf{S}$ to itself in $\mathbf{C}$ is a covariant functor from $\mathbf{S}$ to $\mathbf{C}$. It is called the **inclusion functor**.

Finally, we will be interested in the special case of **full subcategories**, that is subcategories $\mathbf{S}$ of categories $\mathbf{C}$ such that if s,t are objects of $\mathbf{S}$ then every morphism f between s and t in $\mathbf{C}$ is also a morphism between s and t in $\mathbf{S}$. Of course, $\mathbf{S}$ is a full subcategory of $\mathbf{C}$ if and only if the inclusion functor is a full functor.

Example A.2.10.

1. A first obvious example of functor is the **identity functor** which sends the objects and the morphisms of a category $\mathbf{C}$ to themselves. We denoted it by $I_{\mathbf{C}}$.
2. If we go back to Definition A.1.1, let us consider the category **DLat** whose objects are bounded distributive lattices and whose morphisms are Boolean morphisms. The category **Bool** is then a subcategory of **DLat**. Moreover, if we consider the category **Bool′** whose objects are Boolean algebras and whose morphisms are Boolean morphsisms h such that $h(\neg a) = \neg h(a)$, then **Bool′** is a full subcategory of **DLat**. It follows that **Bool** and **Bool′** are actually the same category.
3. The map which sends a Boolean algebra B to the Stone space $\mathrm{Ult}(B)$ and a Boolean morphisms $h : B \longrightarrow C$ to the continuous function $h^\star : \mathrm{Ult}(C) \longrightarrow \mathrm{Ult}(B) : x \longmapsto h^{-1}(x)$ is a functor, denoted Ult, from the category **Bool** to the category **Stone**.

   On the other hand, the map which sends a Stone space X to the Boolean algebra $\mathrm{Clop}(X)$ and a continuous function $f : X \longrightarrow Y$ to the Boolean morphism $f_\star : \mathrm{Clop}(Y) \longrightarrow \mathrm{Clop}(X) : O \longmapsto f^{-1}(O)$ is a functor, denoted Clop, from the category **Stone** to the category **Bool**. Of course, there is more to say about the functors Ult and Clop than just observing they are indeed functors. In order to deal with them in depth, we present the required theory in the next section.

A.3 Adjunctions, equivalences and dualities

Definition A.3.1. Let T and S be two functors from $\mathbf{C}$ to $\mathbf{D}$, a **natural transformation** τ from T to S is map that sends an object c in $\mathbf{C}$ to a morphism τ_c in $\mathbf{D}$, called the **component** of the natural transformation τ in c, such that for every morphism f in $\mathbf{C}(c, c')$ we have $\tau_c \circ T(c) = S(c) \circ \tau_c$, i.e. the following diagram is commutative

$$\begin{tikzcd} T(c) \arrow[r, "\tau_c"] \arrow[d, "T(f)"'] & S(c) \arrow[d, "S(f)"] \\ T(c') \arrow[r, "\tau_{c'}"'] & S(c') \end{tikzcd}$$

A **natural isomorphism** is a natural transformation such that every of its component is an isomorphism of $\mathbf{D}$.

Definition A.3.2. Two categories $\mathbf{C}$ and $\mathbf{D}$ are **equivalent** (resp. **dually equivalent**) if there exist covariant (resp. contravariant) functors T from $\mathbf{C}$ to $\mathbf{D}$ and S from $\mathbf{D}$ to $\mathbf{C}$ such that there exists a natural isomorphism between $S \circ T$ and $I_{\mathbf{C}}$ just as between $T \circ S$ and $I_{\mathbf{D}}$. In that case, we say that the functors T and S establish a **equivalence** (resp. a **duality**) between $\mathbf{C}$ and $\mathbf{D}$.

Example A.3.3. The fact that **Bool** and **Stone** are dually equivalent comes as no surprise. Indeed, consider the map τ sending a Boolean algebra B to the map

$$\eta_B : B \longrightarrow \mathrm{Clop}(\mathrm{Ult}(B)) : b \longmapsto \{x \in \mathrm{Ult}(B) \mid b \in x\}.$$

We already know, thanks to Theorems A.1.16 and A.1.18, that η_B is an isomorphism and that the diagram

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "h"'] & \mathrm{Clop}(\mathrm{Ult}(B)) \arrow[d, "\mathrm{Clop}(\mathrm{Ult}(h))"] \\ C \arrow[r, "\eta_C"'] & \mathrm{Clop}(\mathrm{Ult}(C)) \end{tikzcd}$$

is commutative for every h in $\mathbf{Bool}(B, C)$.

To find the other natural isomorphism, it suffices to sends a Stone space X to the continuous function

$$\varepsilon_X : X \longrightarrow \mathrm{Ult}(\mathrm{Clop}(X)) : x \longmapsto \{O \in \mathrm{Clop}(X) \mid x \in O\}.$$

We thus have the following theorem.

Theorem A.3.4. *The category* **Bool** *and the category* **Stone** *are dually equivalent. This will be denoted by the following diagram.*

$$\mathbf{Bool} \text{ --- } \mathbf{Stone}$$

Definition A.3.5. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. An **adjunction** from $\mathbf{D}$ to $\mathbf{C}$ is a triple (T, S, φ) where T is a covariant functor from $\mathbf{D}$ to $\mathbf{C}$, S a covariant functor from $\mathbf{C}$ to $\mathbf{D}$ and φ is a map which sends every pair (c, d), where c is an objects of $\mathbf{C}$ and d one of $\mathbf{D}$, to a bijection $\varphi_{c,d}$

$$\mathbf{C}(T(d), c) \cong^{\varphi_{c,d}} \mathbf{D}(d, S(c)) \tag{A.3}$$

such that the following diagrams are commutative

$$\begin{tikzcd} \mathbf{C}(T(d), c) \arrow[r, "\varphi"] \arrow[d, "k_\star"'] & \mathbf{D}(d, S(c)) \arrow[d, "S(k)_\star"] \\ \mathbf{C}(T(d), c') \arrow[r, "\varphi"'] & \mathbf{D}(d, S(c')) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C}(T(d), c) \arrow[r, "\varphi"] \arrow[d, "T(h)^\star"'] & \mathbf{D}(d, S(c)) \arrow[d, "h^\star"] \\ \mathbf{D}(T(d'), c) \arrow[r, "\varphi"'] & \mathbf{D}(d', S(c)) \end{tikzcd}$$

where k is a morphism of $\mathbf{C}(c, c')$, h is a morphism of $\mathbf{D}(d, d')$ and we consider the notation of Example A.2.8. Using the notations of [75], we will denote the adjunction by $T \dashv S$.

In particular, T is said to be a **left adjoint** for S and S to be a **right adjoint** for T.

Remark A.3.6. In (A.3), consider the case where the object c is $T(d)$ for some object d of D. We then obtain

$$\mathbf{C}(T(d), T(d)) \cong \mathbf{D}(d, S(T(d))).$$

Since $1_{T(d)}$ is a morphism of $\mathbf{C}(T(d), T(d))$, it follows that $\varphi(1_{T(d)})$ is a morphism of $\mathbf{D}(d, S(T(d)))$. We call this morphism the **unit** of the adjunction and denote it by η_d.

On the other hand, the **co-unit** of the adjunction, denoted by ε_c, is the morphism

$$\varphi^{-1}(1_{S(c)}) : T(S(c)) \longrightarrow c.$$

It is clear that the roles of unit and co-unit in Stone duality are fulfilled by η and ε.

We end this short presentation of category theory with another characterisation of (dually) equivalent categories.

Definition A.3.7. An **adjoint equivalence** between categories is an adjunction (T, S, φ) such that its unit and its co-unit are natural isomorphisms.

Theorem A.3.8. *Let $T : \boldsymbol{D} \longrightarrow \boldsymbol{C}$ be a functor. The following are equivalent:*

1. *$\boldsymbol{D}$ and $\boldsymbol{C}$ are equivalent categories,*
2. *T is a full and faithful functor such that for every c in $\mathbf{C}$, there exists d in $\mathbf{D}$ such that c and $T(d)$ are isomorphic,*
3. *there exists a functor $S : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ such that T and S form an adjoint equivalence.*

Proof. See [52], Theorem IV.4.1. □

Appendix B

Some extensions of Stone duality

It is possible to generalise Stone duality to categories weaker than **Bool** and **Stone**. For instance, we already saw in Example A.2.10 that **Bool** is a subcategory of **DLat**, whose objects are bounded distributive lattices not necessarily complemented. One may ask not only if there exists a topological category playing for **DLat** a similar role to the one played by **Stone** for **Bool**, but also if this newer duality somehow extends Stone's one. This topological category exists and is the category **Priest** of Priestley spaces as proved by Priestley in [56] and [57].

On the other hand, the category **Stone** could be weakened by dropping the zero dimensional property and therefore forming the category **KHaus** of compact Hausdorff spaces equipped with continuous functions.

It was proved by de Vries in [26] that **KHaus** is dually equivalent to the category of compingent algebras, more commonly called de Vries algebras. A de Vries algebra is a hybrid structure, both algebraic and relational, a concept that popped under different names through history. We can for instance quote contact algebras, subordination algebras or quasi-modal algebras. We will explore these different concepts in other chapters as the scope of this chapter is to expose the extensions of Stone duality we just discussed.

The common idea behind Stone, Priestley and de Vries dualities is that topological spaces are characterised by a particular subfamily of their topology (namely the clopen, the increasing clopen and the regular open sets). However, topological spaces can also be characterised by their ring of real and complex continuous functions. This fact leads to the duality between **KHaus** and **ubal** of [7], to the Gelfand-Neumark duality (see [37]) between **KHaus** and $C^\star$-algebras (see [46, Chapter IV] for proofs and relevant definitions).

All the extensions of Stone duality we considered until now in this introduction share a common feature: weakening one of the categories involved. However, we can seen extensions from another angle: we keep the Boolean algebras and add to them supplementary structures. Instances of this kind of extensions are of course subordination algebras, but also modal algebras, which are respectively dually equivalent to subordination and modal spaces.

B.1 Priestley duality

The layout of Priestley duality is quite analogue to the layout of Stone's one. Some differences are nevertheless to note. First of all, the proper prime filters of a bounded distributive lattice are not ultrafilters anymore, as it was previously the case, as seen in Theorem A.1.5. Secondly, for a bounded distributive lattice L, if X_L denotes the set of proper prime filters of L, the set

$$\eta(a) = \{x \in X_L \mid x \ni a\}$$

may fail to be a clopen set for some $a \in L$ in the Stone topology. We thus move to the following construction.

Construction B.1.1. Let L be a bounded distributive lattice. We endow X_L with the topology generated by

$$\mathcal{A} = \{\eta(a) \mid a \in L\} \cup \{X_L \setminus \eta(a) \mid a \in L\}.$$

With this definition, we retrieve the fact that $\eta(a)$ is a clopen set for every $a \in L$ but one can remark that the characterisation of clopen sets given in Proposition A.1.14 for the Boolean case is no longer on the agenda as $X_L \setminus \eta(a)$ is a clopen set and, unless there exists $b \in L$ such that $a \wedge b = 0$ and $b \vee a = 1$, that is $b = \neg a$, there is no $b \in L$ such that $X_L \setminus \eta(a) = \eta(b)$.

Fortunately, by ordering the elements of X_L by inclusion, we obtain the following Proposition.

Proposition B.1.2. *Let L be a bounded distributive lattice. The increasing clopen sets of X_L are exactly the sets $\eta(a)$ for $a \in L$.*

Proof. It is obvious by construction that $\eta(a)$ is an increasing clopen set of X_L for every $a \in L$.

On the other hand, let O be an increasing clopen set. We can consider that O is a proper not empty subset of X_L, as otherwise, we would have $O = \eta(1)$ or $O = \eta(0)$ and the proof would be conclude. Thus we can consider prime filters x and y such that $x \in O$ and $y \in O^c$. As O is increasing, we have that $x \not\leq y$, such that there exists $a_{xy} \in L$ such that $a_{xy} \in x$ and $a_{xy} \notin y$, that is $x \in \eta(a_{xy})$ and $y \in X_L \setminus \eta(a_{xy})$.

Let us fix one $y_0 \notin O$, we have that

$$O \subseteq \cup\{\eta(a_{xy_0}) \mid x \in O\}.$$

With Proposition B.1.4, we will know that X_L is compact, ensuring us that O, as a clopen set, is compact. Therefore, there exist $x_1, \ldots, x_n \in O$ such that

$$O \subseteq \eta(a_{x_1y_0}) \cup ... \cup \eta(a_{x_ny_0}) = \eta(a),$$

for $a = a_{x_1y_0} \vee ... \vee a_{x_ny_0}$, which concludes the proof.

□

Corollary B.1.3. *Let L be a bounded distributive lattice. The decreasing clopen sets of X_L are exactly the sets $X_L \setminus \eta(a)$ for $a \in L$.*

Proposition B.1.4. *Let L be a bounded distributive lattice, then X_L is a compact Hausdorff space.*

Proof. We start by proving that X_L is indeed compact. Consider a cover of X_L

$$\theta = \{\eta(s) \mid s \in S\} \cup \{X_L \setminus \eta(t) \mid t \in T\}$$

with $S, T \subseteq L$. Let F denote the filter generated by T and I the ideal generated by S.

In a first phase, we prove that $F \cap I \neq \emptyset$. If this not so, there exists a prime filter x such that $F \subseteq x$ and $I \cap x = \emptyset$. More precisely, we have $T \subseteq x$ and $S \cap x = \emptyset$. It follows that for every $t \in T$, $x \notin X_L \setminus \eta(t)$ and that for every $s \in S$, $x \notin \eta(s)$. This is absurd since θ is a cover of X_L.

In a second phase, take $a \in F \cap I$. By definition of generated filter and ideal, there exist elements $s_1, ..., s_n \in S$ and $t_1, ..., t_m \in T$ such that

$$t_1 \wedge ... \wedge t_m \leq a \leq s_1 \vee ... \vee s_n.$$

Therefore, we obtain

$$\begin{aligned} X_L &= \eta(a) \cup X_L \setminus \eta(a) \\ &= \eta(s_1 \vee ... \vee s_n) \cup X_L \setminus \eta(t_1 \wedge ... \wedge t_m) \\ &= \eta(s_1) \cup ... \cup \eta(s_n) \cup X_L \setminus \eta(t_1) \cup ... \cup X_L \setminus \eta(t_m) \end{aligned}$$

as required.

We will now prove that X_L is Hausdorff. Let x, y be two distinct elements of X_L, then we can consider, without loss of generality, that $x \not\leq y$. Henceforth, there exists $a \in L$ such that $a \in x$ and $a \notin y$, that is such that $x \in \eta(a)$ and $y \in X_L \setminus \eta(a)$. Thus we have the conclusion. □

Remark B.1.5. It turns out that X_L is a little bit more than just a compact Hausdorff space. Indeed, in our proof of Proposition B.1.4, the open sets used to separate x from y have the interesting properties to be clopen, respectively increasing and decreasing and each other's complements. . Thus, consider the following definition.

Definition B.1.6. A topological space X equipped with an order is a **Priestley space** if X is compact and if for every $x, y \in X$ such that $x \not\leq y$ there exists an increasing clopen set O such that $x \in O$ and and $y \notin O$. This latter property is known as **totally order-disconnected**.

Proposition B.1.7. *Let X be an ordered topological space.*

1. *If X is totally order-disconnected, then its order is closed in X^2.*
2. *If the order of X is closed in X^2, then X is a Hausdorff space.*
3. *If $(X, \tau, \leq)$ is an ordered space, then*

   $$\tau^{\uparrow} = \{\omega \in \tau \mid \uparrow\omega = \omega\} \text{ and } \tau^{\downarrow} = \{\omega \in \tau \mid \downarrow\omega = \omega\}$$

   are topologies on X.
4. *If $(X, \tau, \leq)$ is a Priestley space, then $\tau^{\uparrow}$ is generated by the set $\mathcal{I}$ of increasing clopen sets, $\tau^{\downarrow}$ by the set $\mathcal{D}$ of decreasing clopen sets and τ by the set*

   $$\mathcal{A} = \{O \cap U \mid O \in \mathcal{I},\ U \in \mathcal{D}\}.$$

Proof. Only the fourth assertion requires a non-trivial proof. Let us prove that τ is generated by $\mathcal{A}$. Let ω be an open set of X and $x \in \omega$. For every $y \in \omega^c$, we have $x \not\leq y$ or $y \not\leq x$. In any case, there exists a clopen sets such that $x \in O_{x,y}$ and $y \notin O_{x,y}$ which is respectively increasing or decreasing.

We then have

$$\omega^c \subseteq \cup\{O^c_{x,y} \mid y \notin \omega\}$$

and, by compactness of ω^c, there exist $y_1, ..., y_n$ in ω^c such that

$$\omega^c \subseteq \cup\{O^c_{x,y_i} \mid 1 \leq i \leq n\}.$$

It suffices now to denote
$$O_x = \cap\{O_{x,y_i} \mid 1 \leq i \leq n\}$$
to obtain that $\omega = \cup O_x$ and conclude the proof.

The reasoning to prove that $\tau^{\uparrow}$ is generated by $\mathcal{I}$ and $\tau^{\downarrow}$ by $\mathcal{D}$ is practically identical with the small difference that for $y \notin \omega^c$ we only have one of $x \not\leq y$ and $y \not\leq x$. □

Theorem B.1.8. *Let L be a bounded distributive lattice. The topological space X_L of prime filters of L is a Priestley space.*

Proof. This is just a rewriting of Proposition B.1.4 taking into consideration Remark B.1.5. □

With the previous theorem, we established a functor from the category **DLat** of bounded distributive lattices to the category **Priest** of Priestley spaces (although only for the objects). To find the functor in the other direction, a closer look at Proposition B.1.2 gives us the right approach.

Theorem B.1.9. *Let X be a Priestley space. The set L_X of increasing clopen sets of X ordered by inclusion is a bounded distributive lattice.*

Proof. It is a simple verification. □

Now we end the definition of the functors between **Priest** and **DLat** by describing their actions on morphisms.

Theorem B.1.10.

1. *Let L and M be two bounded distributive lattices and $h : L \longrightarrow M$ a lattice morphism, then*
$$h^\star : X_M \longrightarrow X_L : x \longmapsto h^{-1}(x)$$
is an increasing continuous function.

2. *Let X and Y be two Priestley spaces and $f : X \longrightarrow Y$ an increasing continuous function, then*
$$f_\star : L_Y \longrightarrow L_X : O \longmapsto f^{-1}(O)$$
is a lattice morphism.

Proof. The proof follows the same lines as the one used in Stone's case. □

Definition B.1.11.

1. We denote by **Priest** the category of Priestley spaces equipped with increasing continuous functions.

2. We denote by **DLat** the category of bounded distributive lattice equipped with lattice morphisms.

3. We denote by $\uparrow$Clop the (contravariant) functor from **Priest** to **DLat** which sends a Priestley space X to its lattice of increasing clopen sets X_L and a function $f \in \mathbf{Priest}(X, Y)$ to the morphism $f^{-1} \in \mathbf{DLat}(L_Y, L_X)$.

4. We denote by Prim the (contravariant) functor from **DLat** to **Priest** which sends a bounded distributive lattice L to its set of prime filters X_L and a morphism $h \in \mathbf{DLat}(L, M)$ to the increasing function $h^{-1} \in \mathbf{Priest}(X_M, X_L)$.

Lemma B.1.12. *A $f : X \longrightarrow Y$ morphism in* **Priest** *is an isomorphism if and only if f is a homeomorphism such that for $x, y \in X$, $x \leq y$ if and only if $f(x) \leq f(y)$.*

Proof. It is straightforward. □

Definition B.1.13. An increasing continuous functions satisfying the conditions of Lemma B.1.12 is said to be an **order homeomorphism**.

Before continuing, a reasonable question would be to ask if, since **DLat** extends **Bool**, **Priest** does extend **Stone**? It is enough to consider that a Stone space is a Priestley space ordered by the equality to see that indeed **Stone** is a full subcategory of **Priest**.

Theorem B.1.14. *1. Let L be a bounded distributive lattice. The map*

$$\eta : L \longrightarrow \uparrow\mathrm{Clop}(\mathrm{Prim}(L)) : a \longmapsto \eta(a)$$

is a lattice isomorphism.

2. Let X be a Priestley space. The function

$$\varepsilon : X \longrightarrow \mathrm{Prim}(\uparrow\mathrm{Clop}(X)) : x \longmapsto \{O \in L_X \mid x \in O\}$$

is an order homeomorphism.

3. The functors Prim *and* $\uparrow$Clop *establish a duality between* **Priest** *and* **DLat**.

Proof. 1. By Proposition B.1.2, we know that η is well-defined and onto. Moreover, since it is not hard to prove that η is a lattice morphism, it only remains to prove that η is one-to-one.

Suppose, without loss of generality, that a, b are elements of L such that $a \not\leq b$. It follows that $\uparrow a \cap \downarrow b = \emptyset$, implying the existence of a prime filter x such that $\uparrow a \subseteq x$ and $\downarrow b \cap x = \emptyset$. Henceforth, we have $x \in \eta(a)$, while $x \notin \eta(b)$.

2. First of all, we have to prove that $\varepsilon(x)$ is indeed an element of the set $\mathrm{Prim}(\uparrow\mathrm{Clop}(X))$ but it is purely routine.

Then, we have

$$\varepsilon^{-1}(\eta(O)) = O \text{ and } \varepsilon^{-1}(\eta(O)^c) = O^c,$$

for every $O \in \uparrow\mathrm{Clop}(X)$. Consequently, the function ε is continuous. Moreover, since the clopen sets of $\uparrow\mathrm{Clop}(X)$ are increasing by definition, it is clear that $x \leq y$ implies $\varepsilon(x) \subseteq \varepsilon(y)$.

Furthermore, since X is totally order disconnected, $x \not\leq y$ implies that $x \in O$ and $y \notin O$ for some increasing clopen set O. It follows that $\varepsilon(x) \not\subseteq \varepsilon(y)$ and therefore that ε is one-to-one. Considering that ϵ has already been proven to be increasing, one can note that, in particular, we showed that $x \leq y$ if and only $\varepsilon(x) \leq \varepsilon(y)$.

Finally, suppose that ε is not onto. Then, there exists a prime filter $\mathfrak{F}$ such that $\mathfrak{F} \notin \varepsilon(X)$. But, since $\mathrm{Prim}(\uparrow\mathrm{Clop}(X))$ is a regular topological space and $\varepsilon(X)$ is one of its closed subset, there exists an open set ω of $\mathrm{Prim}(\uparrow\mathrm{Clop}(X))$ such that $\mathfrak{F} \in \omega$ and $\varepsilon(X) \subseteq \omega^c$. Hence, by Proposition B.1.7, we can consider that

$$\omega = \eta(O) \cap \eta(U)^c$$

for some $O, U \in \uparrow\mathrm{Clop}(X)$. In particular, it means that

$$\emptyset = \varepsilon^{-1}(\eta(O) \cap \eta(U)^c) = O \cap U^c,$$

which means that $O \subseteq U$. Therefore, $\mathfrak{F} \in \eta(O)$ implies $\mathfrak{F} \in \eta(U)$. This contradicts $\mathfrak{F} \in \eta(U)^c \cap \eta(O)$. Thus, ε is a bijective continuous function between compact Hausdorff spaces and so is a homeomorphism which completes our proof.

3. What remains to prove is that ε and η are natural transformations. The proof is identical to the one of A.1.18 for Stone duality and is left to the reader.

$\square$

B.2 Remarks on Priestley duality

The functors presented here for the Priestley duality are not the usual ones since it is more common to work with prime ideals and decreasing clopen sets (see for instance [22]) instead of prime filters and increasing clopen sets.

However, on one hand, there is a correspondence between prime ideals and prime filters of a bounded distributive lattice, and on the other hand, there is another correspondence between decreasing clopen sets and increasing clopen sets of a Priestley space (namely the complement set gives both correspondences). Therefore, we can easily interchange a concept with another.

Finally, we have the theorem which states that Priestley duality indeed extends Stone duality.

Theorem B.2.1. *1. Let B, C be Boolean algebras and $h : B \longrightarrow C$ a morphism in* **Bool**. *Then,* $\mathrm{Prim}(B) = \mathrm{Ult}(B)$ *is a Stone space and* $\mathrm{Prim}(h) = \mathrm{Ult}(h)$ *is a morphism of* **Stone**.

2. Let X, Y be Stone spaces and $f : X \longrightarrow Y$ a morphism in **Stone**. *Then,* $\uparrow\mathrm{Clop}(X) = \mathrm{Clop}(X)$ *is a Boolean algebra and* $\uparrow\mathrm{Clop}(f) = \mathrm{Clop}(f)$ *is a morphism in* **Bool**.

Completing the picture of Theorem A.3.4, we obtain the following diagram.

$$\begin{tikzcd}
\mathbf{DLat} \arrow[rrr, dash] & & & \mathbf{Priest} \\
& \mathbf{Bool} \arrow[ul] \arrow[r, dash] & \mathbf{Stone} \arrow[ur] &
\end{tikzcd}$$

Proof. Rather than an authentic proof, we just remark that for a Boolean algebra prime filters and ultrafilters are equivalent notions and that, since the order in a Stone space is the equality, the clopen sets of a Stone space are increasing. $\square$

B.3 Modal algebras

Modal algebras are used as models for modal logic in a similar way as Boolean algebra are models for classical logic. We redirect the reader to [16] or [9] for more details on the subject.

Definition B.3.1. 1. A **modal algebra** is a pair $\mathfrak{B} = (B, \Diamond)$ where B is a Boolean algebra and $\Diamond$ is a map $B \longrightarrow B$ such that :

(a) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ for all $a, b \in B$,

(b) $\Diamond 0 = 0$.

Note that $\Box$ is a common shortcut for $\neg\Diamond\neg$.

2. Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras, a map $h : A \longrightarrow B$ is a **modal morphism** if it is a Boolean morphism such that $h(\Diamond a) = \Diamond h(a)$ for every $a \in A$.

3. We denote by **ModAlg** the category whose objects are modal algebras and whose morphisms are modal morphisms.

Hence, for a modal algebra $\mathfrak{B} = (B, \Diamond)$, its Boolean part B possesses a Stone dual X, the main mission now is to determine what will become the modal part $\Diamond$ of $\mathfrak{B}$. It could be interesting to note that another family of objects is used to provide modal logic with models: modal spaces (often called descriptive frames).

Definition B.3.2. 1. A **modal space** is a pair $\mathfrak{X} = (X, R)$ where $\mathfrak{X}$ is a Stone space and R is a **modal accessibility relation**, that is a closed binary relation on X such that for every clopen subset O of X, the set

$$R(-, O) := \{x \in X \mid \exists y \in O : x\ R\ y\}$$

is still clopen.

2. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be modal spaces, a function $f : X \longrightarrow Y$ is a **modal function** if it is a continuous function such that, for all $x, y \in X$ and $z \in Y$
   (a) $x\ R\ y$ implies $f(x)\ R\ f(y)$,
   (b) $f(x)\ R\ z$ implies that there exists $y \in X$ such that $x\ R\ y$ and $f(y) = z$.
3. We denote by **ModSp** the category whose objects are modal spaces and whose morphisms are modal functions.

Remark B.3.3. An interesting property about the modal accessibility relation of a modal space (X, R) is that for a closed subset F of X, the sets

$$R(F, -) := \{x \in X \mid \exists y \in F : y\ R\ x\}$$

and $R(-, F)$ (defined analogously) are still closed, while for an open subset O of X, the sets $R(O^c, -)^c$ and $R(-, O^c)^c$ are open.

Let us prove the first assertion and let us suppose that $x \notin R(F, -)$. Then, for every $y \in F$, we have that $(y, x) \notin R$. As R is closed, there exist open sets O_y, U_y such that

$$(y, x) \in O_y \times U_y \subseteq R^c.$$

In particular, it follows that $F \subseteq \cup O_y$, and by compactness, there exist $y_1, ..., y_n \in F$ with $F \subseteq O := \cup O_{y_i}$. Hence, it suffices to notice that

$$x \in \cup U_{y_i} \subseteq R(F, -)^c$$

to conclude.

The way a modal space is used as model for modal logic clearly enlightens us about the future of $\Diamond$ in the duality. Indeed, let us consider a modal space $\mathfrak{X} = (X, R)$ and a valuation v on $\mathfrak{X}$, that is a map from a set Var of variables to $\mathrm{Clop}(X)$, then it is possible to extend v to the set of the modal formula $\Diamond p$ as follows

$$v(\Diamond p) = R(-, v(p)).$$

In other words, it means that for $x, y \in X$

$$x\ R\ y \Rightarrow (y \in v(p) \Rightarrow x \in v(\Diamond p)).$$

Using Remark B.3.3, it is not hard to prove that we have actually

$$x\ R\ y \Leftrightarrow (\forall v \text{ valuation})(y \in v(p) \Rightarrow x \in v(\Diamond p)).$$

Finally, we have to remember that $v(p)$ and $v(\Diamond p)$ are clopen sets of X, and thus elements of its dual to come up with the following theorem.

Theorem B.3.4. *1. If $\mathfrak{B} = (B, \Diamond)$ is a modal algebra, then the pair $\mathfrak{X} = (\mathrm{Ult}(B), R_\Diamond)$ with $R_\Diamond$ defined as*

$$x \, R_\Diamond \, y \Leftrightarrow (\forall a \in B)(a \in y \Rightarrow \Diamond a \in x) \Leftrightarrow \Diamond y \subseteq x \tag{B.1}$$

is a modal space.

2. *If $\mathfrak{X} = (X, R)$ is a modal space, then $\mathfrak{B} = (\mathrm{Clop}(X), \Diamond_R)$ with $\Diamond_R$ defined as*

$$\Diamond_R O = R(-, O)$$

is a modal algebra.

3. *If $h \in \mathbf{ModAlg}(A, B)$, then $\mathrm{Ult}(h) \in \mathbf{ModSp}(\mathrm{Ult}(B), \mathrm{Ult}(A))$.*

4. *If $f \in \mathbf{ModSp}(X, Y)$, then $\mathrm{Clop}(f) \in \mathbf{ModAlg}(\mathrm{Clop}(Y), \mathrm{Clop}(X))$.*

5. *The categories* **ModSp** *and* **ModAlg** *are dually equivalent*

B.4 Bounded Archimedean ℓ-algebras

We saw that Stone spaces could be characterised by their set of clopen subsets. In a similar way, we saw in Chapter 1 that compact Hausdorff spaces can be characterised by their regular open subsets.

Now, we will discuss another way to characterise compact Hausdorff spaces: through their set of real continuous functions via real Gelfand-Neumark duality (see [37] and [71]).

As announced in the introduction, we will merely mention the definitions and theorems of [7].

Definition B.4.1. 1. An **ℓ-ring** is an algebra $(A, \cdot, +, \wedge, \vee, 0, 1)$ such that :

(a) $(A, \cdot, +, 0, 1)$ is a ring,

(b) $(A, \wedge, \vee)$ is a lattice,

(c) $a \leq b$ implies $a + c \leq b + c$,

(d) $0 \leq a, b$ implies $0 \leq ab$.

We say that an ℓ-ring A is

(a) **Archimedean** if for each $a, b \in A$ $na \leq b$ for each $n \in \mathbb{N}$ implies $a \leq 0$.

(b) **bounded** if for each $a \in A$ there exists $n \in \mathbb{N}$ such that

$$a \leq \underbrace{1 + \ldots + 1}_{n\text{times}}.$$

(c) a **bal** if it is a bounded Archimedean ℓ-ring which is also an $\mathbb{R}$-algebra such that $0 \leq a \in A$ and $r \in \mathbb{R}^+$ implies $0 \leq ra$.

2. For two bals A and B, a map $\alpha : A \longrightarrow B$ is a **bal morphism** if it is a lattice morphism and an $\mathbb{R}$-algebra morphism.

3. We denote by **bal** the category whose objects are bals and whose morphisms are bal morphisms.

Notation B.4.2. Let A be a bal and $a \in A$. We denote by $|a|$ the **absolute value** of a, that is $a \vee -a$.

Definition B.4.3. Let A be a bal. A subset I of A is an ℓ-ideal if it is a ring ideal such that for $a, b \in A$, $|a| \leq |b|$ and $b \in I$ implies $a \in I$.

Notation B.4.4.

1. For a bal A, we denote by $\mathrm{MaxId}_\ell(A)$ its set of maximal ℓ-ideals for the inclusion.
2. For a compact Hausdorff space X, we denote by $C(X, \mathbb{R})$ its set of real continuous functions (that is continuous functions from X to $\mathbb{R}$).

Proposition B.4.5.

1. *For a bal A, the set $\mathrm{MaxId}_\ell(A)$ endowed with the topology generated by the subsets of the form*

$$\omega(a) = \{I \in \mathrm{MaxId}_\ell(A) \mid I \not\ni a\},$$

for $a \in A$, is a compact Hausdorff space.

2. *Let $\alpha \in \mathbf{bal}(A, B)$, then*

$$\mathrm{MaxId}_\ell(\alpha) : \mathrm{MaxId}_\ell(B) \longrightarrow \mathrm{MaxId}_\ell(A) : I \longmapsto \alpha^{-1}(I)$$

is a continuous function.

Proposition B.4.6.

1. *For a compact Hausdorff space X, the set $C(X, \mathbb{R})$ equipped with pointwise operations and order is a bal.*
2. *Let $f \in \mathbf{KHaus}(X, Y)$, then*

$$C(f) : C(Y, \mathbb{R}) \longrightarrow C(X, R) : g \longmapsto g \circ f$$

is a bal morphism.

With the previous two propositions, one could think that **bal** and **KHaus** are dually equivalent categories. However, the next proposition will show us that this is not entirely the case.

Definition B.4.7. Let A be an object of **bal** and $a \in A$. The **uniform norm** of a is defined by

$$||a|| = \inf\{\lambda \in \mathbb{R} \mid |a| \leq \lambda \cdot 1\}.$$

Proposition B.4.8. *If X is a compact Hausdorff space then its ring of continuous functions $C(X, \mathbb{R})$ is complete with respect to its uniform norm, that is every Cauchy sequence in $C(X, R)$ is convergent.*

Definition B.4.9. We denote by **ubal** the full subcategory of **bal** whose objects are uniformly complete bals.

We are now ready to state the duality of this section.

Theorem B.4.10. *The categories* **ubal** *and* **KHaus** *are dually equivalent.*

Appendix C

Contact - Subordination - quasi-modal operator

We consider in this appendix four structures on Boolean algebras: subordination relation, already encountered in Chapter 2, pre-contact relations and (dual) quasi-modal operators. The four structures substantially bear the same information.

C.1 Interconnections between definitions

Definition C.1.1 ([5]). Let B be a Boolean algebra. A **subordination relation** on B is a binary relation $\prec$ on B such that

S1. $0 \prec 0$ and $1 \prec 1$,

S2. $a \prec b, c$ implies $a \prec b \wedge c$,

S2'. $a, b \prec c$ implies $a \vee b \prec c$,

S3. $a \leq b \prec c \leq d$ implies $a \prec d$.

A **subordination relation** is a pair $\mathfrak{B} = (B, \prec)$ where B is a Boolean algebra and $\prec$ is a subordination relation on B.

Definition C.1.2 ([14]). Let B be a Boolean algebra. A **quasi-modal operator** on B is a map

$$\triangledown : B \longrightarrow \mathcal{I}(B)$$

such that :

1. $\triangledown(a \wedge b) = \triangledown(a) \cap \triangledown(b)$,
2. $\triangledown 1 = B$.

A **quasi-modal algebra** is a pair $\mathfrak{B} = (B, \triangledown)$ where B is a Boolean algebra and $\triangledown$ is a quasi-modal operator on B.

Definition C.1.3. Let B be a Boolean algebra. A **dual quasi-modal operator** on B is a map

$$\triangle : B \longrightarrow \mathcal{F}(B)$$

such that:

1. $\triangle(a \vee b) = \triangle(a) \cap \triangle(b)$,
2. $\triangle 0 = B$.

A **dual quasi-modal algebra** is a pair $\mathfrak{B} = (B, \triangle)$ where B is a Boolean algebra and $\triangle$ is a dual quasi-modal operator on B.

Definition C.1.4 ([28])**.** Let B be a Boolean algebra. A **pre-contact relation** on B is a binary relation $\mathcal{C}$ on B such that

C1. $1 \not{\mathcal{C}}\, 0$ and $0 \not{\mathcal{C}}\, 1$,

C2. $a\, \mathcal{C}\, b \vee c$ implies $a\, \mathcal{C}\, b$ or $a\, \mathcal{C}\, c$,

C2'. $a \vee b\, \mathcal{C}\, c$ implies $a\, \mathcal{C}\, c$ or $b\, \mathcal{C}\, c$,

C3. $a \geq b\, \mathcal{C}\, c \leq d$ implies $a\, \mathcal{C}\, d$.

A **pre-contact algebra** is a pair $\mathfrak{B} = (B, \mathcal{C})$ where B is a Boolean algebra and $\mathcal{C}$ a pre-contact relation.

Theorem C.1.5. *Let B be a Boolean algebra, $\prec$ a subordination relation on B, $\triangledown$ a quasi-modal operator on $\mathfrak{B}$, $\triangle$ a dual quasi-modal operator on B and $\mathcal{C}$ a pre-contact relation on $\mathfrak{B}$. Then*

1. *the map $\triangledown_{\prec} : B \longrightarrow \mathcal{I}(B)$ defined by $\triangledown(a) = \prec(-, a)$ is a quasi-modal operator,*
2. *the map $\triangle_{\prec} : B \longrightarrow \mathcal{F}(B)$ defined by $\triangle(a) = \prec(a, -)$ is a quasi-modal operator,*
3. *the relation $\prec_{\triangledown}$ defined by $a \prec_{\triangledown} b$ iff $a \in \triangledown b$ is a subordination relation,*
4. *the relation $\prec_{\triangle}$ defined by $a \prec_{\triangle} b$ iff $b \in \triangle a$ is a subordination relation,*
5. *the relation $\mathcal{C}_{\prec}$ defined by $a\, \mathcal{C}_{\prec}\, b$ iff $a \not\prec \neg b$ is a pre-contact relation,*
6. *the relation $\prec_{\mathcal{C}}$ defined by $a \prec_{\mathcal{C}} b$ iff $a \not{\mathcal{C}}\, \neg b$ is a subordination relation.*

Moreover, we have

7. *$a \prec b$ if and only if $a \prec_{\triangle_{\prec}} b$ if and only if $a \prec_{\triangledown_{\prec}} b$ if and only if $a \prec_{\mathcal{C}_{\prec}} b$,*
8. *$\triangle(a) = \triangle_{\prec_{\triangle}}(a)$,*
9. *$\triangledown(a) = \triangledown_{\prec_{\triangledown}}(a)$,*
10. *$a\, \mathcal{C}\, b$ if and only if $a\, \mathcal{C}_{\prec_{\mathcal{C}}}\, b$.*

Proof. One just has to use the definitions to prove the theorem. $\square$

Note that in Theorem C.1.5, we chose to use $\prec$ as a common ground. Of course, through their associated subordination relations, it is possible to inter-define quasi-modal operator, dual quasi-modal operator and pre-contact relation. In short, we have the following table of correspondences.

Subordination	Pre-contact	Quasi-modal	Dual quasi-modal
$a \prec b$	$a \not{\mathcal{C}}\, \neg b$	$a \in \triangledown b$	$b \in \triangle a$

Finally, note that the axiomatisation of pre-contact algebra given here in Definition C.1.4 is not the usual one. Indeed, we wrote the axioms C1 - C3 so that they are the exact correspondent of the axioms S1 - S3 of definition C.1.1, meaning that $(B, \prec)$ satisfies the axiom Si if and only if $(B, \mathcal{C}_{\prec})$ satisfies the axiom Ci. In the next section, we will discuss equivalent axiomatisation of both subordination and pre-contact algebras, as well as giving translation for additional axioms.

C.2 Equivalent and additional axioms

Definition C.2.1. Let B be a Boolean algebra and $\prec$, $\mathcal{C}$ some binary relations on B. We have the following axioms

S4.	$a \prec b$ implies $a \leq b$	C4.	$b, c \geq a \neq 0$ implies $b \,\mathcal{C}\, c$
S5.	$a \prec b$ implies $\neg b \prec \neg a$	C5.	$a \,\mathcal{C}\, b$ implies $b \,\mathcal{C}\, a$.
S6.	$a \prec b \prec c$ implies $\exists c$: $a \prec c \prec b$	C6.	$(\forall c)(a \,\mathcal{C}\, c$ or $\neg c \,\mathcal{C}\, b)$ implies $a \,\mathcal{C}\, b$
S7.	$a \neq 0$ implies $\exists b \neq 0$: $b \prec a$	C7.	$b \,\mathcal{C}\, a$ for all $b \neq 0$ implies $a = 1$

Note that, once again, we gave the axioms in order to ensure that $(B, \prec)$ satisfies Si if and only if $(B, \mathcal{C}_\prec)$ satisfies Ci.

Proposition C.2.2. *Let B be a Boolean algebra and $\prec$, $\mathcal{C}$ some binary relations on B.*

1. *If $(B, \prec)$ satisfies S3, then $(B, \prec)$ satisfies S1 if and only if it satisfies*

   eS1. $0 \prec a \prec 1$ *for all* $a \in B$.

   Similarly, if $(B, \mathcal{C})$ satisfies C3, then $(B, \mathcal{C})$ satisfies C1 if and only if it satisfies

   eC1. $0 \not\mathcal{C}\, a$ *and* $a \not\mathcal{C}\, 0$ *for all* $a \in B$.

   Of course, we also have that $(B, \prec)$ satisfies eS1 if and only $(B, \mathcal{C})$ satisfies eC1.

2. *If $(B, \prec)$ satisfies eS1 and S4, then it satisfies S6 and S7 if and only if it satisfies*

   S6-7. $a \prec b \neq 0$ *implies* $\exists c \neq 0$: $a \prec c \prec b$.

3. *$(B, \mathcal{C})$ satisfies C7 if and only if it satisfies*

   aC7. $a \neq 1$ *implies* $\exists b \neq 0$: $b \not\mathcal{C}\, a$.

   Moreover, if it satisfies C5, then it satisfies aC7 if and only if it satisfies

   bC7. $a \neq 1$ *implies* $\exists b \neq 0$: $a \not\mathcal{C}\, b$.

   Finally, if $(B, \mathcal{C})$ satisfies C2 and C2', then it satisfies bC7 if and only if it satisfies

   cC7. $a \not\leq b$ *implies* $\exists c$: $a \,\mathcal{C}\, c$ *and* $b \not\mathcal{C}\, c$.

   Moreover, if $(B, \mathcal{C})$ satisfies eC1 and C4, then it satisfies cC7 if and only if it satisfies

   dC7. $\mathcal{C}(a, -) = \mathcal{C}(b, -)$ *implies* $a = b$.

   Finally, $(B, \mathcal{C}_\prec)$ satisfies dC7 if and only if $(B, \prec)$ satisfies

   dS7. $\prec(a, -) = \prec(b, -)$.

4. *If $(B, \mathcal{C})$ satisfies C3, then C4 is equivalent to*

   eC4. $a \neq 0$ *implies* $a \,\mathcal{C}\, a$.

   Similarly, if $(B, \prec)$ satisfies S3, then S4 is equivalent to

   eS4. $a \neq 0$ *implies* $a \not\prec \neg a$.

   Finally, it is clear that $(B, \prec)$ satisfies eS4 if and only if $(B, \mathcal{C}_\prec)$ satisfies eC4.

5. *If $(B, \prec)$ satisfies S1, S6 and S3 then it satisfies S4 if and only it satisfies*

   eS4. $a = \vee\{b : b \prec a\}$.