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HYDROELASTIC ANALYSIS
USING THE FINITE ELEMENT FORMULATION

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1. THE FINITE ELEMENT FORMULATION FOR FLUID-STRUCTURE PROBLEMS

1.1 INTRODUCTION

To the best of our knowledge, the first purely lagrangian consistent formulation was proposed by Debongnie in [7]. At this time, even though natural, this idea as well as the notion of displacement potential coming out of this formulation led to some surprises. These rationalisation efforts led to the final formulation of the corrected Tong's formulation in 1978 [8] and to the corresponding formulation for the compressible case in 1984. Similar results were obtained by Morand in 1987 [10].

1.2 FLUID DESCRIPTION

1.2.1 Introduction

The motions of interest are vibrations around an equilibrium position. The mass conservation of the fluid suggests a Lagrangian description. As the motions considered are small, the equations can be linearized. This can be easily done using an appropriate parametrisation. Let a_i be the initial coordinates of a fluid particle. At time t, this particle position is

$$x_i(t) = a_i + \lambda(t)\, u_i(\mathbf{a})$$

This description is particularly adapted for a modal analysis where λ can always be taken to a small value compared to unity and must satisfy the equation

$$\ddot{\lambda}(t) + \omega^2\, \lambda(t) = 0$$

where ω is the frequency.

In the following, the derivation by a_i is noted D_i and ∂_i represents a derivation by x_i. J represents the Jacobian $\partial x / \partial a$. In the reference position corresponding to static equilibrium, the fluid takes a volume V, the wet surface and the free surface are noted S_f and Γ respectively. The out of equilibrium variables are noted V^*, S_f^* and Γ^*.

1.2.2 Linearisation of some auxiliary values

It is useful to make some linearisation once for all. First, the jacobian J is the determinant of the matrix defined by

$$D_j x_i = D_j(a_i + \lambda u_i) = \delta_{ij} + \lambda D_j u_i$$

The exact value of the Jacobian is

$$J = 1 + \lambda D_i u_i + \frac{\lambda^2}{2}(D_i u_i D_j u_j - D_j u_i D_i u_j) + \lambda^3 dtm(D_j u_i)$$

If this expression is limited to the first order term, it becomes

$$J \simeq 1 + \lambda D_i u_i \tag{2.1}$$

The linearized expression of the inverse matrix $(\partial_j a_i)$ is also useful. At the equilibrium, $\partial_j a_i = \delta_{ij}$. Thus its expression must look like :

$$\partial_j a_i = \delta_{ji} + \lambda \alpha_{ji} + O(\lambda^2)$$

The α_{ji} are obtained by solving the following equation :

$$\delta_{ki} = D_k x_j \partial_j a_i = (\delta_{kj} + \lambda D_k u_j)[\delta_{ji} + \lambda \alpha_{ji} + O(\lambda^2)]$$
$$= \delta_{ki} + \lambda D_k u_i + \lambda \alpha_{ki} + O(\lambda^2)$$

This gives

$$\delta_{ki} = -D_k u_i$$

and

$$\partial_j a_i \simeq \delta_{ji} - \lambda D_j u_i \tag{2.2}$$

1.2.3 Motion equations

The motions equations are

$$\rho_f \ddot{\lambda} u_i = \rho_f g_i + D_j t_{ji} \tag{2.3}$$

where ρ_f is the reference density, g_i the gravity acceleration vector and t_{ji} the Piola stresses of the fluid. For a non-viscous fluid, these stresses are the expression of the normal pressure which only works for a volume increase. Thus the virtual work corresponding to an elementary initial volume dV is

$$t_{ji} D_j \delta u_i dV = -p \, \delta(dV^*) = -p \, \delta J dV$$

If M_{ij} represents the cofactor of the $D_j x_i$ term of the jacobian matrix, the variation of the jacobian is written

$$\delta J = M_{ji} D_j \delta u_i = J \partial_i a_j D_j \delta u_i$$

in accordance with Cramer inversion formula

$$\partial_i a_j = \frac{1}{J} M_{ji}$$

and the expression of the virtual work equilibrium can be written :

$$t_{ji} D_j \delta u_i dV = -p J \partial_i a_j D_j \partial u_i dV$$

so

$$t_{ji} = -p J \partial_i a_j$$

This last expression can be linearized if the pressure is expressed as function of the pressure at the equilibrium p_0

$$p = p_0 + \lambda q$$

Using expressions (2.1) and (2.2), the following expression for the Piola stresses is obtained :

$$t_{ij} \simeq -(p_0 + \lambda q)(1 + \lambda D_k u_k)(\delta_{ij} - \lambda D_i u_j)$$
$$= -p_0 \delta_{ij} - \lambda[(q + p_0 D_k u_k)\delta_{ij} - p_0 D_i u_j]$$

The motion equation (2.3) becomes

$$-\rho_f \omega^2 \lambda u_i = \rho_f g_i - D_i p_0 - \lambda[D_i(q + p_0 D_k u_k) - D_j(p_0 D_i u_j)]$$

so
- for the zero order : $D_i p_0 = \rho_f g_i$ $\tag{2.4}$
- for the first order : $\rho_f \omega^2 u_i = D_i(q + p_0 D_k u_k) - D_j(p_0 D_i u_j)$ $\tag{2.5}$

1.2.4 The incompressibility condition

The incompressibility condition is expressed by

$$J = 1$$

The first order expression is

$$1 + \lambda D_i u_i \simeq 1$$

this leads to

$$D_i u_i = 0 \tag{2.6}$$

1.2.5 The displacement potential

Expanding equation (2.5) leads to

$$\rho_f \omega^2 u_i = D_i q + D_i p_0 D_k u_k + p_0 D_{ik} u_k - D_j p_0 D_i u_j - p_0 D_{ij} u_j$$

Including the results of equation (2.4), this gives

$$\rho_f \omega^2 u_i = D_i q - \rho_f g_i D_k u_k - \rho_f g_j D_i u_j$$

This expression can be further simplified using the incompressibility condition (2.6) :

$$\rho_f \omega^2 u_i = D_i(q - \rho_f g_j u_j)$$

A displacement potential φ can thus be defined

$$u_i = D_i \varphi \tag{2.7}$$

This potential is related to the pressure q by the following expression

$$q = \rho_f(g_j u_j + \omega^2 \varphi) = \rho_f(g_i D_i \varphi + \omega^2 \varphi) \tag{2.8}$$

Moreover it must satisfy the incompressibility condition

$$D_i u_i = \nabla^2 \varphi = 0 \tag{2.9}$$

1.2.6 The free surface condition

On the free surface, the pressure must be equal to zero. Thus
- for the zero order : $p_0|_\Gamma = 0$ $\hspace{2cm}$ (2.10)
- for the first order : $g_i D_i \varphi + \omega^2 \varphi = 0$ $\hspace{2cm}$ (2.11)

As the free surface remains horizontal, the gravity always acts perpendicularly to it and therefore

$$g_i D_i \varphi = -g \frac{\partial \varphi}{\partial n}$$

Taking into account the above remark, the condition can be expressed as

$$\begin{cases} \frac{\partial \varphi}{\partial n} = \eta \\ g\eta - \omega^2 \varphi = 0 \end{cases} \tag{2.12}$$

1.2.7 Conditions on the wet surface of the shell

The connexion condition between the fluid and the structure being very delicate, a separate section has been devoted to it. It can already be mentioned that concerning the potential, the normal displacement must be connected. The mathematical form of this condition is

$$\frac{\partial \varphi}{\partial n} = w_i n_i$$

on the shell. This condition even though intuitive is not obvious and further explanations are given below in section 1.4.

1.3 STRUCTURE DESCRIPTION

1.3.1 Introduction

It is clear that the structure analysis will be done as usual in a Lagrangian formalism. But it is essential to note that at the fluid-structure interface, the spatial coordinates x_i are the only continuous ones. The material coordinates are not continuous. Indeed, a particular point of the shell sees during the motion a certain number of fluid particles. The material coordinates related to the structure must be distinguished from the ones related to the fluid. b_i represents the coordinates of a point related to the shell at equilibrium. During the motion, the point undergoes a displacement λw_i and thus the instantaneous coordinates are

$$x_i = b_i + \lambda w_i$$

A derivative by b_i is noted ∇_i, ∂_i represents a derivative by x_i and $\mathcal{J}$ is the jacobian $\partial x / \partial b$.

1.3.2 Motion equations

The motion equations are similar to the equations used for the fluid :

$$\rho_s \ddot{\lambda} w_i = \rho_s g_i + \nabla_j t_{ji}$$

Here, however, the Kirchhoff-Trefftz stresses are preferred to the Piola ones. These stresses are related to each other by the following relation

$$t_{ji} \lambda D_j \delta w_i = s_{ji} \delta \gamma_{ij} = s_{jl}(\delta_{li} + \lambda \nabla_l w_i) \lambda \nabla_j \delta w_i$$

or

$$t_{ji} = s_{jl}(\delta_{li} + \lambda \nabla_l w_i)$$

The stresses are linearized as follows :

$$s_{ij} = s_{ij}{}^0 + \lambda \sigma_{ij}$$

The relation between the stresses is thus

$$t_{ji} \simeq s_{ji}{}^0 + \lambda(\sigma_{ji} + s_{jl}{}^0 \nabla_l w_i) \qquad (3.1)$$

or
- for the zero order : $\nabla_j s_{ji}{}^0 + \rho_s g_i = 0$ $\qquad (3.2)$
- for the first order : $-\rho_s \omega^2 w_i = \nabla_j(\sigma_{ji} + s_{jl}{}^0 \nabla_l w_i)$ $\qquad (3.3)$

1.3.3 Constitutive equations

A linear relation between the Kirchhoff-Trefftz stresses σ_{ij} and the Green tensor is adopted :

$$\lambda\sigma_{ij} = C_{ijkl}\,\gamma_{kl}$$

where

$$\gamma_{kl} = \frac{\lambda}{2}(\nabla_k w_l + \nabla_l w_k) + \frac{\lambda^2}{2}\nabla_k w_i \nabla_l w_i$$

Keeping the terms up to the first order, one obtains

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \tag{3.4}$$

where

$$\varepsilon_{kl} = \frac{1}{2}(\nabla_k w_l + \nabla_l w_k) \tag{3.5}$$

1.3.4 Variational principle for the structure

Multiplying equation (3.3) by δw_i and integrating it on the structure gives :

$$-\int_{V_s} \rho_s \omega^2 w_i \delta w_i dV_s - \int_{V_s} \nabla_j(\sigma_{ij} + s_{jl}{}^0\nabla_l w_i)\delta w_i dV_s = 0$$

Integrating by parts this equation gives

$$-\int_{V_s} \rho_s \omega^2 w_i \delta w_i dV_s - \int_S (\sigma_{ij} + s_{jl}{}^0\nabla_l w_i)n_j \delta w_i dS$$
$$+ \int_{V_s} (\sigma_{ij}\delta\varepsilon_{ij} + s_{jl}{}^0\nabla_l w_i \nabla_j \delta w_i)dV_s = 0$$

The second term represents the virtual work on the surface. On the fixed part of the surface S_0, this term cancels; on S_1 where the dead load f_1 are imposed, the virtual work has no first order term. Finally, the variational principle can be written

$$\delta\{\frac{1}{2}\int_{V_s}(C_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl} + s_{ij}{}^0\nabla_i w_k \nabla_j w_k - \rho_s\omega^2 w_i w_i)dV_s\} - \delta\tau_f = 0 \tag{3.6}$$

where $\delta\tau_f$ represents the virtual work on the wet surface.

1.4 THE FLUID-STRUCTURE CONNEXION

1.4.1 Introduction

It is now time to examine more precisely the connexion between the shell and the structure. Two things must be determined : the kinematic conditions and the equilibrium conditions.

1.4.2 Kinematic conditions

The exact condition is that during the motion, the fluid particle which stands in x_i on the shell must have a normal speed equal to the normal speed of the shell at this point. But as mentioned earlier, distinction between the material coordinates of the fluid and of the structure must be made. The fluid particle which is at coordinates x_i at time t was at the reference state at point a. The structure particle which is in contact with the fluid particle at time t was not necessarily in contact at the reference state. b represents the initial position of the structure particle. The kinematic condition is thus :

$$\dot{\lambda} u_i[\mathbf{a}(\mathbf{x})] n_i^*(x) = \dot{\lambda} w_i[\mathbf{b}(\mathbf{x})] n_i^*(\mathbf{x})$$

could the displacements of the shell w_i and the fluid u_i be both expressed in term of b and thus transform the kinematic expression into $[u_i(\mathbf{b}) - w_i(\mathbf{b})] n_i^* = 0$?
As at time t, the two particles are in contact,

$$x_i = a_i + \lambda u_i(\mathbf{a})$$
$$= b_i + \lambda w_i(\mathbf{b})$$

and thus

$$a_i - b_i = \lambda(w_i(\mathbf{b}) - u_i(\mathbf{a})) \tag{4.1}$$

Developing $u_i(\mathbf{a})$ in Taylor series around point b and limiting the series to the first order term

$$u_i(\mathbf{a}) = u_i(\mathbf{b}) + (D_j u_i)_b (a_j - b_j)$$
$$= u_i(\mathbf{b}) + (D_j u_i)_b \lambda(w_j(\mathbf{b}) - u_j(\mathbf{a}))$$

Using this development, the kinematic condition becomes :

$$\dot{\lambda}[u_i(\mathbf{b}) + \lambda(D_j u_i)_b(w_j(\mathbf{b}) - u_j(\mathbf{a})) - w_i(\mathbf{b})] n_i^* = 0$$

The second term of this expression is of second order. The linearized expression is thus :

$$[u_i(\mathbf{b}) - w_i(\mathbf{b})] n_i^* = 0$$

As the correction of the normal is of the second order too, the kinematic condition finally becomes :

$$[u_i(\mathbf{b}) - w_i(\mathbf{b})] n_i = 0$$

1.4.3 Work done by the pressure on the interface

The expression of the virtual work $\delta\tau_f$ done by the fluid pressure on the shell must be determined. Let b be a point of the shell of coordinates x_i at time t. The infinitesimal surface dS^* around this point undergoes a force equal to

$$p[\mathbf{a}(\mathbf{x})] n_i^* dS^*$$

where n_i^* is the moved normal pointing to the exterior of the fluid domain. The notation $p[\mathbf{a}(\mathbf{x})]$ reminds that the pressure at point b is due to the fluid particle initially located at point $\mathbf{a}(\mathbf{x})$ and not at point b. The pressure at point a is

$$p(\mathbf{a}) = p_0(\mathbf{a}) + \lambda q(\mathbf{a})$$

$p_0(\mathbf{a})=$ hydrostatic pressure at the equilibrium
$q(\mathbf{a}) =$ pressure coming from the perturbation

Combining expression (4.1) with the Taylor development of $u_i(\mathbf{a})$, the a_i coordinates can be expressed as

$$a_i = b_i + \lambda w_i(\mathbf{b}) - \lambda[u_i(\mathbf{b}) + (D_j u_i)_b \lambda(w_j(\mathbf{b}) - u_j(\mathbf{a}))]$$
$$= b_i + \lambda[w_i(\mathbf{b}) - u_i(\mathbf{b})] + O(\lambda^2)$$

Developing $p_0(\mathbf{a})$ and $q(\mathbf{a})$ in Taylor series gives

$$p_0(\mathbf{a}) = p_0(\mathbf{b}) + (D_i p_0)_b (a_i - b_i)$$
$$= p_0(\mathbf{b}) + \lambda (D_i p_0)_b (w_i(\mathbf{b}) - u_i(\mathbf{b})) + O(\lambda^2)$$
$$q(\mathbf{a}) = q(\mathbf{b}) + \lambda (D_i q)_b (w_i(\mathbf{b}) - u_i(\mathbf{b})) + O(\lambda^2)$$

The combination of both expressions gives the expression of the pressure $p(\mathbf{a})$

$$p(\mathbf{a}) = p_0(\mathbf{b}) + \lambda (D_i p_0)_b [w_i(\mathbf{b}) - u_i(\mathbf{b})] + \lambda q(\mathbf{b}) + O(\lambda^2)$$

Introducing the motion equation $D_i p_0 = \rho_f g_i$ and the relation between dynamic pressure q and potential φ : $q = \rho_f(g_i u_i + \omega^2 \varphi)$ in the above equation gives :

$$p(\mathbf{a}) = p_0(\mathbf{b}) + \lambda \rho_f g_i w_i(\mathbf{b}) + \lambda \rho_f \omega^2 \varphi \qquad (4.2)$$

The validity of this assertion is demonstrated considering for instance the rigid body rotation of the shell around a point such that the free surface remains still. In this case, when a point goes down, the pressure it undergoes increases in proportion to its vertical displacement, the fluid remaining still.

In order to evaluate $n_i^* dS^*$, let ψ be an arbitrary function defined in the structure and on its surface. Green's integration gives

$$-\int_{S^*} \psi n_i^* dS^* = \int_{V^*} \partial_i \psi dV^* = \int_V \mathcal{J} \partial_i b_j \nabla_j \psi dV$$

Making use of the classical Jacobi identity [9]

$$\nabla_j(\mathcal{J} \partial_i b_j) = 0$$

the integral on V becomes

$$\int_V \nabla_j(\mathcal{J} \partial_i b_j \psi) dV = -\int_S n_j \mathcal{J} \partial_i b_j \psi dS$$

As ψ was chosen arbitrary, the following expression stands :

$$n_i^* dS^* = \mathcal{J} \partial_i b_j n_j dS$$

As
$$\mathcal{J} = 1 + \lambda \nabla_k w_k + O(\lambda^2)$$
$$\partial_i b_j = \delta_{ij} - \lambda \nabla_i w_j + O(\lambda^2)$$
this last expression becomes :

$$n_i^* dS^* = [n_i + \lambda(\nabla_k w_k n_i - \nabla_i w_j n_j)] dS \qquad (4.3)$$

The coefficient multiplying λ in this expression can be transformed into

$$\nabla_p w_q n_r (\delta_{pq} \delta_{ri} - \delta_{pi} \delta_{qr}) = e_{jpr} e_{jqi} n_r \nabla_p w_q \qquad (4.4)$$

This term involves only tangential derivatives, which is in agreement with common sense. Gathering the results from equations (4.2),(4.3) and (4.4) gives

$$\mathcal{A} = \int_{S_f^*} p(\mathbf{a}) n_i^* \delta w_i dS^*$$
$$= \int_{S_f} [p_0 + \lambda \rho_f g_k w_k + \lambda \rho_f \omega^2 \varphi][n_i + \lambda e_{jpr} e_{jqi} n_r \nabla_p w_q] \delta w_i dS$$
$$= \int_{S_f} p_0 n_i \delta w_i dS + \lambda \int_{S_f} [\rho_f \omega^2 \varphi n_i \delta w_i + \rho_f g_k w_k \delta w_i n_i + p_0 e_{jpr} e_{jqi} n_r \nabla_p w_q \delta w_i] dS \qquad (4.5)$$

The last term of this expression can be integrated by parts according to Stokes-Ampere's theorem. As p_0 is null on $S_f \cap \Gamma$, the integral over this part of the boundary vanishes.

$$\frac{1}{2} \int_{S_f} p_0 e_{jpr} e_{jqi} n_r \nabla_p w_q \delta w_i dS = -\frac{1}{2} \int_{S_f} e_{jpr} e_{jqi} n_r \nabla_p p_0 w_q \delta w_i dS$$

$$-\frac{1}{2} \int_{S_f} p_0 e_{jpr} e_{jqi} n_r w_q \nabla_p \delta w_i dS \qquad (4.6)$$

▷ The first integral of the right-hand side can be transformed into

$$= -\frac{1}{2} \int_{S_f} (\delta_{pq} \delta_{ri} - \delta_{pi} \delta_{rq}) \rho_f g_p n_r w_q \delta w_i dS$$

$$= -\frac{1}{2} \int_{S_f} \rho_f g_p w_p n_i \delta w_i dS - \frac{1}{2} \int_{S_f} \rho_f g_p \delta w_p n_r w_r dS \qquad (4.7)$$

▷ The second integral is equivalent to

$$= -\frac{1}{2} \int_{S_f} p_0 (\delta_{pq} \delta_{ri} - \delta_{pi} \delta_{rq}) n_r w_q \nabla_p \delta w_i dS$$

$$= -\frac{1}{2} \int_{S_f} p_0 (n_i w_p \nabla_p \delta w_i - n_r w_r \nabla_i \delta w_i) dS \qquad (4.8)$$

▷ Finally, the second half of the last term of the right-hand side of (4.5) is explicitly

$$= -\frac{1}{2} \int_{S_f} p_0 (\delta_{pq} \delta_{ri} - \delta_{pi} \delta_{qr}) n_r \nabla_p w_q \delta w_i dS$$

$$= -\frac{1}{2} \int_{S_f} p_0 (n_r \delta w_r \nabla_p w_q - n_q \delta w_i \nabla_i w_q) dS \qquad (4.9)$$

Gathering the results from (4.5) to (4.9), $\mathcal{A}$ takes the form

$$\mathcal{A} = \int_{S_f} p_0 n_i \delta w_i dS + \lambda \int_{S_f} \left[\rho_f \omega^2 \varphi n_i \delta w_i + \frac{1}{2} \rho_f g_k w_k \delta w_i n_i + \frac{1}{2} \rho_f g_k \delta w_k w_i n_i \right.$$

$$- \frac{1}{2} p_0 (n_i w_j \nabla_j \delta w_i + n_i \delta w_j \nabla_j w_i) + \frac{1}{2} p_0 (n_i w_i \nabla_j \delta w_j + n_i \delta w_i \nabla_j w_j) \bigg] dS$$

$$= \int_{S_f} p_0 n_i \delta w_i dS + \lambda \int_{S_f} \rho_f \omega^2 \varphi n_i \delta w_i dS$$

$$+ \frac{\lambda}{2} \delta \int_{S_f} \left[\rho_f g_k w_k n_i w_i + p_0 (n_i w_i \nabla_j w_j - n_i w_j \nabla_j w_i) \right] dS$$

The first term is of order zero and thus of no interest in this linearized study. The second term is the expression of the virtual work $\delta \tau_f$:

$$\delta \tau_f = -\delta U_f + \int_{S_f} \rho_f \omega^2 \varphi n_i \delta w_i dS$$

where U_f is the energy introduced by the fluid pressure

$$U_f = -\frac{1}{2} \int_{S_f} \left[\rho_f g_k w_k n_i w_i + p_0 (n_i w_i \nabla_j w_j - n_i w_j \nabla_j w_i) \right] dS \qquad (4.10)$$

The structure accepts thus the variational equation

$$\delta U(w) - \omega^2 \delta T(w) - \delta P_1(w) - \omega^2 \int_{S_f} \rho_f \varphi n_i \delta w_i dS = 0 \qquad (4.11)$$

with

$$U = U_e + U_g + U_f$$

where

$$U_e = \frac{1}{2} \int_{V_s} C_{ijkl} \, \varepsilon_{ij}(w)\varepsilon_{kl}(w)dV_s$$

$$U_g = \frac{1}{2} \int_{V_s} s_{ij}{}^0 \nabla_i w_k \nabla_j w_k dV_s$$

$$U_f = (4.10)$$

So on the wet surface of the structure S_f, the influence of the fluid is a coupling between φ and $n_i w_i$ and a supplementary potential energy involving only *the structural displacements*.

Three facts argue for an expression including only the structural displacements.
  (a) Similar terms have been obtained in 1978 using a direct energy approach [8] when the fluid-structure interaction module of SAMCEF was developed.
  (b) Morand [10] arrives at the same terms. His integral including the p_0 term is

$$\frac{1}{2} \int_{S_f} p_0 \mathbf{n_1}(\mathbf{w})\mathbf{w} dS$$

where $\mathbf{n_1}$ is defined by

$$n^* dS^* = [\mathbf{n} + \mathbf{n_1}(\mathbf{w}) + O(w^2)]dS$$

This implies, using (4.3) that

$$(n_1)_i = \nabla_k w_k n_i - \nabla_i w_j n_j$$

So the present formulation, the SAMCEF formulation, as well as the one found by Morand are identical and will be called the modified Tong's formulation.
  (c) A detailed study of the behaviour of rigid body modes of fluid-structure systems shows that the modified Tong's formulation leads to results in perfect agreement with the physics of the problem.

1.4.4 Variational principle for the fluid-structure system

Integrating the free surface equation on its domain and taking its variation gives

$$\delta \int_\Gamma \frac{1}{2} \rho_f g \eta^2 dS - \omega^2 \int_\Gamma \rho_f \varphi \delta \eta dS = 0 \qquad (4.12)$$

Inside the fluid, the variational principle is

$$\delta \int_{V_f} \frac{1}{2} \rho_f D_i \varphi D_i \varphi dV - \int_{S_f} \rho_f n_i w_i \delta \varphi dS - \int_\Gamma \rho_f \eta \delta \varphi dS = 0 \qquad (4.13)$$

Adding together equations (4.11),(4.12) and substracting from this sum equation (4.13) multiplied by ω^2 gives :

$$\delta\left(U + \frac{1}{2} \int_\Gamma \rho_f g \eta^2 dS\right) - \omega^2 \delta\left(T - \frac{1}{2} \int_{V_f} \rho_f D_i \varphi D_i \varphi dV + \int_{S_f} \rho_f n_i w_i \varphi dS\right.$$

$$\left. + \int_\Gamma \rho_f \eta \varphi dS\right) = 0 \qquad (4.14)$$

This is the modified Tong's principle [6,8,10]. Can φ be eliminated from the principle ? A priori not because the Neumann problem for the potential does not admit a unique solution. Thus a certain linear form of φ must be set to zero (in the discretized form of the principle, the potential is set to zero at one point). But doing this, the results from the variation $\delta\varphi = C^{ste}$ is lost. This variation is

$$c^{ste}\left\{ \int_{S_f} \rho_f n_i w_i dS + \int_\Gamma \rho_f \eta dS\right\} = 0 \qquad (4.15)$$

which is precisely the existence condition of the Neumann problem. In compensation of this fixation, it is necessary to insure the global incompressibility with a Lagrangian multiplier λ. The principle becomes

$$\delta\{(U + \frac{1}{2}\int_\Gamma \rho_f g\eta^2 dS) - \omega^2(T - \frac{1}{2}\int_{V_f} \rho_f D_i\varphi D_i\varphi dV + \int_{S_f} \rho_f n_i w_i \varphi dS$$

$$+ \int_\Gamma \rho_f \eta\varphi dS) + \lambda(\int_{S_f} \rho_f n_i w_i dS + \int_\Gamma \rho_f \eta dS)\} = 0 \qquad (4.16)$$

The elimination of φ gives a relation

$$\varphi = \Phi(n_i w_i, \eta) \qquad (4.17)$$

This particular potential can also be chosen as $\delta\varphi$ in (4.13). This gives

$$\int_{V_f} \rho_f D_i\varphi D_i\varphi dV = \int_{S_f} \rho_f \varphi n_i w_i dS + \int_\Gamma \rho_f \varphi\eta dS$$

and finally, the principle takes the form :

$$\delta\{(U + \frac{1}{2}\int_\Gamma \rho_f g\eta^2 dS) - \omega^2(T + \frac{1}{2}\int_{S_f} \rho_f \varphi n_i w_i dS + \frac{1}{2}\int_\Gamma \rho_f \eta\varphi dS)$$

$$+ \lambda(\int_{S_f} \rho_f n_i w_i dS + \int_\Gamma \rho_f \eta dS)\} = 0$$

if condition (4.17) is fulfilled. This condition must only be known on S_f and Γ as the principle above indicates. One way to get this condition is with an integral method.

1.4.5 Discretized form of the variational principle

The shell displacement w, the potential φ and the free surface normal displacement η can be respectively discretized by

$$w = W(x)q \qquad\qquad \varphi = a^T(x)f \qquad\qquad \eta = b^T(x)y \qquad (4.18)$$

The different terms of the principle take the discretized forms :

$$U = \frac{1}{2}q^T K_s q \qquad\qquad \text{where } K_s = 2U(W)$$

$$\frac{1}{2}\int_\Gamma \rho_f g\eta^2 dS = \frac{1}{2}y^T K_f y \qquad\qquad \text{where } K_f = \int_\Gamma \rho_f g bb^T dS$$

$$T = \frac{1}{2}q^T M_s q \qquad\qquad \text{where } M_s = 2T(W)$$

$$\frac{1}{2}\int_{S_f} \rho_f \varphi n_i w_i dS = \frac{1}{2}f^T Aq \qquad\qquad \text{where } A = \int_{S_f} \rho_f a n_i W_i dS$$

$$\frac{1}{2}\int_\Gamma \rho_f \eta\varphi dS) = \frac{1}{2}f^T By \qquad\qquad \text{where } B = \int_\Gamma \rho_f ab^T dS$$

If f represents the particular e vector where all the elements are equal to 1

$$\int_{S_f} \rho_f n_i w_i dS = e^T Aq$$

$$\int_\Gamma \rho_f \eta dS = e^T By$$

The discretized principle takes the form :

$$\delta\{\frac{1}{2}(q^T K_s q + y^T K_f y) - \frac{\omega^2}{2}(q^T M_s q + f^T Aq + f^T By) + \lambda(e^T Aq + e^T By)\} = 0$$

The relation (4.17) which gives the potential as a function of the normal displacements on the boundary takes the following discretized form :

$$f = Qq + Yy \tag{4.19}$$

This can be introduced in the above principle :

$$f^T Aq = q^T Q^T Aq + y^T Y^T Aq = q^T \frac{1}{2}(Q^T A + A^T Q)q + y^T Y^T Aq$$

$$f^T Bq = q^T Q^T By + y^T Y^T By = q^T Q^T By + y^T \frac{1}{2}(Y^T B + B^T Y)y$$

and it becomes

$$\delta\{\frac{1}{2}(q^T K_s q + y^T K_f y) - \frac{\omega^2}{2}(q^T M_{qq}q + 2y^T M_{yq}q + y^T M_{yy}y) + \lambda(e^T Aq + e^T By)\} = 0$$

where

$$M_{qq} = M_s + \frac{1}{2}(Q^T A + A^T Q)$$

$$M_{yq} = \frac{1}{2}Y^T A + \frac{1}{2}B^T Q$$

$$M_{yy} = \frac{1}{2}(Y^T B + B^T Y)$$

In matrix form, this can be written

$$\begin{bmatrix} K_s & 0 & Ae \\ 0 & K_f & Be \\ e^T A & e^T B & 0 \end{bmatrix} \begin{bmatrix} q \\ y \\ \lambda \end{bmatrix} - \omega^2 \begin{bmatrix} M_{qq} & M_{qy} & 0 \\ M_{yq} & M_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ y \\ \lambda \end{bmatrix} = 0$$

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2. ON THE REPRESENTATION OF RIGID BODY MODES WITH THE SAMCEF FORMULATION FOR HYDROELASTIC PROBLEMS

2.1 INTRODUCTION

The question to see wether the SAMCEF formulation of hydroelastic problems respects the rigid body modes is often raised. It appears very often that some of the rigid rotation modes are lost for the free-free problem. The present report tries to give better understanding of this problem.

2.2 POSITION OF THE FREE-FREE PROBLEM

The hydroelastic problems differ from the conventional elastic problem in the unavoidable role the gravity acceleration takes. Consequently, a relaxed state does not exist. At equilibrium, reactions materialized in practice by the engine thrust inevitably appear.

At equilibrium, the structure is submitted to the stresses s_{ij}^0 and the fluid is submitted to the hydrostatic pressure p_0. These efforts comes from the gravity and must be balanced by forces $\bar{t}_i$ on a portion of its surface S_2. If V_f represents the fluid volume, V_s the structure domain and S_f the interface between them, the equilibrium conditions can be written under the variational form as follows :

- for the structure

$$\int_{V_s} s_{ij}^0 D_j \delta u_i dV - \int_{V_s} \rho_s g_i \delta u_i dV + \int_{S_f} p_0 n_i^* \delta u_i dS - \int_{S_2} \bar{t}_i \delta u_i dS = 0$$

where n_i^* is the normal to the wet surface and pointing towards the exterior of *the shell*.

- for the fluid

$$\int_{V_f} p_0 D_i \delta u_i dV - \int_{V_f} \rho_f g_i \delta u_i dV + \int_{S_f} p_0 n_i \delta u_i dS = 0$$

where n_i is the normal to the wet surface, but pointing towards the exterior of the *fluid*.

Of course a direct relation exists between the normals : $n_i^* = -n_i$ and both conditions can advantageously be added together in order to cancel the term involving the pressure on surface S_f :

$$\int_{V_s} s_{ij}^0 D_j \delta u_i dV + \int_{V_f} p_0 D_i \delta u_i dV - \int_{V_s} \rho_s g_i \delta u_i dV - \int_{V_f} \rho_f g_i \delta u_i dV - \int_{S_2} \bar{t}_i \delta u_i dS = 0$$

For a rigid virtual displacement of the general form :

$$\delta u_i = \delta a_i + \delta w_{ij} x_j \qquad \delta w_{ij} = -\delta w_{ji} \qquad\qquad (2.1)$$

the equilibrium condition becomes :

$$\int_{V_s} s_{ij}^0 \delta w_{ij} dV + \int_{V_f} p_0 \delta w_{ii} dV - \int_{V_s} \rho_s g_i (\delta a_i + \delta w_{ij} x_j) dV$$

$$- \int_{V_f} \rho_f g_i (\delta a_i + \delta w_{ij} x_j) dV - \int_{S_2} \bar{t}_i (\delta a_i + \delta w_{ij} x_j) dS = 0$$

The first two integrals are strictly null because on one hand s_{ij}^0 is symmetric and δw_{ij} anti-symmetric, and on the other hand $\delta w_{ii} = 0$.
Separating the terms in δa_i and in δw_{ij} gives

$$\int_{V_s} \rho_s g_i dV + \int_{V_f} \rho_f g_i dV + \int_{S_2} \bar{t}_i dS = 0 \qquad (2.2)$$

(translation equilibrium)

$$\int_{V_s} \rho_s g_i x_j dV + \int_{V_f} \rho_f g_i x_j dV + \int_{S_2} \bar{t}_i x_j dS = 0 \qquad i \neq j \qquad (2.3)$$

(rotation equilibrium)

2.3 BEHAVIOUR OF THE ELASTIC ENERGY TERM

The linear elastic energy

$$U_l = \int_{V_s} \frac{1}{2} C_{ijkl} \, \epsilon_{ij} \epsilon_{kl} dV$$

is automatically equal to zero for a rigid body mode.

2.4 BEHAVIOUR OF THE GEOMETRIC STIFFNESS ENERGY TERM

The term

$$U_g = \frac{1}{2} \int_{V_s} s_{ij}^0 D_i u_m D_j u_m dV$$

is equivalent to

$$U_g = \frac{1}{2} \int_{S_f} n_i^* s_{ij}^0 u_m D_j u_m dS + \frac{1}{2} \int_{S_2} n_i s_{ij}^0 u_m D_j u_m dS - \frac{1}{2} \int_{V_s} D_i s_{ij}^0 u_m D_j u_m dV$$

$$- \frac{1}{2} \int_{V_s} s_{ij}^0 D_{ij} u_m dV$$

For a rigid displacement, $D_{ij} u_m = 0$

Taking into account the following conditions :

$$\begin{array}{ll} n_i^* s_{ij}^0 = -p_0 n_j^* & \text{on } S_f \\ n_i s_{ij}^0 = \bar{t}_j & \text{on } S_2 \\ D_i s_{ij}^0 = -\rho_s g_j & \text{in } V_s \end{array}$$

the geometric stiffness term becomes

$$U_g = -\frac{1}{2} \int_{S_f} p_0 n_j^* u_m D_j u_m dS + \frac{1}{2} \int_{S_2} \bar{t}_j u_m D_j u_m dS + \frac{1}{2} \int_{V_s} \rho_s g_j u_m D_j u_m dV$$

If the rigid displacement is of the general form

$$u_i = a_i + w_{ij} x_j \qquad w_{ij} = -w_{ji}$$

It can also be written as follows :

$$U_g = -\frac{1}{2} \int_{S_f} p_0 n_j^* u_m w_{mj} dS + \frac{1}{2} \int_{S_2} \bar{t}_i u_m w_{mj} dS + \frac{1}{2} \int_{V_s} \rho_s g_j u_m w_{mj} dV$$

2.5 BEHAVIOUR OF THE HYDROSTATIC PRESSURE TERM

The general form of this term

$$U_p = -\frac{1}{2} \int_{S_f} p_0(n_i u_i D_j u_j - n_i u_j D_j u_i)dS$$

simpliflies into

$$\frac{1}{2} \int_{S_f} p_0 n_i u_j D_j u_i dS = \frac{1}{2} \int_{S_f} p_0 n_i u_j w_{ij} dS = -\frac{1}{2} \int_{S_f} p_0 n_i u_j w_{ji} dS$$

in the rigid motion case.
It balances exactly the first term of U_g.

2.6 BEHAVIOUR OF TONG'S TERM

Tong's term has for expression :

$$U_T = -\frac{1}{2} \int_{S_f} \rho_f g_k u_k n_i u_i dS$$

Extending the shell displacement inside the fluid domain which is obviously possible for a rigid body mode, this term becomes

$$U_T = \frac{1}{2} \int_{\Gamma} \rho_f g_k u_k n_i u_i dS - \frac{1}{2} \int_{V_f} \rho_f g_k u_k D_i u_i dV - \frac{1}{2} \int_{V_f} \rho_f g_k u_i D_i u_k dV$$

So for a rigid body mode, if the orientation of the gravity acceleration **g** and the following relation on Γ, $n_i u_i = u_3$ are taken into account, the U_T term transforms as follows :

$$U_T = -\frac{1}{2} \int_{\Gamma} \rho_f g u_3^2 dS + \frac{1}{2} \int_{V_f} \rho_f g_j u_i w_{ij} dV$$

2.7 BEHAVIOUR OF THE FREE SURFACE TERM

Its expression is

$$U_\Gamma = \frac{1}{2} \int_{\Gamma} \rho_f g \eta^2 dS$$

Regarding the term of the stiffness including the normal displacement of the free surface η, the constraint necessary to insure the incompressibility of the fluid must be added with the associated Lagrangian multiplier :

$$\frac{1}{2} \int_{\Gamma} \rho_f g \eta^2 dS + \lambda \left(\int_{S_f} \rho_f n_i u_i dS + \int_{\Gamma} \rho_f \eta dS \right)$$

The variation of η gives

$$\rho_f g \eta + \lambda \rho_f = 0$$

This expression leads to a uniform value of η on the free surface given by $\eta = -\dfrac{\lambda}{g}$. To determine this constant value of η, one can transform the linear constraint into :

$$\eta \Gamma = - \int_{S_f} n_i u_i dS$$

where Γ is here the *area* of Γ.
As a rigid body mode is always incompressible, the volume conservation can also be expressed by :

$$- \int_{S_f} n_i u_i dS = \int_{\Gamma} u_3 dS$$

Thus

$$\eta = \frac{1}{\Gamma} \int_{\Gamma} u_3 dS$$

which means that η is *the mean displacement* u_3 on the free surface.

2.8 BEHAVIOUR OF THE TOTAL POTENTIAL ENERGY

The total potential energy is :

$$U = U_l + U_g + U_p + U_T + U_\Gamma = \frac{1}{2}\int_{S_2} \bar{t}_j u_i w_{ij}\,dS + \frac{1}{2}\int_{V_s} \rho_s g_j u_i w_{ij}\,dV$$
$$+\frac{1}{2}\int_{V_f} \rho_f g_j u_i w_{ij}\,dV + \frac{1}{2}\int_{\Gamma} \rho_f g(\eta^2 - u_3^2)\,dS$$

Let us consider first the terms which do not involve the free surface. If U_Γ^* represents the last term of the equation above, the equation can be written as :

$$U - U_\Gamma^* = \frac{1}{2}\int_{S_2} \bar{t}_j(a_i + w_{ik}x_k)w_{ij}\,dS + \frac{1}{2}\int_{V_s} \rho_s g_j(a_i + w_{ik}x_k)w_{ij}\,dV$$
$$+\frac{1}{2}\int_{V_f} \rho_f g_j(a_i + w_{ik}x_k)w_{ij}\,dV$$
$$= \frac{1}{2}a_i w_{ij}[\int_{S_2} \bar{t}_j\,dS + \int_{V_s} \rho_s g_j\,dV + \int_{V_f} \rho_f g_j\,dV]$$
$$+\frac{1}{2}w_{ik}w_{ij}[\int_{S_2} \bar{t}_j x_k\,dS + \int_{V_s} \rho_s g_j x_k\,dV + \int_{V_f} \rho_f g_j x_k\,dV]$$

According to the equilibrium conditions (2.2) and (2.3), the $a_i w_{ij}$ coefficients are equal to zero. The $w_{ik}w_{jk}$ terms however are only null when $j \neq k$. In order to examine the $j = k$ terms, the classical notation here below is used

$$w_1 = w_{23} \quad , \quad w_2 = w_{31} \quad , \quad w_3 = w_{12}$$

Moreover, as $\mathbf{g} = -g\mathbf{e_3}$ the following expression is true

$$\int_{V_s} \rho_s g_3 x_3\,dV + \int_{V_f} \rho_f g_3 x_3\,dV = -Mgz_G$$

where M is the overall mass of the system and Z_g the height of the center of mass. The expression can thus be transformed into :

$$U - U_\Gamma^* = \frac{1}{2}(w_2^2 + w_3^2)\int_{S_2} \bar{t}_1 x_1\,dS + \frac{1}{2}(w_1^2 + w_3^2)\int_{S_2} \bar{t}_2 x_2\,dS$$
$$+\frac{1}{2}(w_1^2 + w_2^2)[\int_{S_2} \bar{t}_3 x_3\,dS - Mgz_G]$$

A simple explanation of the different terms of this expression can be found if a distinction is made between the $\bar{t}_i$ which act in the direction of the e_i axis noted $\bar{t}_i^+$ in the remainder and the ones which act in the opposite direction noted $\bar{t}_i^-$. The resulting forces are distinguished too

$$F_i^+ = \int_{S_2} \bar{t}_i^+\,dS \quad , \quad F_i^- = \int_{S_2} |\bar{t}_i^-|\,dS$$

and are applied to the coordinates :

$$X_i^+ = \frac{1}{F_i^+}\int_{S_2} \bar{t}_i^+ x_i\,dS \quad , \quad X_i^- = \frac{1}{F_i^-}\int_{S_2} |\bar{t}_i^-| x_i\,dS$$

The expression thus becomes

$$U - U_\Gamma^* = \frac{1}{2}(w_2^2 + w_3^2)(F_1^+ X_1^+ - F_1^- X_1^-) + \frac{1}{2}(w_1^2 + w_3^2)(F_2^+ X_2^+ - F_2^- X_2^-)$$
$$+\frac{1}{2}(w_1^2 + w_2^2)(F_3^+ X_3^+ - F_3^- X_3^- - Mgz_G)$$

Consider the first term of this equation. For a rotation ω_2, the load F_1^+ moves to

$$X_1^+ \cos w_2 \approx X_1^+ (1 - \frac{w_2^2}{2})$$

while the load F_1^- moves to

$$X_1^- \cos w_2 \approx X_1^- (1 - \frac{w_2^2}{2})$$

The resulting variation of energy is given by

$$\frac{w_2^2}{2}(F_1^+ X_1^+ - F_1^- X_1^-)$$
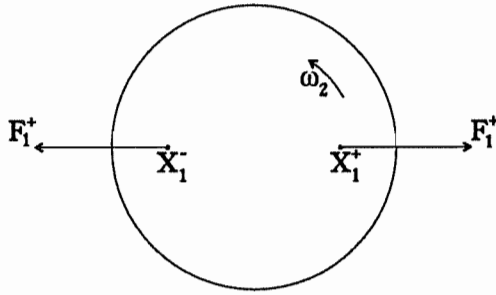
which is positive for the case represented in the above figure and becomes negative if $X_1^+ < X_1^-$. This comes from the fact that the load position is stable in the case represented and unstable in the other one. The term including $(w_1^2 + w_2^2)$ is of similar type.

In particular, for a *filled* tank held by a pivot at height X_3 to be in a stable position, the center of mass must be located below the pivot axis. These conclusions are natural.
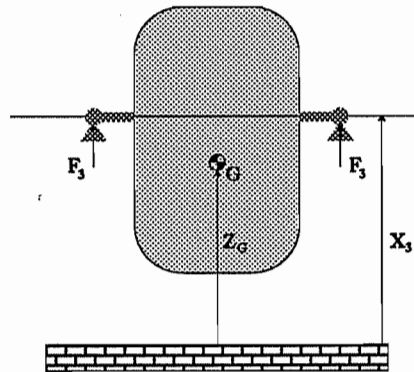
Let us examine the free surface term. If the fluid is frozen, the displacements would be equal to u_3. In practice, the free surface settles down to an horizontal position at the mean value of u_3.

If h represents the free surface height at the equilibrium point, an added potential energy equal to the weight of the fluid above the height h multiplied by the mean height can be associated to an elementary surface dS : $\rho g u_3 dS(h + \frac{u_3}{2})$
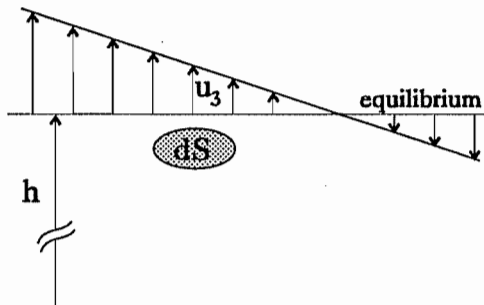
Integrating this elementary potential energy on the free surface gives the total energy

$$EP(u_3) = \int_\Gamma \rho g u_3 (h + \frac{u_3}{2}) dS$$

In the η position, the potential energy takes the similar form

$$EP(\eta) = \int_\Gamma \rho g \eta (h + \frac{\eta}{2}) dS$$

The difference between these two is

$$EP(\eta) - EP(u_3) = \int_\Gamma \rho g h(\eta - u_3) dS + \frac{1}{2} \int_S \rho g (\eta^2 - u_3^2) dS = U_\Gamma^*$$

because the first term is null as η is the mean value of u_3 on the free surface. This supplementary term of the potential energy has a destabilizing effect because the horizontal position of the free surface always corresponds to a minimum of the potential energy.

2.9 CONCLUSION

In most current problems, $F_1^+ = F_1^- = F_2^+ = F_2^- = 0$ and the rigid mode of rotation around the vertical axis is usually conserved. For the other two rotation modes, the potential energy is usually not equal to zero. The contribution of the free surface is negative, and the one from the interior of the fluid depends upon the center of rotation, quite difficult to determine a priori.

2.10 REMARKS ON THE PRACTICAL APPLICATION

The above conclusions assumes naturally that the geometric stiffness U_g and the pressure term U_p are both correctly calculated. That is not necessarily the case. The conclusions are still valid if U_p and U_g are *exact for a rigid mode*. In particular, for the U_p evaluation, the introduction of an average pressure on the element is *not sufficient*. The case of U_g is more complex. Let us examine the simple case of the plate. For a rigid mode

$$\gamma_{11}^{(2)} = \frac{1}{2}[(D_1u_1)^2 + (D_1u_2)^2 + (D_1u_3)^2] = \frac{1}{2}(w_{21}^2 + w_{31}^2)$$

$$\gamma_{22}^{(2)} = \frac{1}{2}[(D_2u_1)^2 + (D_2u_2)^2 + (D_2u_3)^2] = \frac{1}{2}(w_{12}^2 + w_{23}^2)$$

$$\gamma_{33}^{(2)} = \frac{1}{2}[(D_3u_1)^2 + (D_3u_2)^2 + (D_3u_3)^2] = \frac{1}{2}(w_{13}^2 + w_{23}^2)$$

$$\gamma_{13}^{(2)} = \frac{1}{2}[D_1u_1D_3u_1 + D_1u_2D_3u_2 + D_1u_3D_3u_3] = \frac{1}{2}w_{21}w_{23}$$

$$\gamma_{23}^{(2)} = \frac{1}{2}[D_2u_1D_3u_1 + D_2u_2D_3u_2 + D_2u_3D_3u_3] = \frac{1}{2}w_{12}w_{13}$$

$$\gamma_{12}^{(2)} = \frac{1}{2}[D_1u_1D_2u_1 + D_1u_2D_2u_2 + D_1u_3D_2u_3] = \frac{1}{2}w_{31}w_{32}$$

As the equilibrium is obtained assuming $\sigma_{33}^0 = 0$, the geometric stiffness takes the form

$$\frac{1}{2}\int_S [N_{11}^0(w_{21}^2 + w_{31}^2) + N_{22}^0(w_{12}^2 + w_{23}^2) + 2N_{12}^0 w_{31}w_{32} + 2Q_1^0 w_{21}w_{23} + 2Q_2^0 w_{12}w_{13}]dS$$

with

$$w_{21} = \frac{1}{2}(D_1u_2 - D_2u_1) \quad , \quad w_{31} = \frac{1}{2}(D_1u_3 - D_3u_1) \quad , \quad w_{32} = \frac{1}{2}(D_2u_3 - D_3u_2)$$

or any other equivalent expression as for a rigid mode, the following relations apply

$$D_1u_2 + D_2u_1 = 0 \quad , \quad D_1u_3 + D_3u_1 = 0 \quad , \quad D_2u_3 + D_3u_2 = 0$$

In particular, using $w_{31} = D_1u_3, \quad w_{32} = D_2u_3$, the expression of U_g becomes

$$U_g = \frac{1}{2}\int_S \{N_{11}^0[(D_1u_3)^2 + \frac{1}{4}(D_1u_2 - D_2u_1)^2]$$

$$+ N_{22}^0[(D_2u_3)^2 + \frac{1}{4}(D_1u_2 - D_2u_1)^2] + 2N_{12}^0 D_1u_3D_2u_3$$

$$- Q_1^0(D_1u_2 - D_2u_1)D_2u_3 - Q_2^0(D_2u_1 - D_1u_2)D_1u_3\}dS$$

Thus the classical Bryan's terms

$$U_g(Bryan) = \frac{1}{2}\int_S [N_{11}^0(D_1u_3)^2 + N_{22}^0(D_2u_3)^2 + 2N_{12}^0 D_1u_3D_2u_3]dS$$

are *not sufficient*. On the other hand, the stresses can be approximated by the mean stresses over the volume as the rotations are constant for a rigid mode.

2.11 CASE WHERE U_p AND U_g ARE NEGLECTED

U_p and U_g are often neglected. In this case, the energy expression simplifies into

$$U = U_T + U_\Gamma = \frac{1}{2}\int_{V_f} \rho_f g_j u_i w_{ij}\, dV + \frac{1}{2}\int_\Gamma \rho_f g(\eta^2 - u_3^2)\, dS$$

For a rigid translation mode where $w_{ij} = 0$ and $\eta = u_3$, the energy is equal to zero. For a rotation w_{12} around the vertical axis, $\eta = u_3 = 0$, $g_2 = 0$ and consequently the energy is also null. However, for a rotation around the axis e_1 or e_2,

$$U - U_\Gamma^* = -\frac{1}{2}(w_1^2 + w_2^2)M_f g z_{Gf}$$

where M_f is the *fluid mass* and Z_{Gf} its center of mass. This term cancels if the rotation is made around the fluid center of mass. Furthermore, on Γ, $u_3 = -w_1 x_2 + w_2 x_1$ and consequently $\eta = -w_1 \bar{x}_2 + w_2 \bar{x}_1$, if $\bar{x}_1$ and $\bar{x}_2$ represent respectively the mean values of x_1 and x_2 on the free surface. U_Γ^* takes the form

$$U_\Gamma^* = -\frac{1}{2}w_1^2\int_\Gamma \rho_f g(x_2^2 - \bar{x}_2{}^2)dS - \frac{1}{2}w_2^2\int_\Gamma \rho_f g(x_1^2 - \bar{x}_1{}^2)dS$$

$$+ w_1 w_2 \int_\Gamma \rho_f g(x_1 x_2 - \bar{x}_1 \bar{x}_2)dS$$

which can be written

$$U_\Gamma^* = -\frac{1}{2}w_1^2\int_\Gamma \rho_f g(x_2 - \bar{x}_2)^2 dS - \frac{1}{2}w_2^2\int_\Gamma \rho_f g(x_1 - \bar{x}_1)^2 dS$$

$$+ w_1 w_2 \int_\Gamma \rho_f g(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)dS$$

This expression is negative definite and values of z_G such that certain rotations do not have any energy exist

$$\begin{vmatrix} I_2 + z_G & -I_{12} \\ -I_{12} & I_1 + z_G \end{vmatrix} = 0$$

with the following notation

$$I_2 = \frac{1}{M_f g}\int_\Gamma \rho_f g(x_2 - \bar{x}_2)^2 dS$$

$$I_1 = \frac{1}{M_f g}\int_\Gamma \rho_f g(x_1 - \bar{x}_1)^2 dS$$

$$I_{12} = \frac{1}{M_f g}\int_\Gamma \rho_f g(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)dS$$

Explicitly, this determinant is

$$z_G^2 + z_G(I_1 + I_2) + I_1 I_2 - I_{12}^2 = 0$$

the roots are given by

$$z_G = \frac{-(I_1 + I_2) \pm \sqrt{(I_1 + I_2)^2 - 4I_1 I_2 + 4I_{12}^2}}{2}$$

$$= \frac{-(I_1 + I_2) \pm \sqrt{(I_1 - I_2)^2 + 4I_{12}^2}}{2}$$

For circular or square tanks where $I_1 = I_2 = I$ and $I_{12} = 0$ the center of mass height is $z_G = -\frac{I}{2}$. If the tanks are hung such that this value is obtained, the energy of rotation will be null. This

hanging point is located *above* the center of mass of the fluid. As this point does not coincide with the center of rotation of the rigid modes orthogonalised to the four preceeding ones, the energy can either be positive or negative. A good idea of the behaviour of the system can be made for the simple case of a spherical tank (modern water tower). The fluid always tends to align its center of mass below the center of the sphere. This system is similar to a pendulum hung at point O where the destabilizing effect is clearly seen.
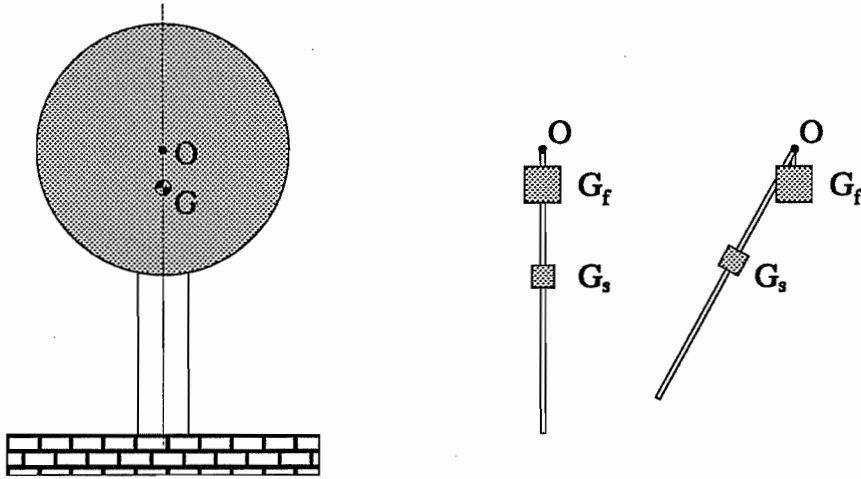
2.12 REFERENCES

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2. J.F. DEBONGNIE   *On a purely lagrangian formulation of sloshing and fluid-induced vibrations of tanks.* Comp. Meth. Appl. Mech. and Engng., 58 (1986), pp. 1-18

3. J.F. DEBONGNIE   *Sur la représentation des modes rigides dans la formulation SAMCEF des problèmes hydroélastiques.* rapport L.T.A.S. VF-68 , juin 1990

3. DYNAMIC ANALYSIS
   OF FLUID-STRUCTURE
   PROBLEMS : RESULTS

3.1 INTRODUCTION

The validation of the RAYON code for fluid-structure interactions calculations will be made through two different sets of tests.

The first one includes the study of the sloshing modes for a set of tanks. For some of them i.e. cylindrical and cubic tanks, an analytical solution to the equations governing the sloshing modes can easily be found due to the simple formulation of the boundary conditions. The mathematical developments which lead to these particular solutions are included in section 3.3.1 and a comparison between these and the computation is made in order to evaluate the accuracy achievable with the finite element calculation.

The second set of tests investigates the frequency domain of hydroelastic modes for the tanks used in the first part of the study.

The problems selected for these tests are coming from the intermediate report referenced ESA/90/04 provided by STRACO.

3.2 MODELISATION OF COUPLED FLUID-STRUCTURE PROBLEMS USING THE FINITE ELEMENT APPROACH

3.2.1 Introduction

The complexity of the modelisation of fluid-structure problems comes from the modelisation of the interfaces i.e. interface between the fluid and the shell on one hand and the fluid and the free surface on the other hand. Indeed an interface element is necessary to insure the relationship between the displacement potential inside the fluid domain and the field of displacement in the structure domain. Similarly, the interface element between the free surface and the fluid insures the coupling between the potential inside the fluid and the displacement on its free boundary. As the degree of discretisation for the fluid potential needs to be higher than the degree of the displacement field, the creation of these interface elements is a tedious task for the user. The pre-processor of the dynamic analysis module (DYNAM) takes care of this work.

The introduction in the data file including the modelisation parameters of the ".FLU" command mentions to the program that the model includes both structure and fluid elements. The pre-processor then creates from a simplified modelisation of the fluid and the modelisation of the structure the complete model necessary to run its analysis. This task includes the creation of additional interface nodes on the fluid side, the creation of the interface elements as well as the creation of the "tanks nodes" (nodes to which the Lagrange multipliers which take into account the incompressibility constraints are associated). A node renumbering is then performed to keep the frontwidth to an acceptable size. This procedure is detailed in the fourth user's manual of SAMCEF.

Different types of elements are available in the SAMCEF library to model the fluid and the structure. A brief description of the elements adopted for the models is made in the following sections.

3.2.2 Fluid elements

▷ Isoparametric volume (referenced in SAMCEF as elements 61, 93 and 94).

The fluid element is an isoparametric "brick" element entirely defined by its eight corner nodes. The displacement potential ϕ field is approximated in the element by means of a polynomial of degree "l" ranging from 2 to 4. The field parameters are expressed as functions of the local values of potential on the corner nodes and on $l-1$ equidistant nodes on each edge. Therefore the potential can be expressed by

$$\phi = \sum_{i=1}^{n} N_i \ \phi_i$$

where

N_i are the interpolation functions corresponding to the n corner or interface nodes of the element.

ϕ_i are the components of the potential at corner or interface nodes i.

The degree of the fluid is the degree desired for the potential. If not specified by the user, it is one degree higher than the overall degree of the problem. A representation of the three types of elements is given below for fluid elements of degree 3.
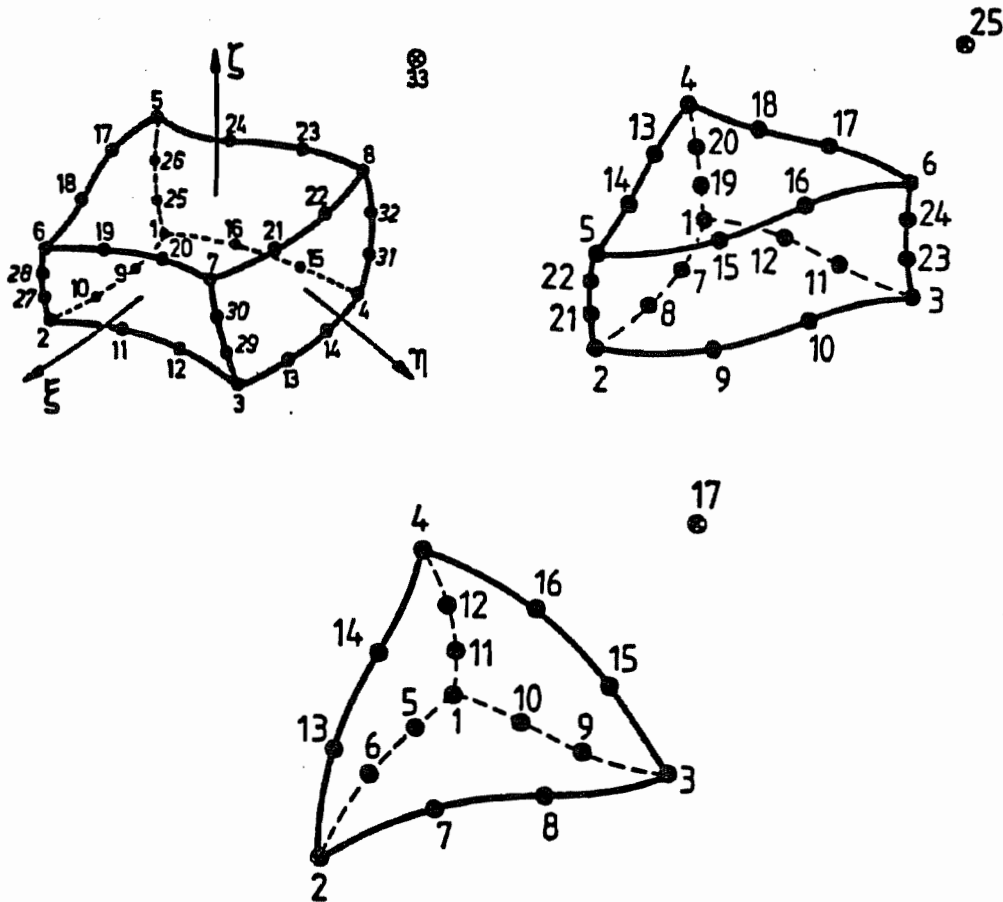
figure 3.2.1

fluid type elements of degree 3 (types 61, 94, 93)

▷ Toroid with triangular section (referenced in SAMCEF as element 41)

This element is designed for the study of systems with geometry of revolution. The Fourier expansion offers an efficient way to solve such problems.
A double discretisation of the structure is done :
- Circonferentially, the field is expanded in a Fourier series.

- In the meridian plane, each coefficient is discretized in finite elements in the classical way.

In the Fourier expansion, the components of the displacements and rotations are expressed in cylindrical coordinates. Two types of developments are used for the displacements u and the rotations ϕ : a cosine development for u_r, u_z and ϕ_θ and a sine development for u_θ, ϕ_r and ϕ_z :

$$a(r, z, \theta) = \sum_{n=0}^{\infty} \sum_{m=0}^{1} a^{nm}(r, z) \quad \cos(n\theta + m\frac{\pi}{2})$$

$$b(r, z, \theta) = \sum_{n=0}^{\infty} \sum_{m=0}^{1} b^{nm}(r, z) \quad \sin(n\theta + m\frac{\pi}{2})$$

For each term [n,m] of the development, the amplitudes u_r^{nm}, u_z^{nm}, u_θ^{nm}, ϕ_r^{nm}, ϕ_z^{nm} et ϕ_θ^{nm} are discretized in finite elements.

It might be useful to remind the physical meaning of the different [n,m] analyses † :

Analysis [0,0]	purely axisymmetric breathing and elongation aloung oz
Analysis [0,1]	torsion
Analysis [1,0]	bending about oy
Analysis [1,1]	bending about ox
Analyses n > 1	different breathing modes

In the particular case of element 41, the displacement potential is decomposed as a Fourier series :

$$\phi(r, z, \theta) = \sum_{n=0}^{\infty} \sum_{m=0}^{1} \phi^{nm}(r, z) \quad \cos(n\theta + m\frac{\pi}{2})$$

For each harmonics, a separate study is performed where ϕ^{nm} is given for an element of degree k $(k = 1, 2, 3, 4)$ by :

$$\phi^{nm}(r, z) = \sum_{l=0}^{k} \sum_{j=0}^{l} \alpha_{jl} \quad r^j \quad z^{l-j}$$

3.2.3 Structure elements

▷ Hybrid triangle and quadrangle from Marguerre (referenced in SAMCEF as elements 56 and 55)

These are non-conforming elements which superpose a Marguerre membrane with an hybrid plate :
- The plate is a hybrid element obtained from Kirchhoff's theory and defined by the reference quadrilateral. The reference plane includes the first node of the element and has for normal the cross product of the vectors joigning the mid-points of opposite sides.
- The membrane is a kinematic hypo-parametric element from the Marguerre theory of "serendipity" type.

The elements of degree two have three d.o.f. for each corner node and five d.o.f. for the interface nodes. These interface degrees of freedom are the mean displacements and the mean symmetric and antisymmetric normal rotations. The two elements are represented below with their d.o.f. for degree 2 elements.

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† These couples [n,m] are presented in the frequencies tables which appear in the following sections with the modes they are associated to
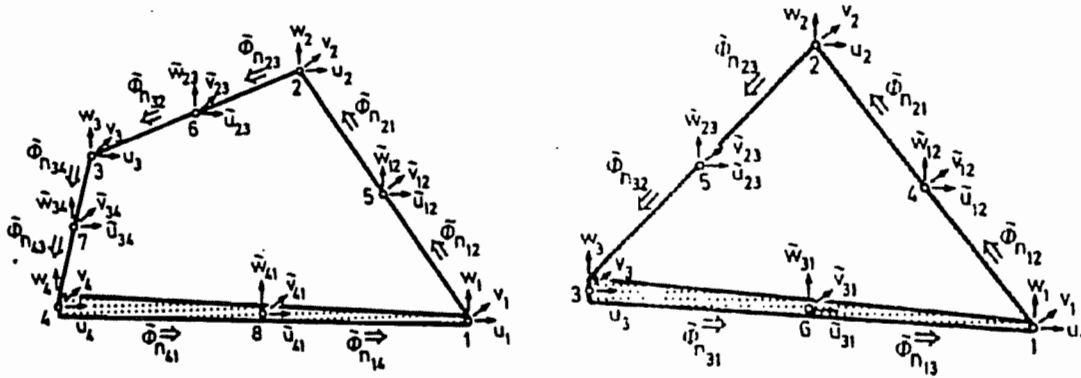
figure 3.2.2

Structure type elements of degree 2 (types 56, 55)

▷ conical axisymmetric shell (referenced in SAMCEF as element 18)

It is a two-node element where the displacement are calculated as a function of the displacement of the middle line. In the local axis defined in the figure below, these displacements can be expressed as :

$$\begin{cases} u_{x'}^{nm}(x',z') & = & u^{nm}(x') & +z' & \alpha^{nm}(x') \\ u_{\theta}^{nm}(x',z') & = & v^{nm}(x') & +z' & \beta^{nm}(x') \\ u_{z'}^{nm}(x',z') & = & w^{nm}(x') \end{cases}$$

where u^{nm}, v^{nm} and w^{nm} are the displacement fields and α^{nm}, β^{nm} are the rotation fields. A third rotation is added to insure the connexion between elements.
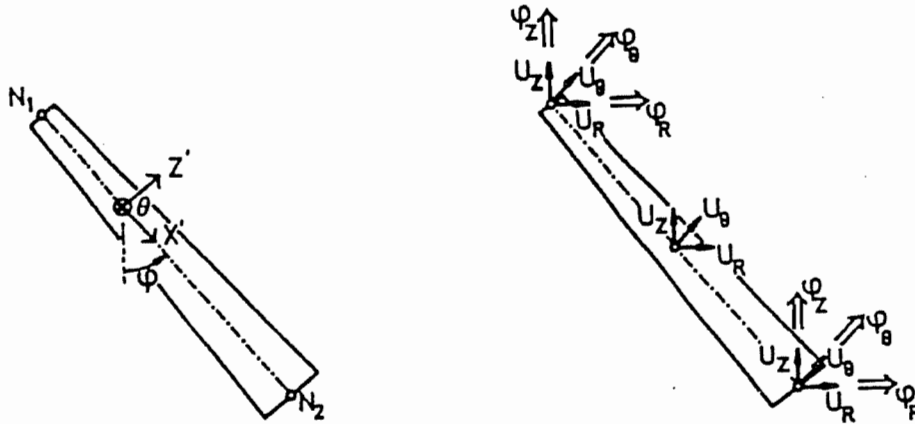
figure 3.2.3

Local axes and d.o.f. of Fourier structure type element (type 18)

3.3 RESULTS

3.3.1 Analytical results for two simple sloshing problems

▷ Parallelepipedic tank

a. Motion equations

$$\Delta\phi = 0$$
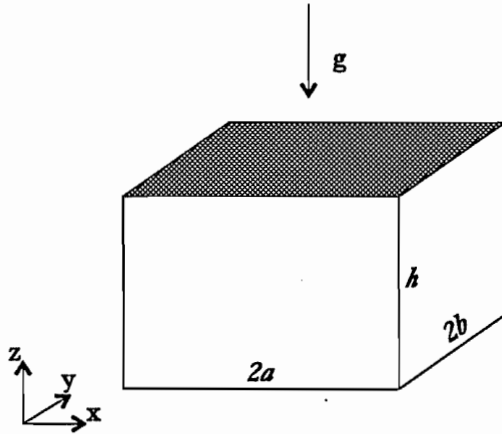
b. Boundary conditions

null displacement along the rigid walls :

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=\pm a} = 0$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=\pm b} = 0$$

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = 0$$

free surface equation :

$$g\frac{\partial\phi}{\partial z} + \left.\frac{\partial^2\phi}{\partial t^2}\right|_{z=h} = 0$$

c. Solution

The potential takes the form :

$$\phi^{mn}(x, y, z) = cos[\frac{n\pi}{2a}(x + a)] \ cos[\frac{m\pi}{2b}(y + b)] \ \frac{cosh[K^{mn}(z + h)]}{cosh[K^{mn}h]}$$

and the pulsation :

$$\omega^{mn\,2} = K^{mn} \quad g \quad tanh(K^{mn}h)$$

where

$$K^{mn} = \sqrt{(\frac{n\pi}{2a})^2 + (\frac{m\pi}{2b})^2}$$

for $m, n = 0, 1, 2, 3, \ldots$

▷ Cylindrical tank

a. Motion equations

$$\Delta\phi = 0$$
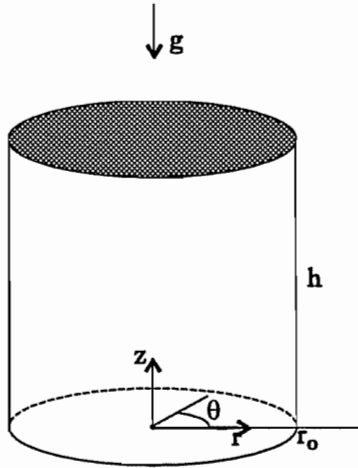
b. Boundary conditions

null displacement along the rigid walls :

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = 0$$

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=r_0} = 0$$

free surface equation :

$$g\frac{\partial\phi}{\partial z} + \left.\frac{\partial^2\phi}{\partial t^2}\right|_{z=h} = 0$$

3. Solution

- If the initial perturbation is such that the revolution symmetry is kept during the motion :
  The potential takes the form :

$$\phi(r,z,t) = \sum_{m=0}^{\infty} \alpha_m sin(\omega_m t) \; cosh(\frac{y+h}{r_0}\lambda_m) \; J_0\left(\frac{\lambda_m r}{r_0}\right)$$

where the λ_m are given by

$$J_1'(\lambda_m) = 0$$

and the pulsation :

$$\omega_m^2 = \frac{\lambda^m}{r_0} \; g \; tanh(\frac{\lambda_m h}{r_0})$$

- If the initial perturbation is such that the revolution symmetry is suppressed during the motion :
  The potential takes the form :

$$\phi(r,z,t) = \sum_{m=0}^{\infty} \alpha_m sin(\omega_m t) \; cosh(\frac{y+h}{r_0}\lambda_m) \; J_1\left(\frac{\lambda_m r}{r_0}\right) \; cos\theta$$

where the λ_m are given by

$$J_0'(\lambda_m) = -J_1(\lambda_m) = 0$$

and the pulsation :

$$\omega_m^2 = \frac{\lambda^m}{r_0} \; g \; tanh(\frac{\lambda_m h}{r_0})$$

3.3.2 Sloshing modes

Problem 1 :	Sloshing modes of a cubic tank
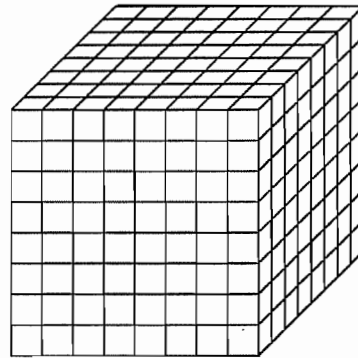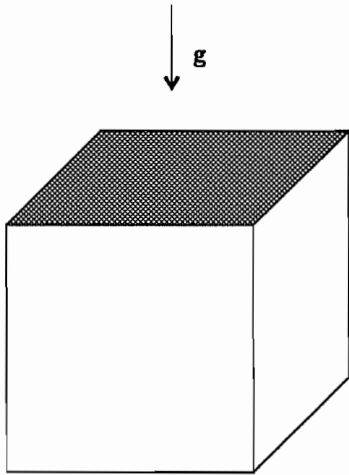
- Geometry.
  tank side : 0.2 m

- Physical properties.
  fluid density : 1000 kg/m^3
  gravity acceleration : 10 m/s^2

- Boundary counditions.
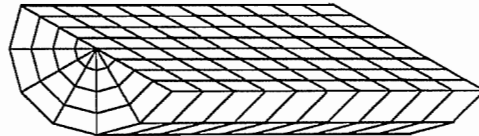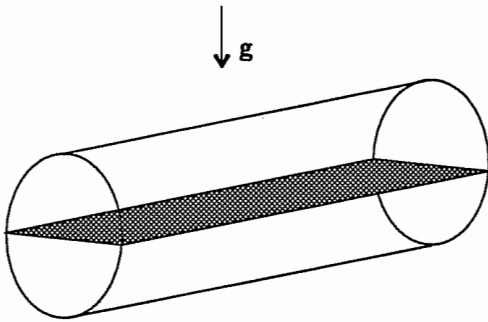 rigid walls

- Model characteristics.
  fluid elements of degree 3
  free surface elements of degree 2
  number of d.o.f. : 4842

- Results.

Modes	Frequencies (Hz)	
	FE model	Analytical
1	1.9909	1.9909
2	1.9909	1.9909
3	2.3718	2.3718
4	2.8209	2.8209
5	2.8209	2.8209

| Modes	Frequencies (Hz)	
	FE model	Analytical
6	2.9828	2.9828
7	2.9828	2.9828
8	3.3549	3.3547
9	3.4556	3.4549
10	parasitic	3.4549

• Mode shapes visualisation.
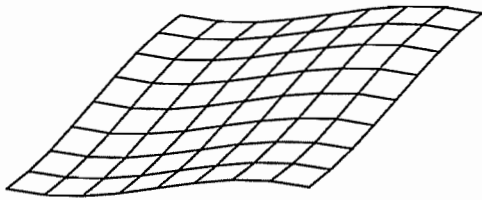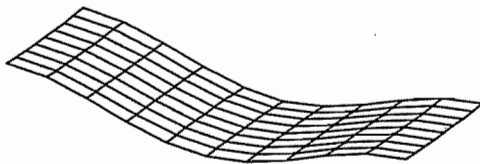

figure 3.3.1
Mode 1
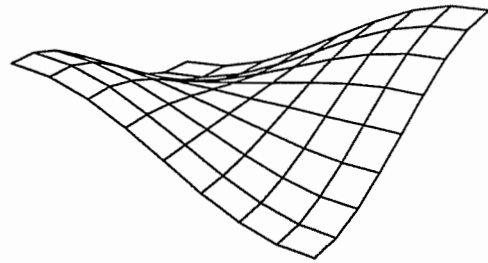
figure 3.3.2
Mode 3
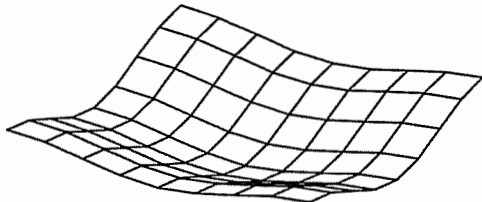
figure 3.3.3
Mode 4
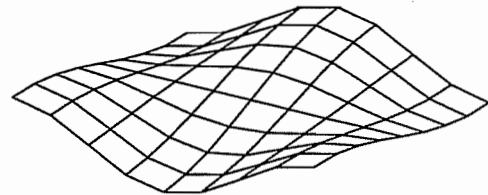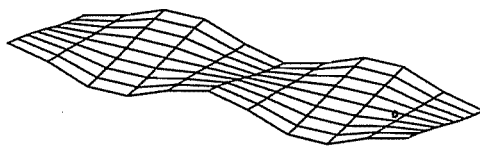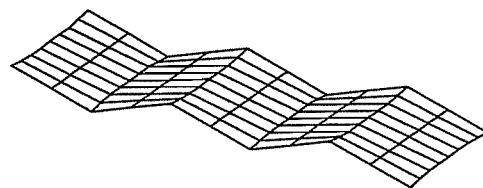
figure 3.3.4
Mode 6
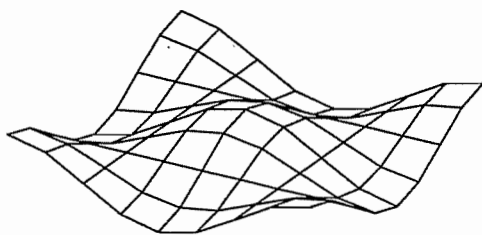
figure 3.3.5
Mode 8
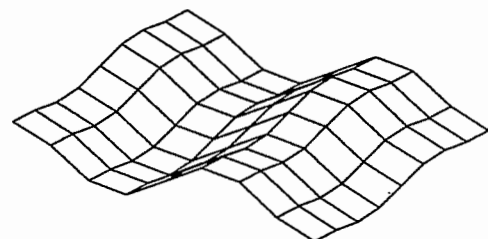
figure 3.3.6
Mode 9

Problem 2 :   Sloshing modes of a cubic tank with horizontal baffle
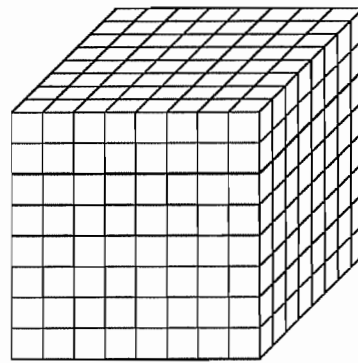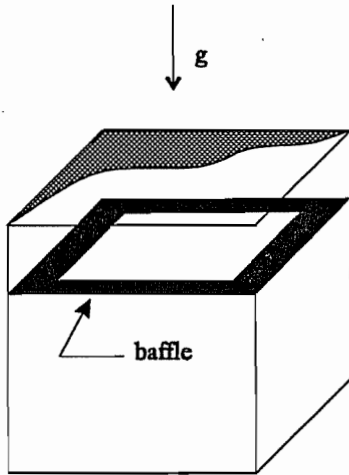
- Geometry.

    tank side : 0.2 m
    baffle fixed at 3/4 of the height
    baffle width : 0.025 and 0.05 m

- Physical properties.

    fluid density : 1000 kg/m^3
    gravity acceleration : 10 m/s^2

- Boundary counditions.

    rigid walls
    rigid baffle

- Model properties.

    fluid elements of degree 3
    free surface elements of degree 2
    number of d.o.f. : 5309

- Results.

	Frequencies (Hz)			Frequencies (Hz)	
	Baffle (0.025m)	Baffle (0.05m)		Baffle (0.025m)	Baffle (0.05m)
Modes	FE model	FE model	Modes	FE model	FE model
1	1.9029	1.7319	6	2.9409	2.9069
2	1.9029	1.7319	7	2.9413	2.9071
3	2.2773	2.1564	8	3.3351	3.3209
4	2.7745	2.7158	9	3.4425	3.4348
5	2.7770	2.7171	10	*parasitic*	3.4348

● Mode shapes visualisation (baffle width equal to 0.05m).
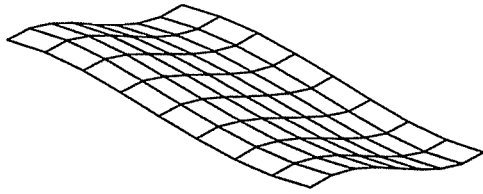
figure 3.3.7

Modes 1 and 2
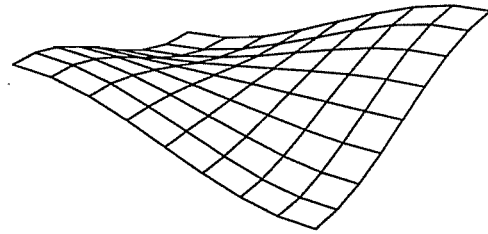
figure 3.3.8

Mode 3
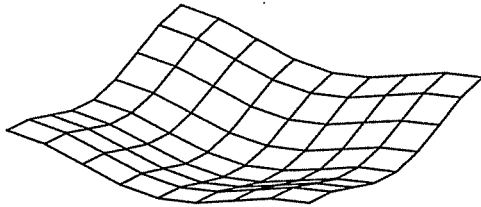
figure 3.3.9

Modes 4 and 5
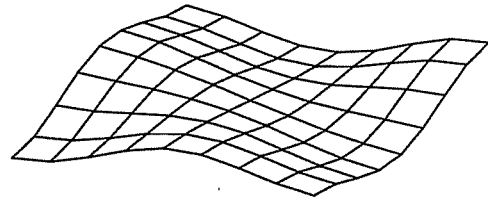
figure 3.3.10

Modes 6 and 7
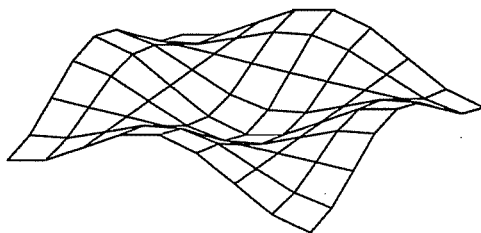
figure 3.3.11

Mode 8
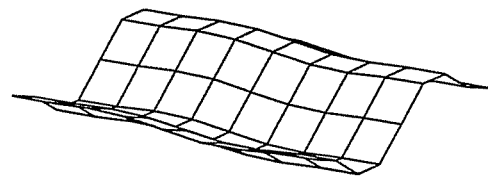
figure 3.3.12

Modes 9 and 10

Problem 3 :   Sloshing modes of a partitioned tank
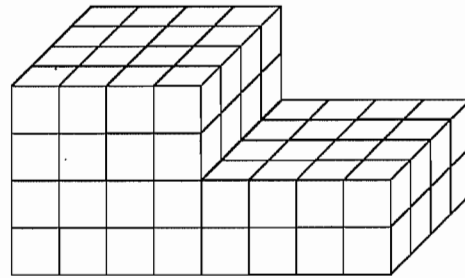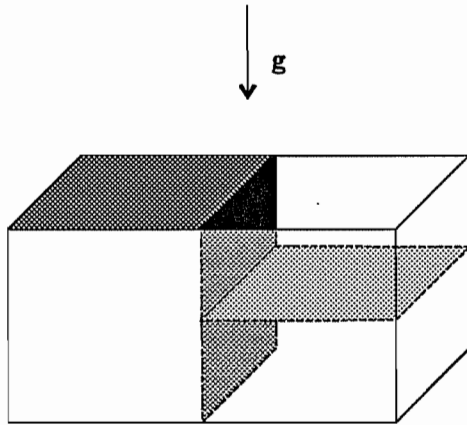
- Geometry.

  tank length : 0.4 m
  tank width : 0.2 m
  fluid height in the left part : 0.2 m
  fluid height in the right part : 0.1 m

- Physical properties.

  fluid density : 1000 kg/m^3
  gravity acceleration : 10 m/s^2

- Boundary counditions.

  rigid walls

- Model characteristics.

  fluid elements of degree 3
  free surface elements of degree 2
  number of d.o.f. : 1270

- Results.

	Frequencies (Hz)	
Modes	FE model	Analytical
1	1.9103	1.9103
2	1.9134	1.9103
3	1.9910	1.9909
4	1.9936	1.9909
5	2.3462	2.3444

	Frequencies (Hz)	
Modes	FE model	Analytical
6	2.3734	2.3718
7	2.8187	2.8157
8	2.8223	2.8157
9	2.8239	2.8209
10	2.8274	2.8209

Problem 4 : Sloshing modes of a parallelepipedic tank with a vertical baffle

— plate
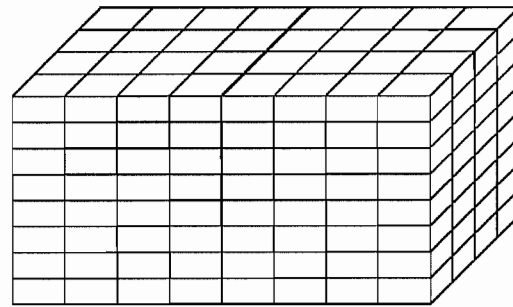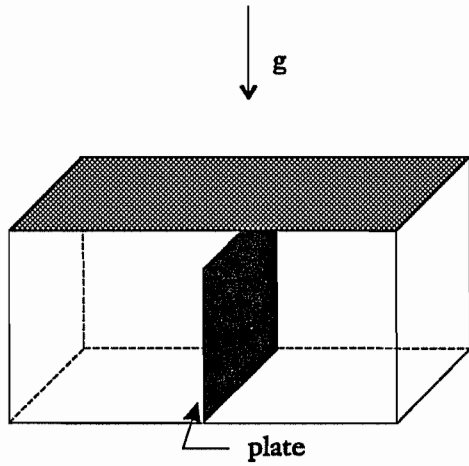
- Geometry.

    tank length : 0.4 m
    tank width : 0.2 m
    fluid height : 0.2 m
    baffle height : 0.15 m

- Physical properties.

    . fluid density : 1000 kg/m^3
    gravity acceleration : 10 m/s^2

- Boundary counditions.

    rigid walls
    rigid baffle

- Model characteristics.

    fluid elements of degree 3
    free surface elements of degree 2
    number of d.o.f. : 2855

- Results.

Modes	Frequencies (Hz) FE model
1	1.1162
2	1.9905
3	1.9910
4	1.9936
5	2.3462

Modes	Frequencies (Hz) FE model
6	2.3734
7	2.8187
8	2.8223
9	2.8239
10	2.8274

Problem 5 :	Sloshing modes of a cylindrical tank
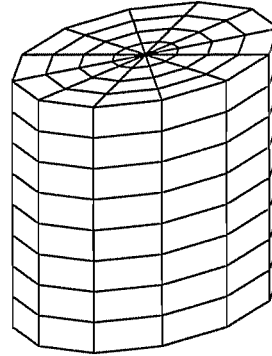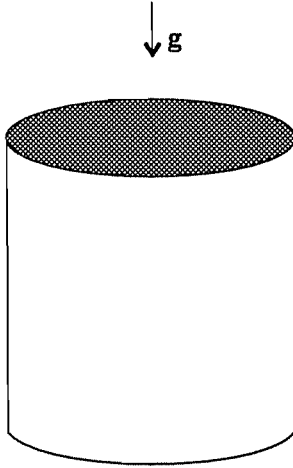
• Geometry.

   tank radius : 0.1 m

   tank height : 0.2 m

• Physical properties.

   fluid density : 1000 kg/m^3

   gravity acceleration : 10 m/s^2

• Boundary counditions.

   rigid walls

• Model characteristics.

   fluid elements of degree 3

   free surface elements of degree 2

   number of d.o.f. : 3098

• Results.

		Frequencies (Hz)		
Modes		3-D model	Fourier	Analytical
1,2	$[1,0],[1,1]$	2.1830	2.1582	2.1582
3,4	$[2,0],[2,1]$	2.8126	2.7814	2.7814
5	$[0,0]$	3.1704	3.1155	3.1154
6,7	$[3,0],[3,1]$	3.2992	3.2622	3.2622
8,9	$[4,0],[4,1]$	3.7171	3.6701	3.6701
10,11	$[1,0],[1,1]$	3.7360	3.6753	3.6749

Problem 6 :	Sloshing modes of an horizontal cylindrical tank

- Geometry.

    tank length : 0.6 m
    tank radius : 0.1 m

- Physical properties.

    fluid density : 1000 kg/m^3
    gravity acceleration : 10 m/s^2

- Boundary counditions.

    rigid walls

- Model characteristics.

    fluid elements of degree 3
    free surface elements of degree 2
    number of d.o.f. : 2320

- Results.

	Frequencies (Hz)
Modes	FE model
1	0.7011
2	1.3025
3	1.7798
4	1.8280
5	1.8800

	Frequencies (Hz)
Modes	FE model
6	2.0184
7	2.1599
8	2.2091
9	2.4220
10	2.4759

● Mode shapes visualisation.
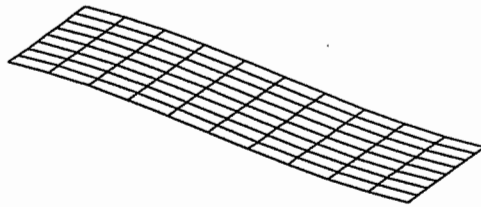
figure 3.3.13
Mode 1

figure 3.3.14
Mode 2
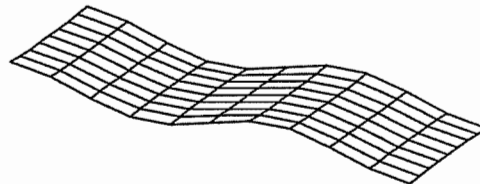
figure 3.3.15
Mode 3
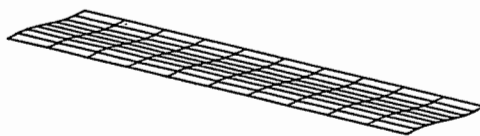
figure 3.3.16
Mode 4
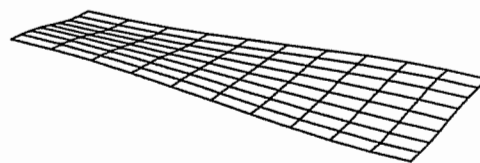
figure 3.3.17
Mode 5

- Mode shapes visualisation.
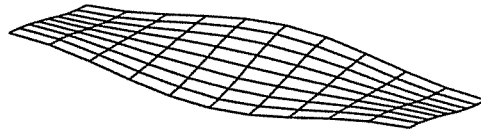
figure 3.3.18
Mode 6
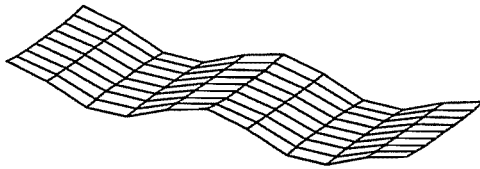
figure 3.3.19
Mode 7
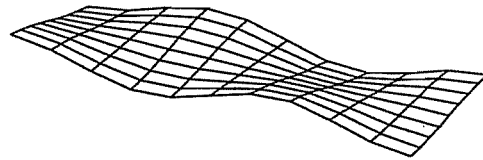
figure 3.3.20
Mode 8

figure 3.3.21
Mode 9

figure 3.3.22
Mode 10

Problem 7 :   Sloshing modes of a partially filled spherical tank
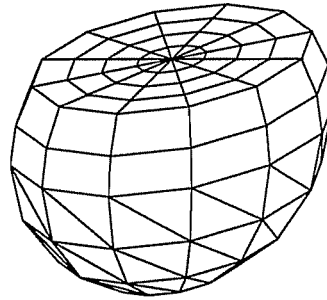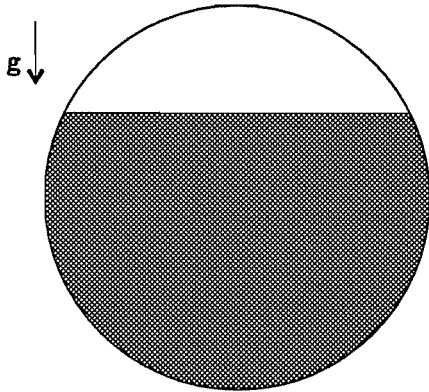
- Geometry.

  tank radius : 0.1 m

  fluid height : 0.15 m

- Physical properties.

  fluid density : 1000 kg/m^3

  gravity acceleration : 10 m/s^2

- Boundary counditions.

  rigid walls

- Model characteristics.

  fluid elements of degree 3

  free surface elements of degree 2

  number of d.o.f. : 2468

- Results.

Modes		Frequencies (Hz)	
		3-D model	Fourier
1, 2	[1, 0], [1, 1]	2.4639	2.4364
3, 4	[2, 0], [2, 1]	3.1937	3.1597
5	[0, 0]	3.4811	3.4051
6, 7	[3, 0], [3, 1]	3.7381	3.6965
8, 9	[1, 0], [1, 1]	4.0852	4.0115
10, 11	[4, 0], [4, 1]	4.2028	4.1454

Problem 8 :   Sloshing modes of a cubic tank with a vertical baffle

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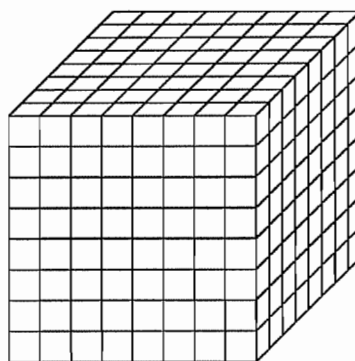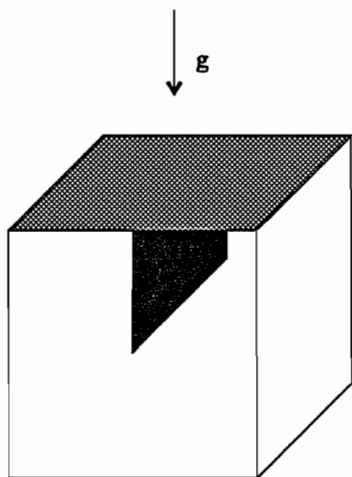
- Geometry.

  tank side : 0.2 m
  baffle height : 0.1 m
  baffle thickness : 1 mm

- Physical properties.

  fluid density : 1000 kg/m^3
  gravity acceleration : 10 m/s^2
  shell density : 7800 kg/m^3
  Young modulus : 2.1e11 N/m^2
  Poisson coefficient : 0.3

- Boundary couditions.

  rigid walls

- Model characteristics.

  fluid elements of degree 3
  free surface elements of degree 2
  number of d.o.f. : 5546

- Results.

Modes	Frequencies (Hz) FE model
1	1.3054
2	1.9909
3	2.0107
4	2.8000
5	2.8209

Modes	Frequencies (Hz) FE model
6	2.8209
7	2.8216
8	2.9717
9	2.9828
10	3.3543

● Mode shapes visualisation.
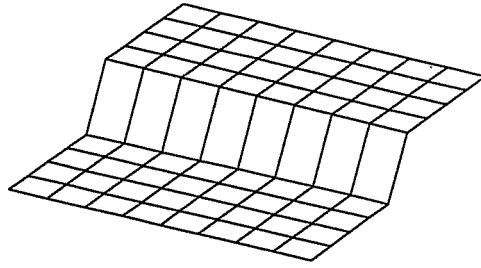
figure 3.3.23
Mode 1
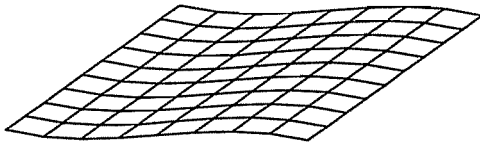
figure 3.3.24
Mode 2
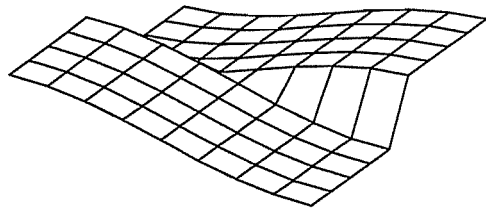
figure 3.3.25
Mode 3
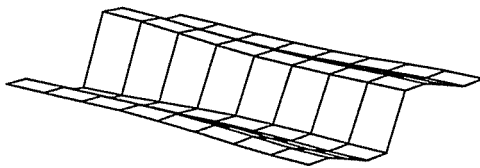
figure 3.3.26
Mode 4
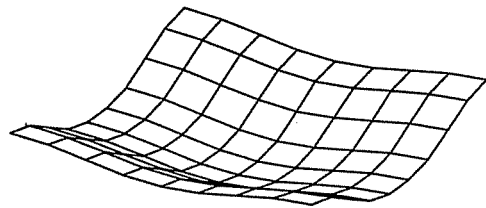
figure 3.3.27
Mode 5
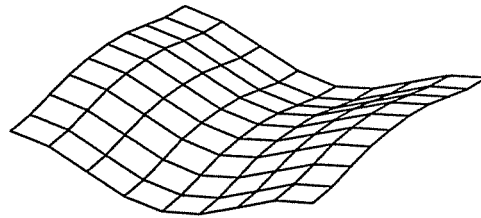
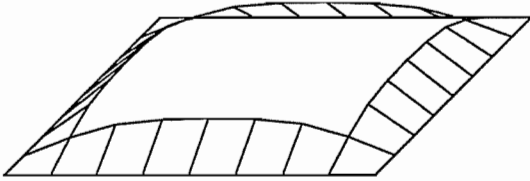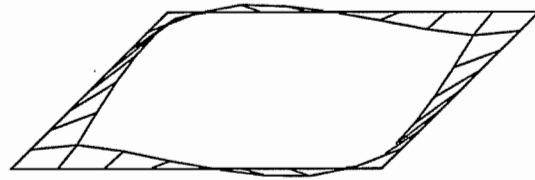figure 3.3.28
Mode 6
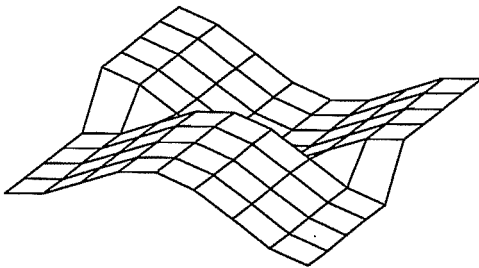
figure 3.3.29
Mode 7
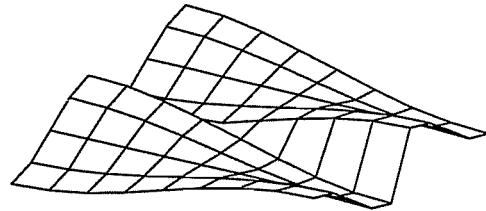
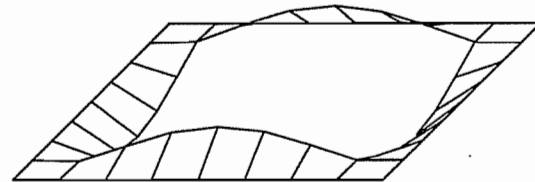figure 3.3.30
Mode 8
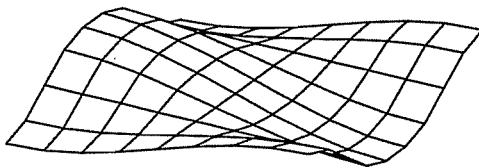
figure 3.3.31
Mode 9
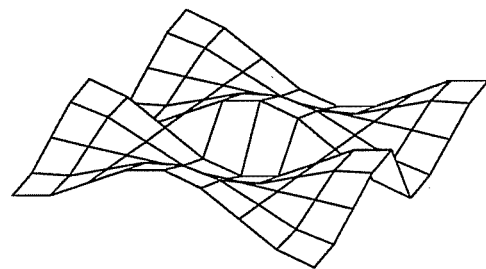
figure 3.3.32
Mode 10

3.3.3 Hydroelastic modes

Problem 1 :	Hydroelastic modes of a cubic tank
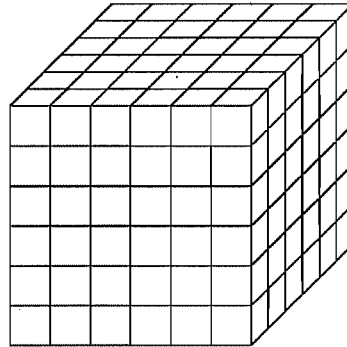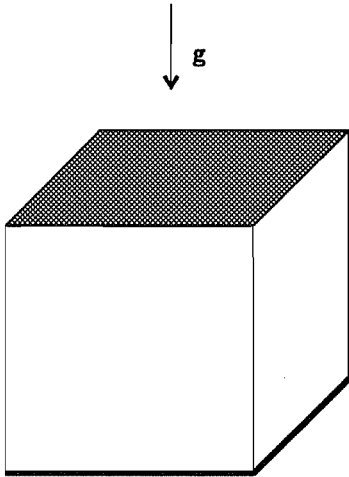
- Geometry.

    tank side : 0.2 m
    shell thickness : 1 mm

- Physical properties.

    fluid density : 1000 kg/m^3
    gravity acceleration : 10 m/s^2
    shell density : 2800 kg/m^3
    Young modulus : 7e10 N/m^2
    Poisson coefficient : 0.3

- Boundary counditions.

    base perimeter clamped
    only 1 overall d.o.f. for the free
    surface

- Model characteristics.

    fluid elements of degree 3
    free surface elements of degree 1
    shell elements of degree 2
    number of d.o.f. : 4355

- Results.

	Frequencies (Hz)
Modes	FE model
1	19.4905
2	20.0760
3	21.8258
4	21.8258
5	43.6180

	Frequencies (Hz)
Modes	FE model
6	55.4632
7	55.4632
8	57.3482
9	66.7854
10	80.7046

Problem 2 :   Hydroelastic modes of a cubic tank with horizontal baffle
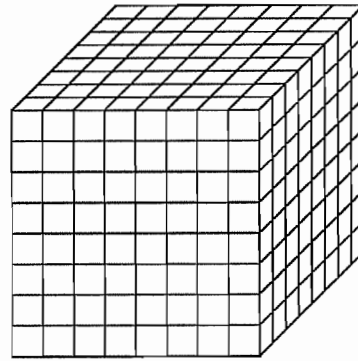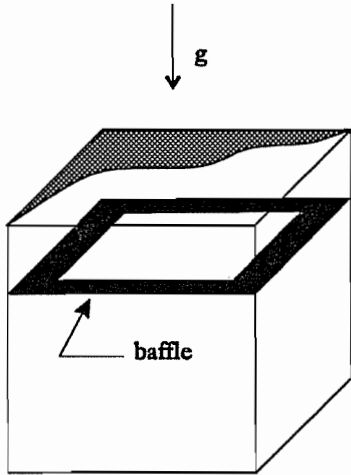
• Geometry.

  tank side : 0.2 m
  shell thickness : 1 mm
  baffle width : 0.025 m

• Physical properties.

  fluid density : 1000 kg/m^3
  gravity acceleration : 10 m/s^2
  shell density : 2800 kg/m^3
  Young modulus : 7e10 N/m^2
  Poisson coefficient : 0.3

• Boundary couindtions.

  rigid walls
  baffle clamped to the wall
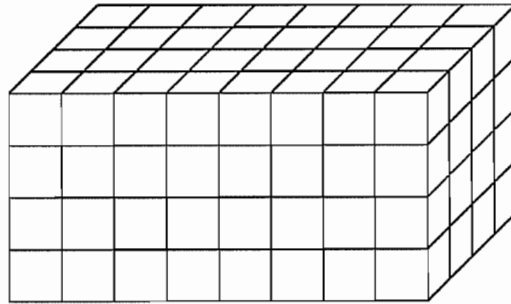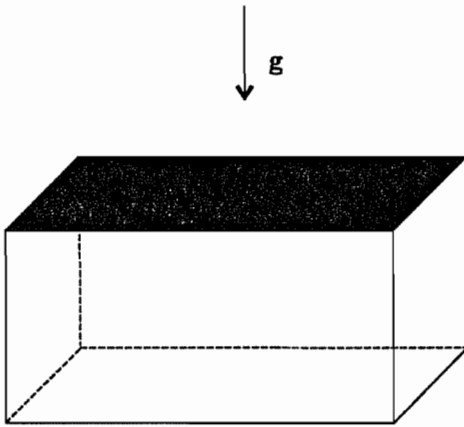 only 1 overall d.o.f. for the free
  surface

• Model characteristics.

  fluid elements of degree 3
  free surface elements of degree 1
  shell elements of degree 2
  number of d.o.f. : 5108

• Results.

	Frequencies (Hz)
Modes	FE model
1	606.39
2	633.02
3	633.02
4	678.05
5	702.01

|	Frequencies (Hz)
Modes	FE model
6	771.82
7	771.82
8	869.64
9	885.54
10	1009.64

• Mode shapes visualisation.

figure 3.3.33
Mode 1

figure 3.3.34
Mode 2
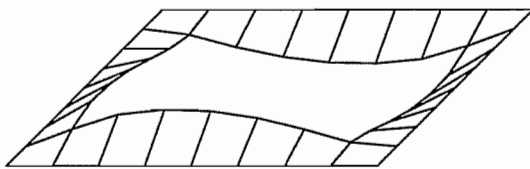
figure 3.3.35
Mode 3

figure 3.3.36
Mode 4

Problem 3 :   Hydroelastic modes of a parallelepipedic tank with vertical baffle

— plaque
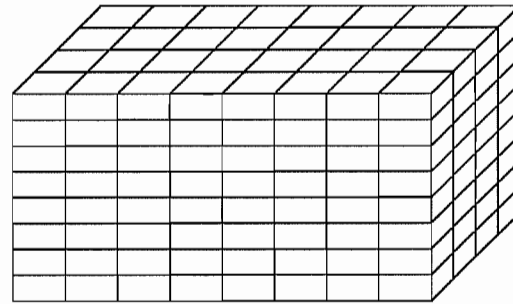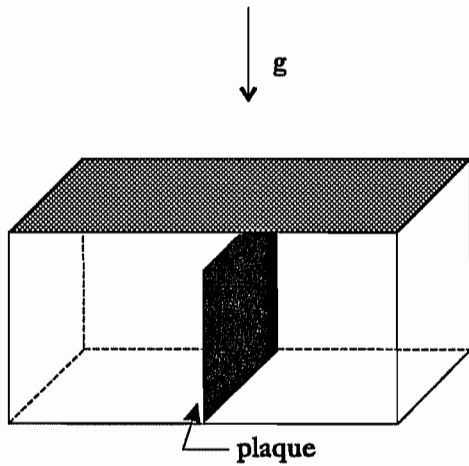
- Geometry.
    tank width : 0.2 m
    tank length : 0.4 m
    tank height : 0.2 m
    baffle height : 0.15 m
    shell thickness : 1 mm

- Physical properties.
    fluid density : 1000 kg/m^3
    gravity acceleration : 10 m/s^2
    shell density : 2800 kg/m^3
    Young modulus : 7e10 N/m^2
    Poisson coefficient : 0.3

- Boundary counditions.
    rigid walls
    baffle clamped to the bottom
    only 1 overall d.o.f. for the free
    surface

- Model characteristics.
    fluid elements of degree 3
    free surface elements of degree 1
    shell elements of degree 2
    number of d.o.f. : 2992

- Results.

Modes	Frequencies (Hz) FE model
1	5.349
2	15.274
3	43.937
4	49.780
5	71.837

Modes	Frequencies (Hz) FE model
6	117.758
7	124.816
8	176.766
9	204.669
10	218.104

| Problem 4 : | Hydroelastic modes of a rigid tanks with an elastic top |

- Geometry.

  tank width : 0.5 m
  tank length : 1 m
  tank height : 0.5 m
  shell thickness : 5 mm

- Physical properties.

  fluid density : 1225 kg/m^3
  gravity acceleration : 10 m/s^2
  shell density : 7800 kg/m^3
  Young modulus : 2.65e10 N/m^2
  Poisson coefficient : 0.3

- Boundary counditions.

  rigid walls
  elastic top simply supported

- Model characteristics.

  fluid elements of degree 3
  shell elements of degree 2
  number of d.o.f. : 1716

- Results.

	Frequencies (Hz)
Modes	FE model
1	12.005
2	24.608
3	33.994
4	42.498
5	44.121

	Frequencies (Hz)
Modes	FE model
6	57.429
7	70.563
8	79.461
9	90.508
10	101.419

● Mode shapes visualisation.
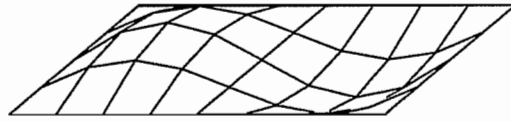
figure 3.3.37
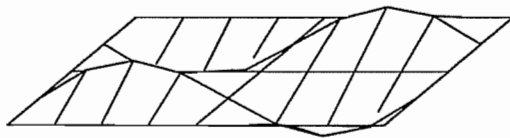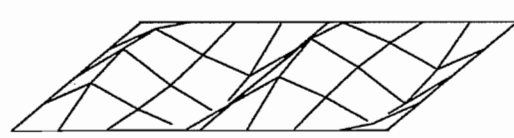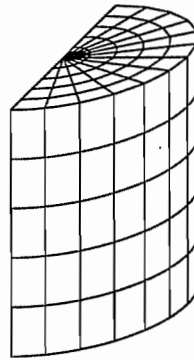Mode 1
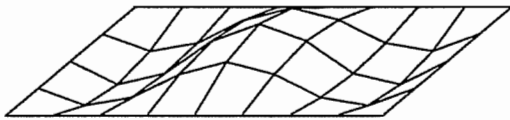
figure 3.3.38
Mode 2
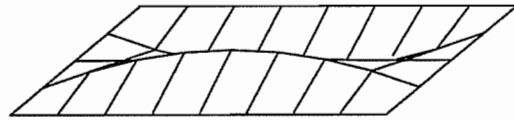
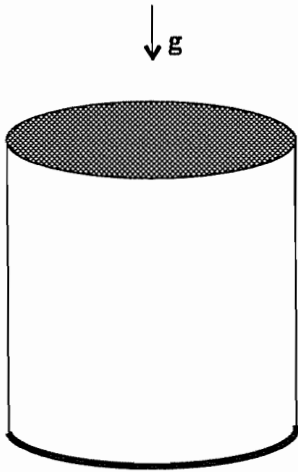figure 3.3.39
Mode 3

figure 3.3.40
Mode 4

figure 3.3.41
Mode 5

Problem 5 :   Hydroelastic modes of a cylindrical tank

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- Geometry.

  tank radius : 0.1 m
  tank height : 0.2 m
  shell thickness : 1 mm

- Physical properties.

  fluid density : 1000 kg/m^3
  gravity acceleration : 10 m/s^2
  shell density : 7800 kg/m^3
  Young modulus : 21e10 N/m^2
  Poisson coefficient : 0.3

- Boundary counditions.

  base perimeter clamped
  only 1 overall d.o.f. for the free
  surface

- Model characteristics.

  fluid elements of degree 3
  free surface elements of degree 1
  shell elements of degree 2
  number of d.o.f. : 4030

- Results.

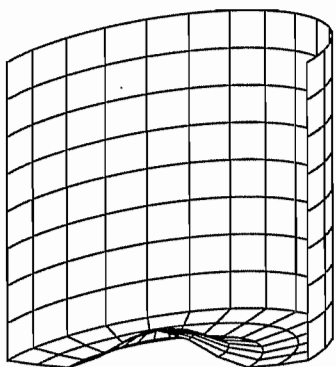|		Frequencies (Hz)	
Modes		3-D model	Fourier
1	[0, 0]	64.638	64.267
2, 3	[1, 0], [1, 1]	229.805	256.909
4, 5	[4, 0], [4, 1]	278.108	306.057
6, 7	[3, 0], [3, 1]	315.804	358.202
8, 9	[5, 0], [5, 1]	357.525	382.806
10	[0, 0]	444.802	510.354

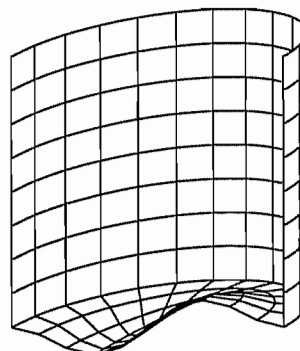- Mode shapes visualisation.

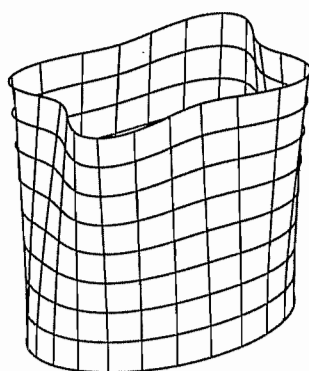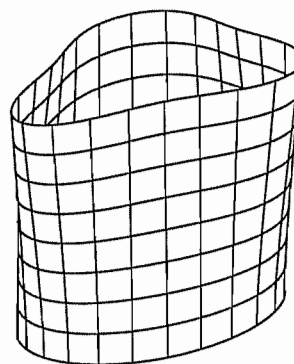figure 3.3.42
Mode 1

figure 3.3.43
Modes 2 and 3

figure 3.3.44
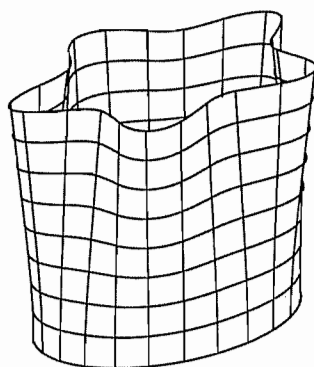Modes 4 and 5

figure 3.3.45
Modes 6 and 7

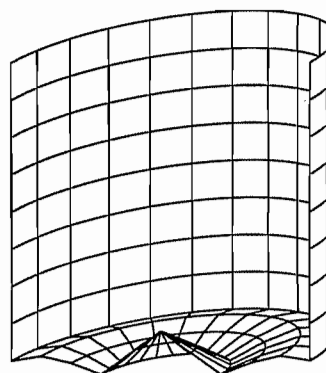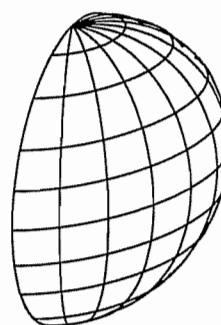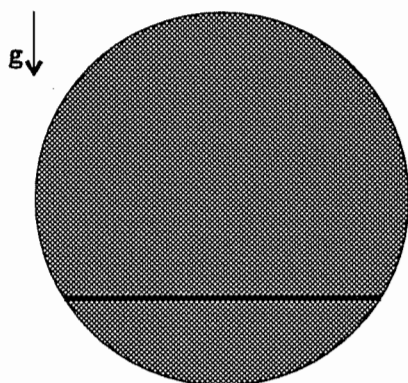figure 3.3.46
Modes 8 and 9

figure 3.3.47
Mode 10

Problem 6 :	Hydroelastic modes of a spherical tank

- Geometry.

    tank radius : 0.1 m
    shell thickness : 1 mm

- Physical properties.

    fluid density : 1000 kg/m^3
    gravity acceleration : 10 m/s^2
    shell density : 7800 kg/m^3
    Young modulus : 21e10 N/m^2
    Poisson coefficient : 0.3

- Boundary counditions.

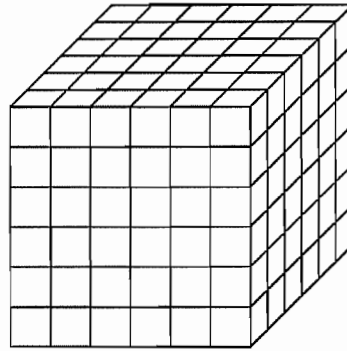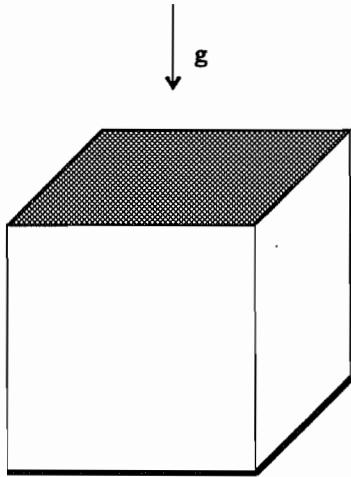   clamped on the circle located at 0.5 m
    from the bottom

- Model characteristics.

    fluid elements of degree 3
    shell elements of degree 2
    number of d.o.f. : 3011

- Results.

		Frequencies (Hz)	
Modes		3-D model	Fourier
1, 2	$[1, 0], [0, 1]$	1207.02	1310.08
3	$[0, 0]$	1868.48	2067.33
4, 5	$[2, 0], [2, 1]$	2930.75	2980.93
6, 7	$[1, 0], [1, 1]$	3269.05	3061.53
8	$[0, 0]$	3376.12	3432.45
9, 10	$[1, 0], [1, 1]$	3491.71	3471.66

Problem 7 : Hydroelastic modes of a cubic tank

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● Geometry.

tank side : 0.2 m
shell thickness : 0.1 mm

● Physical properties.

fluid density : 1000 kg/m^3
gravity acceleration : 10 m/s^2
shell density : 2800 kg/m^3
Young modulus : 7e10 N/m^2
Poisson coefficient : 0.3

● Boundary counditions.

base perimeter clamped
only 1 overall d.o.f. for the free
surface

● Model characteristics.

fluid elements of degree 3
free surface elements of degree 1
shell elements of degree 2
number of d.o.f. : 4355

● Results.

Modes	Frequencies (Hz) FE model
1	0.6306
2	0.6861
3	0.6861
4	1.5653
5	1.7053
6	1.7053

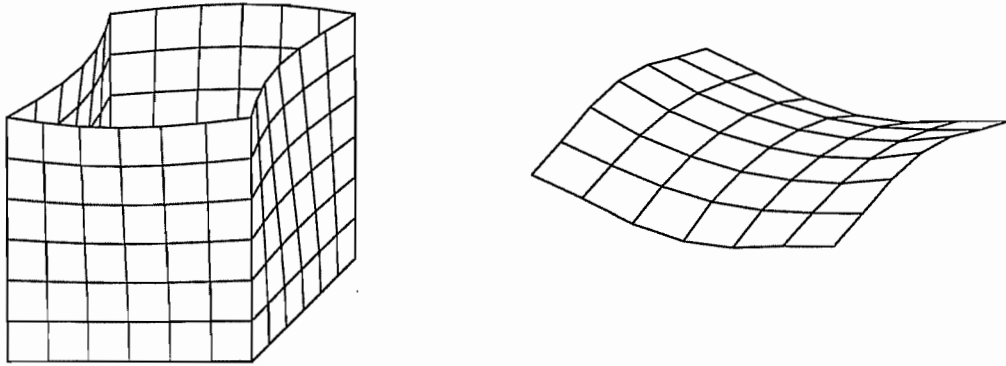| Modes	Frequencies (Hz) FE model
7	1.8177
8	1.9903
9	2.1063
10	2.1063
11	2.4794
12	2.5067

● Mode shapes visualisation.

figure 3.3.48
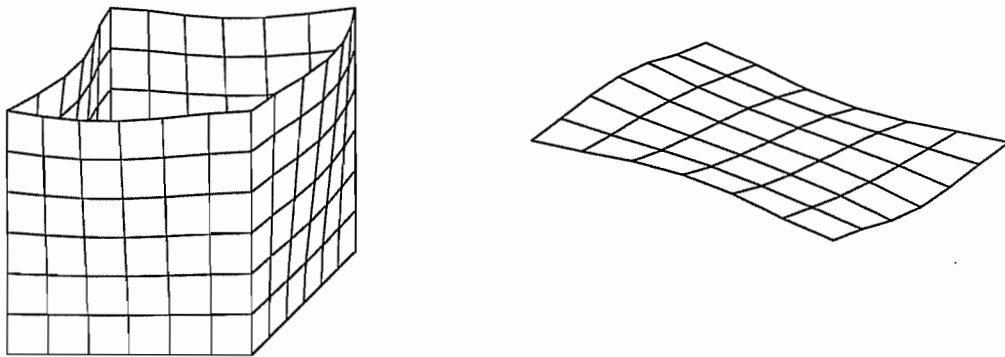*Mode 1 : structure and free surface*

figure 3.3.49
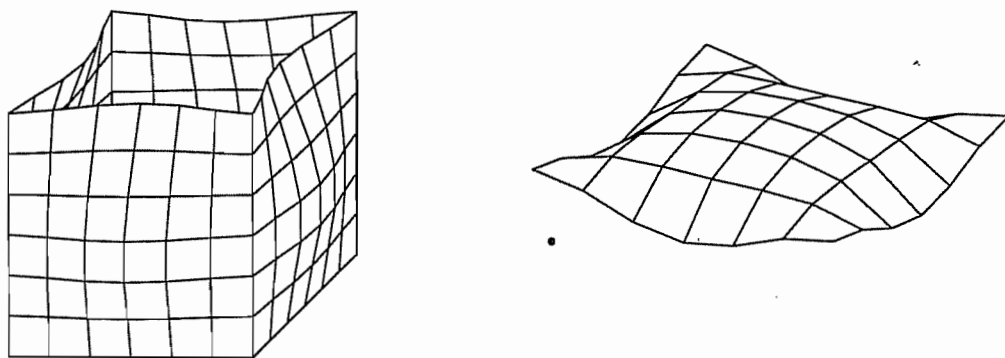*Modes 2 and 3 : structure and free surface*

figure 3.3.50
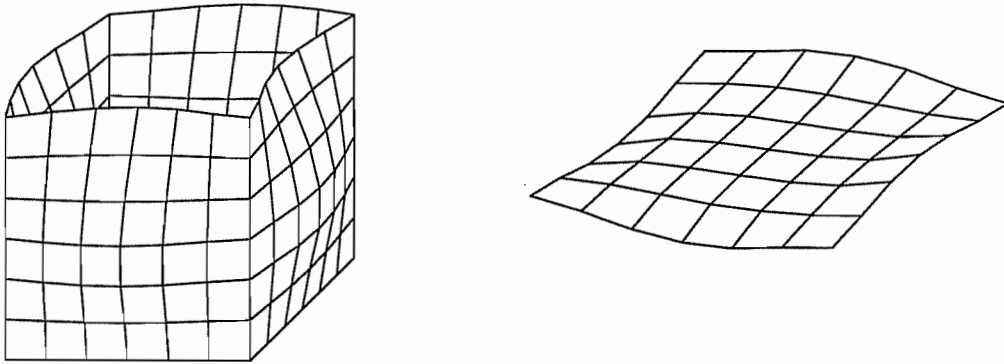*Mode 4 : structure and free surface*

figure 3.3.51
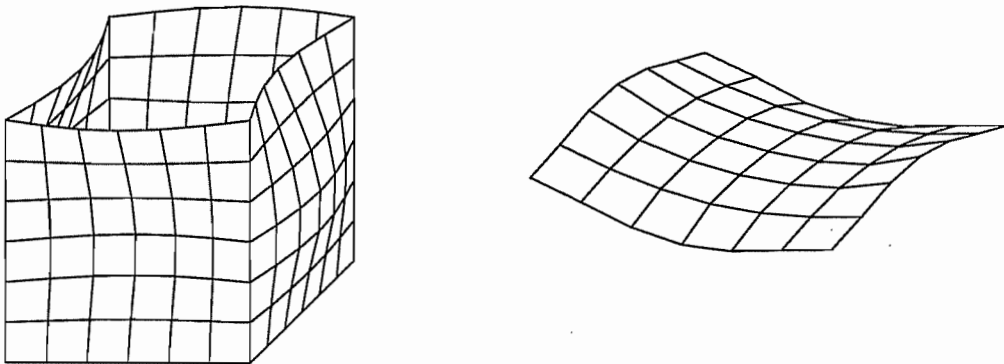*Modes 5 and 6 : structure and free surface*

figure 3.3.52
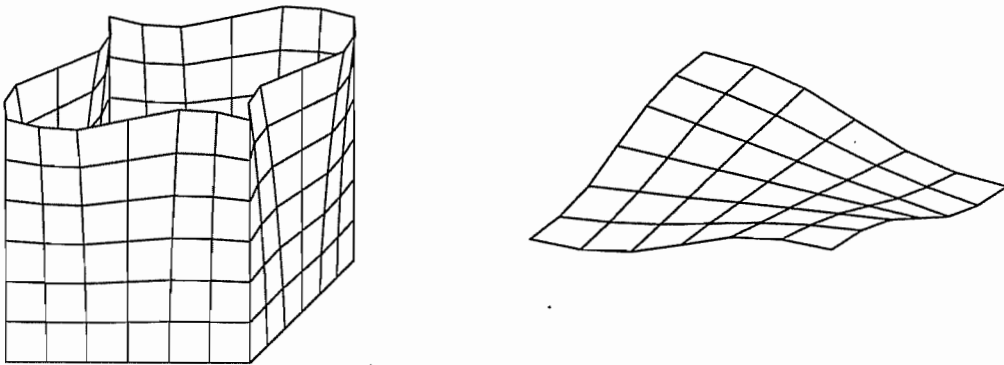*Mode 7 : structure and free surface*

figure 3.3.53
Mode 8 : structure and free surface

3.3.4 Comments about the results.

(a)   The Lanczos method used for the solution of the eigenvalue problem computes the eigenfrequencies in increasing order. The sloshing frequencies which generally appear first in the spectrum are numerous. A mean to get directly to the hydroelastic frequencies is to reduce the number of free surface degrees of freedom. In the problems treated above, the d.o.f. of the free surface are linked together and only one overall vertical displacement is kept in order to get the hydroelastic frequencies.

(b)   The parasitic frequencies appearing in some cases come from a well known problem inherent to the Lanczos method. The inaccuracy appearing during re-orthogonalisation can lead to the appearance of a linear combination of solutions already computed. The threshold of the error coefficient is put to 10^{-2}.

(c)   The difference between the results obtained with the 3-D analysis and those obtained with the Fourier analysis mainly comes from the circonferential discretisation of the shell. In most axisymmetric problems, the angles of the shell element were limited to an opening of 18 degrees with an interface node on the arc. In order to be convinced of this fact, tests were made on the empty axisymmetric tanks. The order of magnitude of the difference was found to be the same.

3.3.5 Evaluation of the potential energy for the rigid body modes of a partially filled spherical tank
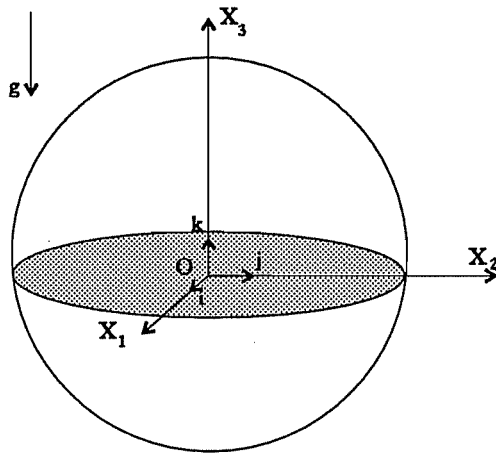
notations

Oijk cartesian system of coordinates
x_1, x_2, x_3 cartesian coordinates of a point M of the sphere
S_f is the wet surface of the tank
Γ is the free surface of the tank
w_i are the components of the displacements of the shell

As mentionned page 1.10, the potential energy has four contributions :

$$U = U_e + U_g + U_f + U_\Gamma$$

▷ The elastic energy term :

$$U_e = \frac{1}{2} \int_{V_s} C_{ijkl}\, \epsilon_{ij}\, \epsilon_{kl} dV_s$$

▷ The geometric stiffness term :

$$U_g = \frac{1}{2} \int_{V_s} s_{ij}^0\, \nabla_i w_k\, \nabla_j w_k dV_s$$

▷ The Tong's term and the hydrostatic pressure term :

$$U_f = \frac{1}{2} \int_{S_f} [\rho_f g_k w_k n_i w_i + p_0(n_i w_i \nabla_j w_j - n_i w_j \nabla_j w_i)] dS_f$$

▷ The free surface term :

$$U_\Gamma = \frac{1}{2} \int_\Gamma \rho_f g \eta^2 dS$$

The evaluation of the different terms can be done using the considerations of section 2 of this report.
A rigid displacement **u** can be expressed as the combination of a translation and a rotation :

$$u_i = a_i + x_j \omega_{ij}$$

and the different contributions of the potential energy transform into :

▷ $U_e = 0$ by definition of the rigid body mode

▷ $U_\Gamma = 0$ In this particular case, the free surface remains in its initial position

▷ The geometric stiffness term takes the following expression if use of the equilibrium equations is made (see section 2.4) :

$$U_g = -\frac{1}{2} \int_{S_f} p_0 n_i^* u_m \omega_{mj} dS + \frac{1}{2} \int_{S_2} \bar{t}_i u_m \omega_{mj} dS + \frac{1}{2} \int_{V_s} \rho_s g_j u_m \nabla_j u_m dV$$

▷ The term of potential energy coming from the fluid can also be expressed as a function of the rigid displacements u_i :

$$U_f = \frac{1}{2} \int_{S_f} [\rho_f g_k u_k n_i u_i + p_0 (n_i u_i \nabla_j u_j - n_i u_j \nabla_j u_i)] dS$$

which, according to sections 2.5 and 2.6, simplifies into

$$U_f = -\frac{1}{2} \int_\Gamma \rho_f g u_3^2 dS - \frac{1}{2} \int_{V_f} \rho_f g_k u_i \nabla_i u_k dV - \frac{1}{2} \int_{S_f} p_0 n_i u_j \omega_{ji} dS$$

The first term of this expression vanishes as the free surface remains still. The last term of this expression cancels exactly the first term of the geometric stiffness.

Finally, the expression of the total potential energy becomes :

$$\dot{U} = \frac{1}{2} \int_{V_s} \rho_s g_j u_m \nabla_j u_m dV - \frac{1}{2} \int_{V_f} \rho_f g_k u_i \nabla_i u_k dV$$

⋆ For a translation mode, $u_i = a_i$ and the total potential energy is equal to zero.

⋆ For a rotation around the vertical axis **k**, the displacement takes the form :

$$\mathbf{u} = \Omega \mathbf{k} \times (x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k})$$
$$= -\Omega(x_2 \mathbf{i} - x_1 \mathbf{j})$$

This also leads to a null overall potential energy because on one hand, the displacement does not depend upon the x_3 coordinate and on the other hand the displacement does not have a vertical component.

3.3.6 Detection of rigid body modes for fluid-structure problems using SAM-CEF.

The rigid body modes detection for fluid-structure systems is possible with SAMCEF but at the price of certain calculation artifices. The difficulty comes from the term in p_0 and the geometric stiffness term. In the SAMCEF code, it appears finally that the term including the hydrostatic pressure is not introduced in any version. It was thought up to now that it was not worth introducing this term considering its order of magnitude. The importance of this term grows with a decreasing Young's modulus of the structure but, in this case, it is partly compensated by the geometric stiffness term. The other difficulty comes from the combination of the geometric stiffness matrix and the stiffness matrix. Indeed, two different models are necessary. The first

one gives the geometric stiffness and only involves the shell on which the hydrostatic pressure is introduced, the fluid being not modelled. The second model involves both the shell and the fluid and therefore the d.o.f. numbers appearing in the two models do not coincide. There is thus no automatic procedure to sum the linear and geometric stiffness matrices while keeping the good ordering of the d.o.f.

How is it still possible to get the rigid body modes then ? An artifice must be used. In order to get the rigid modes, the gravity acceleration must be taken to a very low value (order of 10^{-7}). In this case, the Tong's term and the free surface term are nearly equal to zero. The geometric stiffness term can be separated in different contributions : one contribution compensates the term of hydrostatic pressure and the other one is a function of gravity and thus can also be neglected. So the only remaining contribution is the elastic part of the potential energy which is identically null and the rigid modes can be extracted. Unfortunately, this does not proof the correctness of the formulation eventhough the rigid modes may be detected.

To summarize, in order to detect the rigid modes with SAMCEF, the following data must be modified in the input file :

  ⋆ the gravity acceleration must be small enough in order to cancel all the terms mentioned above
  ⋆ a sufficient number of d.o.f. must be kept (command .RET) in order to be able to represent the rigid modes.
  ⋆ if the rotations around the axis perpendicular to the gravity acceleration are of interest, the center of gravity of the filled tank must be fixed in translation.