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Generalized T_u^p spaces: On the trail of Calderón and Zygmund

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Abstract

We generalize the T_u^p spaces introduced by Calderón and Zygmund and show that most of the results obtained in their study of the pointwise estimates for solutions of elliptic partial differential equations and systems can be generalized in this framework with L^p-conditions.

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1. Introduction

The T^p_u spaces were introduced in essence by Calderón and Zygmund [8]: for a point x_0 of the d-dimensional Euclidean space $\mathbb{R}^d$, $p \in [1,\infty]$ and a number $u \geq -d/p$, $T^p_u(x_0)$ denotes the class of function $f \in L^p(\mathbb{R}^d)$ such that there exists a polynomial P of degree strictly less than u with the property that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq Cr^u \tag{1.1}$$

for a constant C (which does not depend on r), where $B(x_0,r)$ denotes the open ball centered at x_0 with radius r and $\|f\|_{L^p(E)}$ stands for the usual norm of the space $L^p(E)$ of the measurable functions on E for which the pth power of the absolute value is (Lebesgue) integrable. If $f \in T^p_u(x_0)$ also satisfies

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} = o(r^u) \quad \text{as } r \to 0^+,$$

where P is a polynomial of degree less than or equal to u (where we have used the usual Bachmann–Landau notations), then f is said to belong to $t^p_u(x_0)$. To emphasize the fact that the integral mean value is involved in the definition of these spaces, let us point out that $f \in T^p_u(x_0)$ with $p < \infty$ means that we have $(⨍_{B(x_0,r)} |f - P|^p)^{1/p} \leq Cr^u$. In their seminal paper [8], the authors use these spaces to obtain pointwise estimates for solutions of elliptic partial differential equations $\mathcal{E}f = g$. More precisely, the functions $f \in T^p_u(x_0)$ form a linear space with norm $\|\cdot\|_{T^p_u(x_0)}$ defined by the sum of $\|\cdot\|_{L^p}$, the absolute values of the coefficients of P and the infimum of the constants C in (1.1). The main theorem can be stated as follows: if all the coefficients of the differential operator $\mathcal{E}$ are of class $T^\infty_u(x_0)$, if all components f_j and g_k are of class L^p and $g_k \in T^p_v$ with $p \in (1,\infty)$, $-d/p \leq v \leq u$, $v \notin \mathbb{Z}$, then there exists a constant C for which, using Euler's notation for the derivatives (i.e. $D_j f$ designates the derivative of f following the jth component)

$$\|D^\alpha f_j\|_{T^q_{v+m-|\alpha|}(x_0)} \leq C\Big(\sum_k \|g_k\|_{T^p_v(x_0)} + \sum_j \|f_j\|_{W^p_m}\Big) \tag{1.2}$$

for all j, $|\alpha| \leq m$, where q is a number satisfying

- $p \leq q \leq \infty$ if $1/p < (m - |\alpha|)/d$,
- $p \leq q < \infty$ if $1/p = (m - |\alpha|)/d$,
- $1/p \leq 1/q \leq 1/p - (m - |\alpha|)/d$ otherwise.

Moreover, if g belongs to $t^p_v(x_0)$, then $D^\alpha f$ belongs to $t^q_{v+m-|\alpha|}(x_0)$. Another theorem states that if $\mathcal{E}$ is elliptic almost everywhere on a set of positive measure whose points x_0 satisfy $\mu(x_0) > c$ for some constant $c > 0$, if the coefficients of $\mathcal{E}$ are in $T^\infty_u(x_0)$ and $g \in T^p_v(x_0)$ for almost every x_0 and if $f \in L^p_m$, then $D^\alpha f$ belongs to $t^q_{v+m-|\alpha|}(x_0)$ for almost every x_0. Let us remark that there is a common misunderstanding when stating the hypothesis of this main theorem: the coefficients of $\mathcal{E}$ have to belong to $T_u^\infty(x_0)$ (see [8, p. 72], where T_u is defined as T_u^∞); the case where these coefficients belong to $T_u^p(x_0)$ with $p < \infty$ is not considered in [8].

This seminal paper illustrates the fact that classical regularity spaces have played a significant role in numerous parts of mathematics over the years. Still, it has become clear that there are appreciable advantages to be gained by adding to such spaces features that allow to more finely tune their regularity properties. Roughly speaking, the general idea consists in adding spaces in-between existing spaces; such generalizations have been investigated in connection with applications in embeddings, entropy numbers, probability theory, signal analysis, spectral theory and theory of stochastic processes for instance (see e.g. [17, 12, 10, 5, 11, 22, 30, 28] and references therein). Let us mention that there are other ways to obtain more general spaces (one can for example use weighted spaces [2]).

As a notable case, the versatility of the generalized Besov spaces (see e.g. [25, 9]) is particularly clear. They can be obtained from the usual Sobolev spaces using some kind of real interpolation and most of the properties of the usual spaces are preserved in the generalized version (see e.g. [24]). Lately, such a generalization has been used for the detection of the law of the iterated logarithm in signals using multifractal formalisms (see [28, 18, 21]). However, none of these approaches fully takes advantage of the versatility of the generalized Besov spaces: a pointwise counterpart of such spaces is missing.

The idea is to generalize the pointwise Hölder spaces in order to be able to consider non-locally-bounded functions (as in [31, 16]) and deal with logarithmic corrections (as in [28, 21]). In this work, we introduce such spaces, following Calderón and Zygmund in their study of local behaviors of solutions of elliptic PDE's [8]. The general idea consists in replacing the power function $r \mapsto r^u$ appearing in (1.1) with $r \mapsto \phi(r)$ $(r > 0)$, where ϕ is a function satisfying some basic properties, to obtain the generalized spaces T_ϕ^p and t_ϕ^p respectively; typically, such a function ϕ could be $r \mapsto r^u |\ln r|$ for the detection of the logarithmic corrections (such an idea is exploited in [10, 30] in the case of the Bessel potential spaces) or more generally $r \mapsto r^u \psi(r)$, where ψ is any weakly varying function, i.e. a strictly positive function satisfying

$$\lim_{t \to 0} \frac{\psi(rt)}{\psi(t)} = 1$$

for any $r > 0$ (see [19] for example). Such a choice is natural and observed in many financial models that are derived from the Brownian motion (e.g. the geometric Brownian motion used in the Black and Scholes model [15], the Hull and White one-factor model [4], etc.).

Before investigating these spaces from a fractal point of view [23], we must first explore their properties as regularity spaces and show that they are still related to some notion of smoothness. To do so, we follow the ideas of Calderón and Zygmund and show that most of the properties established in [8] still hold for the generalized versions T_ϕ^p and t_ϕ^p; we thus introduce here some generalizations of the results obtained in [8]. In particular, we obtain the main theorem of [8] (see inequality (1.2) and Theorem 3.4.6) to the case where the coefficients of $\mathcal{E}$ are of class $T_u^p(x_0)$ (and even $T_\phi^p(x_0)$). As already mentioned, one of the most remarkable aspects of the spaces of generalized smoothness is that most of the properties of the usual spaces are preserved in the generalized version (see e.g. [19, 20, 18, 21, 24], where it is shown that the usual characterizations still hold in the general settings); this crucial feature of such generalizations is also observed here. The spaces defined here are thus a natural extension of the usual spaces; moreover they enlighten the role of the power function in the classical theory.

Although the machinery applied is standard, some subtle arguments must be occasionally introduced to obtain the natural results presented here, since the convergence of an expression depending on $\phi(r)$ is often less obvious than the same statement involving a power function. Let us also remark that we use here conditions based on L^p-norms instead of L^∞-norms (as already stated for the main theorem of [8]); this is indeed the main source of difficulty, the use of the generalized space being quite natural.

This paper is organized as follows. First we introduce the generalized spaces T^p_ϕ and t^p_ϕ, using Boyd functions and, in the subsequent sections, give some basic properties of such spaces (completeness, density, embeddings, etc.). Next, we give a generalization of Whitney's extension theorem, before studying the Bessel operator. We also investigate the estimations that can be made if the derivatives belong to the spaces T^p_ϕ or t^p_ϕ. We end this work by studying the action of the convolution integral operator on T^p_ϕ and show how these spaces can be utilized to examine the regularity of the solutions of an elliptic partial differential equation.

The notations used here are rather standard; $\mathcal{D}(\mathbb{R}^d)$ will stand for the class of infinitely differentiable functions with compact support on $\mathbb{R}^d$.

2. Spaces of generalized smoothness

2.1. The spaces T^p_ϕ and t^p_ϕ. The generalization of T^p_u spaces that we shall introduce relies on the notion of Boyd function.

Definition 2.1.1. A function $\phi : (0,+\infty) \to (0,+\infty)$ is a *Boyd function* if $\phi(1) = 1$, ϕ is continuous and, for all $x \in (0,+\infty)$,

$$\overline{\phi}(x) := \sup_{y>0} \frac{\phi(xy)}{\phi(y)} < \infty. \tag{2.1}$$

We denote by $\mathcal{B}$ the set of Boyd functions.

If $\phi \in \mathcal{B}$, then

- $\overline{\phi}$ is submultiplicative; this follows from the fact that
$$\frac{\phi(xyz)}{\phi(z)} = \frac{\phi(xz)}{\phi(z)} \frac{\phi(xzy)}{\phi(xz)} \le \overline{\phi}(x)\overline{\phi}(y) \quad \text{for any } x, y, z > 0,$$
- $\overline{\phi}$ is Lebesgue-measurable, since ϕ is continuous,
- $\overline{\phi}(x) \ge \phi(x)$ and $\overline{\phi}(1/x) \ge 1/\phi(x)$ for any $x > 0$.

The fact that $\overline{\phi}$ is submultiplicative allows us to introduce the following notion (see e.g. [9]):

Definition 2.1.2. The *lower* and *upper Boyd indices* of the function $\phi \in \mathcal{B}$ are respectively defined by

$$\underline{b}(\phi) := \sup_{x\in(0,1)} \frac{\log \overline{\phi}(x)}{\log x} = \lim_{x\to 0} \frac{\log \overline{\phi}(x)}{\log x},$$

$$\overline{b}(\phi) := \inf_{x\in(1,+\infty)} \frac{\log \overline{\phi}(x)}{\log x} = \lim_{x\to +\infty} \frac{\log \overline{\phi}(x)}{\log x}.$$

The change of supremum and infimum into limits in the previous equalities comes from a classical result (see e.g. [14, Theorem 7.6.2]). Let us point out that we have $-\infty < \underline{b}(\phi) \le \overline{b}(\phi) < +\infty$, since if b is defined as

$$b(x) := \frac{\log \overline{\phi}(x)}{\log x},$$

then $b(x) \ge b(1/x)$ for $x > 1$.

Proposition 2.1.3. *Let $\phi \in \mathcal{B}$, $\varepsilon > 0$ and $R > 0$; there exist $C_1, C_2, C_3, C_4 > 0$ such that*

$$C_1 r^{\overline{b}(\phi)+\varepsilon} \le \phi(r) \le C_2 r^{\underline{b}(\phi)-\varepsilon} \quad \textit{for all } r \in (0, R], \tag{2.2}$$

$$C_3 r^{\underline{b}(\phi)-\varepsilon} \le \phi(r) \le C_4 r^{\overline{b}(\phi)+\varepsilon} \quad \textit{for all } r \in [R, +\infty). \tag{2.3}$$

Proof. Let us prove (2.2). There exists $R_0 \in (0, 1)$ such that, for all $r \in (0, R_0)$,

$$\underline{b}(\phi) - \frac{\log \overline{\phi}(r)}{\log r} \le \varepsilon,$$

which implies that, for such r,

$$\overline{\phi}(r) \le r^{\underline{b}(\phi)-\varepsilon}. \tag{2.4}$$

Similarly, there exists $R_1 > 1$ such that, for all $r \in (R_1, \infty)$,

$$\overline{\phi}(r) \le r^{\overline{b}(\phi)+\varepsilon}. \tag{2.5}$$

Now, using (2.1), we have

$$\overline{\phi}(1/r)^{-1} \le \phi(r) \le \overline{\phi}(r) \tag{2.6}$$

for all $r > 0$ and from (2.4)–(2.6), we get

$$r^{\overline{b}(\phi)+\varepsilon} \le \phi(r) \le r^{\underline{b}(\phi)-\varepsilon}$$

for $0 < r \le \min\{R_0, 1/R_1\}$. If $R \le \min\{R_0, 1/R_1\}$, one can take $C_1 = C_2 = 1$; otherwise we can use the continuity of the functions

$$r \mapsto \frac{\phi(r)}{r^{\overline{b}(\phi)+\varepsilon}} \quad \text{and} \quad r \mapsto \frac{\phi(r)}{r^{\underline{b}(\phi)-\varepsilon}}$$

on the compact set $[\min\{R_0, 1/R_1\}, R]$ to find two constants $C_1, C_2 > 0$ such that (2.2) holds. Inequality (2.3) can be obtained by an analogous reasoning. ∎

Remark 2.1.4. Inequality (2.4) can be extended in the following way: for all $\varepsilon > 0$ and $R > 0$, there exists $C > 0$ such that, for all $r \in (0, R]$,

$$\overline{\phi}(r) \le C r^{\underline{b}(\phi)-\varepsilon}.$$

If $R > R_0$, we can use the submultiplicativity of $\overline{\phi}$ to see that, for all $r \in (0, R]$,

$$\overline{\phi}(r) \leq \overline{\phi}\left(\frac{R}{R_0}\right)\overline{\phi}\left(\frac{R_0}{R}r\right) \leq \overline{\phi}\left(\frac{R}{R_0}\right)\left(\frac{R_0}{R}\right)^{\underline{b}(\phi)-\varepsilon} r^{\underline{b}(\phi)-\varepsilon}.$$

Similarly, we can extend inequality (2.5) using the same approach: for all $\varepsilon > 0$ and $R > 0$, there exists $C > 0$ such that, for all $r \in [R, \infty)$,

$$\overline{\phi}(r) \leq C r^{\overline{b}(\phi)+\varepsilon}.$$

As a corollary to this remark, we have the following result (see e.g. [9], [25]), showing that the Boyd indices give an integrability criterion for Boyd functions.

Proposition 2.1.5. *Let $\phi \in \mathcal{B}$. If $\overline{b}(\phi) < 0$, then $\int_1^{+\infty} \overline{\phi}(x)/x\, dx < \infty$, and if $\underline{b}(\phi) > 0$, then $\int_0^1 \overline{\phi}(x)/x\, dx < \infty$.*

We can now introduce the spaces T^p_ϕ and t^p_ϕ.

Definition 2.1.6. Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty]$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$. A function $f \in L^p(\mathbb{R}^d)$ belongs to the space $T^p_\phi(x_0)$ if there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ and a constant $C > 0$ such that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C\phi(r) \quad \forall r > 0. \tag{2.7}$$

Moreover, if we also have

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \in o(\phi(r)) \quad \text{as } r \to 0^+, \tag{2.8}$$

we say that f belongs to $t^p_\phi(x_0)$.

Remark 2.1.7. In the previous definition, the condition $\underline{b}(\phi) > -d/p$ is there to ensure that the spaces T^p_ϕ are not degenerate: if $r^{-d/p} < C\phi(r)$ is satisfied in a neighborhood of the origin, then any function belongs to $T^p_\phi(x_0)$; this inequality is never satisfied if $-d/p < \underline{b}(\phi)$. This condition could be relaxed in Definition 2.1.6, but the interest of such an extended definition is not obvious.

Remark 2.1.8. Let us highlight the fact that $t^p_\phi(x_0)$ is a "true subspace" of $T^p_\phi(x_0)$; indeed, under the assumptions of the previous definition, if $f \in L^p(\mathbb{R}^d)$ is such that there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ for which

$$\phi(r)^{-1} r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \to 0 \quad \text{as } r \to 0^+,$$

then there exists $R > 0$ such that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq \phi(r)$$

for all $r \leq R$. Moreover, for $r \geq R$, we have

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq r^{-d/p}\|f\|_{L^p(\mathbb{R}^d)} + C_R(1 + r^n)$$

and an application of Proposition 2.1.3 shows that the right-hand side can be bounded from above by $\phi(r)$, which means that $f \in T^p_\phi(x_0)$.

Let us study the basic properties of the spaces T^p_ϕ.

Proposition 2.1.9. *If $f \in T^p_\phi(x_0)$, then the polynomial P in (2.7) is unique.*

Proof. Of course, if $\underline{b}(\phi) \leq 0$, the polynomial appearing in (2.7) must be 0. Now, if $\underline{b}(\phi) > 0$, let us suppose that there exist two polynomials P and P' of degree strictly less than $\underline{b}(\phi)$ and $C, C' > 0$ such that, for all $r > 0$,

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C\phi(r),$$
$$r^{-d/p}\|f - P'\|_{L^p(B(x_0,r))} \leq C'\phi(r).$$

Now, if we define $Q := P - P'$, then Q is a polynomial of degree $n < \underline{b}(\phi)$. So, if $\varepsilon > 0$ is such that $n < \underline{b}(\phi) - \varepsilon$, then by Proposition 2.1.3 there exists $C'' > 0$ such that

$$r^{-d/p}\|Q\|_{L^p(B(x_0,r))} \leq C'' r^{\underline{b}(\phi)-\varepsilon}.$$

But, if Q is a non-zero polynomial, then the left-hand side must decrease at most like r^n, which contradicts this last inequality. ∎

Remark 2.1.10. If $\phi \in \mathcal{B}$ and if $f \in T^p_\phi(x_0)$ for some $p \in [1, \infty]$, then, in particular, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Suppose that $\underline{b}(\phi) > 0$ (otherwise, the polynomial P in (2.7) is identically zero) and recall (see [3]) that almost every $x \in \mathbb{R}^d$ is then a Lebesgue point of f, which means that

$$\lim_{r \to 0^+} r^{-d}\|f - f(x)\|_{L^1(B(x,r))} = 0.$$

If x_0 is a Lebesgue point of f and if P is of degree strictly less than $\underline{b}(\phi)$ such that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C\phi(r) \qquad \forall r > 0,$$

then we also have

$$r^{-d}\|f - P\|_{L^1(B(x_0,r))} \leq C_d r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq C'\phi(r)$$

for all $r > 0$. From the previous relations, we have

$$\begin{aligned} |f(x_0) - P(x_0)| &\leq C_d r^{-d}\|f(x_0) - P(x_0)\|_{L^1(B(x_0,r))} \\ &\leq r^{-d}\|f(x_0) - f\|_{L^1(B(x_0,r))} + r^{-d}\|f - P\|_{L^1(B(x_0,r))} \\ &\quad + r^{-d}\|P - P(x_0)\|_{L^1(B(x_0,r))} \\ &\leq r^{-d}\|f(x_0) - f\|_{L^1(B(x_0,r))} + C'\phi(r) + C_d \sum_{1 \leq |\alpha| < \underline{b}(\phi)} \left|\frac{D^\alpha P(x_0)}{\alpha!}\right| r^{|\alpha|}. \end{aligned}$$

But, as $\underline{b}(\phi) > 0$, Proposition 2.1.3 implies that $\phi(r)$ converges to 0 as r tends to 0^+. As a consequence, since x_0 is supposed to be a Lebesgue point of f, the last upper bound in the previous inequality tends to 0 as r tends to 0^+, which implies $f(x_0) = P(x_0)$.

Let $f \in T^p_\phi(x_0)$ and let

$$P := \sum_{|\alpha| < \underline{b}(\phi)} \frac{D^\alpha P(x_0)}{\alpha!}(x - x_0)^\alpha$$

be the polynomial which appears in (2.7). Let us set

$$|f|_{T^p_\phi(x_0)} := \sup_{r>0} \phi(r)^{-1} r^{-d/p}\|f - P\|_{L^p(B(x_0,r))}$$

and

$$\|f\|_{T^p_\phi(x_0)} := \|f\|_{L^p(\mathbb{R}^d)} + \sum_{|\alpha| < \underline{b}(\phi)} \frac{|D^\alpha P(x_0)|}{\alpha!} + |f|_{T^p_\phi(x_0)}.$$

PROPOSITION 2.1.11. *Let $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$. Then $(T^p_\phi(x_0), \|\cdot\|_{T^p_\phi(x_0)})$ is a Banach space.*

Proof. It is straightforward to show that $\|\cdot\|_{T^p_\phi(x_0)}$ is a norm on $T^p_\phi(x_0)$.

Let us now consider a Cauchy sequence $(f_j)_{j\in\mathbb{N}_0}$ in $(T^p_\phi(x_0), \|\cdot\|_{T^p_\phi(x_0)})$. For $j \in \mathbb{N}_0$, let us denote by P_j the polynomial of degree strictly less than $\underline{b}(\phi)$ such that, for all $r > 0$,

$$r^{-d/p}\|f_j - P_j\|_{L^p(B(x_0,r))} \le |f_j|_{T^p_\phi(x_0)}\phi(r).$$

Let $f \in L^p(\mathbb{R}^d)$ and $c_\alpha \in \mathbb{C}$ ($|\alpha| < \underline{b}(\phi)$) satisfy $f_j \to f$ in $L^p(\mathbb{R}^d)$ and $D^\alpha P_j(x_0)/\alpha! \to c_\alpha$ in $\mathbb{C}$ for all $|\alpha| < \underline{b}(\phi)$. Let us then define

$$P := \sum_{|\alpha|<\underline{b}(\phi)} c_\alpha (x - x_0)^\alpha.$$

For all $q \in \mathbb{N}_0$, we have

$$\begin{aligned}\phi(r)^{-1}r^{-d/p}&\|(f - f_q) - (P - P_q)\|_{L^p(B(x_0,r))}\\ &= \phi(r)^{-1}r^{-d/p}\lim_{s\to\infty}\|(f_s - f_q) - (P_s - P_q)\|_{L^p(B(x_0,r))}\\ &\le \limsup_{s\to\infty}\|f_q - f_s\|_{T^p_\phi(x_0)} < \infty.\end{aligned}$$

Taking the supremum over $r > 0$ gives us

$$|f - f_q|_{T^p_\phi(x_0)} \le \limsup_{s\to\infty}\|f_q - f_s\|_{T^p_\phi(x_0)} < \infty$$

and passing to the limit for $q \to +\infty$ allows us to get

$$\lim_{q\to+\infty}|f - f_q|_{T^p_\phi(x_0)} = 0,$$

which is enough to conclude the proof, as the finiteness of $|f|_{T^p_\phi(x_0)}$ follows from the triangle inequality. ∎

PROPOSITION 2.1.12. *Let $x_0 \in \mathbb{R}^d$, $p \in [1,\infty]$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$. Then $t^p_\phi(x_0)$ is a closed subspace of $T^p_\phi(x_0)$.*

Proof. Let $(f_j)_{j\in\mathbb{N}_0}$ be a sequence of functions in $t^p_\phi(x_0)$ for which there exists $f \in T^p_\phi(x_0)$ such that $f_j \to f$ in $T^p_\phi(x_0)$ and let us show that $f \in t^p_\phi(x_0)$. Let P and P_j ($j \in \mathbb{N}_0$) be polynomials of degree strictly less than $\underline{b}(\phi)$ such that

$$r^{-d/p}\|f_j - P_j\|_{L^p(B(x_0,r))} \le |f_j|_{T^p_\phi(x_0)}\phi(r) \quad \forall j \in \mathbb{N}_0$$

and

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \le |f|_{T^p_\phi(x_0)}\phi(r).$$

If we set $R := f - P$ and $R_j := f_j - P_j$, we know that

$$\sup_{r>0}\phi(r)^{-1}r^{-d/p}\|R_j - R\|_{L^p(B(x_0,r))} \le \|f_j - f\|_{T^p_\phi(x_0)} \to 0 \quad \text{as } j \to \infty$$

and

$$\phi(r)^{-1}r^{-d/p}\|R_j\|_{L^p(B(x_0,r))} \to 0 \quad \text{as } r \to 0^+.$$

Given $\varepsilon > 0$, let $J \in \mathbb{N}_0$ be such that $j \ge J$ implies

$$\sup_{r>0}\phi(r)^{-1}r^{-d/p}\|R_j - R\|_{L^p(B(x_0,r))} < \varepsilon/2.$$

There also exists ρ_J such that, for all $r \in (0, \rho_J]$,

$$\phi(r)^{-1} r^{-d/p} \|R_J\|_{L^p(B(x_0,r))} < \varepsilon/2.$$

As a consequence, for such r,

$$\phi(r)^{-1} r^{-d/p} \|R\|_{L^p(B(x_0,r))} < \varepsilon,$$

which proves that $f \in t^p_\phi(x_0)$. ■

There is an obvious link between the classical spaces C^k of the k-times continuously differentiable functions and the spaces $t^p_\phi(x_0)$, given by the following remark.

Remark 2.1.13. Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty]$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$. First, if $\overline{b}(\phi) < 0$ and $f \in C^0(V)$, where V is an open neighborhood of x_0, then $f \in t^p_\phi(x_0)$. Indeed, if $R > 0$ is such that $B(x_0, R) \subseteq V$ then there exists $C > 0$ such that $|f| \leq C$ on $B(x_0, R)$ and, for $r \in (0, R]$, we have

$$r^{-d/p} \|f\|_{L^p(B(x_0,r))} \leq C.$$

It follows from Proposition 2.1.3 that

$$r^{-d/p} \|f\|_{L^p(B(x_0,r))} \in o(\phi(r)) \quad \text{as } r \to 0^+.$$

Also, if there exists $n \in \mathbb{N}_0$ such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n + 1$ and $f \in C^{n+1}(V)$, then again $f \in t^p_\phi(x_0)$. Let P be the Taylor expansion of order n of f at x_0. There exists $C > 0$ such that $|f - P| \leq C(\cdot - x_0)^{n+1}$ on $B(x_0, R)$. Therefore

$$r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \leq C r^{n+1} \quad \text{for } r \in (0, R],$$

and the conclusion follows again from Proposition 2.1.3.

2.2. A density result. Let φ be a non-negative, real-valued function in $\mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \varphi(x)\, dx = 1 \quad \text{and} \quad \operatorname{supp}(\varphi) \subset \overline{B(0,1)}.$$

Let f be a function which belongs to $L^p(\mathbb{R}^d)$ for some $p \in [1, \infty)$ and, for $\lambda > 0$, define f_λ by

$$f_\lambda := \lambda^d \varphi(\lambda \cdot) * f. \tag{2.9}$$

It is well-known that $f_\lambda \in L^p(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and $\|f_\lambda - f\|_{L^p(\mathbb{R}^d)} \to 0$ as $\lambda \to \infty$. Let us show that if $f \in t^p_\phi(x_0)$, then under some basic assumptions on ϕ, the convergence also holds in $T^p_\phi(x_0)$.

Proposition 2.2.1. *Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty)$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$ and either $\underline{b}(\phi) \leq 0$ or there exists $n \in \mathbb{N}_0$ such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n + 1$. If a function f belongs to $t^p_\phi(x_0)$, then $\|f_\lambda - f\|_{T^p_\phi(x_0)} \to 0$ as $\lambda \to \infty$.*

Proof. Without loss of generality, we can suppose that $x_0 = 0$. Let us first consider the case where there exists $n \in \mathbb{N}_0$ such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n + 1$. Given $\lambda > 0$, define $R_\lambda := f_\lambda - P_\lambda$ where P_λ is the Taylor expansion of order n of f_λ at 0. Let $R := f - P$, where P is a polynomial of degree n, be such that

$$\phi(r)^{-1} r^{-d/p} \|R\|_{L^p(B(0,r))} \to 0 \quad \text{as } r \to 0^+.$$

For $r > 0$, we have

$$r^{-d}\|R\|_{L^1(B(0,r))} \le C_d r^{-d/p}\|R\|_{L^p(B(0,r))} \le \varepsilon(r)\phi(r),$$

where $\varepsilon(r) \to 0$ as $r \to 0^+$. We can make the assumption that $\varepsilon(r)$ is decreasing to 0 as $r \to 0^+$.

Let us remark that, for $|\alpha| \le n$, $D^\alpha P_\lambda(0) \to D^\alpha P(0)$ as $\lambda \to \infty$. Indeed, for $\lambda > 0$,

$$\begin{aligned} D^\alpha P_\lambda(0) &= D^\alpha f_\lambda(0) \\ &= \int_{\mathbb{R}^d} \lambda^d \varphi(-\lambda y) D^\alpha P(y)\,dy + \int_{\mathbb{R}^d} (-1)^{|\alpha|}\lambda^{d+|\alpha|} D^\alpha\varphi(-\lambda y) R(y)\,dy. \end{aligned}$$

The first term of the right-hand side tends to $D^\alpha P(0)$ as λ tends to infinity and for the second term, we have

$$\left|\int_{\mathbb{R}^d} (-1)^{|\alpha|}\lambda^{d+|\alpha|} D^\alpha\varphi(-\lambda y)R(y)\,dy\right| \le C_\varphi \lambda^{d+|\alpha|}\int_{B(0,1/\lambda)} |R(y)|\,dy \le \varepsilon\Big(\frac{1}{\lambda}\Big)\lambda^{|\alpha|}\phi\Big(\frac{1}{\lambda}\Big),$$

which proves, since $|\alpha| < \underline{b}(\phi)$, that $\int_{\mathbb{R}^d}(-1)^{|\alpha|}\lambda^{d+|\alpha|}D^\alpha\varphi(-\lambda y)R(y)\,dy$ tends to 0 as $\lambda \to \infty$.

Given $r > 0$ and $\lambda > 0$, let us now estimate the quantity $\|R_\lambda\|_{L^p(B(x_0,r))}$. For all $x \in \mathbb{R}^d$, we have

$$\begin{aligned} R_\lambda(x) &= f_\lambda(x) - P_\lambda(x) \\ &= \int_{\mathbb{R}^d}\Big(\lambda^d\varphi(\lambda(x-y)) - \sum_{|\alpha|\le n}\lambda^{d+|\alpha|}\frac{D^\alpha\varphi(-\lambda y)}{\alpha!}x^\alpha\Big)(P(y)+R(y))\,dy \end{aligned}$$

and since

$$\int_{\mathbb{R}^d}\Big(\lambda^d\varphi(\lambda(x-y))P(y) - \sum_{|\alpha|\le n}\lambda^{d+|\alpha|}\frac{D^\alpha\varphi(-\lambda y)}{\alpha!}P(y)x^\alpha\Big)\,dy$$

is equal to $\lambda^d\varphi(\lambda\cdot) * P$ (which is a polynomial of degree n) minus its Taylor expansion of order n at 0, this last integral is equal to 0. Therefore,

$$R_\lambda(x) = \int_{\mathbb{R}^d}\Big(\lambda^d\varphi(\lambda(x-y)) - \sum_{|\alpha|\le n}\lambda^{d+|\alpha|}\frac{D^\alpha\varphi(-\lambda y)}{\alpha!}x^\alpha\Big)R(y)\,dy.$$

It follows, by Young's inequality, that

$$\begin{aligned} \|R_\lambda\|_{L^p(B(0,r))} &\le C_\varphi\|R\|_{L^p(B(0,2r))} + \sum_{\alpha\le n}\lambda^{d+|\alpha|}\|RD^\alpha\varphi(-\lambda\cdot)\|_{L^1(B(0,1/\lambda))}\|\cdot^\alpha\|_{L^p(B(0,r))} \\ &\le C'_\varphi\Big(r^{d/p}\varepsilon(2r)\phi(r) + \sum_{\alpha\le n}\varepsilon\Big(\frac{1}{\lambda}\Big)\lambda^{|\alpha|}\phi\Big(\frac{1}{\lambda}\Big)r^{d/p+|\alpha|}\Big) \end{aligned}$$

for all $r \ge 1/\lambda$. But, as $\phi(1/\lambda) \le \phi(r)\overline{\phi}\big(\frac{1}{r\lambda}\big)$ and $\frac{1}{r\lambda} \le 1$, we have, thanks to Remark 2.1.4,

$$\overline{\phi}\Big(\frac{1}{r\lambda}\Big)(r\lambda)^{|\alpha|} \le C_\delta (r\lambda)^{-(\underline{b}(\phi)-\delta-|\alpha|)} \le C_\delta,$$

where $\delta > 0$ has been chosen such that $\underline{b}(\phi) - \delta - n \ge 0$. Consequently, given $r, \lambda > 0$ such that $r \ge 1/\lambda$, we have

$$\|R_\lambda\|_{L^p(B(0,r))} \le Cr^{d/p}\varepsilon(2r)\phi(r). \tag{2.10}$$

On the other hand, if $r < 1/\lambda$, Taylor's formula provides

$$\left|\lambda^d \varphi(\lambda(x-y)) - \sum_{|\alpha|\leq n} \lambda^{d+|\alpha|} \frac{D^\alpha \varphi(-\lambda y)}{\alpha!} x^\alpha\right| \leq C_\varphi (\lambda|x|)^{n+1}\lambda^d,$$

which implies

$$|R_\lambda(x)| \leq C_\varphi (\lambda|x|)^{n+1}\lambda^d \int_{B(0,2/\lambda)} |R(y)|\, dy \leq C_{\varphi,d}(\lambda|x|)^{n+1}\varepsilon\Big(\frac{2}{\lambda}\Big)\phi\Big(\frac{2}{\lambda}\Big)$$

for all $x \in B(0,r)$. Therefore,

$$\|R_\lambda\|_{L^p(B(0,r))} \leq C r^{d/p}(\lambda r)^{n+1}\varepsilon\Big(\frac{2}{\lambda}\Big)\phi\Big(\frac{1}{\lambda}\Big).$$

Now, using the second part of Remark 2.1.4, we can write

$$(\lambda r)^{n+1}\phi\Big(\frac{1}{\lambda}\Big) \leq \phi(r)(\lambda r)^{n+1}\overline{\phi}\Big(\frac{1}{r\lambda}\Big) \leq C_{\delta'}\phi(r)(r\lambda)^{n+1-\overline{b}(\phi)-\delta'} \leq C_{\delta'}\phi(r),$$

where $\delta' > 0$ has been chosen such that $n+1-\overline{b}(\phi)-\delta' \geq 0$. As a consequence, given $R, \lambda > 0$ such that $r < 1/\lambda$, we have

$$\|R_\lambda\|_{L^p(B(0,r))} \leq C r^{d/p}\varepsilon\Big(\frac{2}{\lambda}\Big)\phi(r). \tag{2.11}$$

From (2.10) and (2.11), we have

$$\phi(r)^{-1}r^{-d/p}\|R_\lambda\|_{L^p(B(0,r))} \leq C\Big(\varepsilon(2r) + \varepsilon\Big(\frac{2}{\lambda}\Big)\Big)$$

for all $r, \lambda > 0$, which naturally implies

$$\phi(r)^{-1}r^{-d/p}\|R - R_\lambda\|_{L^p(B(0,r))} \leq C\Big(\varepsilon(2r) + \varepsilon\Big(\frac{2}{\lambda}\Big)\Big). \tag{2.12}$$

Let us now remark that if we fix $\rho > 0$ and choose $\eta > 0$ such that

$$\underline{b}(\phi) - \eta > n,$$

then, from Proposition 2.1.3, we have

$$\begin{aligned}
&\phi(r)^{-1}r^{-d/p}\|R - R_\lambda\|_{L^p(B(0,r))} \\
&\quad \leq \phi(r)^{-1}r^{-d/p}\|f - f_\lambda\|_{L^p(B(0,r))} + C_d \sum_{|\alpha|\leq n} \frac{|D^\alpha P(x_0) - D^\alpha P_\lambda(x_0)|}{\alpha!}\phi(r)^{-1}r^{|\alpha|} \\
&\quad \leq C_\rho r^{-\underline{b}(\phi)+\eta-d/p}\|f - f_\lambda\|_{L^p(\mathbb{R}^d)} + C_{d,\rho} \sum_{|\alpha|\leq n} \frac{|D^\alpha P(x_0) - D^\alpha P_\lambda(x_0)|}{\alpha!} r^{-\underline{b}(\phi)+\eta+|\alpha|} \\
&\quad \leq C_\rho \rho^{-\underline{b}(\phi)+\eta-d/p}\|f - f_\lambda\|_{L^p(\mathbb{R}^d)} + C_{d,\rho} \sum_{|\alpha|\leq n} \frac{|D^\alpha P(x_0) - D^\alpha P_\lambda(x_0)|}{\alpha!} \rho^{-\underline{b}(\phi)+\eta+|\alpha|}
\end{aligned}$$

for all $r > \rho$. As we know that $\|f - f_\lambda\|_{L^p(\mathbb{R}^d)} \to 0$ and $D^\alpha P_\lambda(0) \to D^\alpha P(0)$ as $\lambda \to \infty$ for all $|\alpha| \leq n$, we get

$$\sup_{r\geq\rho} \phi(r)^{-1}r^{-d/p}\|R - R_\lambda\|_{L^p(B(0,r))} \to 0 \quad \text{as } \lambda \to \infty. \tag{2.13}$$

Combining (2.12) and (2.13) leads to

$$\sup_{r>0} \phi(r)^{-1} r^{-d/p} \|R - R_\lambda\|_{L^p(B(0,r))} \to 0 \quad \text{as } \lambda \to \infty, \tag{2.14}$$

since otherwise there exists $\xi > 0$ such that for all $\Lambda > 0$ there exists $\lambda > \Lambda$ for which

$$\sup_{r>0} \phi(r)^{-1} r^{-d/p} \|R - R_\lambda\|_{L^p(B(0,r))} \geq \xi,$$

which makes us able to build a sequence $(\lambda_j)_{j\in\mathbb{N}_0}$ that converges to ∞ and satisfies

$$\sup_{r>0} \phi(r)^{-1} r^{-d/p} \|R - R_{\lambda_j}\|_{L^p(B(0,r))} \geq \xi \quad \text{for all } j.$$

In particular, given $j \in \mathbb{N}_0$, there exists $r_j > 0$ such that

$$\phi(r_j)^{-1} r_j^{-d/p} \|R - R_{\lambda_j}\|_{L^p(B(0,r_j))} \geq \xi/2. \tag{2.15}$$

As $\lambda_j \to \infty$, there exists $J_1 \in \mathbb{N}_0$ such that for all $j \geq J_1$, $\varepsilon(2/\lambda_j) < \xi/(4C)$, where $C > 0$ is the constant appearing in (2.12). Moreover, there also exists $\rho > 0$ such that, for any $r \in (0, \rho]$, $\varepsilon(2r) < \xi/(4C)$. From (2.13), we know that there exists $J_2 \in \mathbb{N}_0$ such that, for all $j \geq J_2$,

$$\sup_{r>\rho} \phi(r)^{-1} r^{-d/p} \|R - R_{\lambda_j}\|_{L^p(B(0,r))} < \xi/2. \tag{2.16}$$

Therefore, if $j \geq \max\{J_1, J_2\}$, (2.16) implies $r_j \leq \rho$ and, by (2.12) and (2.15), we finally get a contradiction.

If we now assume that $\overline{b}(\phi) \leq 0$, then $R = f$ and $R_\lambda = f_\lambda$. Therefore, by Young's inequality, we have

$$\|R_\lambda\|_{L^p(B(0,r))} \leq C_\varphi \|R\|_{L^p(B(0,2r))} \leq C\varepsilon(2r)\phi(r).$$

If $r \leq 1/\lambda$, let us recall that $\varepsilon(2r) \leq \varepsilon(2/\lambda)$. As a consequence, (2.10)–(2.12) still hold and we can conclude the proof in the same way, using the fact that

$$\phi(r)^{-1} r^{-d/p} \|R - R_\lambda\|_{L^p(B(0,r))} = \phi(r)^{-1} r^{-d/p} \|f - f_\lambda\|_{L^p(B(0,r))}$$

and $\underline{b}(\phi) > -d/p$. ∎

The last proposition has the following useful corollary.

Corollary 2.2.2. *Under the assumptions of Proposition 2.2.1, the space $\mathcal{D}(\mathbb{R}^d)$ is a dense subspace of $t^p_\phi(x_0)$.*

Proof. Let us consider $f \in t^p_\phi(x_0)$ and the sequence $(f_j)_{j\in\mathbb{N}_0}$ of functions defined by

$$f_j := f\chi_{\overline{B(0,2^j)}} \quad (j \in \mathbb{N}_0).$$

By Lebesgue's theorem, it is clear that $f_j \to f$ in $L^p(\mathbb{R}^d)$; we will show that $f_j \in t^p_\phi(x_0)$ ($j \in \mathbb{N}_0$) and that the convergence also holds in $T^p_\phi(x_0)$.

Let P be the polynomial of degree strictly less than $\underline{b}(\phi)$ such that

$$\phi(r)^{-1} r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \to 0 \quad \text{as } r \to 0^+.$$

First, as $f_j = f$ on $B(x_0, 1)$, we have

$$\phi(r)^{-1} r^{-d/p} \|f_j - P\|_{L^p(B(x_0,r))} \to 0 \quad \text{as } r \to 0^+$$

for any $j \in \mathbb{N}_0$. Therefore, given $j \in \mathbb{N}_0$, $f_j \in t^p_\phi(x_0)$ and

$$\|f - f_j\|_{T^p_\phi(x_0)} = \|f - f_j\|_{L^p(\mathbb{R}^d)} + \sup_{r>0} \phi(r)^{-1} r^{-d/p} \|f_j - f\|_{L^p(B(x_0,r))}.$$

On the one hand, if $r \in (0, 2^j]$ then

$$\phi(r)^{-1} r^{-d/p} \|f_j - f\|_{L^p(B(x_0,r))} = 0$$

and, on the other hand, if $r > 2^j$, by Proposition 2.1.3,

$$\begin{aligned}\phi(r)^{-1} r^{-d/p} \|f_j - f\|_{L^p(B(x_0,r))} &\leq C r^{-(\underline{b}(\phi)-\varepsilon+d/p)} \|f_j - f\|_{L^p(\mathbb{R}^d)} \\ &\leq C 2^{-j(\underline{b}(\phi)-\varepsilon+d/p)} \|f_j - f\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

where $\varepsilon > 0$ satisfies $\underline{b}(\phi) - \varepsilon + d/p \geq 0$ and $C > 0$ satisfies $r^{(\underline{b}(\phi)-\varepsilon)} \leq C\phi(r)$ for all $r \geq 1$. Therefore,

$$\|f - f_j\|_{T^p_\phi(x_0)} \leq \|f - f_j\|_{L^p(\mathbb{R}^d)} + C 2^{-j(\underline{b}(\phi)-\varepsilon+d/p)} \|f_j - f\|_{L^p(\mathbb{R}^d)} \to 0 \quad \text{as } j \to \infty,$$

which provides the convergence in $T^p_\phi(x_0)$.

The conclusion then follows from Proposition 2.2.1. ■

2.3. Some embeddings

Notation 2.3.1. Given $\phi, \psi \in \mathcal{B}$, we will write $\phi \preccurlyeq \psi$ to mean that there exists $R, C > 0$ such that, for all $r \in (0, R)$, we have $\phi(r) \leq C\psi(r)$.

Of course, by continuity, one has $\phi \preccurlyeq \psi$ if and only if, for all $R > 0$, there exists $C > 0$ such that $\phi(r) \leq C\psi(r)$ for all $r \in (0, R)$.

Proposition 2.3.2. *Let $\phi, \psi \in \mathcal{B}$. If $\overline{b}(\psi) < \underline{b}(\phi)$ then $\phi \preccurlyeq \psi$. Conversely, if $\phi \preccurlyeq \psi$, then $\underline{b}(\psi) \leq \overline{b}(\phi)$.*

Proof. Let us first assume that $\overline{b}(\psi) < \underline{b}(\phi)$ and let $\varepsilon > 0$ be such that

$$\overline{b}(\psi) + \varepsilon < \underline{b}(\phi) - \varepsilon.$$

By Proposition 2.1.3, given $R > 0$, there exists $C > 0$ such that, for all $r \in (0, R)$,

$$\phi(r) \leq C r^{\underline{b}(\phi)-\varepsilon} \leq C' r^{\overline{b}(\psi)+\varepsilon} \leq C'' \psi(r),$$

which means $\phi \preccurlyeq \psi$.

If we now assume $\phi \preccurlyeq \psi$ then, in particular, there exists $C > 0$ such that, for all $r \in (0, 1)$,

$$\overline{\phi}(1/r)^{-1} \leq C\overline{\psi}(r).$$

Therefore, for such r, we have

$$\frac{\log(\overline{\phi}(1/r))}{\log(1/r)} \geq \frac{\log(C)}{\log(r)} + \frac{\log(\overline{\psi}(r)}{\log(r)}$$

and letting $r \to 0^+$ gives $\overline{b}(\phi) \geq \underline{b}(\psi)$. ■

Proposition 2.3.3. *Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty]$ and $\phi, \psi \in \mathcal{B}$ be such that either $\overline{b}(\psi) < 0$ or there exists $n \in \mathbb{N}_0$ for which $n < \underline{b}(\psi) \leq \overline{b}(\psi) < n+1$. If $\phi \preccurlyeq \psi$, then $T^p_\phi(x_0) \hookrightarrow T^p_\psi(x_0)$. Moreover, if $\phi(r) \in o(\psi(r))$ as $r \to 0^+$, then $T^p_\phi(x_0) \hookrightarrow t^p_\psi(x_0)$.*

Proof. Let $f \in T^p_\phi(x_0)$; there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ such that

$$r^{-d/p}\|f-P\|_{L^p(B(x_0,r))} \le |f|_{T^p_\phi(x_0)}\phi(r) \quad \forall r>0.$$

Let $Q=0$, $k=l=0$ if $\overline{b}(\psi)<0$ and

$$Q=\sum_{|\alpha|\le n}\frac{D^\alpha P(x_0)}{\alpha!}(\cdot - x_0)^\alpha,$$

$k=n+1$, $l=n$ if $n\in\mathbb{N}_0$ satisfies $n<\underline{b}(\psi)\le\overline{b}(\psi)<n+1$. For any $r\le 1$, we obviously have, by Proposition 2.1.3,

$$\begin{aligned} r^{-d/p}\|f-Q\|_{L^p(B(x_0,r))} &\le r^{-d/p}\|f-P\|_{L^p(B(x_0,r))}+r^{-d/p}\|P-Q\|_{L^p(B(x_0,r))}\\ &\le |f|_{T^p_\phi(x_0)}\phi(r)+C_d\|f\|_{T^p_\phi(x_0)}r^k \le C_{\phi,\psi}\|f\|_{T^p_\phi(x_0)}\psi(r), \end{aligned}$$

while for $r>1$,

$$\begin{aligned} r^{-d/p}\|f-Q\|_{L^p(B(x_0,r))} &\le r^{-d/p}\|f\|_{L^p(B(x_0,r))}+r^{-d/p}\|Q\|_{L^p(B(x_0,r))}\\ &\le r^{-d/p}\|f\|_{L^p(\mathbb{R}^d)}+C_{d,p}\|f\|_{T^p_\phi}r^l \le C_\phi\|f\|_{T^p_\phi(x_0)}\psi(r), \end{aligned}$$

which leads to the first part of the proposition.

The second part comes from the inequality

$$r^{-d/p}\|f-Q\|_{L^p(B(x_0,r))} \le |f|_{T^p_\phi(x_0)}\phi(r)+C_d\|f\|_{T^p_\phi(x_0)}r^k,$$

valid for all $0<r\le 1$ and the relations $\phi(r)\in o(\psi(r))$ and $r^k\in o(\phi(r))$. ∎

Proposition 2.3.4. *Let $x_0\in\mathbb{R}^d$, $p_1,p_2\in[1,\infty]$ and p_3 be such that*

$$0\le\frac{1}{p_3}:=\frac{1}{p_1}+\frac{1}{p_2}\le 1$$

and let $\phi\in\mathcal{B}$ be such that there exists $n\in\mathbb{N}_0$ for which $n<\underline{b}(\phi)\le\overline{b}(\phi)<n+1$. Given $f_1\in T^{p_1}_\phi(x_0)$ and $f_2\in T^{p_2}_\phi(x_0)$, we have $f_1f_2\in T^{p_3}_\phi(x_0)$ with

$$\|f_1f_2\|_{T^{p_3}_\phi(x_0)}\le C_{d,p_1,p_2,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)}.$$

Moreover, if $f_1\in t^{p_1}_\phi(x_0)$ and $f_2\in t^{p_2}_\phi(x_0)$, then $f_1f_2\in t^{p_3}_\phi(x_0)$.

Proof. We know that, given $k\in\{1,2\}$, there exists a polynomial P_k of degree less than or equal to n such that $R_k:=f_k-P_k$ satisfies

$$r^{-d/p_k}\|R_k\|_{L^{p_k}(B(x_0,r))}\le|f_k|_{T^{p_k}_\phi(x_0)}\phi(r). \tag{2.17}$$

Therefore, if we denote by P the sum of the terms of degree less than or equal to n in P_1P_2, we have

$$f_1f_2=P_1P_2+R_1P_2+R_2f_1=P+P_1P_2-P+R_1P_2+R_2f_1.$$

Let $R:=P_1P_2-P+R_1P_2+R_2f_1$; clearly,

$$\sum_{|\alpha|\le n}\frac{|D^\alpha P(x_0)|}{\alpha!}\le\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)}.$$

Let us first consider $r\le 1$; by Proposition 2.1.3, since

$$|P_1P_2(x)-P(x)|\le(x-x_0)^{n+1}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)}$$

for $x \in B(x_0, r)$, we have

$$\begin{aligned} r^{-d/p_3}\|P_1P_2 - P\|_{L^{p_3}(B(x_0,r))} &\leq C_{d,p_3}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)}r^{n+1} \\ &\leq C_{d,p_1,p_2,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)}\phi(r). \end{aligned}$$

Also, for all $x \in B(x_0, r)$, since $|P_k(x)| \leq \|f_k\|_{T^{p_k}_\phi(x_0)}$ ($k \in \{1, 2\}$),

$$\begin{aligned} r^{-d/p_3}\|R_1P_2\|_{L^{p_3}(B(x_0,r))} &\leq r^{-d/p_2}\|P_2\|_{L^{p_2}(B(x_0,r))}r^{-d/p_1}\|R_1\|_{L^{p_1}(B(x_0,r))} \\ &\leq C_{d,p_2}\|f_2\|_{T^{p_2}_\phi(x_0)}|f_1|_{T^{p_1}_\phi(x_0)}\phi(r). \end{aligned}$$

Using again Proposition 2.1.3, we get

$$\begin{aligned} r^{-d/p_1}\|f_1\|_{L^{p_1}(B(x_0,r))} &\leq r^{-d/p_1}\|f_1 - P_1\|_{L^{p_1}(B(x_0,r))} + r^{-d/p_1}\|P_1\|_{L^{p_1}(B(x_0,r))} \\ &\leq |f_1|_{T^{p_1}_\phi(x_0)}\phi(r) + C_{d,p_1}\|f_1\|_{T^{p_1}_\phi(x_0)}r^n \leq C_{d,p_1,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}, \end{aligned}$$

and thus

$$\begin{aligned} r^{-d/p_3}\|f_1R_2\|_{L^{p_3}(B(x_0,r))} &\leq r^{-d/p_1}\|f_1\|_{L^{p_1}(B(x_0,r))}r^{-d/p_2}\|R_2\|_{L^{p_2}(B(x_0,r))} \\ &\leq C_{d,p_1,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\phi(x_0)}\phi(r). \end{aligned}$$

As a consequence, we can write, for $r < 1$,

$$r^{-d/p_3}\|R\|_{L^{p_3}(B(x_0,r))} \leq C_{d,p_1,p_2,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)}\phi(r). \tag{2.18}$$

If now we consider $r > 1$, as $|R| \leq |f_1|\,|f_2| + |P|$, we get

$$r^{-d/p_3}\|R\|_{L^{p_3}(B(x_0,r))} \leq r^{-d/p_3}\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)} + C_{d,p}r^n\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\phi(x_0)},$$

so that (2.18) still holds in this case, by Proposition 2.1.3.

Finally, if $f_1 \in t^{p_1}_\phi(x_0)$ and $f_2 \in t^{p_2}_\phi(x_0)$, we can write

$$r^{-d/p_k}\|R_k\|_{L^{p_k}(B(x_0,r))} \leq \varepsilon_k(r)\phi(r),$$

with $\varepsilon_k(r) > 0$ for $r > 0$ and $\varepsilon_k(r) \to 0$ as $r \to 0^+$ ($k \in \{1, 2\}$). Replacing $|f_k|_{T^{p_k}_\phi(x_0)}$ with $\varepsilon_k(r)$ in the preceding relations, one gets

$$\phi(r)^{-1}r^{-d/p_3}\|R\|_{L^{p_3}(B(x_0,r))} \to 0^+ \quad \text{as } r \to 0^+,$$

which is sufficient to conclude the proof. ∎

Corollary 2.3.5. *Let $x_0 \in \mathbb{R}^d$, $p_1, p_2 \in [1, \infty]$ and p_3 be such that*

$$0 \leq \frac{1}{p_3} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1,$$

and let ϕ, ψ be functions in $\mathcal{B}$ such that $\underline{b}(\phi) > 0$, $\underline{b}(\psi) \geq -d/p_2$, $\phi \preccurlyeq \psi$ and either $\underline{b}(\psi) \leq 0$ or $n < \underline{b}(\psi) \leq \overline{b}(\psi) < n+1$ for some $n \in \mathbb{N}_0$. If $f_1 \in T^{p_1}_\phi(x_0)$ and $f_2 \in T^{p_2}_\psi(x_0)$, then $f_1f_2 \in T^{p_3}_\psi(x_0)$, with

$$\|f_1f_2\|_{T^{p_3}_\psi(x_0)} \leq C_{d,p_1,p_2,\phi,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\psi(x_0)}.$$

Moreover, if $f_1 \in t^{p_1}_\phi(x_0)$ and $f_2 \in t^{p_2}_\psi(x_0)$, then $f_1f_2 \in t^{p_3}_\psi(x_0)$.

Proof. If $\underline{b}(\psi) \le 0$, the embedding is obvious since $T^p_\phi(x_0) \hookrightarrow t^p_0(x_0)$ and so, for $r > 0$,

$$\begin{aligned} r^{-d/p_3}\|f_1 f_2\|_{L^{p_3}(B(x_0,r))} &\le r^{-d/p_1}\|f_1\|_{L^{p_1}(B(x_0,r))} r^{-d/p_2}\|f_2\|_{L^{p_2}(B(x_0,r))} \\ &\le C_{p_1,\phi,0}\|f_1\|_{T^{p_1}_\phi(x_0)} |f_2|_{T^{p_2}_\psi(x_0)}\psi(r). \end{aligned}$$

Otherwise $\underline{b}(\psi) > 0$ and $f_1 \in T^{p_1}_\psi(x_0)$ with

$$\|f_1\|_{T^{p_1}_\psi(x_0)} \le C_{\phi,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}.$$

Using the previous proposition, we get $f_1 f_2 \in T^{p_3}_\psi(x_0)$ and

$$\|f_1 f_2\|_{T^{p_3}_\psi(x_0)} \le C_{d,p,\psi}\|f_1\|_{T^{p_1}_\psi(x_0)}\|f_2\|_{T^{p_2}_\psi(x_0)} \le C_{d,p,\phi,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\psi(x_0)},$$

which allows us to conclude the proof. The second part can be obtained using the usual arguments. ■

Proposition 2.3.6. *Let $p_1, p_2 \in [1,\infty]$ and p_3 be such that $0 \le 1/p_3 := 1/p_1 + 1/p_2 \le 1$ and let $\phi, \varphi \in \mathcal{B}$ be such that $-d/p_2 \le \underline{b}(\varphi)$, $0 < \underline{b}(\phi)$. Let also $f_1 \in T^{p_1}_\phi(x_0)$, $f_2 \in T^{p_2}_\varphi(x_0)$, where x_0 is a Lebesgue point of f_1. Finally let $\psi \in \mathcal{B}$ be such that $\underline{b}(\psi) > -d/p_2$, $\phi \preccurlyeq \psi$ and*

- *$\overline{b}(\psi) - \underline{b}(\varphi) < \underline{b}(\phi)$ if $\underline{b}(\phi) \le 1$,*
- *$\overline{b}(\psi) - \underline{b}(\varphi) < 1$ if $\underline{b}(\phi) > 1$ and either $\overline{b}(\psi) < 1$ or there exists $n \in \mathbb{N}$ for which $n < \underline{b}(\psi) \le \overline{b}(\psi) < n+1$.*

There exists a polynomial P of degree strictly less than $\underline{b}(\psi)$ such that, for all $r > 0$,

$$r^{-d/p_3}\|(f_1 - f_1(x_0))f_2 - P\|_{L^{p_3}(B(x_0,r))} \le C_{p_1,p_2,\phi,\varphi,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\varphi(x_0)}\psi(r).$$

Consequently, if $f_2 \in L^{p_3}(\mathbb{R}^d)$, then $(f_1 - f_1(x_0))f_2$ belongs to $T^{p_3}_\psi(x_0)$, with

$$\|(f_1 - f_1(x_0))f_2\|_{T^{p_3}_\psi(x_0)} \le C_{p_1,p_2,\phi,\varphi,\psi}\|f_1\|_{T^{p_1}_\varphi(x_0)}\big(\|f_2\|_{T^{p_2}_\varphi(x_0)} + \|f_2\|_{L^{p_3}(\mathbb{R}^d)}\big).$$

Proof. We use here the same notations as in the proof of Proposition 2.3.4 and set $g_1 := f_1 - f_1(x_0)$. Let us first consider the case $\underline{b}(\phi) \le 1$; P_1 must be a constant and, by Remark 2.1.10, we have $P_1 = f_1(x_0)$, which allows us to write

$$r^{-d/p_1}\|g_1\|_{L^{p_1}(B(x_0,r))} \le |f_1|_{T^{p_1}_\phi(x_0)}\phi(r). \tag{2.19}$$

Let us consider each case separately. If $\underline{b}(\varphi) \le 0$, then

$$r^{-d/p_2}\|f_2\|_{L^{p_2}(B(x_0,r))} \le |f_2|_{T^{p_2}_\varphi(x_0)}\varphi(r).$$

Therefore, if $\psi \in \mathcal{B}$ is such that $\overline{b}(\psi) < \underline{b}(\phi) + \underline{b}(\varphi)$, then, by choosing $\varepsilon > 0$ such that $\overline{b}(\psi) + \varepsilon < \underline{b}(\phi) + \underline{b}(\varphi) - 2\varepsilon$, we get, by Proposition 2.1.3,

$$\begin{aligned} &r^{-d/p_3}\|g_1 f_2\|_{L^{p_3}(B(x_0,r))} \\ &\quad\le r^{-d/p_1}\|g_1\|_{L^{p_1}(B(x_0,r))} r^{-d/p_2}\|f_2\|_{L^{p_2}(B(x_0,r))} \le |f_1|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\varphi(x_0)}\phi(r)\varphi(r) \\ &\quad\le C|f_1|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\varphi(x_0)} r^{\underline{b}(\phi)+\underline{b}(\varphi)-2\varepsilon} \le C'\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\varphi(x_0)}\psi(r) \end{aligned}$$

for $0 < r \leq 1$, where $C, C' > 0$ only depend on ϕ, φ and ψ. If $r > 1$, as $-d/p_2 < \underline{b}(\psi)$, we can use Proposition 2.1.3 to get

$$\begin{aligned}
r^{-d/p_3}\|g_1 f_2\|_{L^{p_3}(B(x_0,r))} &\leq r^{-d/p_3}\|f_1 f_2\|_{L^{p_3}(B(x_0,r))} + r^{-d/p_3}|f_1(x_0)|\,\|f_2\|_{L^{p_3}(B(x_0,r))}\\
&\leq r^{-d/p_3}\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)} + C_{p_2,p_3} r^{-d/p_2}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{L^{p_2}(B(x_0,r))}\\
&\leq C_{p_1,p_2,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\varphi(x_0)}\psi(r).
\end{aligned}$$

If $\underline{b}(\varphi) > 0$, let us consider $\psi \in \mathcal{B}$ such that $\underline{b}(\psi) > -d/p_2$, $\overline{b}(\psi) < \underline{b}(\phi) + \underline{b}(\varphi)$ and $\phi \preccurlyeq \psi$. For $0 < r \leq 1$, Proposition 2.1.3 allows us to write

$$\begin{aligned}
&r^{-d/p_3}\|g_1 f_2\|_{L^{p_3}(B(x_0,r))}\\
&\quad\leq r^{-d/p_3}\|g_1 P_2\|_{L^{p_3}(B(x_0,r))} + r^{-d/p_3}\|g_1 R_2\|_{L^{p_3}(B(x_0,r))}\\
&\quad\leq C_{d,p_2}|f_1|_{T^{p_1}_\phi(x_0)}\phi(r)\Bigg(\sum_{|\alpha|<\underline{b}(\varphi)}\frac{|D^\alpha P_2(x_0)|}{\alpha!}\Bigg) + |f_1|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\varphi(x_0)}\phi(r)\varphi(r)\\
&\quad\leq C_{p_2,\phi,\varphi,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p}_\varphi(x_0)}\psi(r).
\end{aligned}$$

Again, the previous inequality holds for $r > 1$ as well.

Let us now investigate the case $\underline{b}(\phi) > 1$. For $0 < r \leq 1$ we have, as we know that $P_1(x_0) = f_1(x_0)$,

$$\begin{aligned}
r^{-d/p_1}\|g_1\|_{L^{p_1}(B(x_0,r))} &\leq |f_1|_{T^{p_1}_\phi(x_0)}\phi(r) + C_{d,p_1}\Bigg(\sum_{1\leq|\alpha|<\underline{b}(\phi)}\frac{|D^\alpha P_1(x_0)|}{\alpha!}\Bigg)r\\
&\leq C_{p_1,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}r.
\end{aligned}$$

Obviously, this inequality still holds for $r > 1$. If $\underline{b}(\varphi) \leq 0$, then for all $\psi \in \mathcal{B}$ such that $\underline{b}(\psi) > -d/p_2$ and $\overline{b}(\psi) < \underline{b}(\varphi) + 1$, we have, by Proposition 2.1.3,

$$\begin{aligned}
r^{-d/p_3}\|g_1 f_2\|_{L^{p_3}(B(x_0,r))} &\leq C_{p_1,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\varphi(x_0)}\varphi(r)r\\
&\leq C_{p_1,\phi,\varphi,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\varphi(x_0)}\psi(r)
\end{aligned}$$

for $0 \leq r < 1$. As $\underline{b}(\psi) > -d/p_3$, this inequality is also satisfied for $r > 1$. If $\underline{b}(\varphi) > 0$, let us consider $\psi \in \mathcal{B}$ with $\underline{b}(\psi) > -d/p_2$, $\overline{b}(\psi) < \underline{b}(\varphi)+1$ and $\phi \preccurlyeq \psi$. On the one hand, if $\overline{b}(\psi) < 1$, Proposition 2.1.3 implies

$$\begin{aligned}
r^{-d/p_3}\|g_1 f_2\|_{L^{p_3}(B(x_0,r))} &\leq r^{-d/p_3}\|g_1 P_2\|_{L^{p_3}(B(x_0,r))} + r^{-d/p_3}\|g_1 R_2\|_{L^{p_3}(B(x_0,r))}\\
&\leq C_{p_1,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\varphi(x_0)}r\\
&\qquad + C_{p_1,\phi}\|f_1\|_{T^{p_1}_\phi(x_0)}|f_2|_{T^{p_2}_\varphi(x_0)}\varphi(r)r\\
&\leq C_{\phi,\varphi,\psi}\|f_1\|_{T^{p_1}_\phi(x_0)}\|f_2\|_{T^{p_2}_\varphi(x_0)}\psi(r)
\end{aligned}$$

for $0 < r \leq 1$; again one can easily check that this inequality also holds for $r > 1$. On the other hand, if $n \in \mathbb{N}$ is such that $n < \underline{b}(\psi) \leq \overline{b}(\psi) < n+1$, let us define P as the sum of terms of degree less than or equal to n in $(P_1 - f_1(x_0))P_2$; we have

$$g_1 f_2 = (P_1 - f_1(x_0))P_2 + R_1 P_2 + R_2 g_1 = P + (P_1 - f_1(x_0))P_2 - P + R_1 P_2 + R_2 g_1.$$

By setting $R := (P_1 - f_1(x_0))P_2 - P + R_1P_2 + R_2g_1$, Proposition 2.1.3 gives

$$\begin{aligned} r^{-d/p_3}\|R\|_{L^{p_3}(B(x_0,r))} &\le r^{-d/p_3}\|g_1f_2\|_{L^{p_3}(B(x_0,r))} + r^{-d/p_3}\|P\|_{L^{p_3}(B(x_0,r))} \\ &\le C_{p_3,\psi}\|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)}\psi(r) \\ &\quad + C_{p_3,p_2,\psi}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)}\psi(r) \\ &\quad + C_{d,p_3}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}r^n \\ &\le C_{\psi,p_1,p_2}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}\psi(r) \end{aligned}$$

for $r > 1$, while for $0 < r < 1$ we have

$$\begin{aligned} r^{-d/p_3}\|R_1P_2\|_{L^{p_3}(B(x_0,r))} &\le C_{d,p_2}|f_1|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}\phi(r) \\ &\le C_{p_2,\phi,\psi}|f_1|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}\psi(r), \end{aligned}$$

$$\begin{aligned} r^{-d/p_3}\|R_2g_1\|_{L^{p_3}(B(x_0,r))} &\le C_{p_1,\phi}\|f_1\|_{T_\phi^{p_1}(x_0)}|f_2|_{T_\varphi^{p_2}(x_0)}\varphi(r)r \\ &\le C_{p_1\phi,\varphi,\psi}\|f_1\|_{T_\phi^{p_1}(x_0)}|f_2|_{T_\varphi^{p_2}(x_0)}\psi(r) \end{aligned}$$

and

$$\begin{aligned} r^{-d/p_3}\|(P_1 - f_1(x_0))P_2 - P\|_{L^{p_3}(B(x_0,r))} &\le C_{d,p_3}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}r^{n+1} \\ &\le C_{p_1,p_2,\psi}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}\psi(r). \end{aligned}$$

This proves that there exists a constant $C_{p_1,p_2\phi,\varphi,\psi} > 0$ such that, for all $r > 0$,

$$r^{-d/p_3}\|g_1f_2 - P\|_{L^{p_3}(B(x_0,r))} \le C_{p_1,p_2,\phi,\varphi,\psi}\|f_1\|_{T_\phi^{p_1}(x_0)}\|f_2\|_{T_\varphi^{p_2}(x_0)}\psi(r).$$

If $f_2 \in L^{p_3}(\mathbb{R}^d)$, then

$$\|g_1f_2\|_{L^{p_3}(\mathbb{R}^d)} \le \|f_1\|_{L^{p_1}(\mathbb{R}^d)}\|f_2\|_{L^{p_2}(\mathbb{R}^d)} + |f_1(x_0)|\,\|f_2\|_{L^{p_3}(\mathbb{R}^d)},$$

which gives the conclusion. ■

Proposition 2.3.7. *Let $x_0 \in \mathbb{R}^d$, $p_1, p_2 \in [1,\infty]$ be such that $p_1 \le p_2$ and let $\phi \in \mathcal{B}$ be such that $-d/p_2 < \underline{b}(\phi)$. If $f \in T_\phi^{p_2}(x_0) \cap L^{p_1}(\mathbb{R}^d)$, then $f \in T_\phi^{p_1}(\mathbb{R}^d)$, with*

$$\|f\|_{T_\phi^{p_1}(\mathbb{R}^d)} \le \|f\|_{T_\phi^{p_2}(x_0)} + \|f\|_{L^{p_1}(\mathbb{R}^d)}.$$

Moreover, in this case, $f \in t_\phi^{p_2}(x_0)$ implies $f \in t_\phi^{p_1}(x_0)$.

Proof. Let P be the polynomial of degree strictly less than $\underline{b}(\phi)$ such that, for $r > 0$,

$$r^{-d/p_2}\|f - P\|_{L^{p_2}(B(x_0,r))} \le |f|_{T_\phi^{p_2}(x_0)}\phi(r).$$

For such r, we have

$$\begin{aligned} r^{-d/p_1}\|f - P\|_{L^{p_1}(B(x_0,r))} &\le r^{-d/p_1}C_{d,p_1,p_2}r^{d/p_1-d/p_2}\|f - P\|_{L^{p_2}(B(x_0,r))} \\ &\le C_{d,p_1,p_2}|f|_{T_\phi^{p_2}(x_0)}\phi(r), \end{aligned}$$

which is sufficient to conclude the proof, as $f \in L^{p_1}(\mathbb{R}^d)$.

The second part can be obtained using the same arguments as usual. ■

2.4. Generalization of Whitney's extension theorem. In this section, we show that some uniform conditions on a closed set E involving the spaces T^p_ϕ and t^p_ϕ imply the belonging to the spaces $B_\phi(E)$ and $b_\phi(E)$ respectively, which we define below. Then, we show that a function which has such properties can be extended in an open neighborhood of E into a function which satisfies generalized Hölderian condition type (see [19]).

In what follows we will heavily need the following lemma. Its proof can be found in [35] for example.

Lemma 2.4.1. *Given $n \in \mathbb{N}_0$, there exists a function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with support in $\overline{B(0,1)}$ such that, for any polynomial P of degree less than or equal to n and any $\varepsilon > 0$,*

$$\varphi_\varepsilon * P = P.$$

We now introduce the spaces $B_\phi(E)$ and $b_\phi(E)$ of functions which admit a formal Taylor expansion on a set $E \subset \mathbb{R}^d$ for which the behavior can be characterized by a Lipschitz-type condition given by a function $\phi \in \mathcal{B}$.

Definition 2.4.2. Let E be a subset of $\mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$. A bounded function f on E belongs to the space $B_\phi(E)$ if there exist $C, M > 0$ such that, for all $x_0 \in E$, there exist a polynomial P_{x_0} of degree strictly less than $\underline{b}(\phi)$,

$$P_{x_0} := \sum_{|\alpha| < \underline{b}(\phi)} \frac{f_\alpha(x_0)}{\alpha!}(\cdot - x_0)^\alpha,$$

such that $f_0(x_0) = f(x_0)$, $|f_\alpha(x_0)| \leq M$ for all $|\alpha| < \underline{b}(\phi)$ and

$$|D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| \leq C\phi(|x - x_0|)|x - x_0|^{-|\alpha|}$$

for all $x \in E$ satisfying $x \neq x_0$ and all $|\alpha| < \underline{b}(\phi)$.

Definition 2.4.3. Let E be a subset of $\mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$. A function f defined on E belongs to the space $b_\phi(E)$ if, for any $x_0 \in E$, there exists a polynomial P_{x_0} of degree strictly less than $\underline{b}(\phi)$,

$$P_{x_0} := \sum_{|\alpha| < \underline{b}(\phi)} \frac{f_\alpha(x_0)}{\alpha!}(\cdot - x_0)^\alpha,$$

for which $f_0(x_0) = f(x_0)$ and

$$\lim_{\substack{x \to x_0 \\ x \in E}} \phi(|x - x_0|)^{-1}|x - x_0|^{|\alpha|}|D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| = 0 \quad \text{uniformly in } x_0 \in E.$$

Proposition 2.4.4. *Let E be a closed subset of $\mathbb{R}^d$ and let $\phi \in \mathcal{B}$ satisfy $\underline{b}(\phi) > 0$.*

(1) *If there exists $M > 0$ such that $f \in T^p_\phi(x_0)$ with $\|f\|_{T^p_\phi(x_0)} \leq M$ for all $x_0 \in E$, then $f \in B_\phi(E)$ (in the sense that f is equal almost everywhere to a function in $B_\phi(E)$).*
(2) *If $f \in t^p_\phi(x_0)$ for all $x_0 \in E$, with (2.8) holding uniformly in $x_0 \in E$, then $f \in b_\phi(E)$.*

Proof. Let us prove (1). We know that for any $x_0 \in E$, there exists a polynomial P_{x_0} of degree strictly less than $\underline{b}(\phi)$ such that $R_{x_0} := f - P_{x_0}$ satisfies

$$r^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,r))} \leq M\phi(r), \tag{2.20}$$

for $r > 0$, with $|D^\alpha P_{x_0}(x_0)|/\alpha! \le M$ for all $|\alpha| < \underline{b}(\phi)$. Moreover, in the light of Remark 2.1.10, one can modify f on a negligible set in order to have $f(x_0) = P_{x_0}(x_0)$ for all $x_0 \in E$. In particular $|f(x_0)| \le M$ for all $x_0 \in E$ and f is bounded on E.

Let us take a function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ as in Lemma 2.4.1, let x, x_0 be distinct points of E and set $\varepsilon := |x - x_0|$. Let us define, for $|\alpha| < \underline{b}(\phi)$,

$$I_\alpha := D^\alpha(\varphi_\varepsilon * f)(x).$$

On the one hand, we have

$$\begin{aligned} I_\alpha &= D^\alpha(\varphi_\varepsilon * (P_{x_0} + R_{x_0}))(x) = (\varphi_\varepsilon * D^\alpha P_{x_0})(x) + (D^\alpha \varphi_\varepsilon \star R_{x_0})(x) \\ &= D^\alpha P_{x_0}(x) + (D^\alpha \varphi_\varepsilon * R_{x_0})(x), \end{aligned}$$

and, on the other hand,

$$I_\alpha = D^\alpha P_x(x) + (D^\alpha \varphi_\varepsilon * R_x)(x).$$

So we get, for $|\alpha| < \underline{b}(\phi)$,

$$\begin{aligned} D^\alpha P_x(x) &= D^\alpha P_{x_0}(x) + (D^\alpha \varphi_\varepsilon * (R_{x_0} - R_x)(x) \\ &= D^\alpha P_{x_0}(x) + \int_{B(x,\varepsilon)} \varepsilon^{-d+|\alpha|} D^\alpha \varphi\Big(\frac{x-y}{\varepsilon}\Big)(R_{x_0}(y) - R_x(y))\, dy. \end{aligned}$$

Setting $C_\varphi := \sup_{|\alpha|<\underline{b}(\phi)} \|D^\alpha \varphi\|_\infty$, we finally get, for $|\alpha| < \underline{b}(\phi)$,

$$\begin{aligned} |D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| &\le C_\varphi \varepsilon^{-|\alpha|}\big(\varepsilon^{-d}\|R_{x_0}\|_{L^1(B(x,\varepsilon))} + \varepsilon^{-d}\|R_x\|_{L^1(B(x,\varepsilon))}\big) \\ &\le C_\varphi C_d \varepsilon^{-|\alpha|}\big((2\varepsilon)^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,2\varepsilon))} + \varepsilon^{-d/p}\|R_x\|_{L^p(B(x,\varepsilon))}\big) \\ &\le C\phi(|x - x_0|)|x - x_0|^{-\alpha}, \end{aligned}$$

where the constant $C > 0$ only depends on C_φ, M, d and ϕ.

For (2), let us consider

$$r^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,r))} \in o(\phi(r)) \quad \text{as } r \to 0^+$$

uniformly in $x_0 \in E$, instead of (2.20). Since

$$|D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| \le C_\varphi C_d \varepsilon^{-|\alpha|}\big((2\varepsilon)^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,2\varepsilon))} + \varepsilon^{-d/p}\|R_x\|_{L^p(B(x,\varepsilon))}\big)$$

for all $x, x_0 \in E$ with $0 < \varepsilon = |x - x_0|$, we conclude that, given $C > 0$, there exists $\eta > 0$ such that if $0 < |x - x_0| < \eta$ ($x, x_0 \in E$) then

$$|D^\alpha P_x(x) - D^\alpha P_{x_0}(x)| \le C\phi(|x - x_0|)|x - x_0|^{-\alpha},$$

which means that $f \in b_\phi(E)$. ∎

The theorem concluding this section relies on the following lemma, which establishes the existence of a smooth function on a neighborhood of a closed subset E which is comparable to the distance from E (see e.g. [35, 8]).

Lemma 2.4.5. *Let $E \subset \mathbb{R}^d$ be a closed set and $U = \{x \in \mathbb{R}^d : d(x,E) < 1\}$. There exist $\delta \in C^\infty(U \setminus E)$ and $C > 0$ such that*

$$C^{-1} d(x,E) \le \delta(x) \le C d(x,E) \quad \forall x \in U \setminus E$$

and

$$|D^\alpha \delta(x)| \le C(\alpha) d(x,E)^{1-|\alpha|} \quad \forall x \in U \setminus E,\ |\alpha| \ge 0.$$

We will also need the following combinatorial lemma, which can easily be proved by induction on $l \in \mathbb{N}_0$.

LEMMA 2.4.6. *Let $l \in \mathbb{N}_0$.*

- *If $l = 0 \mod 4$, then*

$$-\frac{1}{2}\binom{l}{l/2} = \sum_{j=0}^{l/2-1} (-1)^j \binom{l}{j} = \sum_{j=l/2+1}^{l} (-1)^j \binom{l}{j}.$$

- *If $l = 1 \mod 4$, then*

$$\binom{l-1}{(l-1)/2} = \sum_{j=0}^{(l-1)/2} (-1)^j \binom{l}{j} = -\sum_{j=\frac{l-1}{2}+1}^{l} (-1)^j \binom{l}{j}.$$

- *If $l = 2 \mod 4$, then*

$$\frac{1}{2}\binom{l}{l/2} = \sum_{j=0}^{l/2-1} (-1)^j \binom{l}{j} = \sum_{j=l/2+1}^{l} (-1)^j \binom{l}{j}.$$

- *If $l = 3 \mod 4$, then*

$$-\binom{l-1}{(l-1)/2} = \sum_{j=0}^{(l-1)/2} (-1)^j \binom{l}{j} = -\sum_{j=(l-1)/2+1}^{l} (-1)^j \binom{l}{j}.$$

For the following result, we need the notion of finite (forward) difference for a function f (see e.g. [29]): set $\Delta_h^1 f(x) = f(x+h) - f(x)$ and, given $n \in \mathbb{N}$, $\Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x)$.

THEOREM 2.4.7. *Let $E \subset \mathbb{R}^d$ be a closed set, $U = \{x \in \mathbb{R}^d : d(x,E) < 1\}$, $n \in \mathbb{N}_0$ and $\phi \in \mathcal{B}$ be such that $n < \underline{b}(\phi)$. If $f \in T_\phi^p(x_0)$ satisfies $\|f\|_{T_\phi^p(x_0)} \leq M$ for some $M > 0$ and all $x_0 \in E$, then there exists $F \in C^n(U)$ such that $F = f$ almost everywhere on E.*

Moreover, if $m \in \mathbb{N}_0$ is such that $n < \underline{b}(\phi) \leq \overline{b}(\phi) < m$, then there exists $C > 0$ such that for any $x \in U$ and any $h \in \mathbb{R}^d \setminus \{0\}$ for which $[x, x + (m-n)h] \subset U$, we have

$$|\Delta_h^{m-n} D^\alpha F(x)| \leq C\phi(|h|)|h|^{-n} \quad \textit{for any } |\alpha| = n.$$

Proof. Let us consider the functions φ and δ from Lemmata 2.4.1 and 2.4.5 respectively. We know that we can modify f on a set of measure zero so that $f \in B_\phi(E)$. Let us define the function F on U by

$$F(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \delta(x)^{-d} \int_{\mathbb{R}^d} \varphi((x-y)\delta(x)^{-1}) f(y)\, dy & \text{otherwise.} \end{cases}$$

Obviously $F \in C^\infty(U \setminus E)$. Let $\overline{x} \in U \setminus E$ and $x_0 \in E$ be such that $|\overline{x} - x_0| = d(\overline{x}, E)$. As $x_0 \in E$, there exists a polynomial P_{x_0} of degree less than or equal to n such that $R_{x_0} := f - P_{x_0}$ satisfies

$$r^{-d/p}\|R_{x_0}\|_{L^p(B(x_0,r))} \leq M\phi(r) \quad \text{for all } r > 0.$$

For any $x \in U \setminus E$, by setting

$$\Phi_\alpha(x, \cdot) = D_x^\alpha(\delta(x)^{-d}\varphi((x - \cdot)\delta(x)^{-1}),$$

we have, by Lemma 2.4.1,

$$D^\alpha F(x) = D^\alpha P_{x_0}(x) + \int_{\mathbb{R}^d} \Phi_\alpha(x,y) R_{x_0}(y)\,dy.$$

One can easily check (by induction) that $\Phi_\alpha(x,\cdot)$ is of the form

$$\delta(x)^{-d-k} D^\alpha \varphi((x-\cdot)\delta^{-1}(x))(x-\cdot)^\gamma P(x),$$

where $P(x)$ is a product of derivatives of the function δ evaluated at x with t factors and whose sum of orders is equal to w and where $k+w-t-|\gamma| = |\alpha|$. Thanks to the property of the function δ, we have $|P(x)| \le Cd(x,E)^{t-w}$, $\delta(x)^{-d-k} \le C^* d(x,E)^{-d-k}$ and

$$|D^\alpha \varphi((x-\cdot)\delta^{-1}(x))(x-y)^\gamma| \le C_{\gamma,\alpha} d(x,E)^{|\gamma|},$$

as $D^\alpha \varphi((x-\cdot)\delta^{-1}(x))(x-\cdot)^\gamma$ does not vanish if $|x-\cdot| \le \delta(x)$. We thus have

$$\left| \int_{\mathbb{R}^d} \Phi_\alpha(x,y) R_{x_0}(y)\,dy \right| \le C_1 d(x,E)^{-d-|\alpha|} \int_{B(x,\delta(x))} |R_{x_0}(y)|\,dy$$

for all $\alpha \in \mathbb{N}_0^d$ and $x \in U \setminus E$. As there exists $C' > 0$ such that $\delta(x) \le C' d(x,E)$ for all $x \in U \setminus E$, we can write

$$\begin{aligned}
|D^\alpha F(\overline{x}) - D^\alpha P_{x_0}(\overline{x})| &\le C_1 d(\overline{x},E)^{-d-|\alpha|} \int_{B(\overline{x},C'd(\overline{x},E))} |R_{x_0}(y)|\,dy \\
&\le C_1 d(\overline{x},E)^{-|\alpha|} d(\overline{x},E)^{-d} \int_{B(x_0,(C'+1)d(\overline{x},E))} |R_{x_0}(y)|\,dy \\
&\le C_2 M \phi(d(\overline{x},E)) d(\overline{x},E)^{-|\alpha|} \\
&= C_2 M \phi(|\overline{x}-x_0|)(|\overline{x}-x_0|)^{-|\alpha|},
\end{aligned}$$

where $C_2 > 0$ is a constant which only depends on φ, ϕ, C_1, C' and d. Moreover, since $f \in B_\phi(E)$, we know that $P_{x_0}(x_0) = f(x_0)$ and for all $x_1 \in E$ such that $x_1 \ne x_0$, $D^\alpha P_{x_0}(x_0) = D^\alpha P_{x_1}(x_0) + R_\alpha(x_0,x_1)$, where R_α satisfies

$$|R_\alpha(x_0,x_1)| \le C\phi(|x_0-x_1|)(|x_0-x_1|)^{-|\alpha|} \quad \text{for all } |\alpha| \le n. \tag{2.21}$$

Therefore, thanks to Taylor's formula, we have, for $|\alpha| \le n$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
D^\alpha P_{x_0}(x) &= \sum_{|\beta| \le n-|\alpha|} \frac{1}{\beta!} D^{\alpha+\beta} P_{x_0}(x_0)(x-x_0)^\beta \\
&= \sum_{|\beta| \le n-|\alpha|} \frac{1}{\beta!} (D^{\alpha+\beta} P_{x_1}(x_0) + R_{\alpha+\beta}(x_0,x_1))(x-x_0)^\beta \\
&= \sum_{|\beta| \le n-|\alpha|} \frac{1}{\beta!} \Bigg(\sum_{|\gamma| \le n-(|\alpha|+|\beta|)} \frac{1}{\gamma!} D^{\alpha+\beta+\gamma} P_{x_1}(x_1)(x_0-x_1)^\gamma + R_{\alpha+\beta}(x_0,x_1) \Bigg)(x-x_0)^\beta
\end{aligned}$$

and

$$\begin{aligned}\sum_{|\beta|\le n-|\alpha|}\frac{1}{\beta!}&\sum_{|\gamma|\le n-(|\alpha|+|\beta|)}\frac{1}{\gamma!}D^{\alpha+\beta+\gamma}P_{x_1}(x_1)(x_0-x_1)^\gamma(x-x_0)^\beta\\
&=\sum_{|\gamma|\le n-|\alpha|}\frac{1}{\gamma!}\sum_{|\beta|\le n-(|\alpha|+|\gamma|)}\frac{1}{\beta!}D^{\alpha+\beta+\gamma}P_{x_1}(x_1)(x-x_0)^\beta(x_0-x_1)^\gamma\\
&=\sum_{|\gamma|\le n-|\alpha|}\frac{1}{\gamma!}D^{\alpha+\gamma}P_{x_1}(x-x_0+x_1)(x_0-x_1)^\gamma=D^\alpha P_{x_1}(x).\end{aligned}$$

Finally, we have

$$D^\alpha P_{x_0}(x)=D^\alpha P_{x_1}(x)+\sum_{|\beta|\le n-|\alpha|}\frac{1}{\beta!}R_{\alpha+\beta}(x_0,x_1)(x-x_0)^\beta$$

for all $x_0,x_1\in E$ and $x\in\mathbb{R}^d$. In particular, for $|\alpha|\le n$,

$$|D^\alpha P_{x_0}(\overline{x})-D^\alpha P_{x_1}(\overline{x})|\le C\sum_{|\beta|\le n-|\alpha|}\phi(|x_0-x_1|)|x_0-x_1|^{-|\alpha|-|\beta|}|\overline{x}-x_0|^{|\beta|},$$

and as $|\overline{x}-x_0|\le|\overline{x}-x_1|$, we have $|x_0-x_1|\le 2|\overline{x}-x_1|$. Therefore,

$$\begin{aligned}\phi(|x_0-x_1|)&|x_0-x_1|^{-|\alpha|-|\beta|}\\
&\le\phi(|\overline{x}-x_1|)|\overline{x}-x_1|^{-|\alpha|-|\beta|}\overline{\phi}\Big(\frac{|x_0-x_1|)}{|\overline{x}-x_1|}\Big)\Big(\frac{|x_0-x_1|)}{|\overline{x}-x_1|}\Big)^{-|\alpha|-|\beta|},\end{aligned}$$

and as $|\alpha|+|\beta|\le n<\underline{b}(\phi)$, Remark 2.1.4 implies that

$$\overline{\phi}\Big(\frac{|x_0-x_1|)}{|\overline{x}-x_1|}\Big)\Big(\frac{|x_0-x_1|)}{|\overline{x}-x_1|}\Big)^{-|\alpha|-|\beta|}$$

is bounded (by a constant which only depends on ϕ). We thus have

$$|D^\alpha P_{x_0}(\overline{x})-D^\alpha P_{x_1}(\overline{x})|\le C\phi(|\overline{x}-x_1|)|\overline{x}-x_1|^{-|\alpha|}\quad\text{for } |\alpha|\le n.$$

This inequality and the upper bound obtained for $D^\alpha F(\overline{x})-D^\alpha P_{x_0}(\overline{x})$ give

$$|D^\alpha F(\overline{x})-D^\alpha P_{x_1}(\overline{x})|\le C(\phi(|\overline{x}-x_0|)|\overline{x}-x_0|^{-|\alpha|}+\phi(|\overline{x}-x_1|)|\overline{x}-x_0|^{-|\alpha|})\quad\text{for all } x_1\in E,$$

and as $|\overline{x}-x_0|\le|\overline{x}-x_1|$, we get, as before,

$$|D^\alpha F(\overline{x})-D^\alpha P_{x_1}(\overline{x})|\le C\phi(|\overline{x}-x_1|)|\overline{x}-x_1|^{-|\alpha|}.\tag{2.22}$$

Let F_α be the function defined on U by

$$F_\alpha(x):=\begin{cases}D^\alpha P_x(x) & \text{if } x\in E,\\ D^\alpha F(x) & \text{otherwise.}\end{cases}$$

We have proved that, for $|\alpha|\le n$, $F_\alpha\in C^\infty(U\setminus E)$ and for $x\in E$ and $h\neq 0$ such that $x+h\in U$, we have

$$F_\alpha(x+h)=\sum_{|\beta|\le n-|\alpha|}D^{\alpha+\beta}P_x(x)h^\beta+R_\alpha(x,x+h),\tag{2.23}$$

where

$$|R_\alpha(x,x+h)|\le C\phi(|h|)|h|^{-|\alpha|},$$

with a uniform constant. More precisely, if h is such that $x + h \in E$, the previous inequality is satisfied because f belongs to $B_\phi(E)$; otherwise $x + h \in U \setminus E$ and the inequality follows from (2.22). This is sufficient to show that $F \in C^n(U)$ and $D^\alpha F = F_\alpha$ on U for all $|\alpha| \leq n$. Indeed, (2.23) implies that F_α is continuous on E, and so on U. Given $n \geq 1$, let us fix $x \in E$; if $h \in \mathbb{R} \setminus \{0\}$ is sufficiently small, for $j \in \{1, \dots, d\}$, we have

$$F(x + he_j) - F(x) = \sum_{|\beta|=1}^{n} D^\beta P_x(x)(he_j)^\beta + R_0(x, x + h),$$

which allows us to write

$$\begin{aligned}\left| \frac{F(x + he_j) - F(x)}{h} - F_{e_j}(x) \right| &\leq \sum_{|\beta|=2}^{n} |D^\beta P_x(x)|\, |h|^{|\beta|-1} + \frac{|R_0(x, x + h)|}{|h|} \\ &\leq \sum_{|\beta|=2}^{n} |D^\beta P_x(x)|\, |h|^{|\beta|-1} + C \frac{\phi(|h|)}{|h|}\end{aligned}$$

and, as the right-hand side tends to 0 as h tends to 0, we conclude, since $1 \leq n < \underline{b}(\phi)$, that F is differentiable at x and $D_j F(x) = F_{e_j}(x)$. If we now assume that F is $(n-1)$-times continuously differentiable at x, with $D^\alpha F(x) = F_\alpha(x)$ for every $|\alpha| \leq n - 1$, we have, for $|\alpha| = n - 1$, $h \in \mathbb{R} \setminus \{0\}$ sufficiently small and $j \in \{1, \dots, d\}$,

$$\begin{aligned}\left| \frac{F_\alpha(x + he_j) - F_\alpha(x)}{h} - F_{\alpha+e_j}(x) \right| &\leq \sum_{|\beta|=1} |D^{\alpha+\beta} P_x(x)|\, |h|^{|\beta|-1} + \frac{|R_\alpha(x, x + h)|}{|h|} \\ &\leq \sum_{|\beta|=1} |D^{\alpha+\beta} P_x(x)|\, |h|^{|\beta|-1} + C \frac{\phi(|h|)}{|h|^n}\end{aligned}$$

and we conclude, in the same way, that F_α is differentiable at x with $D_j F_\alpha(x) = F_{\alpha+e_j}(x)$.

Let us now prove that if $n < \underline{b}(\phi) \leq \overline{b}(\phi) < m$, then there exists $C > 0$ such that, for all $x \in U$ and $h \in \mathbb{R}^d$ such that $[x, x + mh] \subset U$, we have

$$|\Delta_h^{m-n} D^\alpha F(x)| \leq C \phi(|h|) |h|^{-n},$$

for all $|\alpha| = n$. So far, we know from (2.21) and (2.22) that the following inequality holds for all $|\alpha| = n$, $x \in U$ and y in E satisfying $x \neq y$:

$$|F_\alpha(x) - F_\alpha(y)| \leq C \phi(|x - y|) |x - y|^{-n}.$$

If $x \in U$ and $h \in \mathbb{R}^d \setminus \{0\}$ are such that there exists $k \in \{0, \dots, m - n\}$ for which $x + kh \in E$, we can use Lemma 2.4.6 to obtain, setting $l = m - n$,

$$\begin{aligned}|\Delta_h^l D^\alpha F(x)| &= \left| \sum_{j=0}^{l} (-1)^j \binom{l}{j} D^\alpha F(x + jh) \right| \\ &= \left| \sum_{j=0}^{l} (-1)^j \binom{l}{j} (D^\alpha F(x + jh) - D^\alpha F(x + kh)) \right| \\ &\leq \sum_{j=0}^{l} \binom{l}{j} C \phi(|(j - k)h|)\, |(j - k)h|^{-n} \leq C' \phi(|h|) |h|^{-n}.\end{aligned}$$

Let us now consider the case where $x+kh \in U \setminus E$ for all $k \in \{0,\dots,l\}$; let us first suppose that $d(x,E) \le (l+1)|h|$ and take $x_0 \in E$ such that $|x_0 - x| = d(x,E)$. Of course $|x_0 - x| \le (l+1)|h|$ and, for all $j \in \{0,\dots,l\}$, we have $|x_0 - (x+jh)| \le (2l+1)|h|$. As before,

$$\begin{aligned}|\Delta_h^l D^\alpha F(x)| &\le \sum_{j=0}^{l} \binom{l}{j} |D^\alpha F(x+jh) - D^\alpha F(x_0))| \\ &\le C \sum_{j=0}^{l} \binom{l}{j} \phi(|x+jh-x_0)|)\, |x+jh-x_0)|^{-n}\end{aligned}$$

and, for all $j \in \{0,\dots,l\}$,

$$\phi(|x+jh-x_0)|)|x+jh-x_0|^{-n} \le \phi(|h|)\, |h|^{-n} \overline{\phi}\Big(\frac{|x+jh-x_0)|}{|h|}\Big)\Big(\frac{|x+jh-x_0|}{|h|}\Big)^{-n}.$$

That being said, we have $|x+jh-x_0|/h \le 2l+1$ and so, by Remark 2.1.4,

$$\overline{\phi}\Big(\frac{|x+jh-x_0|}{|h|}\Big)\Big(\frac{|x+jh-x_0|}{|h|}\Big)^{-n} \le C,$$

where the constant C only depends on ϕ and l. Therefore, we can write

$$|\Delta_h^l D^\alpha F(x)| \le C' \phi(|h|)|h|^{-n}.$$

It remains to consider the case where $x+kh \in U \setminus E$ for all $k \in \{0,\dots,l\}$ and $(l+1)|h| < d(x,E)$. As before, let x_0 stand for a point in E such that $|x_0 - x| = d(x,E)$. We already know that, for any $y \in U \setminus E$,

$$D^\alpha F(y) = D^\alpha P_{x_0}(y) + \int_{\mathbb{R}^d} \Phi_\alpha(y,\xi) R_{x_0}(\xi)\, d\xi.$$

The function $y \mapsto \int_{\mathbb{R}^d} \Phi_\alpha(y,\xi) R_{x_0}(\xi)\, d\xi$ belongs to $C^\infty(U \setminus E)$ and, for all $\beta \in \mathbb{N}_0^d$,

$$D^\beta \int_{\mathbb{R}^d} \Phi_\alpha(y,\xi) R_{x_0}(\xi)\, d\xi = \int_{\mathbb{R}^d} \Phi_{\alpha+\beta}(y,\xi) R_{x_0}(\xi)\, d\xi.$$

As the segment $[x, x+lh]$ is included in $U \setminus E$, we know, by Taylor's formula, that there exist points x_β with $|\beta| = l$ on the segment $[x, x+lh]$ such that

$$\begin{aligned}\Delta_h^l D^\alpha F(x) = \Delta_h^l \int_{\mathbb{R}^d} \Phi_\alpha(x,\xi) R_{x_0}(\xi)\, d\xi &= \sum_{|\beta|=l} h^\beta \int_{\mathbb{R}^d} \Phi_{\alpha+\beta}(x_\beta,\xi) R_{x_0}(\xi)\, d\xi \\ &= \sum_{|\beta|=l} h^\beta \int_{B(x_\beta, C d(x_\beta,E))} \Phi_{\alpha+\beta}(x_\beta,\xi) R_{x_0}(\xi)\, d\xi,\end{aligned}$$

where C is a constant such that $\delta(y) \le C d(y,E)$ for all $y \in U \setminus E$. Moreover, for such y, we have already obtained

$$|\Phi_{\alpha+\beta}(y)| \le C' d(y,E)^{-d-(|\alpha|+|\beta|)} = C' d(y,E)^{-d-m}.$$

If $|\beta| = l$, since $x_\beta \in [x, x+lh]$, we have

$$d(x_\beta, E) \ge d(x,E) - |x - x_\beta| \ge (l+1)|h| - l|h| = |h|$$

and so, if $\xi \in B(x_\beta, Cd(x_\beta, E))$,

$$\begin{aligned}|\xi - x_0| &\le |\xi - x_\beta| + |x_\beta - x| + |x - x_0| \le Cd(x_\beta, E) + l|h| + d(x, E)\\ &\le Cd(x_\beta, E) + ld(x_\beta, E) + d(x_\beta, E) + l|h| \le C'' d(x_\beta, E).\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\int_{B(x_\beta, Cd(x_\beta,E))} \Phi_{\alpha+\beta}(x_\beta, \xi) R_{x_0}(\xi)\, d\xi\right| &\le C' d(x_\beta, E)^{-d-m} \int_{B(x_0, C''d(x_\beta,E))} |R_{x_0}(\xi)|\, d\xi\\ &\le C' M d(x_\beta, E)^{-m} \phi(d(x_\beta, F))\\ &\le C' M \phi(|h|)|h|^{-m} \overline{\phi}\Big(\frac{d(x_\beta, E)}{|h|}\Big)\Big(\frac{d(x_\beta, E)}{|h|}\Big)^{-m}.\end{aligned}$$

Now, as $d(x_\beta, E)/|h| \ge 1$ and $\overline{b}(\phi) < m$, we know that

$$\overline{\phi}\Big(\frac{d(x_\beta, E)}{|h|}\Big)\Big(\frac{d(x_\beta, E)}{|h|}\Big)^{-m}$$

is bounded by a constant which only depends on ϕ and m. We can thus write

$$|\Delta^l_h D^\alpha F(x)| \le C' \phi(|h|)|h|^{-n},$$

which is what we needed to conclude the proof. ∎

THEOREM 2.4.8. *Let $E \subset \mathbb{R}^d$ be a closed set, $U = \{x \in \mathbb{R}^d : d(x, E) < 1\}$, $n \in \mathbb{N}_0$ and $\phi \in \mathcal{B}$ be such that $n < \underline{b}(\phi)$. If $f \in t^p_\phi(x_0)$ for all $x_0 \in E$, with (2.8) holding uniformly in $x_0 \in E$, then there exists $F \in C^n(U)$ such that $F = f$ almost everywhere on E.*

Moreover, if $m \in \mathbb{N}_0$ is such that $n < \underline{b}(\phi) \le \overline{b}(\phi) < m$, then, for all $|\alpha| = n$, $x \in E$, and $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $0 < |h| \le \eta$ for which $[x, x+(m-n)h] \subset E$,

$$|\Delta^{m-n}_h D^\alpha F(x)| \le \varepsilon \phi(|h|)|h|^{-n}.$$

Proof. The proof is essentially the same as the previous one, using this time the fact that $f \in b_\phi(E)$ and

$$r^{-d/p} \|R_{x_0}\|_{L^p(B(x_0,r))} \in o(\phi(r)) \quad \text{as } r \to 0^+,$$

uniformly in $x_0 \in E$. ∎

3. Applications to operators

3.1. The Bessel operator. In this section we look at the action of the Bessel operator of order s,

$$\mathcal{J}^s f := \mathcal{F}^{-1}\big((1 + |\cdot|^2)^{-s/2} \mathcal{F} f\big) \quad (s \in \mathbb{R},\ f \in \mathcal{S}'),$$

on the spaces $T^p_\phi(x_0)$ and $t^p_\phi(x_0)$. If $\phi \in \mathcal{B}$ and $s \in \mathbb{R}$, then ϕ_s will denote the function

$$\phi_s : (0, +\infty) \to (0, +\infty), \quad x \mapsto \phi(x)x^s.$$

It is obvious that ϕ_s is again in $\mathcal{B}$ and $\underline{b}(\phi_s) = \underline{b}(\phi) + s$ and $\overline{b}(\phi_s) = \overline{b}(\phi) + s$.

Let us recall that if $0 < s < d+1$ then we have $\mathcal{J}^s f = u_s * f$, where u_s is the function defined for $x \neq 0$ by

$$u_s(x) = \frac{1}{(2\pi)^{\frac{d-1}{2}} 2^{s/2}\Gamma(s/2)\Gamma\left(\frac{d-s+1}{2}\right)} e^{|x|} \int_0^{+\infty} e^{-|x|t}(t+t^2/2)^{\frac{d-s-1}{2}}\, dt.$$

The following inequality holds for all $0 < s < d$ and $\alpha \in \mathbb{N}_0^d$:

$$D^\alpha u_s(x) \leq C_{s,\alpha} e^{-|x|}(1+|x|^{-d+s-|\alpha|}). \tag{3.1}$$

For simplicity, let us introduce the notion of admissible value for a real number.

Definition 3.1.1. Given $\phi \in \mathcal{B}$, a value $s > 0$ is said to be *admissible* (for ϕ) if one of the following two conditions is satisfied:

- $\overline{b}(\phi) + s < 0$,
- there exists $n \in \mathbb{N}_0$ such that $n < \underline{b}(\phi) + s \leq \overline{b}(\phi) + s < n + 1$.

Theorem 3.1.2. *Let $x_0 \in \mathbb{R}^d$, $p \in (1,\infty]$, $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$ and $s > 0$ be an admissible value for ϕ. The operator $\mathcal{J}^s$ maps continuously $T^p_\phi(x_0)$ into $T^q_{\phi_s}(x_0)$, where*

- *$1/p \geq 1/q \geq 1/p - s/d$ if $p < d/s$,*
- *$p \leq q \leq \infty$ if $d/s < p \leq \infty$,*
- *$p \leq q < \infty$ if $d/s = p$.*

Proof. Let $f \in T^p_\phi(x_0)$; we know that there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ such that $R := f - P$ satisfies

$$r^{-d/p}\|R\|_{L^p(B(x_0,r))} \leq |f|_{T^p_\phi(x_0)}\phi(r) \quad \text{for all } r > 0. \tag{3.2}$$

Without loss of generality, we can assume that $x_0 = 0$. We first want to estimate the following two quantities, for all $r > 0$ and $u \in \mathbb{R}$:

$$\int_{B(0,r)} |R(x)|\,|x|^{-u}\, dx \quad \text{and} \quad \int_{\mathbb{R}^d \setminus B(0,r)} |R(x)|\,|x|^{-u}\, dx.$$

For this purpose, let us set

$$\varphi(r) := \int_{B(0,r)} |R(x)|\, dx;$$

from (3.2), we have

$$\varphi(r) \leq C_d |f|_{T^p_\phi(0)} r^d \phi(r). \tag{3.3}$$

Moreover, using the spherical coordinates in $\mathbb{R}^d$, we can write

$$\varphi(r) = \int_0^r \psi(\rho)\, d\rho, \tag{3.4}$$

where

$$\psi(\rho) := \rho^{d-1} \int_0^{2\pi}\int_0^\pi \cdots \int_0^\pi |R(x(\rho,\theta_1,\ldots,\theta_{d-1}))|\, d\Omega_d,$$

and $d\Omega_d$ stands for $\sin^{d-2}(\theta_1)\cdots\sin(\theta_{d-2})d\theta_1\cdots d\theta_{d-1}$. Therefore, for $\varepsilon>0$ we have

$$\begin{aligned}\varphi(r)r^{-u}-\varphi(\varepsilon)\varepsilon^{-u}&=\int_\varepsilon^r \rho^{-u}\psi(\rho)\,d\rho-\int_\varepsilon^r u\rho^{-(u+1)}\varphi(\rho)\,d\rho\\&=\int_{B(0,r)\setminus B(0,\varepsilon)}|R(x)|\,|x|^{-u}\,dx-\int_\varepsilon^r u\rho^{-(u+1)}\varphi(\rho)\,d\rho.\end{aligned}$$

Consequently,

$$\int_{B(0,r)\setminus B(0,\varepsilon)}|R(x)|\,|x|^{-u}\,dx\le\varphi(r)r^{-u}+u\int_0^r\rho^{-(u+1)}\varphi(\rho)\,d\rho.$$

If $\underline{b}(\phi)+d-u>0$, then

$$\begin{aligned}\int_0^r\rho^{-(u+1)}\varphi(\rho)\,d\rho&\le C_d|f|_{T^p_\phi(0)}\int_0^r\rho^{d-u-1}\phi(\rho)\,d\rho\le C_d|f|_{T^p_\phi(0)}\phi(r)\int_0^r\rho^{d-u-1}\overline{\phi}\left(\frac{\rho}{r}\right)d\rho\\&=C_d|f|_{T^p_\phi(0)}\phi(r)r^{d-u}\int_0^1\frac{\overline{\phi}(\xi)\xi^{d-u}}{\xi}\,d\xi\le C_u|f|_{T^p_\phi(0)}\phi(r)r^{d-u},\end{aligned}$$

thanks to Proposition 2.1.5. Hence, for all $r>0$ and $u\in\mathbb{R}$ such that $\underline{b}(\phi)+d-u>0$,

$$\int_{B(0,r)}|R(x)|\,|x|^{-u}\,dx\le C_{d,u}|f|_{T^p_\phi(x_0)}\phi(r)r^{d-u}.\tag{3.5}$$

If we now assume that $\overline{b}(\phi)+d-u<0$, then, for all $N>0$,

$$\int_{B(0,N)\setminus B(0,r)}|R(x)|\,|x|^{-u}\,dx=\varphi(N)N^{-u}-\varphi(r)r^{-u}+u\int_r^N\rho^{-u-1}\varphi(\rho)\,d\rho$$

and, since $\varphi(N)N^{-u}$ tends to 0 as $N\to\infty$, we get, thanks to (3.3) and Proposition 2.1.3,

$$\int_{\mathbb{R}^d\setminus B(0,r)}|R(x)|\,|x|^{-u}\,dx\le C_u|f|_{T^p_\phi(x_0)}\phi(r)r^{d-u},\tag{3.6}$$

using the same technique as before.

Let us assume first that $0<s<d$; we have

$$\mathcal{J}^s f=u_s*P+u_s*R,$$

where u_s*P is a polynomial of degree strictly less than $\underline{b}(\phi)$ whose sum of coefficients is bounded by the sum of the coefficients of P. We thus need to estimate u_s*R. Let us fix $r>0$ and $x\in\mathbb{R}^d$ such that $2|x|<r$; if there exists $n\in\mathbb{N}_0$ for which $n<\underline{b}(\phi)+s\le\overline{b}(\phi)+s<n+1$, by Taylor's formula, we find that

$$\begin{aligned}(u_s*R)(x)=&\int_{B(0,r)}u_s(x-y)R(y)\,dy+\sum_{|\alpha|\le n}\frac{x^\alpha}{\alpha!}\int_{\mathbb{R}^d}D^\alpha u_s(-y)R(y)\,dy\\&-\sum_{|\alpha|\le n}\frac{x^\alpha}{\alpha!}\int_{B(0,r)}D^\alpha u_s(-y)R(y)\,dy\\&+\sum_{|\alpha|=n+1}\int_{\mathbb{R}^d\setminus B(0,r)}D^\alpha u_s(\Theta(x)x-y)R(y)\,dy,\end{aligned}$$

for some $\Theta(x) \in (0,1)$. Using (3.1) and then (3.5), we get, for all $|\alpha| \le n$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} D^\alpha u_s(-y)R(y)\,dy\right| \\
&\le C\Big(\int_{B(0,1)} |y|^{-d+s-|\alpha|}|R(y)|\,dy + \int_{\mathbb{R}^d\setminus B(0,1)} e^{-|y|}|f(y)|\,dy + \int_{\mathbb{R}^d\setminus B(0,1)} e^{-|y|}|P(y)|\,dy\Big) \\
&\le C_{\alpha,s}|f|_{T^p_\phi(0)} + C_p\|f\|_{L^p(\mathbb{R}^d)} + C\sum_{|\beta|<\underline{b}(\phi)} \frac{|D^\beta P(0)|}{\beta!}\int_{\mathbb{R}^d\setminus B(0,1)} e^{-|y|}|y|^\beta\,dy \\
&\le C_{\alpha,s,p,d}\|f\|_{T^p_\phi(0)},
\end{aligned}$$

so that

$$\sum_{|\alpha|\le n} \frac{x^\alpha}{\alpha!}\int_{\mathbb{R}^d} D^\alpha u_s(-y)R(y)\,dy$$

is a polynomial of degree n whose coefficients are bounded by $\|f\|_{T^p_\phi(0)}$. For all $|\alpha| \le n$, we also have, thanks to (3.5),

$$\left|\int_{B(0,r)} D^\alpha u_s(-y)R(y)\,dy\right| \le C\int_{B(0,r)} |y|^{-d+s-|\alpha|}|R(y)|\,dy \le C_\alpha |f|_{T^p_\phi(x_0)}\phi(r)r^{s-|\alpha|}.$$

Now, if $|\alpha| = n+1$ and if $|y| \ge r$, then $|\Theta(x)x - y| \ge |y|/2$ and, assuming that $s < d$,

$$|D^\alpha u_s(\Theta(x)x - y)| \le C|\Theta(x)x - y|^{-d+s-|\alpha|} \le C'|y|^{-d+s-|\alpha|}.$$

From (3.6), we get

$$\left|\int_{\mathbb{R}^d\setminus B(0,r)} D^\alpha u_s(\Theta(x)x - y)R(y)\,dy\right| \le C_\alpha |f|_{T^p_\phi(x_0)}\phi(r)r^{s-|\alpha|}.$$

If we also assume that $1/p - s/d < 0$ and if p' is the conjugate exponent of p, then, from $-(d-s)p' < d$, since

$$|u_s| \le C|\cdot|^{-d+s},$$

we infer that $u_s \in L^{p'}(\mathbb{R}^d)$ and

$$\|u_s\|_{L^{p'}(B(0,2r))} \le Cr^s r^{-d/p} \quad \text{for all } r > 0.$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
\left|\int_{B(0,r)} u_s(x-y)R(y)\,dy\right| &\le \|u_s\|_{L^{p'}(B(0,2r))}\|R\|_{L^p(B(0,r))} \\
&\le Cr^s r^{-d/p}\|R\|_{L^p(B(0,r))} \le C|f|_{T^p_\phi(0)}r^s\phi(r).
\end{aligned}$$

This shows that

$$P' = u_s * P - \sum_{|\alpha|\le n} \frac{\cdot^\alpha}{\alpha!}\int_{\mathbb{R}^d} D^\alpha u_s(-y)R(y)\,dy$$

is a polynomial of degree n such that

$$\|\mathcal{J}^s f - P'\|_{L^\infty(B(0,2r))} \le C_{s,\phi,p,d}|f|_{T^p_\phi(0)}\phi_s(2r), \tag{3.7}$$

which means that $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$. Moreover, by Young's inequality,

$$\|\mathcal{J}^s f\|_{L^\infty(\mathbb{R}^d)} \le \|u_s\|_{L^{p'}(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)}. \tag{3.8}$$

From this relation, (3.7) and the fact that the sum of the coefficients of P' is bounded by $\|f\|_{T^p_\phi(0)}$, we get

$$\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}.$$

If we now assume that $1/p - s/d > 0$, then

$$\left|\int_{B(0,r)} u_s(x-y)R(y)\,dy\right| \le C\int_{\mathbb{R}^d} \frac{|R\chi_{B(0,r)}|}{|x-y|^{d-s}}\,dy = CI_s(|R\chi_{B(0,r)}|) \quad \text{for } r>0,$$

where I_s is the Riesz potential of order s. As a consequence, if q satisfies $1/q = 1/p - s/d$, we have, by the Hardy–Littlewood–Sobolev lemma (see e.g. [32]),

$$\|I_s(R\chi_{B(0,r)})\|_{L^q(\mathbb{R}^d)} \le C\|R\|_{L^p(B(0,r))} \le C|f|_{T^p_\phi(0)} r^{d/p}\phi(r) = C|f|_{T^p_\phi(0)} r^{d/q} r^s \phi(r).$$

This implies

$$r^{-d/q}\|\mathcal{J}^s f - P'\|_{L^q(B(0,2r))} \le C_{s,\phi,p,d}|f|_{T^p_\phi(0)}\phi_s(2r) \quad \text{for } r>0,$$

which means that $\mathcal{J}^s f \in T^q_{\phi_s}(0)$. One more use of the Hardy–Littlewood–Sobolev lemma gives

$$\|\mathcal{J}^s f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}$$

and we obtain, using the same arguments as before,

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}. \tag{3.9}$$

If $\overline{b}(\phi) + s < 0$, let us decompose $(u_s * R)(x)$ as follows:

$$(u_s * R)(x) = \int_{B(0,r)} u_s(x-y)R(y)\,dy + \int_{\mathbb{R}^d\setminus B(0,r)} u_s(x-y)R(y)\,dy.$$

We can use (3.6) again to estimate the second term in this equality; more precisely, we have

$$\left|\int_{\mathbb{R}^d\setminus B(0,r)} u_s(x-y)R(y)\,dy\right| \le C_s|f|_{T^p_\phi(x_0)}\phi(r)r^s.$$

We can now use the same reasoning to show that (3.8) and (3.9) still hold in this case.

Let us extend inequalities (3.8) and (3.9) to all the admissible values of $s>0$. If $s=d$, let $0<\varepsilon<d$ be such that $v := s-\varepsilon$ satisfies $0<v<d$ and $n < \underline{b}(\phi)+v \le \overline{b}(\phi)+v < n+1$. Suppose first that $1/p - s/d > 0$; we have $1/p - v/d > 0$, which implies $\mathcal{J}^v f \in T^r_{\phi_v}(0)$, with $1/r = 1/p - v/d$ and $\|\mathcal{J}^v f\|_{T^r_{\phi_v}(0)} \le C\|f\|_{T^p_\phi(0)}$. From $\mathcal{J}^s f = \mathcal{J}^\varepsilon \mathcal{J}^v f$ and

$$\frac{1}{r} - \frac{\varepsilon}{d} = \frac{1}{p} - \frac{s}{d} > 0,$$

we know that $\mathcal{J}^s f \in T^q_{\phi_s}(0)$ with $1/q := 1/r - \varepsilon/d = 1/p - s/d$ and

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \le C\|\mathcal{J}^v f\|_{T^r_{\phi_v}(0)} \le C\|f\|_{T^p_\phi(0)}.$$

Now, let us suppose that $1/p - s/d < 0$; choosing ε such that $1/p - v/d < 0$, we get $\mathcal{J}^v f \in T^\infty_{\phi_v}(0)$, with $\|\mathcal{J}^v f\|_{T^\infty_{\phi_v}(0)} \le C\|f\|_{T^p_\phi(0)}$ and we obtain $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$, with $\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}$.

Let us consider the case $s = kd + v$ with $k \in \mathbb{N}_0$ and $0 < v \le d$; let us first remark that if $n \in \mathbb{N}_0$ satisfies

$$n < \underline{b}(\phi) + s < \overline{b}(\phi) + s < n+1,$$

then $d \leq n$ implies

$$0 \leq n-d < \underline{b}(\phi)+s-d \leq \overline{b}(\phi)+s-d < n-d+1$$

and $s-d$ is still an admissible value. Otherwise, $n < d$ and so $n+1 \leq d$, which means that we have $\overline{b}(\phi)+s-d<0$ and therefore $s-d$ is also an admissible value. Suppose first that $1/p-s/d>0$; let us prove by induction that $\mathcal{J}^s f \in T^q_{\phi_s}(0)$ with $1/q := 1/p - s/d$ and $\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}$. The case $k=0$ being already known, let us show that if the assertion is true for $k-1$, then it is also true for k ($k \geq 1$). Since $s-d$ is an admissible value, $\mathcal{J}^{s-d} f \in T^r_{\phi_{s-d}}(0)$ with $1/r = 1/p-(s-d)/d$ and

$$\|\mathcal{J}^{s-d} f\|_{T^r_{\phi_{s-d}}(0)} \leq C\|f\|_{T^p_\phi(0)}.$$

As

$$\frac{1}{r}-\frac{d}{d}=\frac{1}{p}-\frac{s}{d}>0,$$

we have $\mathcal{J}^s f \in T^q_{\phi_s}(0)$ with $1/q := 1/p-s/d$ and $\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}$. Now, let us suppose that $1/p-s/d<0$; let us prove by induction that $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}.$$

It remains to show that if the assertion is true for $k-1$, then it is also true for k ($k \geq 1$). If $1/p-(s-d)/d<0$, then $\mathcal{J}^{s-d} f \in T^\infty_{\phi_{s-d}}(0)$ and $\|\mathcal{J}^{s-d} f\|_{T^\infty_{\phi_{s-d}}(0)} \leq C\|f\|_{T^p_\phi(0)}$. From what we have obtained before for the case $s=d$, $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}.$$

Otherwise, if $1/p-(s-d)/d>0$, from the previous point, $\mathcal{J}^{s-d} f \in T^r_{\phi_{s-d}}(0)$ with $1/r := 1/p-(s-d)/d$, $\|\mathcal{J}^{s-d} f\|_{T^r_{\phi_{s-d}}(0)} \leq C\|f\|_{T^p_\phi(0)}$ and

$$\frac{1}{r}-\frac{d}{d}=\frac{1}{p}-\frac{s}{d}<0.$$

The case $s=d$ yields $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$ and $\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}$. Finally, if $1/p-(s-d)/d=0$, let $0<\varepsilon<d$ be such that $s-d+\varepsilon$ is still an admissible value. Since $1/p-(s-d+\varepsilon)/d<0$, we have $\mathcal{J}^{s-d+\varepsilon} f \in T^\infty_{\phi_{s-d+\varepsilon}}(0)$ and $\|\mathcal{J}^{s-d+\varepsilon} f\|_{T^\infty_{\phi_{s-d+\varepsilon}}(0)} \leq C\|f\|_{T^p_\phi(0)}$. We can thus write $\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \leq C\|f\|_{T^p_\phi(0)}$.

Let us now remark that if $f \in T^p_\phi(x_0)$ and $\mathcal{J}^s f \in T^q_{\phi_s}(x_0)$ with $q>p$, then we can define $R_s := \mathcal{J}^s f - P_s$ where P_s is a polynomial of degree strictly less than $\underline{b}(\phi)+s$ such that

$$r^{-d/q}\|R_s\|_{L^q(B(x_0,r)} \leq |\mathcal{J}^s f|_{T^q_{\phi_s}(x_0)}\phi_s(r).$$

If $p \leq p' \leq q$ and $q' \geq 1$ is such that $1/q+1/q'=1/p'$, for $r>0$ we have

$$r^{-d/p'}\|R_s\|_{L^{p'}(B(x_0,r))} \leq C_d r^{-d/p'} r^{d/q'}\|R_s\|_{L^q(B(x_0,r))} \leq C_d|\mathcal{J}^s f|_{T^q_{\phi_s}(x_0)}\phi_s(r),$$

which means that $\mathcal{J}^s f \in T^{p'}_{\phi_s}(x_0)$ (using the estimation made by the same polynomial as the one that gives the belonging to $T^q_{\phi_s}(x_0)$). Moreover, if $0 \leq \theta \leq 1$ is such that

$1/p' = \theta/q + (1-\theta)/p$, we know that

$$\|\mathcal{J}^s f\|_{L^{p'}(\mathbb{R}^d)} \le \|\mathcal{J}^s f\|^{\theta}_{L^q(\mathbb{R}^d)} \|\mathcal{J}^s f\|^{1-\theta}_{L^p(\mathbb{R}^d)} \le C\|\mathcal{J}^s f\|^{\theta}_{L^q(\mathbb{R}^d)} \|f\|^{1-\theta}_{L^p(\mathbb{R}^d)}$$
$$\le \|\mathcal{J}^s f\|_{L^q(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)}.$$

We are finally able to prove the three points of the theorem. If $p < d/s$, let us set $1/p^* := 1/p - s/d$; then $p^* \ge 1$ and from the first part of the proof, $\mathcal{J}^s f \in T^{p^*}_{\phi_s}(0)$ and $\|\mathcal{J}^s f\|_{T^{p^*}_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}$. Now, from the second part, for q satisfying $1/p \ge 1/q \ge 1/p^*$, $\mathcal{J}^s f \in T^{p^*}_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \le C(\|\mathcal{J}^s f\|_{T^{p^*}_{\phi_s}(0)} + \|f\|_{L^p(\mathbb{R}^d)}) \le c\|f\|_{T^p_\phi(0)}.$$

Let us consider the case $p > d/s$. The first part of the proof implies that $\mathcal{J}^s f \in T^\infty_{\phi_s}(0)$ and $\|\mathcal{J}^s f\|_{T^\infty_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}$. Using the second part of the proof, for $p \le q \le \infty$, we deduce that $\mathcal{J}^s f \in T^q_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}.$$

For the case $p = d/s$, let $0 < \varepsilon < s$ such that

$$\frac{1}{p} - \frac{\varepsilon}{d} > \frac{1}{p} - \frac{s}{d} = 0,$$

ε being chosen sufficiently close to s so that it is an admissible value; the first part of the proof gives that $\mathcal{J}^\varepsilon f \in T^q_{\phi_\varepsilon}(0)$ and $\|\mathcal{J}^\varepsilon f\|_{T^q_{\phi_\varepsilon}(0)} \le C\|f\|_{T^p_\phi(0)}$ for q such that $1/p \ge 1/q > 1/p - \varepsilon/d$. Now,

$$\frac{1}{q} - \frac{s-\varepsilon}{d} > \frac{1}{p} - \frac{\varepsilon}{d} - \frac{s-\varepsilon}{d} = 0$$

and, from the first part of the proof, $\mathcal{J}^s f \in T^q_{\phi_s}(0)$ and

$$\|\mathcal{J}^s f\|_{T^q_{\phi_s}(0)} \le C\|f\|_{T^p_\phi(0)}.$$

We can conclude the proof by letting $\varepsilon \to s^-$. ∎

This theorem admits the following corollary, regarding the spaces $t^p_\phi(x_0)$.

Corollary 3.1.3. *Let $x_0 \in \mathbb{R}^d$, $p \in (1,\infty)$, $\phi \in \mathcal{B}$ be such that either $\underline{b}(\phi) > -d/p$ and $\underline{b}(\phi) \le 0$ or there exists $n \in \mathbb{N}_0$ such that $n < \underline{b}(\phi) \le \overline{b}(\phi) < n+1$. Consider an admissible value $s > 0$ for ϕ. If $f \in t^p_\phi(x_0)$, then $\mathcal{J}^s f \in t^q_{\phi_s}(x_0)$, where*

- *$1/p \ge 1/q \ge 1/p - s/d$ if $p < d/s$,*
- *$p \le q \le \infty$ if $d/s < p < \infty$,*
- *$p \le q < \infty$ if $d/s = p$.*

Proof. By Corollary 2.2.2, there exists a sequence of functions $(f_j)_{j\in\mathbb{N}_0}$ in $\mathcal{D}(\mathbb{R}^d) \cap t^p_\phi(x_0)$ such that $f_j \to f$ in $T^p_\phi(x_0)$. For such a function, $\mathcal{J}^s f_j \in C^\infty(\mathbb{R}^d)$ and Remark 2.1.13 implies that $\mathcal{J}^s f_j \in t^r_{\phi_s}(x_0)$ for all $r \in [1,\infty]$. But, for all the values of q that we consider, the preceding theorem implies

$$\|\mathcal{J}^s(f_j - f)\|_{T^q_{\phi_s}(x_0)} \le C\|f_j - f\|_{T^p_\phi(x_0)}.$$

Therefore, $\mathcal{J}^s f_j$ converges to $\mathcal{J}^s f$ in $T^q_{\phi_s}(x_0)$. From Proposition 2.1.12, we know that $t^q_{\phi_s}(x_0)$ is a closed subspace of $T^q_{\phi_s}(x_0)$, which gives us the conclusion. ∎

3.2. Derivatives. In this section, we investigate the estimates that can be made for a function whose derivatives are known to belong to $T^p_\phi(x_0)$ (or $t^p_\phi(x_0)$). For such a task, we will need the following classical lemma of Sobolev spaces theory (see e.g. [35]).

LEMMA 3.2.1. *Let $1 \le p < d$ and q be defined by $1/q := 1/p - 1/d$. There exists $C_{p,d} > 0$ such that, for all $f \in \mathcal{D}(\mathbb{R}^d)$,*

$$\|f\|_{L^q(\mathbb{R}^d)} \le C_{p,d} \sum_{j=1}^d \|D_j f\|_{L^p(\mathbb{R}^d)}.$$

Let us recall that, if $\phi \in \mathcal{B}$, then ϕ_1 is the Boyd function defined by

$$\phi_1(x) = x\phi(x) \qquad \forall x > 0.$$

THEOREM 3.2.2. *Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty)$, $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > -d/p$ and either $\overline{b}(\phi) < -1$ or there exists $n \in \mathbb{N}_0 \cup \{-1\}$ for which $n < \underline{b}(\phi) \le \overline{b}(\phi) < n+1$. If f is such that $D_j f \in T^p_\phi(x_0)$ for all $j \in \{1, \dots, d\}$ then*

(1) *if $1 \le p < d$ and $f \in L^q(\mathbb{R}^d)$ with $1/q := 1/p - 1/d$, then $f \in T^q_{\phi_1}(x_0)$ and*

$$\|f\|_{T^q_{\phi_1}(x_0)} \le C_{p,\phi} \sum_{j=1}^d \|D_j f\|_{T^p_\phi(x_0)}, \tag{3.10}$$

(2) *if $f \in L^q(\mathbb{R}^d)$ where $q \in [1, \infty)$ is such that $1/p \ge 1/q > 1/p - 1/d$, then $f \in T^q_{\phi_1}(x_0)$ and*

$$\|f\|_{T^q_{\phi_1}(x_0)} \le C_{p,\phi} \sum_{j=1}^d \|D_j f\|_{T^p_\phi(x_0)} + \|f\|_{L^q(\mathbb{R}^d)}. \tag{3.11}$$

Moreover, if $D_j f \in t^p_\phi(x_0)$ for all $j \in \{1, \dots, d\}$, then also $f \in t^q_{\phi_1}(x_0)$, with q satisfying (1) or (2).

Proof. Let us first suppose that f belongs to $\mathcal{D}(\mathbb{R}^d)$; for $j \in \{1, \dots, d\}$, let us set

$$k_j : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}, \qquad x \mapsto \frac{1}{\omega_d} \frac{x_j}{|x|^d},$$

where ω_d is the area of the hyper-sphere in $\mathbb{R}^d$. It is easy to check that for $x \ne 0$, we have $\sum_{j=1}^d D_j k_j(x) = 0$.

Let us fix $x \in \mathbb{R}^d$, given $r > 0$, set $\Omega_r := \{y \in \mathbb{R}^d : |x - y| \ge r\}$ and denote by $\partial\Omega_r := \{y \in \mathbb{R}^d : |x - y| = r\}$ the boundary of this set. Using Green's first identity (see e.g. [33]), we get

$$\sum_{j=1}^d \int_{\Omega_r} D_j f(y) k_j(x - y)\, dy = \frac{1}{\omega_d} \int_{\partial\Omega_r} \frac{f(y)}{|x - y|^{d-1}}\, d\sigma,$$

where $d\sigma$ is the surface area on $\partial\Omega_r$. Lebesgue's theorem implies that the right-hand side tends to $f(x)$ as r tends to 0^+, while the left-hand side tends to

$$\sum_{j=1}^d \int_{\mathbb{R}^d} D_j f(y) k_j(x - y)\, dy.$$

Therefore, we have the following representation for f:

$$f = \sum_{j=1}^{d} \int_{\mathbb{R}^d} D_j f(y) k_j(\cdot - y)\, dy. \tag{3.12}$$

Let us prove (2) in the case $q = p$. Let us first deal with the case $\overline{b}(\phi) < -1$; for $r > 0$ and $x \in \mathbb{R}^d$ such that $|x - x_0| \leq r$, we can write

$$f(x) = \sum_{j=1}^{d} (f_{1,j}(x) + f_{2,j}(x)),$$

where we have set

$$f_{1,j}(x) := \int_{B(x_0,2r)} D_j f(y) k_j(x-y)\, dy,$$
$$f_{2,j}(x) := \int_{\mathbb{R}^d \setminus B(x_0,2r)} D_j f(y) k_j(x-y)\, dy.$$

By Young's inequality, we have

$$\begin{aligned} r^{-d/p} \|f_{1,j}\|_{L^p(B(x_0,r))} &\leq r^{-d/p} \|D_j f\|_{L^p(B(x_0,2r))} \|k_j\|_{L^1(B(x_0,3r))} \\ &\leq C\overline{\phi}(2) |D_j f|_{T^p_\phi(x_0)} \phi(r) r. \end{aligned} \tag{3.13}$$

To estimate $r^{-d/p}\|f_{2,j}\|_{L^p(B(x_0,r))}$, let us define the function F_j for $r > 0$ by

$$F_j(r) := \int_{B(x_0,r)} |D_j f(y)|\, dy = \int_0^r \psi_j(\rho)\, d\rho,$$

where we have set, using spherical coordinates in $\mathbb{R}^d$ centered at x_0,

$$\psi_j(\rho) := \rho^{d-1} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} |D_j f(y(\rho, \theta_1, \ldots, \theta_{d-1}))|\, d\Omega_d.$$

We know that, for $r > 0$,

$$r^{-d} F_j(r) \leq C_d |D_j f|_{T^p_\phi(x_0)} \phi(r) \tag{3.14}$$

and, for all $R > 0$, we have

$$F_j(R) R^{1-d} - F_j(2r)(2r)^{1-d} = \int_{2r}^{R} \psi_j(\rho) \rho^{1-d}\, d\rho + \int_{2r}^{R} F_j(\rho)(1-d)\rho^{-d}\, d\rho. \tag{3.15}$$

Thanks to (3.14) and Proposition 2.1.3, since $\overline{b}(\phi) < -1$, $F_j(R)R^{1-d}$ tends to 0 as R tends to $+\infty$. Therefore,

$$\begin{aligned} \int_{2r}^{+\infty} \psi_j(\rho)\rho^{1-d}\, d\rho &\leq (d-1) \int_{2r}^{+\infty} F_j(\rho)\rho^{-d}\, d\rho \leq C_d(d-1)|D_j f|_{T^p_\phi(x_0)} \int_{2r}^{+\infty} \phi(\rho)\, d\rho \\ &\leq C_d(d-1)|D_j f|_{T^p_\phi(x_0)} \overline{\phi}(2)\phi(r) \int_{2r}^{+\infty} \overline{\phi}\left(\frac{\rho}{2r}\right) d\rho \\ &= C_d(d-1)|D_j f|_{T^p_\phi(x_0)} \overline{\phi}(2) 2\phi(r) r \int_1^{+\infty} \overline{\phi}(t)\, dt. \end{aligned}$$

By Proposition 2.1.5, the last integral is bounded and thus

$$\int_{2r}^{+\infty} \psi_j(\rho)\rho^{1-d}\, d\rho \leq C_d C_{\phi,1} |D_j f|_{T^p_\phi(x_0)} \phi_1(r),$$

where

$$C_{\phi,1} := \overline{\phi}(2) \int_1^{+\infty} \overline{\phi}(t)\, dt. \tag{3.16}$$

Since

$$|f_{j,2}(x)| \leq \int_{\mathbb{R}^d \setminus B(x_0,2r)} \frac{|D_j f(y)|}{|x-y|^{d-1}}\, dy \leq \int_{\mathbb{R}^d \setminus B(x_0,2r)} \frac{|D_j f(y)|}{\left(\frac{1}{2}|x_0-y|\right)^{d-1}}\, dy$$
$$= C_d \int_{2r}^{+\infty} \rho^{1-d} \psi_j(\rho)\, d\rho,$$

we finally obtain

$$r^{-d/p} \|f_{j,2}\|_{L^p(B(x_0,r))} \leq C_d C_{\phi,1} |D_j f|_{T^p_\phi(x_0)} \phi_1(r).$$

Inequality (3.11) follows from this estimate and (3.13). Now, let us suppose that we have $-1 < \underline{b}(\phi) \leq \overline{b}(\phi) < 0$ and fix $r > 0$. For any $x \in B(x_0, r)$, we have

$$f(x) - f(x_0) = \sum_{j=1}^d (f_{j,1} + f_{j,2} - f_{j,3})(x),$$

where we have set

$$f_{j,1}(x) := \int_{B(x_0,2r)} D_j f(y) k_j(x-y)\, dy,$$
$$f_{j,2}(x) := \int_{\mathbb{R}^d \setminus B(x_0,2r)} D_j f(y)(k_j(x-y) - k_j(x_0-y))\, dy$$
$$f_{j,3}(x) := \int_{B(x_0,2r)} D_j f(y) k_j(x_0-y)\, dy.$$

Once again, we have

$$r^{-d/p} \|f_{1,j}\|_{L^p(B(x_0,r))} \leq C\overline{\phi}(2) |D_j f|_{T^p_\phi(x_0)} \phi_1(r).$$

Moreover, if $x \in B(x_0, r)$ and $|x_0 - y| \geq 2r$, then, for all $|h| \leq |x - x_0|$, $|x_0 - y + h| \geq |x_0 - y|/2$ and so, by the mean value theorem and the fact that $|D^\alpha k_j(z)| \leq C/|z|^d$ for all $z \neq 0$ and $|\alpha| = 1$,

$$|k_j(x-y) - k_j(x_0-y)| \leq Cr|x_0-y|^{-d}.$$

Therefore,

$$|f_{j,2}(x)| \leq Cr \int_{2r}^{+\infty} \psi_j(\rho) \rho^{-d}\, d\rho$$

and reasoning as before, using this time $\overline{b}(\phi) < 0$, we get

$$r^{-d/p} \|f_{j,2}\|_{L^p(B(x_0,r))} \leq C_d C_{\phi,2} |D_j f|_{T^p_\phi(x_0)} \phi_1(r),$$

where

$$C_{\phi,2} := \overline{\phi}(2) \int_1^{+\infty} \frac{\overline{\phi}(t)}{t}\, dt. \tag{3.17}$$

For the last term, we have

$$|f_{j,3}(x)| \leq r \int_0^{2r} \psi_j(\rho) \rho^{1-d}\, d\rho$$

and using an equality similar to (3.15), we obtain

$$\int_0^{2r} \psi_j(\rho)\rho^{1-d}\,d\rho \le F_j(2r)(2r)^{1-d} + d\int_0^{2r} F_j(\rho)\rho^{-d}\,d\rho.$$

As $-1 < \underline{b}(\phi)$, we have

$$r^{-d/p}\|f_{j,3}\|_{L^p(B(x_0,r))} \le C_d C_{\phi,3}|D_j f|_{T^p_\phi(x_0)}\phi_1(r),$$

where

$$C_{\phi,3} := \overline{\phi}(2)\Big(1 + \int_0^1 \overline{\phi}(t)\,dt\Big). \tag{3.18}$$

Again, (3.11) follows from the estimate made of $r^{-d/p}\|f_{j,k}\|_{L^p(B(x_0,r))}$, for all $r > 0$ and $k \in \{1, 2, 3\}$. Finally, if there exists $n \in \mathbb{N}_0$ such that $n < \underline{b}(\phi) \le \overline{b}(\phi) < n + 1$, let P be the Taylor expansion of f at x_0 of order $n + 1$, set $\widetilde{f} := f - P$ and, for $j \in \{1, \ldots, d\}$, $\widetilde{f}_j := D_j\widetilde{f}$. For $r > 0$, we have

$$\int_{B(x_0,r)} |\widetilde{f}(y)|^p\,dy = \int_0^r\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi |\widetilde{f}(x_0 + y_{(\rho,\theta_1,\ldots,\theta_{d-1})})|^p\rho^{d-1}\,d\Omega_d\,d\rho,$$

where $y_{(\rho,\theta_1,\ldots,\theta_{d-1})}$ is the point defined by

$$[y_{(\rho,\theta_1,\ldots,\theta_{d-1})}]_j := \rho\prod_{k<j}\sin(\theta_k)\cos(\theta_j) \qquad \forall j \in \{0, \ldots, d-1\}$$

and

$$[y_{(\rho,\theta_1,\ldots,\theta_{d-1})}]_d := \rho\prod_{k<d}\sin(\theta_k).$$

Let us set

$$g_j(\theta_1, \ldots, \theta_{d-1}) := \prod_{k<j}\sin(\theta_k)\cos(\theta_j) \quad \text{and} \quad g_d(\theta_1, \ldots, \theta_{d-1}) := \prod_{k<d}\sin(\theta_k).$$

Using Taylor's formula, we have, as $\widetilde{f}(x_0) = 0$,

$$\widetilde{f}(x_0 + y_{(\rho,\theta_1,\ldots,\theta_{d-1})}) = \sum_{j=1}^d\int_0^\rho \widetilde{f}_j(x_0 + y_{(t,\theta_1,\ldots,\theta_{d-1})})g_j(\theta_1, \ldots, \theta_{d-1})\,dt.$$

Therefore, as $|g_j| \le 1$, Hölder's inequality leads to

$$\begin{aligned}
&\int_{B(x_0,r)} |\widetilde{f}(y)|^p\,dy \\
&\qquad \le C_{d,p}\int_0^r\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi \rho^{d-1}\int_0^\rho\sum_{j=1}^d|\widetilde{f}_j(x_0 + y_{(t,\theta_1,\ldots,\theta_{d-1})}|^p\,dt\,\rho^{p-1}\,d\Omega_d\,d\rho \\
&\qquad \le C_{d,p}r^{d+p-2}\sum_{j=1}^d\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi\int_0^r\int_t^r |\widetilde{f}_j(x_0 + y_{(t,\theta_1,\ldots,\theta_{d-1})}|^p\,d\rho\,dt\,d\Omega_d
\end{aligned}$$

$$\leq C_{d,p} r^{d+p-1} \sum_{j=1}^{d} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{r} |\widetilde{f}_j(x_0 + y_{(t,\theta_1,\dots,\theta_{d-1})}|^p \, dt \, d\Omega_d$$
$$= C_{d,p} r^{d+p-1} \sum_{j=1}^{d} \int_{B(x_0,r)} \frac{|\widetilde{f}_j(y)|^p}{|y-x_0|^{d-1}} \, dy.$$

Moreover, using a similar technique as before, we have, for $j \in \{1, \dots, d\}$,

$$\int_{B(x_0,r)} \frac{|\widetilde{f}_j(y)|^p}{|y-x_0|^{d-1}} \, dy \leq |D_j f|^p_{T^p_\phi(x_0)} \phi(r)^p r \left(1 + \int_0^1 \overline{\phi}(t)^p \, dt\right),$$

which allows us to conclude, as $\underline{b}(\phi) > 0$, that

$$r^{-d/p} \|\widetilde{f}\|_{L^p(B(x_0,r))} \leq C_{d,p} C_{\phi,4} \sum_{j=1}^{d} |D_j f|_{T^p_\phi(x_0)} \phi_1(r), \tag{3.19}$$

where

$$C_{\phi,4} := \left(1 + \int_0^1 \overline{\phi}(t)^p \, dt\right)^{1/p}. \tag{3.20}$$

To estimate $\|f\|_{T^p_{\phi_1}(x_0)}$, we need information about $\sum_{|\alpha| \leq n+1} |D^\alpha P(x_0)|/\alpha!$. We have

$$\sum_{0<|\alpha|\leq n+1} \frac{|D^\alpha P(x_0)|}{\alpha!} \leq C \sum_{j=1}^{d} \sum_{0<|\beta|\leq n} \frac{D^\beta P_j(x_0)}{\beta!}, \tag{3.21}$$

where, given $j \in \{1, \dots, d\}$, P_j is the Taylor expansion of $D_j f$ at x_0 of order n. It remains to work on $P(x_0) = f(x_0)$. For this purpose, let us choose $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi = 1$ on $\overline{B(0,1)}$ and $\operatorname{supp}(\varphi) \subseteq \overline{B(0,2)}$. Using (3.12), we obtain

$$f(x_0) = f(x_0)\varphi(x_0 - x_0) = \sum_{j=1}^{d} \Bigg(\int_{\mathbb{R}^d} k_j(x_0 - y) D_j f(y) \varphi(y - x_0) \, dy + \int_{\mathbb{R}^d} k_j(x_0 - y) f(y) D_j \varphi(y - x_0) \, dy \Bigg).$$

For the first term of the right-hand side, we have

$$\left| \int_{\mathbb{R}^d} k_j(x_0 - y) D_j f(y) \varphi(y - x_0) \, dy \right| \leq C_\varphi \int_{B(x_0,2)} \frac{|D_j f(y)|}{|x_0 - y|^{d-1}} \, dy.$$

For $r > 0$, we have

$$r^{-d/p} \|D_j f - P_j\|_{L^p(B(x_0,r))} \leq |D_j f|_{T^p_\phi(x_0)} \phi(r),$$

and so

$$r^{-d/p} \|D_j f\|_{L^p(B(x_0,r))} \leq |D_j f|_{T^p_\phi(x_0)} \phi(r) + C_d \sum_{|\beta|\leq n} \frac{|D^\beta P_j(x_0)|}{\beta!} r^{|\beta|}.$$

Since

$$\sum_{|\beta|\leq n} \frac{|D^\beta P_j(x_0)|}{\beta!} \leq \|D_j f\|_{T^p_\phi(x_0)},$$

we can write, using the same technique as before,

$$\int_{B(x_0,2)} \frac{|D_j f(y)|}{|x_0-y|^{d-1}}\,dy \le C_d C_{\phi,5}\|D_j f\|_{T^p_\phi(x_0)},$$

where

$$C_{\phi,5} := \phi(2) + 2^n + 2\phi(2)\int_0^1 \overline{\phi}(t)\,dt. \tag{3.22}$$

For the second term, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^d} k_j(x_0-y)f(y)D_j\varphi(y-x_0)\,dy\right| &\le \int_{B(x_0,2)\setminus B(x_0,1)} |k_j(x_0-y)|\,|f(y)|\,|D_j\varphi(y-x_0)|\,dy\\ &\le C_\varphi \int_{B(x_0,2)\setminus B(x_0,1)} |f(y)|\,dy \le C_{\varphi,d}\|f\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

which gives

$$|f(x_0)| \le C_{\varphi,d}\Big(C_{\phi,5}\sum_{j=1}^d \|D_j f\|_{T^p_\phi(x_0)} + \|f\|_{L^p(\mathbb{R}^d)}\Big).$$

This relation, (3.19) and (3.21) lead to (3.11). We have thus obtained (2) in the case $p=q$.

Let us now prove (1), still considering a function f from $\mathcal{D}(\mathbb{R}^d)$. As previously, let us denote by φ a function in $\mathcal{D}(\mathbb{R}^d)$ such that $\varphi = 1$ on $\overline{B(0,1)}$ and $\operatorname{supp}(\varphi) \subseteq \overline{B(0,2)}$. If there exists $n \in \mathbb{N}_0 \cup \{-1\}$ such that $n < \underline{b}(\phi) \le \overline{b}(\phi) < n+1$, let P be the Taylor expansion of f at x_0 of order $n+1$; otherwise we set $P=0$. Finally, define $\widetilde{f} := f - P$ and, for $j \in \{1,\ldots,d\}$, $\widetilde{f}_j := D_j\widetilde{f}$. If $1/q := 1/p - 1/d$, thanks to Lemma 3.2.1, we have, for all $r>0$,

$$\begin{aligned} r^{-d/q}\|\widetilde{f}\|_{L^q(B(x_0,r))} &\le r^{-d/q}\left\|\widetilde{f}\varphi\Big(\frac{\cdot - x_0}{r}\Big)\right\|_{L^p(\mathbb{R}^d)}\\ &\le C_{p,d} r^{-d/q}\sum_{j=1}^d\left(\left\|\widetilde{f}_j\varphi\Big(\frac{\cdot - x_0}{r}\Big)\right\|_{L^p(\mathbb{R}^d)} + r^{-1}\left\|\widetilde{f}D_j\varphi\Big(\frac{\cdot - x_0}{r}\Big)\right\|_{L^p(\mathbb{R}^d)}\right)\\ &= C_\varphi C_{p,d}\sum_{j=1}^d \big(r r^{-d/p}\|\widetilde{f}_j\|_{L^p(B(x_0,2r))} + r^{-d/p}\|\widetilde{f}\|_{L^p(B(x_0,2r))}\big).\end{aligned}$$

Moreover, by hypothesis,

$$r r^{-d/p}\|\widetilde{f}_j\|_{L^p(B(x_0,2r))} \le 2^{d/p}\overline{\phi}(2)|D_j f|_{T^p_\phi(x_0)}\phi_1(r) \tag{3.23}$$

and, using what we have proved so far,

$$r^{-d/p}\|\widetilde{f}\|_{L^p(B(x_0,2r))} \le C_{d,p}C_\phi \sum_{j=1}^d |D_j f|_{T^p_\phi(x_0)}\phi_1(r). \tag{3.24}$$

As before,

$$\sum_{|\alpha|\le n+1}\frac{|D^\alpha P(x_0)|}{\alpha!} \le C_{\varphi,d}\Big(C_{\phi,5}\sum_{j=1}^d \|D_j f\|_{T^p_\phi(x_0)} + \|f\|_{L^q(\mathbb{R}^d)}\Big). \tag{3.25}$$

Another use of Lemma 3.2.1 then gives

$$\|f\|_{L^q(\mathbb{R}^d)} \le C_{p,d} \sum_{j=1}^{d} \|D_j f\|_{L^p(\mathbb{R}^d)} \tag{3.26}$$

and (3.10) is proved, thanks to relations (3.23)–(3.26).

Now, let us come back to (2) and investigate the case where $q \ge 1$ is such that $1/p \ge 1/q > 1/p - 1/d$; we still consider a function $f \in \mathcal{D}(\mathbb{R}^d)$. Again, we use (3.12); as $1/p \ge 1/q > 1/p - 1/d$, there exists $p' \in [1, \infty)$ such that $1/q = 1/p + 1/p' - 1$ and, by Young's inequality,

$$\left\| \int_{\mathbb{R}^d} k_j(\cdot - y)\widetilde{f}_j(y)\varphi\Big(\frac{y - x_0}{r}\Big)\, dy \right\|_{L^q(B(x_0,r))} \le C_\varphi \|k_j\|_{L^{p'}(B(x_0,3r))} \|\widetilde{f}_j(y)\|_{L^p(B(x_0,2r))}$$

and

$$\|k_j\|_{L^{p'}(B(x_0,3r))} \le C_{d,p}((3r)^{(d-1)(1-p')+1})^{1/p'} = C_{d,p}(3r)^{d/q-d/p+1},$$

which gives us

$$r^{-d/q}\left\| \int_{\mathbb{R}^d} k_j(\cdot - y)\widetilde{f}_j(y)\varphi\Big(\frac{y - x_0}{r}\Big)\, dy \right\|_{L^q(B(x_0,r))} \le C_{\varphi,d,p}\overline{\phi}(2)|D_j f|_{T^p_\phi(x_0)}\phi_1(r).$$

Similarly, using the first part of the proof, we obtain

$$\begin{aligned}\left\| \int_{\mathbb{R}^d} k_j(\cdot - y)\widetilde{f}(y)r^{-1}D_j\varphi\Big(\frac{y - x_0}{r}\Big)\, dy \right\|_{L^q(B(x_0,r))} &\le C_{\varphi,d,p} r^{-d/p} r^{d/q} \|\widetilde{f}(y)\|_{L^p(B(x_0,2r))} \\ &\le C_{\varphi,d,p} C_\phi \overline{\phi}(2) r^{d/q} \sum_{j=1}^{d} |D_j f|_{T^p_\phi(x_0)}\phi_1(r).\end{aligned}$$

This upper bound and (3.25) lead to (3.11).

Now that the theorem has been proved for functions belonging to $\mathcal{D}(\mathbb{R}^d)$, let us consider a compactly supported function f such that $D_j f \in t^p_\phi(x_0)$, for all $j \in \{1, \ldots, d\}$. Given $\lambda > 0$, let f_λ be the function defined by (2.9) and, for $j \in \{1, \ldots, d\}$, define $f_{\lambda,j} := D_j f_\lambda$. By Proposition 2.2.1, we know that $f_{\lambda,j}$ converges to $D_j f$ in $T^p_\phi(x_0)$ ($j \in \{1, \ldots, d\}$). Inequalities (3.10) and (3.11) imply that $(f_\lambda)_{\lambda>0}$ is a Cauchy sequence in $T^q_{\phi_1}(x_0)$ (with appropriate q) and thus, by Proposition 2.1.11, $(f_\lambda)_{\lambda>0}$ converges in $T^q_{\phi_1}(x_0)$. As f_λ converges to f in $L^q(\mathbb{R}^d)$, we conclude that f_λ converges to f in $T^q_{\phi_1}(x_0)$. Moreover, by passing to the limit, we find that inequalities (3.10) and (3.11) still hold for f. Now, as f_λ belongs to $\mathcal{D}(\mathbb{R}^d)$ and $t^q_{\phi_1}(x_0)$ for all $\lambda > 0$, by Proposition 2.1.12, f also belongs to $t^q_{\phi_1}(x_0)$.

Let us now consider a general function f such that, for all $j \in \{1, \ldots, d\}$, $D_j f$ belongs to $t^p_\phi(x_0)$ and let us again take $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\varphi = 1$ on $\overline{B(0,1)}$ and $\operatorname{supp}(\varphi) \subseteq \overline{B(x_0,2)}$. Given $\varepsilon > 0$, we define

$$f_\varepsilon := f\varphi(\varepsilon(\cdot - x_0)).$$

By assumption, we know that, for all $j \in \{1, \ldots, d\}$, there exists a polynomial P_j of degree strictly less than $\underline{b}(\phi)$ such that

$$\phi(r)^{-1} r^{-d/p}\|D_j f - P_j\|_{L^p(B(x_0,r))} \to 0 \quad \text{as } r \to 0^+.$$

Moreover, since we assume that $f \in L^q(\mathbb{R}^d)$ for some $q \geq p$, it follows that $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$D_j f_\varepsilon = D_j f \varphi(\varepsilon(\cdot - x_0)) + \varepsilon f D_j \varphi(\varepsilon(\cdot - x_0))$$

belongs to $L^p(\mathbb{R}^d)$ for all $\varepsilon > 0$. Of course,

$$\begin{aligned}\phi(r)^{-1} r^{-d/p} \|D_j f_\varepsilon - P_j\|_{L^p(B(x_0,r))} &\leq \phi(r)^{-1} r^{-d/p} \|D_j f \varphi(\varepsilon(\cdot - x_0)) - P_j\|_{L^p(B(x_0,r))} \\ &\quad + \phi(r)^{-1} r^{-d/p} \|\varepsilon f D_j \varphi(\varepsilon(\cdot - x_0))\|_{L^p(B(x_0,r))}.\end{aligned}$$

Now, for r sufficiently small, we have $\varphi(\varepsilon(\cdot - x_0)) = 1$ and $D_j \varphi(\varepsilon(\cdot - x_0)) = 0$ on $B(x_0, r)$ and, for such r,

$$\phi(r)^{-1} r^{-d/p} \|D_j f_\varepsilon - P_j\|_{L^p(B(x_0,r))} \leq \phi(r)^{-1} r^{-d/p} \|D_j f - P_j\|_{L^p(B(x_0,r))},$$

which shows that $D_j f_\varepsilon \in t^p_\phi(x_0)$. As f_ε is compactly supported, the previous case shows that $f_\varepsilon \in t^q_{\phi_1}(x_0)$ (for appropriate q). Let us prove that $D_j f_\varepsilon$ tends to $D_j f$ in $T^p_\phi(x_0)$ as ε tends to 0^+. We have

$$\|D_j f_\varepsilon - D_j f\|_{T^p_\phi(x_0)} = \sup_{r>0} \phi(r)^{-1} r^{-d/p} \|D_j f_\varepsilon - D_j f\|_{L^p(B(x_0,r))} + \|D_j f_\varepsilon - D_j f\|_{L^p(\mathbb{R}^d)}$$

and

$$D_j f_\varepsilon - D_j f = D_j f \big(\varphi(\varepsilon(\cdot - x_0)) - 1\big) + \varepsilon f D_j \varphi(\varepsilon(\cdot - x_0)). \tag{3.27}$$

A simple application of Lebesgue's theorem shows that the L^p-norm of the first term of the right-hand side of (3.27) tends to 0 as ε tends to 0^+, while

$$\begin{aligned}\|\varepsilon f D_j \varphi(\varepsilon(\cdot - x_0))\|_{L^p(\mathbb{R}^d)} &\leq C_\varphi \varepsilon \|f\|_{L^p(\overline{B(x_0,2/\varepsilon)} \setminus B(x_0,1/\varepsilon))} \\ &\leq C_{\varphi,p,q,d} \varepsilon^{1-d/p+d/q} \|f\|_{L^q(\mathbb{R}^d \setminus B(x_0,1/\varepsilon))}.\end{aligned}$$

Since $1 - d/p + d/q \geq 0$ by hypothesis and $\|f\|_{L^q(\mathbb{R}^d \setminus B(x_0,1/\varepsilon))}$ tends to 0 as ε tends to 0^+, so does $\|D_j f_\varepsilon - D_j f\|_{L^p(\mathbb{R}^d)}$. Moreover, for $0 < \varepsilon < 1$, if $0 < r < 1/\varepsilon$, then $D_j f_\varepsilon - D_j f$ vanishes on $B(x_0, r)$. If $r > 1/\varepsilon$, then $r > 1$ and if $\delta > 0$ satisfies $\underline{b}(\phi) - \delta + d/p > 0$, then by Proposition 2.1.3,

$$\phi(r)^{-1} r^{-d/p} \leq C_{\delta,\phi} r^{-(\underline{b}(\phi) - \delta + d/p)} \leq C_{\delta,\phi} \varepsilon^{\underline{b}(\phi) - \delta + d/p},$$

which finally leads to

$$\sup_{r>0} \phi(r)^{-1} r^{-d/p} \|D_j f_\varepsilon - D_j f\|_{L^p(B(x_0,r))} \leq C_{\delta,\phi} \varepsilon^{\underline{b}(\phi) - \delta + d/p} \|D_j f_\varepsilon - D_j f\|_{L^p(\mathbb{R}^d)},$$

so that $D_j f_\varepsilon$ tends to $D_j f$ in $T^p_\phi(x_0)$ as $\varepsilon \to 0^+$. Using again the completeness of the space $T^q_{\phi_1}(x_0)$ and the closedness of $t^q_{\phi_1}(x_0)$, we conclude, by (3.10) and (3.11), that f_ε tends to f in $T^q_{\phi_1}(x_0)$ and $f \in t^q_{\phi_1}(x_0)$. By passing to the limit in (3.10) and (3.11), we conclude that those inequalities still hold for f.

It remains to consider the case of a function f such that, for $j \in \{1, \dots, d\}$, $D_j f$ belongs to $T^p_\phi(x_0)$. Let $\varepsilon > 0$ be such that

$$-d/p < \underline{b}(\phi) - \varepsilon \leq \overline{b}(\phi) - \varepsilon < -1$$

if $\overline{b}(\phi) < -1$, and

$$n < \underline{b}(\phi) - \varepsilon \leq \overline{b}(\phi) - \varepsilon < n + 1$$

if $n \in \mathbb{N}_0 \cup \{-1\}$ satisfies $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$. For such ε, $D_j f \in t^p_{\phi_{-\varepsilon}}(x_0)$ for $j \in \{1, \dots, d\}$ and it follows from the previous case that $D_j f \in t^q_{\phi_{1-\varepsilon}}(x_0)$. Moreover, if $1 \leq p < d$ and $f \in L^q(\mathbb{R}^d)$ with $1/q := 1/p - 1/d$, then $f \in T^q_{\phi_{1-\varepsilon}}(x_0)$ and

$$\|f\|_{T^q_{\phi_{1-\varepsilon}}(x_0)} \leq C_{p,\phi_{-\varepsilon}} \sum_{j=1}^{d} \|D_j f\|_{T^p_{\phi_{-\varepsilon}}(x_0)}. \tag{3.28}$$

Otherwise, if $f \in L^q(\mathbb{R}^d)$ with $q \in [1, \infty)$ satisfying $1/p \geq 1/q > 1/p - 1/d$, then we have $f \in T^q_{\phi_{1-\varepsilon}}(x_0)$ and

$$\|f\|_{T^q_{\phi_{1-\varepsilon}}(x_0)} \leq C_{p,\phi-\varepsilon} \sum_{j=1}^{d} \|D_j f\|_{T^p_{\phi_{-\varepsilon}}(x_0)} + \|f\|_{L^q(\mathbb{R}^d)}. \tag{3.29}$$

Let us analyse the constants defined in (3.16)–(3.18), (3.20) and (3.22). For a chosen $\varepsilon > 0$, we have for example

$$C_{\phi_{-\varepsilon},1} = \overline{\phi_{-\varepsilon}}(2) \int_1^{+\infty} \overline{\phi_{-\varepsilon}}(t)\, dt = \overline{\phi}(2) 2^{-\varepsilon} \int_1^{+\infty} \overline{\phi}(t) t^{-\varepsilon}\, dt \leq C_\phi$$

and a similar reasoning applied to (3.17), (3.18), (3.20) and (3.22) shows that we can find a constant $C > 0$ such that, for ε small enough, the constant $C_{p,\phi_{-\varepsilon}}$ appearing in (3.28) and (3.29) is bounded by $CC_{p,\phi}$. Moreover, since

$$\|D_j f\|_{T^p_{\phi_{-\varepsilon}}(x_0)} \leq \|D_j f\|_{T^p_{\phi}(x_0)},$$

we can conclude the proof by letting $\varepsilon \to 0^+$. ■

3.3. Singular integral operators. Let us now study the action of the convolution singular integral operators on the space $T^p_\phi(x_0)$. This class of operators was particularly studied by Calderón and Zygmund in [6, 7], where the authors proved the following crucial theorem.

THEOREM 3.3.1. *Set, for* $\varepsilon > 0$,

$$\mathcal{K}_\varepsilon f = \int_{\mathbb{R}^d \setminus B(\cdot, \varepsilon)} k(\cdot - y) f(y)\, dy,$$

where

- k *is homogeneous* ([1]) *of degree* $-d$,
- k *has mean value zero on the sphere* $\Sigma = \{x \in \mathbb{R}^d : |x| = 1\}$,
- $k \in L^q(\Sigma)$ *for* $1 < q < \infty$,
- $f \in L^p(\mathbb{R}^d)$ *with* $1 < p < \infty$.

Then there exists $\mathcal{K}f \in L^p(\mathbb{R}^d)$ *such that* $\mathcal{K}_\varepsilon f$ *tends to* $\mathcal{K}f$ *in* $L^p(\mathbb{R}^d)$, *and pointwise almost everywhere as* $\varepsilon \to 0^+$. *Moreover, if we set*

$$\mathcal{K}^* f = \sup_{\varepsilon > 0} |\mathcal{K}_\varepsilon f|,$$

then $\mathcal{K}^* f \in L^p(\mathbb{R}^d)$ *and*

$$\|\mathcal{K}^* f\|_{L^p(\mathbb{R}^d)} \leq C_{p,q} \|k\|_{L^q(\Sigma)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.30}$$

([1]) It means that $k(\lambda x) = \lambda^{-d} k(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$.

REMARK 3.3.2. In the theorem originally stated by Calderón and Zygmund, the integrability assumption made on k is the following: $k + k(-\cdot) \in L\log L(\Sigma)$. This condition is a little less restrictive, since for a finite measure space $(X, \mathscr{A}, \mu)$, we have (see [1] for example)

$$L^q(X, \mathscr{A}, \mu) \hookrightarrow L\log L(X, \mathscr{A}, \mu),$$

for all $1 < q < \infty$. But in what follows, we will only need to consider $k \in L^q(\Sigma)$, with $1 < q < \infty$, in order to take advantage of inequality (3.30).

We will use the following notation:

NOTATION 3.3.3. Given $\phi \in \mathcal{B}$, we set

$$\lceil \overline{b}(\phi) \rceil_{\mathbb{N}_0} := \inf\{k \in \mathbb{N}_0 : \overline{b}(\phi) < k\}.$$

PROPOSITION 3.3.4. *Let $\mathcal{K}$ be the convolution singular integral operator defined by*

$$\mathcal{K}f = \text{p.v.} \int k(\cdot - y) f(y)\, dy,$$

where the kernel $k \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is homogeneous of degree $-d$. Assume also that k has mean value zero on the sphere Σ.

Let $p \in (1, \infty)$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that $-d/p < \underline{b}(\phi)$ and either $\overline{b}(\phi) < 0$ or there exists $n \in \mathbb{N}_0$ for which

$$n < \underline{b}(\phi) \le \overline{b}(\phi) < n + 1. \tag{3.31}$$

If $f \in T^p_\phi(x_0)$, then $\mathcal{K}f \in T^p_\phi(x_0)$ and

$$\|\mathcal{K}f\|_{T^p_\phi(x_0)} \le C_{\phi,p} M \|f\|_{T^p_\phi(x_0)}, \tag{3.32}$$

where

$$M := \sup_{\substack{|x|=1 \\ 0 \le |\alpha| \le \lceil \overline{b}(\phi) \rceil_{\mathbb{N}_0}}} |D^\alpha k(x)|.$$

Moreover, if $f \in t^p_\phi(x_0)$, then also $\mathcal{K}f \in t^p_\phi(x_0)$.

Proof. We can assume, without loss of generality, that $x_0 = 0$. If $f \in T^p_\phi(0)$ then there exists a polynomial P of degree strictly less than $\underline{b}(\phi)$ such that

$$r^{-d/p} \|f - P\|_{L^p(B(x_0,r))} \le |f|_{T^p_\phi(0)} \phi(r) \qquad \text{for all } r > 0,$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be such that $\varphi = 1$ on $\overline{B(0,1)}$ and $\operatorname{supp}(\varphi) \subseteq \overline{B(0,2)}$; we set

$$f_1 := \varphi P \quad \text{and} \quad f_2 := f - f_1.$$

If $\overline{b}(\phi) < 0$, then $f_1 = 0$ and obviously $f_1 \in T^p_\phi(0)$ with $\|f_1\|_{T^p_\phi(0)} \le \|f\|_{T^p_\phi(0)}$. Otherwise, (3.31) holds and if $r \le 1$, $r^{-d/p}\|f_1 - P\|_{L^p(B(x_0,r))} = 0$. If $r > 1$, then, by Proposition 2.1.3,

$$\begin{aligned} r^{-d/p}\|f_1 - P\|_{L^p(B(x_0,r))} &\le r^{-d/p} C_{\varphi,p} \|P\|_{L^p(B(x_0,r))} \\ &\le C_{\varphi,d,p} \sum_{|\alpha| \le n} \frac{|D^\alpha P(0)|}{\alpha!} r^{|\alpha|} \le C_{\varphi,d,p} C_\phi \|f\|_{T^p_\phi(0)} \phi(r), \end{aligned}$$

which means that $f_1 \in T^p_\phi(0)$, with

$$\|f_1\|_{T^p_\phi(0)} \le C_{\varphi,d,p} C_\phi \|f\|_{T^p_\phi(0)}.$$

As a consequence,

$$\|f_2\|_{T^p_\phi(0)} \leq (1 + C_{\varphi,d,p}C_\phi)\|f\|_{T^p_\phi(0)}.$$

Let us now consider $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\operatorname{supp}(\psi) \subseteq \overline{B(0,2)}$ and set, for $\varepsilon > 0$ and $x \in \mathbb{R}^d$,

$$I_\varepsilon(x) = \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} k(y)\psi(x-y)\,dy = \int_{B(0,2+|x|)\setminus B(0,\varepsilon)} k(y)\psi(x-y)\,dy.$$

Using the notation introduced in the proof of Theorem 3.2.2, as k is homogeneous of degree $-d$, we have

$$\begin{aligned}
&\int_{B(0,2+|x|)\setminus B(0,\varepsilon)} k(y)\psi(x)\,dy \\
&\qquad = \psi(x)\int_\varepsilon^{2+|x|}\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi k(y_{(\rho,\theta_1,\ldots,\theta_{d-1})})\rho^{d-1}\,d\Omega_d\,d\rho \\
&\qquad = \psi(x)\int_\varepsilon^{2+|x|}\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi k(y_{(1,\theta_1,\ldots,\theta_{d-1})})\rho^{-1}\,d\Omega_d\,d\rho \\
&\qquad = \psi(x)(\ln(2+|x|)-\ln(\varepsilon))\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi k(y_{(1,\theta_1,\ldots,\theta_{d-1})})\,d\Omega_d = 0,
\end{aligned}$$

as k has mean value zero on Σ. Therefore, for $\varepsilon > 0$ and $x \in \mathbb{R}^d$,

$$I_\varepsilon(x) = \int_{B(0,2+|x|)\setminus B(0,\varepsilon)} k(y)(\psi(x-y)-\psi(x))\,dy.$$

We will use this equality to show that the sequence $(I_\varepsilon)_{\varepsilon>0}$ converges uniformly as $\varepsilon \to 0^+$. Indeed, for all $x \in \mathbb{R}^d$, if $0 < \varepsilon < \varepsilon'$, since for all $y \neq 0$, $|k(y)| \leq M|y|^{-d}$ by the homogeneity of k, we have

$$|I_{\varepsilon'}(x) - I_\varepsilon(x)| \leq M\int_{B(0,\varepsilon')\setminus B(0,\varepsilon)} |y|^{-d}|y|\sup_{|\alpha|=1}\|D^\alpha\psi\|_\infty\,dy = C_{\psi,d}M(\varepsilon'-\varepsilon),$$

which shows that $(I_\varepsilon)_{\varepsilon>0}$ is uniformly Cauchy. It follows that $\mathcal{K}\psi$ is well-defined and I_ε uniformly converges to $\mathcal{K}(\psi)$ as $\varepsilon \to 0^+$. Moreover, for $0 < \varepsilon < 1$, we have

$$\begin{aligned}
|I_\varepsilon(x)| &\leq |I_1(x) - I_\varepsilon(x)| + |I_1(x)| \\
&\leq C_{\psi,d}M(1-\varepsilon) + M\int_{\mathbb{R}^d\setminus B(0,1)} |y|^{-d}|\psi(x-y)|\,dy \\
&\leq C_{\psi,d}M(1-\varepsilon) + M\int_{\mathbb{R}^d} |\psi(y)|\,dy \leq C'_{\psi,d}M,
\end{aligned}$$

so that $\|\mathcal{K}(\psi)\|_{\mathbb{R}^d} \leq C'_{\psi,d}M$. Using the same reasoning, we can show that, for $\varepsilon > 0$ and $\alpha \in \mathbb{N}_0^d$,

$$D^\alpha I_\varepsilon = \int_{\mathbb{R}^d\setminus B(0,\varepsilon)} k(y)D^\alpha\psi(\cdot - y)\,dy,$$

$D^\alpha I_\varepsilon$ uniformly converges to $D^\alpha\mathcal{K}(\psi)$ and $\|\mathcal{K}(D^\alpha\psi)\|_{\mathbb{R}^d} \leq C_{\psi,d,\alpha}M$. As a consequence,

$\mathcal{K}(\psi) \in C^\infty(\mathbb{R}^d)$ with $D^\alpha \mathcal{K}(\psi) = \mathcal{K}(D^\alpha \psi)$. Moreover, if $|x| \geq 3$, then, for $\varepsilon > 0$,

$$\begin{aligned} |I_\varepsilon(x)| &\leq M \int_{\{(x,y)\colon |x-y|>\varepsilon, |y|<2\}} |x-y|^{-d} |\psi(y)|\, dy \\ &\leq M 3^d |x|^{-d} \int_{\mathbb{R}^d} |\psi(y)|\, dy = C_\psi M 3^d |x|^{-d} \end{aligned}$$

and so, by Lebesgue's theorem, $\mathcal{K}(\psi) \in L^p(\mathbb{R}^d)$ with $\|\mathcal{K}(\psi)\|_{L^p(\mathbb{R}^d)} \leq C_{\psi,d,p} M$. Combining all these relations, we can claim, using Remark 2.1.13, that $\mathcal{K}(\psi) \in T^p_\phi(0)$ and there exists $C_{\psi,d,p} > 0$ such that $\|\mathcal{K}(\psi)\|_{T^p_\phi(x_0)} \leq C_{\psi,d,p} M$.

Now, let us apply this result to the function $x \mapsto x^\alpha \varphi(x)$ in order to obtain a constant $C_{\varphi,\alpha,d,p}$ such that $\|\mathcal{K}(\cdot^\alpha \varphi)\|_{T^p_\phi(0)} \leq C_{\varphi,\alpha,d,p} M$, which gives

$$\|\mathcal{K}(f_1)\|_{T^p_\phi(0)} \leq \sum_{|\alpha| \leq n} \frac{|D^\alpha P(x_0)|}{\alpha!} \|\mathcal{K}(\cdot^\alpha \varphi)\|_{T^p_\phi(0)} \leq C_{\varphi,d,p} M \|f\|_{T^p_\phi(0)}. \tag{3.33}$$

For $\|\mathcal{K}(f_2)\|_{T^p_\phi(0)}$, we use Hölder's inequality to get, for $r > 0$,

$$r^{-d} \int_{B(0,r)} |f_2(y)|\, dy \leq C'_{\varphi,d,p} \|f\|_{T^p_\phi(0)} \phi(r)$$

and, as in (3.5) and (3.6), we can write

$$\int_{B(0,r)} |f_2(y)|\, |y|^{-s}\, dy \leq C_{\varphi,d,p,s} \|f\|_{T^p_\phi(0)} \phi(r) r^{d-s} \quad \text{if } \underline{b}(\phi) + d - s > 0 \tag{3.34}$$

and

$$\int_{\mathbb{R}^d \setminus B(0,r)} |f_2(y)|\, |y|^{-s}\, dy \leq C_{\varphi,d,p,s} \|f\|_{T^p_\phi(0)} \phi(r) r^{d-s} \quad \text{if } \overline{b}(\phi) + d - s < 0. \tag{3.35}$$

Let us now consider the case where condition (3.31) holds and fix $r > 0$; for x in $B(0, r/2)$, we have, using Taylor's formula,

$$\begin{aligned} \mathcal{K} f_2(x) = {} & \lim_{\varepsilon \to 0^+} \int_{\{(x,y)\colon |x-y|>\varepsilon,\, |y|\leq r\}} k(x-y) f_2(y)\, dy \\ & + \lim_{\varepsilon \to 0^+} \int_{\{(x,y)\colon |x-y|>\varepsilon,\, |y|>r\}} k(x-y) f_2(y)\, dy \\ = {} & \lim_{\varepsilon \to 0^+} \int_{\{(x,y)\colon |x-y|>\varepsilon,\, |y|\leq r\}} k(x-y) f_2(y)\, dy + \int_{\mathbb{R}^d \setminus B(0,r)} k(x-y) f_2(y)\, dy \\ = {} & \lim_{\varepsilon \to 0^+} \int_{\{(x,y)\colon |x-y|>\varepsilon,\, |y|\leq r\}} k(x-y) f_2(y)\, dy \\ & + \sum_{|\alpha| \leq n} \frac{x^\alpha}{\alpha!} \Big(\int_{\mathbb{R}^d} D^\alpha k(-y) f_2(y)\, dy - \int_{B(0,r)} D^\alpha k(-y) f_2(y)\, dy \Big) \\ & + \sum_{|\alpha| \leq n+1} \frac{x^\alpha}{\alpha!} \int_{\mathbb{R}^d \setminus B(0,r)} D^\alpha k(\Theta(x) x - y) f_2(y)\, dy \end{aligned}$$

for some $\Theta(x) \in (0,1)$.

Thanks to the homogeneity of k, we have $|D^\alpha k(-y)| \leq M |y|^{-d-|\alpha|}$ for $|\alpha| \leq n+1$ and $y \neq 0$. Using (3.34) and Hölder's inequality, we get, if $q \in (1, \infty)$ is the conjugate exponent of p,

$$\begin{aligned}\left|\int_{\mathbb{R}^d} D^\alpha k(-y) f_2(y)\,dy\right| &\leq \int_{B(0,1)} |f_2(y)|\,|y|^{-d-|\alpha|}\,dy + \int_{\mathbb{R}^d\setminus B(0,1)} |f_2(y)|\,|y|^{-d-|\alpha|}\,dy \\ &\leq C\|f\|_{T^p_\phi(0)}\phi(1) + \|f_2\|_{L^p(\mathbb{R}^d)}\|\,|\cdot|^{-d-|\alpha|}\|_{L^q(\mathbb{R}^d\setminus B(0,1))} \\ &\leq C'\|f\|_{T^p_\phi(0)} + C''\|f\|_{T^p_\phi(0)}\end{aligned}$$

for $|\alpha| \leq n$. As a consequence,

$$P' := \sum_{|\alpha|\leq n} \frac{\cdot^\alpha}{\alpha!}\int_{\mathbb{R}^d} D^\alpha k(-y) f_2(y)\,dy$$

is a polynomial whose sum of coefficients is bounded by $C_{\phi,p}\|f\|_{T^p_\phi(0)}$. Similarly, for $|\alpha| \leq n$ we have

$$\left|\int_{B(0,r)} D^\alpha k(-y) f_2(y)\,dy\right| \leq C_{\alpha,d}\phi(r) r^{-\alpha}.$$

Given $x \in B(0, r/2)$ and $|y| \geq r$, we have $|\Theta(x)x - y| \geq |y|/2$ and so, by (3.35),

$$\begin{aligned}\left|\int_{\mathbb{R}^d\setminus B(0,r)} D^\alpha k(\Theta(x)x - y) f_2(y)\,dy\right| &\leq M 2^{d+|\alpha|}\int_{\mathbb{R}^d\setminus B(0,r)} |f_2(y)|\,|y|^{-d-|\alpha|}\,dy \\ &\leq M C_{\alpha,d}\phi(r) r^{-\alpha}\end{aligned}$$

for $|\alpha| = n+1$. Finally, using Theorem 3.3.1, we obtain

$$\begin{aligned}&\left\|\lim_{\varepsilon\to 0^+}\int_{\{(\cdot,y):\,|\cdot-y|>\varepsilon,\,|y|\leq r\}} k(\cdot - y) f_2(y)\,dy\right\|_{L^p(\mathbb{R}^d)} \\ &\qquad \leq C_p M\|f_2\|_{L^p(B(x_0,r))} \leq (1 + C_{\varphi,d,p}C_\phi)M\|f\|_{T^p_\phi(0)}\phi(r) r^{d/p}\end{aligned}$$

and we can conclude that there exists a constant $C_{\phi,p,d} > 0$ such that

$$r^{-d/p}\|\mathcal{K}f_2 - P'\|_{L^p(B(0,r))} \leq C_{\phi,p,d} M\phi(r) \quad \text{for } r > 0.$$

If we now assume $\overline{b}(\phi) < 0$, then, for $r > 0$ and $x \in B(0, r/2)$, we have

$$\mathcal{K}f_2(x) = \lim_{\varepsilon\to 0^+}\int_{\{(x,y):\,|x-y|>\varepsilon,\,|y|\leq r\}} k(x-y) f_2(y)\,dy + \int_{\mathbb{R}^d\setminus B(0,r)} k(x-y) f_2(y)\,dy.$$

We can deal with the first term of the right-hand side just as we did before, while for the second we use the estimate

$$\left|\int_{\mathbb{R}^d\setminus B(0,r)} k(x-y) f_2(y)\,dy\right| \leq M\int_{\mathbb{R}^d\setminus B(0,r)} |y|^{-d}|f_2(y)|\,dy \leq C_d M\phi(r),$$

which follows from (3.35). This leads to

$$r^{-d/p}\|\mathcal{K}f_2\|_{L^p(B(0,r))} \leq C_{\phi,p,d} M\phi(r) \quad \text{for } r > 0.$$

One more use of Theorem 3.3.1 ensures

$$\|\mathcal{K}f_2\|_{L^p(\mathbb{R}^d)} \leq C_p M\|f_2\|_{L^p(\mathbb{R}^d)},$$

which allows us to conclude, with (3.33), that the desired inequality (3.32) holds.

If we moreover assume that f belongs to $t^p_\phi(0)$, then we know that there exists a sequence $(f_j)_{j\in\mathbb{N}_0}$ of functions in $\mathcal{D}(\mathbb{R}^d)$ such that f_j converges to f in $T^p_\phi(0)$ as $j \to \infty$.

By a reasoning similar to the one we made for the function ψ at the beginning of the proof, we can conclude that, for all $j \in \mathbb{N}_0$, $\mathcal{K}f_j$ belongs to $C^\infty(\mathbb{R}^d)$ and so to $t^p_\phi(0)$ as well, by Remark 2.1.13. In addition, it follows from (3.32) that $\mathcal{K}f_j$ converges to $\mathcal{K}f$ in $T^p_\phi(0)$ as j tends to infinity and, as $t^p_\phi(0)$ is a closed subspace, we get $\mathcal{K}f \in t^p_\phi(0)$. ■

COROLLARY 3.3.5. *Denote by $\mathcal{Y}_{l,m}$ the convolution singular integral operator defined by*

$$\mathcal{Y}_{l,m}f := \text{p.v.} \int k_{l,m}(\cdot - y) f(y)\, dy,$$

whose kernel is

$$k_{l,m} := Y_{l,m}\Big(\frac{\cdot}{|\cdot|}\Big)|\cdot|^{-d},$$

where $(Y_{l,m})_{l,m}$ forms a complete system of orthogonal spherical harmonics (for the definition of spherical harmonics, see e.g. [26, 27]*), m being the degree of the harmonic. Under the assumption of Proposition* 3.3.4*, there exist constants $C_p, C_{\phi,p} > 0$ such that*

$$\|\mathcal{Y}_{l,m}f\|_{L^p(\mathbb{R}^d)} \le C_p\|f\|_{L^p(\mathbb{R}^d)}, \qquad \|\mathcal{Y}^*_{l,m}f\|_{L^p(\mathbb{R}^d)} \le C_p\|f\|_{L^p(\mathbb{R}^d)} \tag{3.36}$$

and

$$\|\mathcal{Y}_{l,m}f\|_{T^p_\phi(x_0)} \le C_{\phi,p} m^{\frac{d-2}{2}+\lceil \bar{b}(\phi)\rceil_{\mathbb{N}_0}} \|f\|_{T^p_\phi(x_0)}. \tag{3.37}$$

Proof. Inequalities (3.36) come from (3.30) and the fact that $\|k_{l,m}\|_{L^2(\Sigma)} = 1$. Inequality (3.37) is obtained from (3.32), using the fact that, for $\alpha \in \mathbb{N}^d_0$, we have $|D^\alpha Y_{l,m}| \le C_\alpha m^{\frac{d-2}{2}+|\alpha|}$ on Σ (see [7]). ■

A fundamental example of a convolution singular integral operators is given by the Riesz transform $(\mathcal{R}_j)_{1\le j\le d}$, defined for $j \in \{1,\dots,d\}$ by

$$\mathcal{R}_j f(x) := \text{p.v.} \frac{-i\Gamma\big(\frac{d+1}{2}\big)}{\pi^{\frac{d+1}{2}}} \int \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy.$$

Let us fix $1 < p < \infty$ and $k \ge 1$; it is known that the following facts hold (see e.g. [7, 8]):

- if $f \in W^p_k(\mathbb{R}^d)$, then $\mathcal{R}_j f \in W^p_k(\mathbb{R}^d)$ and $\mathcal{R}_j$ is a continuous operator on $W^p_k(\mathbb{R}^d)$,
- for $l \in \{1,\dots,d\}$ and $f \in W^p_k(\mathbb{R}^d)$, we have $D_l(\mathcal{R}_j f) = \mathcal{R}_j(D_l f)$ and $\mathcal{R}_j(D_l f) = \mathcal{R}_l(D_j f)$,
- if $f \in L^p(\mathbb{R}^d)$, then $\sum_{j=1}^d \mathcal{R}_j^2 f = f$.

The operator

$$\Lambda := i\sum_{j=1}^d \mathcal{R}_j D_j$$

continuously maps $W^p_k(\mathbb{R}^d)$ into $W^p_{k-1}(\mathbb{R}^d)$ and, if $k \ge 2$,

$$\Lambda^2 f = -\Delta f \qquad \text{for all } f \in W^p_k(\mathbb{R}^d).$$

We also have the identity $D_j f = -i\mathcal{R}_j \Lambda f$ for all $f \in W^p_k(\mathbb{R}^d)$. It can also be shown that for all $m \in \mathbb{N}$ such that $2m+1 \ge d$, there exist $a_1,\dots,a_m < 0$ and a positive integrable function h_m with derivatives up to order $2m+1-d$ continuous and bounded such that

$$\Lambda\mathcal{J}f = f + \sum_{j=1}^m a_j \mathcal{J}^{2j} f - h_m * f$$

for all $f \in L^p(\mathbb{R}^d)$ (see [8]).

PROPOSITION 3.3.6. *Let $p \in (1,\infty)$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that either $\overline{b}(\phi) < -1$ or there exists $n \in \mathbb{N}_0 \cup \{-1\}$ for which $n < \underline{b}(\phi) \leq \overline{b}(\phi) < n+1$. The operator $D_j\mathcal{J}$ continuously maps $T^p_\phi(x_0)$ into itself.*

Proof. Let $f \in T^p_\phi(x_0)$; from what precedes, we have

$$D_j\mathcal{J}f = -i\mathcal{R}_j\Lambda\mathcal{J}f = -i\mathcal{R}_j\Big(f + \sum_{j=1}^{m} a_j\mathcal{J}^{2j}f - h_m * f\Big),$$

where m has been chosen sufficiently large so that $h_m \in C^{\lceil\overline{b}(\phi)\rceil_{\mathbb{N}_0}}(\mathbb{R}^d)$. Using Remark 2.1.13, we thus have $h_m * f \in t^p_\phi(x_0)$. Moreover, by Theorem 3.1.2 and Proposition 2.3.3, we know that $\mathcal{J}$ continuously maps $T^p_\phi(x_0)$ into itself. The conclusion is obtained by applying Proposition 3.3.4 to $\mathcal{R}_j$. ■

The decomposition of functions into spherical harmonics will lead us to singular integral operators whose kernel depends on several variables.

DEFINITION 3.3.7. Let $q \in [1,\infty]$, $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$ and $x_0 \in \mathbb{R}^d$. Let $\mathcal{K}$ be the singular integral operator of the form

$$f \mapsto a(\cdot)f(\cdot) + \text{p.v.}\int k(\cdot,\cdot - y)f(y)\,dy,$$

where

- a is a bounded measurable function,
- for all $x \in \mathbb{R}^d$, $k(x,\cdot)$ is homogeneous of degree $-d$, has mean value zero on Σ and belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$.

The *symbol* of $\mathcal{K}$ is the function

$$\sigma(\mathcal{K}) : (x,z) \mapsto a(x) + \widehat{k}(x,z),$$

where, given $x \in \mathbb{R}^d$, $\widehat{k}(x,\cdot)$ is the Fourier transform of $k(x,\cdot)$ (understood in the distribution sense). We know that for all $x \in \mathbb{R}^d$, $\widehat{k}(x,\cdot)$ belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and is homogeneous of degree 0 (see e.g. [13]). We say that $\mathcal{K}$ is in the class $T^q_\phi(x_0)$ if, for all $|\alpha| \leq 2d + \lceil\overline{b}(\phi)\rceil_{\mathbb{N}_0}$ and $z \neq 0$, the function

$$x \mapsto D^\alpha_z\sigma(\mathcal{K})(x,z)$$

is in $T^q_\phi(x_0) \cap L^\infty(\mathbb{R}^d)$, uniformly on Σ. We then define

$$\|\mathcal{K}\|_{T^q_\phi(x_0)} = \max\Big\{\sup_{\substack{|z|=1\\ 0\leq|\alpha|\leq 2d+\lceil\overline{b}(\phi)\rceil_{\mathbb{N}_0}}} \|D^\alpha_z\sigma(\mathcal{K})(\cdot,z)\|_{T^q_\phi(x_0)},\ \sup_{\substack{|z|=1\\ 0\leq|\alpha|\leq 2d+\lceil\overline{b}(\phi)\rceil_{\mathbb{N}_0}}} \|D^\alpha_z\sigma(\mathcal{K})(\cdot,z)\|_{L^\infty(\mathbb{R}^d)}\Big\}.$$

If moreover, for all $|\alpha| \leq 2d + \lceil\overline{b}(\phi)\rceil_{\mathbb{N}_0}$ and $z \neq 0$, the function $x \mapsto D^\alpha_z k(x,z)$ belongs to $t^q_\phi(x_0)$ uniformly on Σ, then we say that $\mathcal{K}$ is in the class $t^q_\phi(x_0)$.

REMARK 3.3.8. Given $x \in \mathbb{R}^d$, $\sigma(\mathcal{K})(x,\cdot)$ is a homogeneous function of degree zero; it is proved in [26, 7] that for $(x,z) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$, we have

$$k(x,z) = \sum_{l,m} a_{l,m}(x) Y_{l,m}\left(\frac{z}{|z|}\right)|z|^{-d}, \qquad \sigma(\mathcal{K})(x,z) = a(x) + \sum_{l,m} a_{l,m}(x)\gamma_m Y_{l,m}\left(\frac{z}{|z|}\right),$$

where $\gamma_m := \frac{i^m \pi^{d/2}\Gamma(m/2)}{\Gamma\left(\frac{m+d}{2}\right)}$ and

$$\begin{aligned} a_{l,m}(x) &:= (-1)^v (m(m+d-2))^{-v} \int_\Sigma Y_{l,m} L^v k(x,\cdot)\, d\sigma \\ &= (-1)^v (m(m+d-2))^{-v} \gamma_m^{-1} \int_\Sigma Y_{l,m} L^v \sigma(\mathcal{K})(x,\cdot)\, d\sigma, \end{aligned}$$

with $LF(z) = |z|^2 \Delta F(z)$ and $v \in \mathbb{N}_0$.

THEOREM 3.3.9. *Let $q \in [1,\infty]$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{B}$ be such that $\underline{b}(\phi) > 0$. Let $\mathcal{K}$ be a singular integral operator of class $T^q_\phi(x_0)$.*

(1) $a_{l,m} \in T^q_\phi(x_0) \cap L^\infty(\mathbb{R}^d)$ *and*

$$\max\{\|a_{l,m}\|_{T^q_\phi(x_0)}, \|a_{l,m}\|_{L^\infty(\mathbb{R}^d)}\} \le C_\phi m^{d/2-2v} \|\mathcal{K}\|_{T^q_\phi(x_0)}.$$

(2) *If $p \in (1,\infty)$ is such that $0 \le 1/p^* := 1/q + 1/p \le 1$ and if $f \in L^p(\mathbb{R}^d)$, then, for almost every $x \in \mathbb{R}^d$, $\mathcal{K}f(x)$ and $\mathcal{Y}_{l,m}f(x)$ exist and the series*

$$a(x)f(x) + \sum_{l,m} a_{l,m}(x)\mathcal{Y}_{l,m}f(x)$$

converges absolutely to $\mathcal{K}f(x)$.

(3) *$\mathcal{K}$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^{p^*}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$: there exists a constant $C_{p,q} > 0$ such that, for all $f \in L^p(\mathbb{R}^d)$,*

$$\max\{\|\mathcal{K}f\|_{L^{p^*}(\mathbb{R}^d)}, \|\mathcal{K}f\|_{L^p(\mathbb{R}^d)}\} \le C_{p,q} \|\mathcal{K}\|_{T^q_\phi(x_0)} \|f\|_{L^p(\mathbb{R}^d)}.$$

(4) *Let $\psi \in \mathcal{B}$ be such that $\underline{b}(\psi) \ge -d/p$, $\phi \preccurlyeq \psi$ and either $\underline{b}(\psi) \le 0$ or $n < \underline{b}(\psi) \le \overline{b}(\psi) < n+1$ for some $n \in \mathbb{N}_0$. Then $\mathcal{K}$ is a bounded operator from $T^p_\psi(x_0)$ to $T^{p^*}_\psi(x_0)$: there exists a constant $C_{p,q,\phi,\psi} > 0$ such that, for all $f \in T^p_\psi(x_0)$,*

$$\|\mathcal{K}f\|_{T^{p^*}_\psi(x_0)} \le C_{p,q,\phi,\psi} \|\mathcal{K}\|_{T^q_\phi(x_0)} \|f\|_{T^p_\psi(x_0)}.$$

(5) *If moreover $\mathcal{K}$ is of class $t^q_\phi(x_0)$, then $a_{l,m}$ belongs to $t^q_\phi(x_0)$ and, for all $f \in t^p_\psi(x_0)$, $\mathcal{K}f$ belongs to $t^{p^*}_\psi(x_0)$.*

Proof. We keep the same notations as in Remark 3.3.8 with $v := d + \lceil \frac{\overline{b}(\phi)-1}{2} \rceil_{\mathbb{N}_0}$.

(1) For all $x \in \mathbb{R}^d$ and $z \in \Sigma$, let us write

$$L^v \sigma(\mathcal{K})(x,z) := \sum_{|\alpha| \le 2v} g_\alpha(z) D^\alpha_z \sigma(\mathcal{K})(x,z),$$

where g_α is a product of powers of z_j ($j \in \{1,\dots,d\}$). From the definition of the class of operators in $T^q_\phi(x_0)$, for $z \in \Sigma$, we have

$$\|L^v \sigma(\mathcal{K})(\cdot,z)\|_{L^q(\mathbb{R}^d)} \le C_v \|\mathcal{K}\|_{T^q_\phi(x_0)}.$$

Let us also recall that $\|Y_{l,m}\|_{L^2(\Sigma)} = 1$. If $q \geq 2$, then, if we denote by q' the conjugate exponent of q, then $q' \leq 2$, and by Hölder's inequality (with the usual modification if $q = \infty$),

$$\begin{aligned}
\|a_{l,m}\|_{L^q(\mathbb{R}^d)} &= (m(m+d-2))^{-v}\gamma_m^{-1}\Big(\int_{\mathbb{R}^d}\Big|\int_\Sigma Y_{l,m}(z)L^v\sigma(\mathcal{K})(x,z)\,d\sigma(z)\Big|^q\,dx\Big)^{1/q}\\
&\leq C_d m^{d/2-2v}\Big(\int_{\mathbb{R}^d}\|Y_{l,m}\|^q_{L^{q'}(\Sigma)}\|L^v\sigma(\mathcal{K})(x,\cdot)\|^q_{L^q(\Sigma)}\,dx\Big)^{1/q}\\
&\leq C_d m^{d/2-2v}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1/q'-1/2}\|Y_{l,m}\|_{L^2(\Sigma)}\Big(\int_\Sigma\int_{\mathbb{R}^d}|L^v\sigma(\mathcal{K})(x,z)|^q\,dx\,d\sigma(z)\Big)^{1/q}\\
&\leq C_{d,v}m^{d/2-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1/2} = C_{d,v}m^{d/2-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}.
\end{aligned}$$

From this, we get $\|a_{l,m}\|_{L^q(\mathbb{R}^d)} \leq Cm^{d/2-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}$ and a similar argument can be applied to obtain the same inequality for $\|a_{l,m}\|_{L^\infty(\mathbb{R}^d)}$.

Now, if $q \leq 2$, we have

$$\begin{aligned}
\|a_{l,m}\|_{L^q(\mathbb{R}^d)} &= (m(m+d-2))^{-v}\gamma_m^{-1}\Big(\int_{\mathbb{R}^d}\Big|\int_\Sigma Y_{l,m}(z)L^v\sigma(\mathcal{K})(x,z)\,d\sigma(z)\Big|^q\,dx\Big)^{1/q}\\
&\leq C_d m^{d/2-2v}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1-1/q}\Big(\int_{\mathbb{R}^d}\int_\Sigma|Y_{l,m}(z)|^q|L^v\sigma(\mathcal{K})(x,z)|^q\,d\sigma(z)\,dx\Big)^{1/q}\\
&\leq C_{d,v}m^{d/2-2v}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1-1/q}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|Y_{l,m}\|_{L^q(\Sigma)}\\
&\leq C_{d,v}m^{d/2-2v}\Big(\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\Big)^{1/2}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|Y_{l,m}\|_{L^2(\Sigma)}\\
&= C_{d,v}m^{d/2-2v}\|\mathcal{K}\|_{T^q_\phi(x_0)}.
\end{aligned}$$

Moreover, for $|\alpha| \leq 2d + \overline{b}(\phi) + 1$ and $z \in \Sigma$, there exists a polynomial

$$P_{\alpha,z} := \sum_{|\beta|\leq n} C^{(\beta)}_{z,\alpha}(\cdot - x_0)^\beta$$

of degree n such that

$$\sum_{|\beta|\leq n}|C^{(\beta)}_{z,\alpha}| \leq \|\mathcal{K}\|_{T^q_\phi(x_0)}$$

and, for $r > 0$,

$$r^{-d/q}\|D^\alpha_z\sigma(\mathcal{K})(\cdot,z) - P_{\alpha,z}\|_{L^q(B(x_0,r))} \leq \|\mathcal{K}\|_{T^q_\phi(x_0)}\phi(r).$$

Thus,

$$P = \sum_{|\beta|\leq n}(-1)^v(m(m+d-2))^{-v}\gamma_m^{-1}\int_\Sigma Y_{l,m}(z)\Big(\sum_{|\alpha|\leq 2v}g_\alpha(z)C^{(\beta)}_{z,\alpha}\Big)\,d\sigma(\cdot - x_0)^\beta$$

is a polynomial of degree n for which

$$\sum_{|\beta|\leq n}\Big|(m(m+d-2))^{-v}\gamma_m^{-1}\int_\Sigma Y_{l,m}(z)\Big(\sum_{|\alpha|\leq 2v}g_\alpha(z)C^{(\beta)}_{z,\alpha}\Big)\,d\sigma\Big| \leq C_\phi m^{d/2-2v}\|\mathcal{K}\|_{T_\phi(x_0)}$$

and, for $r > 0$, we can show, in the same way as before, that

$$r^{-d/q}\|a_{l,n} - P\|_{L^q(B(x_0,r))} \le C_{d,q} m^{d/2-2v} \|\mathcal{K}\|_{T^q_\phi(x_0)} \phi(r).$$

(2) It is well-known that there exists a constant $C_d > 0$ such that, for $m \in \mathbb{N}_0$, the number of spherical harmonics of degree m is bounded by $C_d m^{d-2}$ (see e.g. [27]). Moreover, if $f \in L^p(\mathbb{R}^d)$, from Corollary 3.3.5 we also know that $\|\mathcal{Y}^*_{l,m} f\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}$. From this, using (1), we can claim that if $p^* \ge 1$ is such that $1/p^* := 1/p + 1/q$, then $\sum_{l,m} a_{l,m} \mathcal{Y}^*_{l,m} f$ converges in $L^{p^*}(\mathbb{R}^d)$. As a consequence, for almost every $x \in \mathbb{R}^d$, $\sum_{l,m} a_{l,m}(x) \mathcal{Y}^*_{l,m} f(x)$ is finite.

Let us fix $\varepsilon > 0$ and $x \in \mathbb{R}^d$ such that $|a_{l,m}(x)| \le C_\phi m^{d/2-2v}$; we have

$$\begin{aligned}\int_{\mathbb{R}^d \setminus B(x,\varepsilon)} k(x, x-y) f(y)\, dy &= \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} \sum_{l,m} a_{l,m}(x) Y_{l,m}\left(\frac{x-y}{|x-y|}\right) |x-y|^{-d} f(y)\, dy \\ &= \sum_{l,m} a_{l,m}(x) \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} Y_{l,m}\left(\frac{x-y}{|x-y|}\right) |x-y|^{-d} f(y)\, dy,\end{aligned}$$

because $y \mapsto |x-y|^{-d} f(y)$ is integrable (using Hölder's inequality) on $\mathbb{R}^d \setminus B(x,\varepsilon)$ and

$$\left|\sum_{l,m} a_{l,m}(x) Y_{l,m}\left(\frac{x-\cdot}{|x-\cdot|}\right)\right| \le C_{d,q} \sum_{m \in \mathbb{N}_0} m^{d/2-2v} m^{d-2} m^{(d-2)/2} \|\mathcal{K}\|_{T^q_\phi(x_0)} \le C_{d,q} \|\mathcal{K}\|_{T^q_\phi(x_0)}.$$

Now, if x is a point for which $\sum_{l,m} a_{l,m}(x) \mathcal{Y}^*_{l,m} f(x)$ is finite and $\mathcal{Y}_{l,m} f(x)$ exists for all l, m, then, for $\varepsilon > 0$,

$$\int_{\mathbb{R}^d \setminus B(x,\varepsilon)} Y_{l,m}\left(\frac{x-y}{|x-y|}\right) |x-y|^{-d} f(y)\, dy \le \mathcal{Y}^*_{l,m} f(x),$$

which allows us to let $\varepsilon \to 0^+$ to obtain

$$\mathcal{K} f(x) = a(x) f(x) + \sum_{l,m} a_{l,m}(x) \mathcal{Y}_{l,m} f(x).$$

The conclusion follows from the fact that almost every $x \in \mathbb{R}^d$ is such that the quantity $\sum_{l,m} a_{l,m}(x) \mathcal{Y}^*_{l,m} f(x)$ is finite, $|a_{l,m}(x)| \le C_\phi m^{d/2-2v}$ and $\mathcal{Y}^*_{l,m} f(x)$ exists for all l, m, by countable intersection.

(3) For $f \in L^p(\mathbb{R}^d)$, we have, from (2) and Corollary 3.3.5,

$$\begin{aligned}\|\mathcal{K} f\|_{L^{p^*}(\mathbb{R}^d)} &= \|af + \sum_{l,m} a_{l,m} \mathcal{Y}_{l,m} f\|_{L^{p^*}(\mathbb{R}^d)} \\ &\le \|a\|_{L^q(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)} + \sum_{l,m} \|a_{l,m}\|_{L^q(\mathbb{R}^d)} \|\mathcal{Y}_{l,m} f\|_{L^p(\mathbb{R}^d)} \\ &\le \|a\|_{L^q(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)} + C_{p,q,d} \|\mathcal{K}\|_{T^q_\phi(x_0)} \|f\|_{L^p(\mathbb{R}^d)} \sum_{m \in \mathbb{N}_0} m^{d/2-2v} m^{d-2} \\ &\le C_{p,q} \|\mathcal{K}\|_{T_\phi(x_0)} \|f\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

The upper bound for $\|\mathcal{K} f\|_{L^p(\mathbb{R}^d)}$ can be obtained in the same way.

(4) Again, (2), Proposition 2.3.2, Corollaries 3.3.5 and 2.3.5, for $f \in T^p_\phi(x_0)$, we have

$$\begin{aligned}\|\mathcal{K}f\|_{T^{p^*}_\psi(x_0)} &\le C_{p,q,\phi,\psi}\Big(\|a\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)} + \sum_{l,m}\|a_{l,m}\|_{T^q_\phi(x_0)}\|\mathcal{Y}_{l,m}f\|_{T^p_\psi(x_0)}\Big)\\
&\le C_{p,q,\phi,\psi}(\|a\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)} + \|f\|_{T^p_\psi(x_0)}\|\mathcal{K}\|_{T^q_\phi(x_0)}\sum_{m\in\mathbb{N}_0} m^{d-2}m^{d/2-2v}m^{\frac{d-2}{2}+\lceil\overline{b}(\psi)\rceil_{\mathbb{N}_0}}\Big)\\
&\le C_{p,q,\phi,\psi}\Big(\|a\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)} + \|f\|_{T^p_\psi(x_0)}\|\mathcal{K}\|_{T^q_\phi(x_0)}\sum_{m\in\mathbb{N}_0} m^{d-2}m^{d/2-2v}m^{\frac{d-2}{2}+\lceil\overline{b}(\phi)\rceil_{\mathbb{N}_0}}\Big)\\
&\le C_{p,q,\phi,\psi}\|\mathcal{K}\|_{T^q_\phi(x_0)}\|f\|_{T^p_\psi(x_0)}.\end{aligned}$$

(5) We keep the notations from (1). By definition of the class $t^q_\phi(x_0)$, there exist $\varepsilon > 0$ and $\varepsilon(r)$ converging to 0 as $r \to 0^+$ such that, for $|\alpha| \le 2d + \overline{b}(\phi) + 1$, $z \in \Sigma$ and $r > 0$ sufficiently small, we have

$$r^{-d/q}\|D^\alpha\sigma(\mathcal{K})(\cdot, z) - P_{\alpha,z}\|_{L^q(B(x_0,r)} \le \varepsilon(r)\phi(r).$$

As a consequence, for such r,

$$r^{-d/q}\|a_{l,n} - P\|_{L^q(B(x_0,r))} \le C\varepsilon(r)m^{d/2-2v}\phi(r)$$

and $a_{l,n} \in t^q_\phi(x_0)$. The conclusion comes from the second part of Corollary 2.3.5 and the fact that $t^{p^*}_\psi(x_0)$ is closed. ∎

Remark 3.3.10. Let us come back to the convolution singular integral operators we considered in Theorem 3.3.4. For such an operator, the kernel k is independent of the variable x and $\|\mathcal{K}\|^*_{T^p_\phi(x_0)}$ is bounded by the derivatives of k on Σ. Following the path taken in the last theorem, we can also bound this norm using now the derivatives of $\sigma(\mathcal{K})$. Indeed, as k does not depend on x, neither do $\sigma(\mathcal{K})$ and $a_{l,m}$. Let $p \in (1,\infty)$ and $\phi \in \mathcal{B}$, be as in Theorem 3.3.4, and define

$$v(\phi) := \begin{cases} d & \text{if } \overline{b}(\phi) < 0,\\ d + \lceil\frac{\overline{b}(\phi)-1}{2}\rceil_{\mathbb{N}_0} & \text{otherwise,}\end{cases} \qquad N := \sup_{\substack{|z|=1\\ 0\le|\alpha|\le v(\phi)}} |D^\alpha\sigma(\mathcal{K})(z)|.$$

Using an argument similar to the one used in Theorem 3.3.9, we have

$$|a_{l,m}| \le Cm^{d/2-2v}N \quad \text{for all } l, m.$$

For all $f \in L^p(\mathbb{R}^d)$,

$$\mathcal{K}f = \sum_{l,m} a_{l,m}\mathcal{Y}_{l,m}f \quad \text{almost everywhere,}$$

$\mathcal{K}f \in L^p(\mathbb{R}^d)$ and, if $f \in T^p_\phi(x_0)$, then $\mathcal{K}f \in T^p_\phi(x_0)$ with $\|\mathcal{K}f\|_{T^p_\phi(x_0)} \le C_{p,\phi}N\|f\|_{T^p_\phi(x_0)}$.

3.4. Elliptic partial differential equations

Definition 3.4.1. An *elliptic partial differential equation at $x_0 \in \mathbb{R}^d$ of order $m \in \mathbb{N}$* is a partial differential equation of the form

$$\mathcal{E}f = \sum_{|\alpha|\le m} a_\alpha\, D^\alpha f = g,$$

where, for all $|\alpha| \le m$, a_α is an $s \times r$ matrix of functions and

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ \vdots \\ g_s \end{pmatrix}$$

are vector-valued functions with $f_j \in W^p_m(\mathbb{R}^d)$ for all $j \in \{1, \dots, r\}$; D^α stands for the weak derivative and

$$\mu(x_0) := \inf_{|\xi|=1} \det\Big[\Big(\sum_{|\alpha|=m} a^*_\alpha(x_0)\xi^\alpha\Big)\Big(\sum_{|\alpha|=m} a_\alpha(x_0)\xi^\alpha\Big)\Big] > 0$$

is the *ellipticity constant* of $\mathcal{E}$ at x_0.

In [7], Calderón and Zygmund proved that if $\mathcal{E}$ is elliptic with constant coefficients $(a_\alpha)_{|\alpha|=m}$ all of the same order, then we can write

$$\mathcal{E} = \mathcal{K}\Lambda^m,$$

where $\mathcal{K}$ is an $s \times r$ matrix of convolution singular operators, whose matrix of symbols is

$$\sigma(\mathcal{K})(z) = (-i)^m \sum_{|\alpha|=m} a_\alpha z^\alpha |z|^{-m} \qquad \text{for } z \neq 0.$$

They also showed in [8] that, in this case, there exists an $r \times s$ matrix of convolution singular operators whose matrix of symbols is (2)

$$\sigma(\mathcal{H}) = [\sigma(\mathcal{K})^*\sigma(\mathcal{K})]^{-1}\sigma(\mathcal{K})^*$$

and for which $\mathcal{H}\mathcal{K}$ is the identity operator. From Remark 3.3.10, we can estimate the dual norm of $\mathcal{H}$ on the spaces $T^p_\phi(x_0)$, using the ellipticity constant of $\mathcal{E}$ and $(|a_\alpha|)_{|\alpha|=m}$.

Now, if

$$\mathcal{E}f = \sum_{|\alpha|\le m} a_\alpha\, D^\alpha f = g$$

is a general elliptic partial differential equation at $x_0 \in \mathbb{R}^d$ of order $m \in \mathbb{N}$, we set

$$\mathcal{E}_{x_0} := \sum_{|\alpha|=m} a_\alpha(x_0).$$

By what precedes, we have $\mathcal{E}_{x_0} = \mathcal{K}\Lambda^m$, where $\mathcal{K}$ is a matrix of convolution singular operators for which $\mathcal{H}\mathcal{K}$ is the identity operator. Then, let us define

$$h := \begin{cases} (1-\Delta)^{m/2} f & \text{if } m \text{ is even,} \\ (i+\Lambda)(1-\Delta)^{\frac{m-1}{2}} f & \text{if } m \text{ is odd.} \end{cases}$$

Applying $\mathcal{H}$ on $\mathcal{E}_{x_0} f + (\mathcal{E} - \mathcal{E}_{x_0})f = g$ gives

$$\Lambda^m f = \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f$$

and, as $\Lambda^2 = -\Delta$, we obtain, if m is even,

$$h = \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + [(1-\Delta)^{m/2} - (-\Delta)^{m/2}]f = \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + L_1(D)f,$$

(2) The ellipticity of the equation allows us to take the inverse matrix of $\sigma(\mathcal{K})^*\sigma(\mathcal{K})$.

where $L_1(D)$ is a differential operator of order $m-2$ with constant coefficients. Assuming that m is odd, we get

$$\begin{aligned} h &= \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + \big[(i+\Lambda)(1-\Delta)^{\frac{m-1}{2}} - \Lambda(-\Delta)^{\frac{m-1}{2}}\big]f \\ &= \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + L_2(D)f + \Lambda L_3(D)f, \end{aligned}$$

where $L_2(D)$ (resp. $L_3(D)$) is a differential operator of order $m-1$ (resp. $m-3$) with constant coefficients.

In what follows, we choose as the norm of a vector-valued functions the sum of the norms of its components.

Proposition 3.4.2. *Let $p_1 \in (1,\infty)$ and $p_2 \in [1,\infty]$ be such that*

$$0 \le \frac{1}{p_3} := \frac{1}{p_1} + \frac{1}{p_2} \le 1,$$

$x_0 \in \mathbb{R}^d$ and $\phi, \varphi, \psi \in \mathcal{B}$ be such that

- *$0 < \underline{b}(\phi)$ and the coefficients of $\mathcal{E}$ are functions in $T^{p_1}_\phi(x_0)$ for which x_0 is a Lebesgue point,*
- *$\phi \preccurlyeq \psi$,*
- *$-d/p_2 < \underline{b}(\psi)$ and there exists $n \in \mathbb{Z}$ such that $n < \underline{b}(\psi) \le \overline{b}(\psi) < n+1$ and $g \in T^{p_3}_\psi(x_0)$,*
- *$-d/p_2 < \underline{b}(\varphi)$ and there exists $l \in \mathbb{Z}$ such that $l < \underline{b}(\varphi) \le \overline{b}(\varphi) < l+1$ and $h \in T^{p_2}_\varphi(x_0)$,*
- *$\overline{b}(\psi) - \underline{b}(\varphi) < \min\{\underline{b}(\phi), 1\}$.*

Assume also that there exists $p^* \in [1,p_3]$ such that $f \in W^{p^*}_m(\mathbb{R}^d)$. Then $h \in T^{p_3}_\psi(x_0)$ with

$$\|h\|_{T^{p_3}_\psi(x_0)} \le \|\mathcal{H}g\|_{T^{p_3}_\psi(x_0)} + C_{p_1,p_2,\varphi,\psi,\phi}((1+MN)\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}),$$

where M is the least upper bound of the norm of the coefficients of $\mathcal{E}$ in $T^{p_2}_\phi(x_0)$ and

$$N = \sup_{\substack{|z|=1 \\ 0\le|\alpha|\le v(\psi)}} |D^\alpha \sigma(\mathcal{K})(z)|,$$

where $v(\psi)$ is defined as in Remark 3.3.10.

Proof. Let us first consider the case of m even; we have $f = \mathcal{J}^m h$ and therefore (3),

$$D^\alpha f = (D\mathcal{J})^\alpha \mathcal{J}^{m-|\alpha|} h \qquad \text{for } |\alpha| \le m.$$

As a consequence, for $|\alpha| < m$, we have $\overline{b}(\psi) < \underline{b}(\varphi) + 1$, $\varphi_{m-|\alpha|} \preccurlyeq \psi$ and, by Proposition 3.3.6 and Theorem 3.1.2,

$$\begin{aligned} \|D^\alpha f\|_{T^{p_2}_\psi(x_0)} &\le C_{p_2,\psi}\|\mathcal{J}^{m-|\alpha|}h\|_{T^{p_2}_\psi(x_0)} \le C_{p_2,\varphi,\psi}\|\mathcal{J}^{m-|\alpha|}h\|_{T^{p_2}_{\varphi_{m-|\alpha|}}(x_0)} \\ &\le C_{p_2,\varphi,\psi}\|h\|_{T^{p_2}_\varphi(x_0)}. \end{aligned}$$

If $|\alpha| = m$, Proposition 3.3.6 gives

$$\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} = \|(D\mathcal{J})^\alpha h\|_{T^{p_2}_\varphi(x_0)} \le C_{p_2,\varphi}\|h\|_{T^{p_2}_\varphi(x_0)}.$$

(3) $(D\mathcal{J})^\alpha$ stands for $(D_1\mathcal{J})^{\alpha_1}\dots(D_d\mathcal{J})^{\alpha_d}$.

Let us consider the operators

$$\mathcal{E}_1 = \sum_{|\alpha|<m} a_\alpha D^\alpha \quad \text{and} \quad \mathcal{E}_2 = \sum_{|\alpha|=m} (a_\alpha(x_0) - a_\alpha) D^\alpha;$$

by Corollary 2.3.5, we have

$$\begin{aligned}\|\mathcal{H}\mathcal{E}_1 f\|_{T^{p_3}_\psi(x_0)} &\le C_{p_3,\psi} N \|\mathcal{E}_1 f\|_{T^{p_3}_\psi(x_0)} \le C_{p_1,p_2,\phi,\psi} NM \sum_{|\alpha|<m} \|D^\alpha f\|_{T^{p_2}_\psi(x_0)} \\ &\le C_{p_1,p_2,\phi,\varphi,\psi} NM \|h\|_{T^{p_2}_\varphi(x_0)}.\end{aligned}$$

Let us remark that the assumption $\overline{b}(\psi) - \underline{b}(\varphi) < \min\{\underline{b}(\phi), 1\}$ allows us to use Proposition 2.3.6 to get

$$\begin{aligned}\|\mathcal{H}\mathcal{E}_2 f\|_{T^{p_3}_\psi(x_0)} &\le N \|\mathcal{E}_2 f\|_{T^{p_3}_\psi(x_0)} \\ &\le C_{p_1,p_2,\phi,\psi} NM \sum_{|\alpha|=m} (\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} + \|D^\alpha f\|_{L^{p_3}(\mathbb{R}^d)}) \\ &\le C_{p_1,p_2,\phi,\varphi,\psi} NM (\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}).\end{aligned}$$

Finally, by Proposition 2.3.7, we have

$$\begin{aligned}\|L_1(D) f\|_{T^{p_3}_\psi(x_0)} &\le C \sum_{|\alpha|\le m-2} \|D^\alpha f\|_{T^{p_3}_\psi(x_0)} \\ &\le C_{p_2,p_3} \sum_{|\alpha|\le m-2} \|D^\alpha f\|_{T^{p_2}_\psi(x_0)} + \|D^\alpha f\|_{L^{p_3}(\mathbb{R}^d)} \\ &\le C_{p_2,p_3,\varphi,\psi} (\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}),\end{aligned}$$

which leads to the conclusion.

Let us now assume that m is odd; in this case, we have

$$\begin{aligned}\mathcal{J}^{m+1} h = (i + \Lambda) \mathcal{J}^2 f &\implies (-i + \Lambda) \mathcal{J}^{m+1} h = (1 - \Delta) \mathcal{J}^2 f \\ &\implies (-i + \Lambda) \mathcal{J}^{m+1} h = f\end{aligned}$$

and therefore,

$$D^\alpha f = \Big(i \sum_{j=1}^d \mathcal{R}_j (D_j \mathcal{J}) - i\mathcal{J}\Big)(D\mathcal{J})^\alpha \mathcal{J}^{m-|\alpha|} h \quad \text{for } |\alpha| \le m.$$

Given $|\alpha| < m$ and $j \in \{1, \ldots, d\}$, we have, by Propositions 3.3.4, 3.3.6 and Theorem 3.1.2,

$$\|\mathcal{R}_j (D_j \mathcal{J})(D\mathcal{J})^\alpha \mathcal{J}^{m-|\alpha|} h\|_{T^{p_2}_\psi(x_0)} \le C_{p_2,\varphi,\psi} \|h\|_{T^{p_2}_\varphi(x_0)}.$$

From Theorem 3.1.2 and Proposition 2.3.3, we know that $\mathcal{J}$ maps $T^{p_2}_\psi(x_0)$ continuously into itself and so we also have

$$\|\mathcal{J}(D_j \mathcal{J}) \mathcal{J}^{m-|\alpha|} h\|_{T^{p_2}_\psi(x_0)} \le C_{p_2,\varphi,\psi} \|h\|_{T^{p_2}_\varphi(x_0)}.$$

As a consequence, the inequality

$$\|D^\alpha f\|_{T^{p_2}_\psi(x_0)} \le C_{p_2,\varphi,\psi} \|h\|_{T^{p_2}_\varphi(x_0)}$$

still holds for all $|\alpha| < m$. By a similar reasoning,

$$\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} \le C_{p_2,\varphi}\|h\|_{T^{p_2}_\varphi(x_0)} \quad \text{for } |\alpha| = m.$$

Therefore, the upper bounds for $\|\mathcal{H}\mathcal{E}_1 f\|_{T^{p_3}_\psi(x_0)}$ and $\|\mathcal{H}\mathcal{E}_2 f\|_{T^{p_3}_\psi(x_0)}$ are still satisfied. Finally, we also have

$$\|L_2(D)f\|_{T^{p_3}_\psi(x_0)} \le C_{p_2,p_3,\varphi,\psi}(\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)})$$

and, as $\Lambda = i\sum_{j=1}^d \mathcal{R}_j D_j$, Proposition 3.3.4 implies

$$\|\Lambda L_3(D)f\|_{T^{p_3}_\psi(x_0)} \le C_{p_3,\psi}\sum_{|\alpha|\le m-2}\|D^\alpha f\|_{T^{p_3}_\psi(x_0)} \le C_{p_2,p_3,\varphi,\psi}(\|h\|_{T^{p_2}_\varphi(x_0)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}),$$

which gives the conclusion in this case. ∎

Remark 3.4.3. It is still possible to obtain an inequality of some kind if we consider the case $\varphi(r) = r^{-d/p_2}$.

If $d/p_2 \notin \mathbb{N}_0$, then Theorem 3.1.2 still holds for φ, since the assumption $\underline{b}(\phi) > -d/p$ is just made to guarantee $r^{-d/p} \le C\phi(r)$ for r sufficiently large; it can thus be relaxed in this case. Therefore, Proposition 3.3.6 can also be applied with φ, and the inequalities

$$\|D^\alpha f\|_{T^{p_2}_\psi(x_0)} \le C_{p_2,\varphi,\psi}\|h\|_{T^{p_2}_\varphi(x_0)} \quad \forall|\alpha| < m,$$
$$\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} \le C_{p_2,\varphi}\|h\|_{T^{p_2}_\varphi(x_0)} \quad \forall|\alpha| = m$$

are still valid in this case. Let us also remark that

$$\|h\|_{T^{p_2}_\varphi(x_0)} \le 2\|h\|_{L^{p_2}(\mathbb{R}^d)} \le C_{m,p_2}\|f\|_{W^{p_2}_m(\mathbb{R}^d)}.$$

If $d/p_2 \in \mathbb{N}_0$ with $p_2 < d$, let us consider $|\alpha| < m$; we have

$$D^\alpha f \in W^{p_2}_1(\mathbb{R}^d) \hookrightarrow L^{p*}(\mathbb{R}^d),$$

with $1/p^* := 1/p_2 - 1/d$, by Sobolev's embedding. Therefore, for $r > 0$,

$$\begin{aligned} r^{-d/p_2}\|D^\alpha f\|_{L^{p_2}(B(x_0,r))} &\le C_{d,p_2,p^*} r^{-d/p_2} r^{d(1/p_2 - 1/p*)}\|D^\alpha f\|_{L^{p^*}(B(x_0,r))} \\ &\le C_{d,p_2,p*}\|D^\alpha f\|_{W^{p^*}_1(\mathbb{R}^d)} r^{-d/p^*} \end{aligned}$$

and $D^\alpha f \in T^{p_2}_{-d/p^*}(x_0)$ with

$$\|D^\alpha f\|_{T^{p_2}_{-d/p^*}(x_0)} \le C_{d,p_2,p^*}\|f\|_{W^{p_2}_m(\mathbb{R}^d)}.$$

Moreover, as $\overline{b}(\psi) < -d/p_2 + 1 = -d/p^*$, we get

$$\|D^\alpha f\|_{T^{p_2}_\psi(x_0)} \le C_{d,p_2,p^*,\psi}\|f\|_{W^{p_2}_m(\mathbb{R}^d)}.$$

Of course, for $|\alpha| = m$, we have

$$\|D^\alpha f\|_{T^{p_2}_\varphi(x_0)} \le 2\|D^\alpha f\|_{L^{p_2}(\mathbb{R}^d)} \le 2\|f\|_{W^{p_2}_m(\mathbb{R}^d)}$$

and we now conclude that

$$\|h\|_{T^{p_3}_\psi(x_0)} \le \|\mathcal{H}g\|_{T^{p_3}_\psi(x_0)} + C_{p_1,p_2,\varphi,\psi,\phi}((1+MN)\|f\|_{W^{p_2}_m(\mathbb{R}^d)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}).$$

If $d/p_2 \in \mathbb{N}_0$, let us first prove the following lemma.

Lemma 3.4.4. *If $d > 1$, for $d \le q < \infty$, we have the continuous embedding $W^d_1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$.*

Proof. Let g be a function in $W^d_1(\mathbb{R}^d)$; first let us remark that $g \in L^{\frac{d^2}{d-1}}(\mathbb{R}^d)$. Indeed, $g^d \in L^1(\mathbb{R}^d)$, with

$$\|g^d\|_{L^1(\mathbb{R}^d)} = \|g\|^d_{L^d(\mathbb{R}^d)} \leq \|g\|^d_{W^d_1(\mathbb{R}^d)}$$

and, for $|\alpha| = 1$, by Hölder's inequality,

$$\|D^\alpha g^d\|_{L^1(\mathbb{R}^d)} = \|dg^{d-1} D^\alpha g\|_{L^1(\mathbb{R}^d)} \leq d\|g\|^{d-1}_{L^d(\mathbb{R}^d)} \|D^\alpha g\|_{L^d(\mathbb{R}^d)} \leq d\|g\|^d_{W^d_1(\mathbb{R}^d)}.$$

Therefore, $g^d \in W^1_1(\mathbb{R}^d)$ with $\|g^d\|_{W^1_1(\mathbb{R}^d)} \leq C\|g\|^d_{W^d_1(\mathbb{R}^d)}$ and, as $d > 1$, Sobolev's embedding gives $W^1_1(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{d-1}}(\mathbb{R}^d)$ and finally $g \in L^{\frac{d^2}{d-1}}(\mathbb{R}^d)$ with

$$\|g\|_{L^{\frac{d^2}{d-1}}(\mathbb{R}^d)} \leq C\|g\|_{W^d_1(\mathbb{R}^d)}.$$

Let us prove by induction that any $g \in W^d_1(\mathbb{R}^d)$ belongs to $L^{\frac{(d+k)d}{d-1}}(\mathbb{R}^d)$ with

$$\|g\|_{L^{\frac{d(d+k)}{d-1}}(\mathbb{R}^d)} \leq C_k \|g\|_{W^d_1(\mathbb{R}^d)} \qquad \text{for all } k \in \mathbb{N}_0.$$

Let us suppose that this property holds for some $k \in \mathbb{N}_0$ and let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a sequence of functions in $\mathcal{D}(\mathbb{R}^d)$ such that φ_j converges to g in $W^d_1(\mathbb{R}^d)$. In particular, by induction, φ_j converges to g in $L^{(d+k)\frac{d}{d-1}}(\mathbb{R}^d)$. Let us recall that for $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have (see e.g. [34, Lemma 8.7])

$$\left(\int_{\mathbb{R}^d} |\varphi(x)|^{(d+k+1)\frac{d}{d-1}}\, dx\right)^{\frac{d-1}{d}} \leq \frac{d+k+1}{2}\left(\prod_{l=1}^d \|D_l\varphi\|_{L^d(\mathbb{R}^d)}\right)^{1/d} \left(\int_{\mathbb{R}^d} |\varphi(x)|^{(d+k)\frac{d}{d-1}}\, dx\right)^{\frac{d-1}{d}},$$

which holds if and only if

$$\|\varphi\|^{d+k+1}_{L^{(d+k+1)\frac{d}{d-1}}(\mathbb{R}^d)} \leq \frac{d+k+1}{2}\left(\prod_{l=1}^d \|D_l\varphi\|_{L^d(\mathbb{R}^d)}\right)^{1/d} \|\varphi\|^{d+k}_{L^{(d+k)\frac{d}{d-1}}(\mathbb{R}^d)}.$$

This proves that $(\varphi_j)_{j\in\mathbb{N}_0}$ is a Cauchy sequence in $L^{(d+k+1)\frac{d}{d-1}}(\mathbb{R}^d)$. As a consequence, $g \in L^{(d+k+1)\frac{d}{d-1}}(\mathbb{R}^d)$ with

$$\|g\|_{L^{(d+k+1)\frac{d}{d-1}}(\mathbb{R}^d)} \leq C_k \left(\frac{d+k+1}{2}\right)^{\frac{1}{d+k+1}} \|g\|_{W^d_1(\mathbb{R}^d)}. \blacksquare$$

Let us come back to the above remark. If $|\alpha| < m$, then $D^\alpha f \in W^d_1(\mathbb{R}^d)$, so since $-1 < \overline{b}(\psi) < 0$, we can choose $d \leq q < \infty$ such that $\overline{b}(\psi) < -d/q$. By the above lemma, $D^\alpha f \in L^q(\mathbb{R}^d)$ and

$$\|D^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_q \|D^\alpha f\|_{W^d_1(\mathbb{R}^d)}.$$

It follows that, for $r > 0$,

$$\begin{aligned} r^{-1}\|D^\alpha f\|_{L^d(B(x_0,r))} &\leq C_{d,q} r^{-1} r^{d(1/d-1/q)} \|D^\alpha f\|_{L^q(B(x_0,r))} \\ &\leq C_{d,q} r^{-d/q} \|D^\alpha f\|_{W^d_1(\mathbb{R}^d)}. \end{aligned}$$

Hence, $D^\alpha f \in T^p_{-d/q}(x_0)$ with

$$\|D^\alpha f\|_{T^p_{-d/q}(x_0)} \leq C_{d,q} \|D^\alpha f\|_{W^d_1(\mathbb{R}^d)}.$$

Since $\overline{b}(\psi) < -d/q$, we can write

$$\|D^\alpha f\|_{T^d_\psi(x_0)} \leq C_{\psi,q}\|D^\alpha f\|_{T^d_{-d/q}(x_0)} \leq C_{d,q,\psi}\|D^\alpha f\|_{W^d_1(\mathbb{R}^d)}.$$

The previous reasoning for the case $|\alpha| = m$ is still valid and we get again

$$\|h\|_{T^{p_3}_\psi(x_0)} \leq \|\mathcal{H}g\|_{T^{p_3}_\psi(x_0)} + C_{p_1,p_2,\varphi,\psi,\phi}((1+MN)\|f\|_{W^{p_2}_m(\mathbb{R}^d)} + \|f\|_{W^{p_3}_m(\mathbb{R}^d)}).$$

Definition 3.4.5. Let, $p \in (1,\infty)$, $\phi,\varphi \in \mathcal{B}$ be such that $0 < \underline{b}(\phi)$, $-d/p < \underline{b}(\varphi)$ and there exists $n \in \mathbb{Z}$ such that $n < \underline{b}(\varphi) \leq \overline{b}(\varphi) < n+1$. Let us define k_p as follows:

- if $\underline{b}(\varphi) = \overline{b}(\varphi)$,

$$k_p(\phi,\varphi) := \min\left\{k \in \mathbb{N}_0 : \frac{1}{k}(\underline{b}\left(\varphi) + \frac{d}{p}\right) < \min\{1, \underline{b}(\phi)\}\right\},$$

- if $n < \underline{b}(\varphi) < \overline{b}(\varphi) < n+1$,

$$k_p(\phi,\varphi) := k_p(\phi, \cdot^{\underline{b}(\varphi)}) + \min\left\{k \in \mathbb{N}_0 : \frac{\overline{b}(\varphi) - \underline{b}(\varphi)}{k} < \min\{1, \underline{b}(\phi)\}\right\}.$$

Theorem 3.4.6. *Let $p \in (1,\infty)$, $q \in (1,\infty]$, $x_0 \in \mathbb{R}^d$ and $\phi,\varphi \in \mathcal{B}$ be such that $-d/p < \underline{b}(\varphi)$, $0 < \underline{b}(\phi)$ and there exists $n \in \mathbb{Z}$ such that $n < \underline{b}(\varphi) < \overline{b}(\varphi) < n+1$. Let $\mathcal{E}f = g$ be an elliptic differential equation of order m at x_0 such that the coefficients of $\mathcal{E}$ are functions in $T^q_\phi(x_0)$ for which x_0 is a Lebesgue point. Suppose that*

- $g \in T^{p_1}_\varphi(x_0)$ *with* $1/p_1 := 1/p + 1/q$,
- $\phi \preccurlyeq \varphi$ *and* $\overline{b}(\varphi) \leq \underline{b}(\phi)$ *or* $\overline{b}(\varphi) - \underline{b}(\varphi) \leq \min\{1, \underline{b}(\phi)\}$,
- $0 < 1/p' := k_p(\phi,\varphi)/q + 1/p < 1$,
- $f \in W^p_m(\mathbb{R}^d)$ *and* $p^* := \inf\{s \geq 1 : f \in W^s_m(\mathbb{R}^d)\} \leq p'$.

Then there exists a constant $C_{p',\phi,\varphi,m}$ such that, for all $|\alpha| \leq m$, $D^\alpha f \in T^{q'}_{\varphi_{m-|\alpha|}}(x_0)$ and

$$\|D^\alpha f\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} \leq C_{p',\phi,\varphi}(N(1+MN)^{k_p(\phi,\varphi)-1}\|g\|_{T^q_\varphi(x_0)} + k_p(\phi,\varphi)(1+MN)^{k_p(\phi,\varphi)}(\|f\|_{W^p_m(\mathbb{R}^d)} + \|f\|_{W^{p'}_m(\mathbb{R}^d)})$$

for all $q' \geq 1$ such that

- $1/p' \geq 1/q' \geq 1/p' - (m-|\alpha|)/d$ *if* $1/p' > (m-|\alpha|)/d$,
- $p' \leq q' \leq \infty$ *if* $1/p' < (m-|\alpha|)/d$,
- $p' \leq q' < \infty$ *if* $1/p' = (m-|\alpha|)/d$,

where M is the least upper bound of the norm of the coefficients of $\mathcal{E}$ in $T^q_\phi(x_0)$ and

$$N = \sup_{|z|=10\leq|\alpha|\leq v(\varphi)} |D^\alpha \sigma(\mathcal{K})(z)|.$$

Proof. Let us first suppose that $\underline{b}(\varphi) = \overline{b}(\varphi)$ and set $k = k_p(\phi,\varphi)$. Let us choose $0 \leq \varepsilon < 1$ such that

- $0 < \frac{1-\varepsilon}{k}\left(\underline{b}(\varphi) + \frac{d}{p}\right) \leq \frac{1+\varepsilon}{k}\left(\underline{b}(\varphi) + \frac{d}{p}\right) < \min\{1, \underline{b}(\phi)\}$,
- $-\frac{d}{p} + \frac{j+\varepsilon}{k}\left(\underline{b}(\varphi) + \frac{d}{p}\right) \notin \mathbb{Z}$ for all $j \in \{1,\ldots,k-1\}$.

We can then define, for $j \in \{0, \dots, k\}$, the function ψ_j by

$$\psi_j(r) := \begin{cases} r \mapsto r^{-d/p} & \text{if } j = 0, \\ r^{-d/p}(\varphi(r) r^{d/p})^{\frac{j+\varepsilon}{k}} & \text{if } 1 \le j < k, \\ \varphi & \text{if } j = k. \end{cases}$$

For $0 \le j < k$, we have $\overline{b}(\psi_j) < \underline{b}(\varphi)$ and so $\varphi \preccurlyeq \psi_j$. Moreover, for $1 \le j \le k$,

$$\underline{b}(\psi_j) = \overline{b}(\psi_j) = -\frac{d}{p} + \frac{j+\varepsilon}{k}\left(\underline{b}(\varphi) + \frac{d}{p}\right) \notin \mathbb{Z}.$$

We also have

$$\overline{b}(\psi_1) - \underline{b}(\psi_0) = \frac{1+\varepsilon}{k}\left(\underline{b}(\varphi) + \frac{d}{p}\right) < \min\{1, \underline{b}(\phi)\}$$

and, for $1 \le j < k$,

$$\begin{aligned}\overline{b}(\psi_{j+1}) - \underline{b}(\psi_j) &= -\frac{d}{p} + \frac{j+1+\varepsilon}{k}\left(\overline{b}(\varphi) + \frac{d}{p}\right) + \frac{d}{p} - \frac{j+\varepsilon}{k}\left(\underline{b}(\varphi) + \frac{d}{p}\right) \\ &= \frac{1}{k}\left(\underline{b}(\varphi) + \frac{d}{p}\right) < \min\{1, \underline{b}(\phi)\},\end{aligned}$$

as well as

$$\overline{b}(\psi_k) - \underline{b}(\psi_{k-1}) = \frac{1-\varepsilon}{k}\left(\underline{b}(\varphi) + \frac{d}{p}\right) < \min\{1, \underline{b}(\phi)\}.$$

Given $j \in \{0, \dots, k\}$, let us also define $p_j \in (1, \infty)$ by

$$\frac{1}{p_j} := \frac{j}{q} + \frac{1}{p}.$$

Since $h \in L^p(\mathbb{R}^d)$, $h \in T^{p_0}_{\psi_0}(x_0)$ and $\phi \preccurlyeq \psi_1$, we can write, using Remark 3.4.3,

$$\|h\|_{T^{p_1}_{\psi_1}(x_0)} \le \|\mathcal{H}g\|_{T^{p_1}_{\psi_1}(x_0)} + C_1(1+MN)(\|f\|_{W^p_m(\mathbb{R}^d)} + \|f\|_{W^{p_1}_m(\mathbb{R}^d)}).$$

Now, since f belongs to $W^{p_1}_m$ and the coefficients of $\mathcal{E}$ are in $L^q(\mathbb{R}^d)$, g belongs to $L^{p_2}(\mathbb{R}^d)$ and, from Proposition 2.3.7, also to $T^{p_2}_{\psi_2}(x_0)$. Furthermore, by Proposition 3.4.2, we have

$$\|h\|_{T^{p_2}_{\psi_2}(x_0)} \le \|\mathcal{H}g\|_{T^{p_2}_{\psi_2}(x_0)} + C_0(1+MN)(\|h\|_{T^{p_1}_{\psi_1}(x_0)} + \|f\|_{W^{p_2}_m(\mathbb{R}^d)}).$$

By iterating, we find, for $1 \le j \le k$,

$$\|h\|_{T^{p_j}_{\psi_j}(x_0)} \le \|\mathcal{H}g\|_{T^{p_j}_{\psi_j}(x_0)} + C_j(1+MN)(\|h\|_{T^{p_{j-1}}_{\psi_{j-1}}(x_0)} + \|f\|_{W^{p_j}_m(\mathbb{R}^d)}).$$

Now, for $1 \le j \le k$, we have

$$\begin{aligned}\|\mathcal{H}g\|_{T^{p_j}_{\psi_j}(x_0)} &\le C_{p_j,\psi_j} N \|g\|_{T^{p_j}_{\psi_j}(x_0)} \le C_{p_1,p_j,\psi_j} N \|g\|_{T^{p_1}_{\psi_j}(x_0)} + N\|g\|_{L^{p_1}(\mathbb{R}^d)} \\ &\le C_{p_1,p',\phi} N \|g\|_{T^{p_1}_{\phi}(x_0)}\end{aligned}$$

and

$$\|f\|_{W^{p_j}_m(\mathbb{R}^d)} \le \|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m(\mathbb{R}^d)},$$

which implies the existence of a constant $C_{p,p',\phi,\varphi} > 0$ such that

$$\|h\|_{T^{p'}_{\varphi}(x_0)} \le C_{p,p',\phi,\varphi}\big(N(1+MN)^{k-1}\|g\|_{T^p_{\varphi}(x_0)} + k(1+MN)^k(\|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m(\mathbb{R}^d)})\big).$$

Let us now establish the same inequality under the assumption $n < \underline{b}(\varphi) < \overline{b}(\varphi) < n+1$. If $\overline{b}(\varphi) \leq \underline{b}(\phi)$, then we set $k_1 := k_p(\phi, \cdot^{\,\underline{b}(\varphi)})$ and

$$k_2 := \min\left\{k \in \mathbb{N}_0 : \frac{\overline{b}(\varphi) - \underline{b}(\varphi)}{k} < \min\{1, \underline{b}(\phi)\}\right\}.$$

We also define

$$\psi_j(r) := r^{\underline{b}(\varphi)+\frac{j}{k_2}(\overline{b}(\varphi)-\underline{b}(\varphi))} \quad \text{for } 0 \leq j < k_2 \quad \text{and} \quad \psi_{k_2} := \varphi.$$

For $0 \leq j < k$, we have

$$\overline{b}(\psi_j) = \underline{b}(\varphi) + \frac{j}{k_2}(\overline{b}(\varphi) - \underline{b}(\varphi)) < \overline{b}(\varphi) \leq \underline{b}(\phi),$$

and so $\phi \preccurlyeq \psi_j$. Also,

$$\overline{b}(\psi_{j+1}) - \underline{b}(\psi_j) = \frac{1}{k_2}(\overline{b}(\varphi) - \underline{b}(\varphi)) < \min\{1, \underline{b}(\phi)\}.$$

From the first part of the proof, we can write, if p_0 is defined by $1/p_0 := k_1/q + 1/p$,

$$\begin{aligned}\|h\|_{T^{p_0}_{\psi_0}(x_0)} \leq C_{p,p_0,\phi,\varphi}\big(&N(1+MN)^{k_1-1}\|g\|_{T^q_\varphi(x_0)} \\ &+ k_1(1+MN)^{k_1}(\|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m(\mathbb{R}^d)})\big).\end{aligned}$$

We can proceed as in the first part to get the desired inequality.

Now let us consider the case where $\overline{b}(\varphi) > \underline{b}(\phi)$ and $\overline{b}(\varphi) - \underline{b}(\varphi) < \min\{1, \underline{b}(\phi)\}$. Let us choose α such that $\max\{-d/p, n\} < \alpha < \underline{b}(\varphi)$ and $\overline{b}(\varphi) - \alpha < \underline{b}(\phi)$; in particular, α is not an integer. From the first part of the proof, we know that there exists a constant $C_{p,p'\phi,\varphi} > 0$ such that

$$\begin{aligned}\|h\|_{T^{p''}_\alpha(x_0)} \leq C_{p,p',\phi,\varphi}\big(&N(1+MN)^{k-2}\|g\|_{T^q_\varphi(x_0)} \\ &+ (k-1)(1+MN)^{k-1}(\|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m})\big)\end{aligned}$$

with $1/p'' := (k-1)/q + 1/p$. Now, Proposition 3.4.2 implies

$$\begin{aligned}\|h\|_{T^{p'}_\varphi(x_0)} &\leq C_{p,\phi,\varphi}\big(N\|g\|_{T^q_\varphi(x_0)} + (1+MN)(\|h\|_{T^{p''}_\alpha(x_0)} + \|f\|_{W^{p'}_m(\mathbb{R}^d)})\big) \\ &\leq C_{p,p',\phi,\varphi}\big(N(1+MN)^{k-1}\|g\|_{T^p_\varphi(x_0)} + k(1+MN)^k(\|f\|_{W^{p'}_m(\mathbb{R}^d)} + \|f\|_{W^p_m})\big),\end{aligned}$$

which gives the desired inequality.

Let us now consider $|\alpha| \leq m$ and $q' \geq 1$ as in the assumption. If m is even then

$$\|D^\alpha f\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} = \|\mathcal{J}^{m-|\alpha|}(DJ)^\alpha h\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} \leq C_\varphi\|(DJ)^\alpha h\|_{T^{p'}_\varphi(x_0)} \leq C_\varphi\|h\|_{T^{p'}_\varphi(x_0)},$$

by Theorem 3.1.2 and Proposition 3.3.6. If m is odd, we get

$$\begin{aligned}\|D^\alpha f\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} &= \Big\|\mathcal{J}^{m-|\alpha|}\Big(i\sum_{j=1}^d \mathcal{R}_j(D_j\mathcal{J}) - i\mathcal{J}\Big)(D\mathcal{J})^\alpha h\Big\|_{T^{q'}_{\varphi_{m-|\alpha|}}(x_0)} \\ &\leq C_\varphi\Big\|\Big(i\sum_{j=1}^d \mathcal{R}_j(D_j\mathcal{J}) - i\mathcal{J}\Big)(D\mathcal{J})^\alpha h\Big\|_{T^{p'}_\varphi(x_0)} \leq C_\varphi\|h\|_{T^{p'}_\varphi(x_0)},\end{aligned}$$

by Theorem 3.1.2 and Propositions 3.3.4, 3.3.6. From this, the inequality obtained in the first part of the proof allows us to conclude the proof. ∎

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