WHATEVER HAPPENED TO THE DYER–ROEDER DISTANCE?

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The Universe is not completely homogeneous. Even if it is sufficiently so on large scales, it is very inhomogeneous at small scales, and this has an effect on light propagation, so that the distance as a function of redshift, which in many cases is defined *via* light propagation, can differ from the homogeneous case. Simple models can take this into account. One such model is known as the Dyer–Roeder distance. I sketch the history of this model and some applications, then suggest some reasons why it is still relatively obscure.

Introduction

Classical cosmology aims to determine the cosmological parameters λ_0 and Ω_0 by calculating the dependence of observable quantities, which depend on some sort of distance, on the redshift z for different values of those parameters, which are then fitted for *via* comparing calculations with observations. That includes not only the classical (and classic) tests such as the m–z relation, the angular-size–redshift relation, and number counts (*e.g.,* ref. 1) but also less straightforward calculations such as gravitational-lensing statistics (*e.g.,* ref. 2 and references therein). Usually, the distance is calculated from the redshift assuming an ideal Friedmann–Robertson–Walker (FRW) model. Since a look at the sky shows that the Universe is not homogeneous, at least not on the relevant scales (given by the angular size of an observed object), the question arises to what extent this inhomogeneity can affect light propagation and hence the distance calculated from redshift and thus the derivation of cosmological parameters *via* comparing theory with observations; in particular, under-dense lines of sight correspond to larger angular-size and luminosity distances. The simplest more refined model retains the background geometry and expansion history of an FRW model but separates matter into two components, one smoothly distributed comprising the fraction η of the total density and the other $(1 - \eta)$ consisting of clumps, and considers the case where light from a distance object propagates far from all clumps (and thus through under-dense regions).*

*Since α is almost universally used to denote the gravitationl-lensing bending angle, Kayser *et al.* [3], hereafter KHS, adopted η instead of the more confusing α or $\tilde{\alpha}$ used by some other authors; since then, some authors other than KHS have also used η instead of α or $\tilde{\alpha}$ for the

The plan of this paper is as follows. First I give a brief overview of the history of the Dyer–Roeder distance, then discuss various applications of the Dyer–Roeder distance in observational cosmology, before offering some reasons why the Dyer–Roeder distance is not often taken into account and perhaps doesn't need to be.

Back in the U.S.S.R.

The first attempt to calculate distances in a universe with small-scale inhomogeneities was, as far as I know, that of Zel'dovich[4], though more are probably more familiar with the English translation[5]. I'll refer to both as Z64. In modern notation, Z64 considered the Einstein–de Sitter universe with $\Omega_0 = 1$ and $\lambda_0 = 0$. (I use terminology in which Ω refers to the matter content in units of the critical density; $\Omega = 8\pi G\rho/(3H^2)$, where ρ is the density, G the gravitational constant, and H the Hubble constant. $\lambda = \Lambda/(3H^2)$ is the dimensionless cosmological constant. The suffix 0 refers to the value today, since in general cosmological parameters are time-dependent.) That is, an Einstein–de Sitter universe on large scales, but allowing for inhomogeneities on small scales, though these do not affect the metric (in other words, no back-reaction — more precisely, one considers inhomogeneities so small that the effect on the metric is negligible on large scales, *i.e.,* the universe is still well described by an FRW model overall, though the small-scale effects on light propagation can be non-negligible). The idea is that all matter is in galaxies and one sees distant objects between the galaxies.

The angular-size distance D^{A} is defined as ℓ/θ, where ℓ is the physical length of an object and θ the angle between the two sides of the object as seen by the observer. It is essentially the proper distance at the time the light was emitted, corrected for curvature effects if, unlike the Einstein–de Sitter model, the universe is not flat (see KHS for discussion of various cosmological distances). In a perfectly homogeneous universe, in the general case distances can be calculated *via* elliptic integrals (*e.g.,* ref. 6), though there are simpler solutions for special cases (KHS, Appendix B). Fig. 1 illustrates the definitions in the completely homogeneous case.

Z64 takes a different approach, deriving a differential equation for the separation between two light rays (easily converted to the angular-size distance), which changes due to the expansion of the universe and due to convergence caused by matter between the two rays (*i.e.,* 'in the beam'). If the beam is under-dense, then the convergence is negative. The effect of matter in the beam is calculated by generalizing the deflection of light by a point mass due to the gravitational-lens effect (*e.g.,* ref. 7) to a smooth distribution. It is noted that for $\eta = 0$ the angular-size distance "increases monotonically right up to the [particle] horizon ($\Delta = 1$) where it reaches the value 2/5".† (In this paper, all distances are in units of the Hubble length c/H_0. and, unless otherwise specified, the distance is between the observer at redshift $z = 0$ and an object at a

inhomogeneity parameter.

†$\Delta := 1 - 1/(1 + z)$. For a modern reader, the notation of Z64 is bizarre. See Helbig[8] for a translation into modern notation.

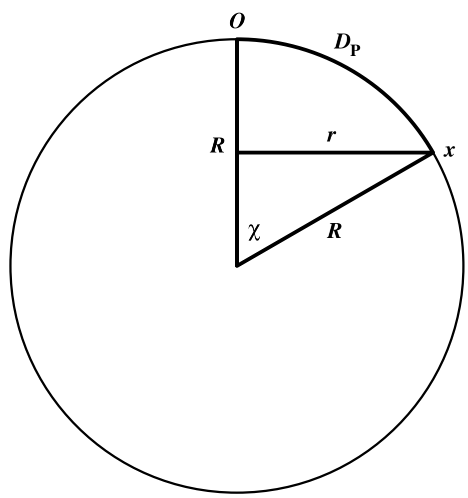

FIG. 1

Although the corresponding definitions are valid for models with k of 0 and -1 (zero and negative curvature, respectively) as well, easiest to visualize are distance definitions for the case $k = +1$ (positive curvature). The universe can be thought of as a curved three-dimensional space, corresponding to the circle. Two dimensions are hence suppressed, so that the two dimensions in the plane of the figure can show the universe and its spatial curvature. R is the scale factor of the universe, as usual chosen to correspond to the radius of curvature. The observer is located at the top of the circle at O and observes an object located at x. D^{P}, the length of the arc, is the proper distance to that object. For $\eta = 1$, the angular-size and luminosity distances (as well as other distances not discussed here such as the proper-motion distance and parallax distance) depend on $r = R\sin(\chi)$ in a relatively simple manner; see KHS for details. (They depend on r and not D^{P} for the same reason that the length of a parallel of latitude is less than $2\pi D^{\mathrm{P}}$ where, in the case of the Earth, D^{P} is the distance along the surface, 'as the crow flies'.) Note that χ is constant in time; one can use it or $\sigma = r/R$, which is also constant in time, as the basis for a so-called co-moving distance.

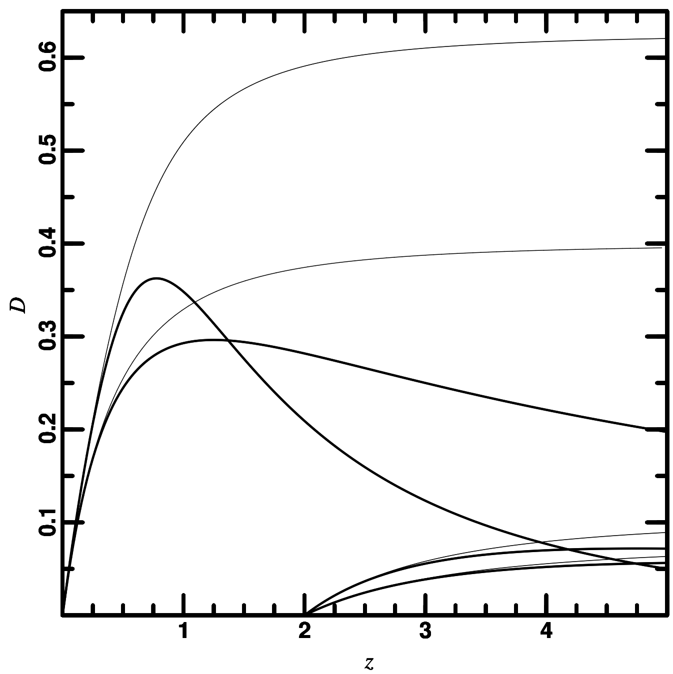

FIG. 2

The angular size distance from the observer ($z_1 = 0$) and from $z_1 = 2$ (lower right) as a function of the redshift z_2 for different cosmological models. Thin curves are for $\eta = 0$, thick for $\eta = 1$. The upper curves near $z = 0$ ($z = 2$ at lower right) are for $\lambda_0 = 2$, the lower for $\lambda_0 = 0$. $\Omega_0 = 1$ for all curves. The angular size distance D is given in units of c/H_0.

redshift $z > 0$ — it is sometimes necessary, especially in gravitational lensing, to discuss distances between two non-zero redshifts (*e.g.,* the lens and the source).) Thus, the maximum in the angular-size distance is at $z = \infty$, in marked contrast to the homogeneous case, where there is a maximum ($D^{\mathrm{A}} \approx 0.296$) at $z = 1.25$ (see Fig. 2).

In addition to the perfect Einstein–de Sitter model and one with all the matter in clumps (galaxies), D^{A} is also calculated for $\Omega_0 = 0$. Since there is no matter at all in such a model, there is no difference between the standard distance (filled-beam) and the empty-beam distance. (It is of course a consistency check that the differential equation for the empty model gives the same result as the standard calculation, *i.e.,* assuming perfect homogeneity.) There are simple analytic solutions for these three cases.:

$$D^{\mathrm{A}} = 2\left((1+z)^{-1} - (1+z)^{-\frac{3}{2}}\right) \tag{1}$$

(Einstein–de Sitter model, filled beam),

$$D^{\mathrm{A}} = \frac{2}{5}\left(1 - (1+z)^{-\frac{5}{2}}\right) \tag{2}$$

(Einstein–de Sitter model, empty beam), and

$$D^{\mathrm{A}} = z\left(1+\frac{z}{2}\right)(1+z)^{-2} = \frac{1}{2}\left(1-(1+z)^{-2}\right) \tag{3}$$

($\Omega_0 = 0$). Note that Eq. (1) is a special case of the Mattig[9] formula

$$D^{\mathrm{A}} = \left(\frac{2}{\Omega_0^2(1+z)^2}\right)\left(\Omega_0 z - (2-\Omega_0)\left(\sqrt{\Omega_0 z+1}-1\right)\right) . \tag{4}$$

Dashevskii & Zel'dovich[10], an English translation of ref. 11, hereafter DZ65, generalize the idea of Z64 to arbitrary values of Ω_0 ($\lambda_0 = 0$ is still assumed). The derivation follows a different route, again an interesting consistency check. In this more general case as well, a completely empty beam puts the maximum in the angular-size distance at $z = \infty$. No analytic solution is given, though one exists[12]; *cf.* KHS, equation (B15) — much more complicated than Eqs. (2) & (3). The angular-size distance for the filled (f) and empty (f_1) beam is plotted as a function of $\Delta = 1 - 1/(1+z)$ for a few values of Ω_0, and for a few more values of Ω_0, Δ_{max} (the value of Δ at which the maximum in the angular-size distance for $\eta = 1$ occurs) and the values of f at Δ_{max} and f_1 at $\Delta = 1$ are tabulated. Several interesting features are pointed out in the text and/or are obvious from the figure (if Ω_0 is not mentioned, then the effect is independent of the value of Ω_0):

(i) The angular-size distance for $\eta = 0$ increases monotonically with redshift.

(ii) The angular-size distance for $\eta = 0$ is less than the light-travel–time distance $c(t_0 - t)$ and larger than the angular-size distance for $\eta = 1$ (at least for $\lambda_0 = 0$).

(iii) The angular-size distance for $\eta = 0$ has its maximum value at $z = \infty$.

(iv) For $\eta = 0$, $\mathrm{d}D^{\mathrm{A}}/\,\mathrm{d}z = 0$ at $z = \infty$.

(v) The angular-size distance for $\eta = 1$ has a maximum at $z < \infty$.

(vi) The value of the maximum of the angular-size distance for $\eta = 1$ increases with decreasing Ω_0.

(vii) The redshift of the maximum of the angular-size distance for $\eta = 1$ increases with decreasing Ω_0.

(viii) The angular-size distance for $\eta = 1$ is 0 at $z = \infty$.

(ix) Both for $\eta = 0$ and $\eta = 1$, the value of D^{A} at any redshift increases with decreasing Ω_0.

(x) For given values of Ω_0 and z, D^{A} for $\eta = 0$ is always larger than D^{A} for $\eta = 1$.

DZ65 note that the claim by Wheeler [13] that the maximum in the angular-size distance for $\eta = 1$ occurs only in the case of a spatially closed universe is wrong. While positive spatial curvature can contribute to the maximum, in most cosmological models it is not the main effect [8]. (Spatial curvature can cause a maximum in the angular-size distance for the same reason that the circumference of a parallel of latitude as a function of longitude has a maximum (at the equator), but in most cosmological models the main reason is that the angular-size distance (ignoring curvature effects) is equivalent to the proper distance at the time of emission. At small redshifts, as the redshift increases, the object was farther away (in proper distance) when the light was emitted, thus the angular-size distance increases with redshift. However, at large redshifts, light was emitted when the proper distance was small, long ago, but, due to the more rapid expansion of the universe in the past, is reaching the observer just now. Referring to Fig. 1, $D^{\mathrm{A}} = Rf(\chi)$, where $f(\chi)$ is $\sin(\chi)$, χ, or $\sinh(\chi)$ for k equal to $+1$, 0, or -1, *i.e.,* positive, zero, or negative spatial curvature, respectively. At small z, χ is small; at large z, R was small when the light was emitted.)

Dashevskii & Slysh [14], hereafter DS66, an English translation of ref. 15, generalize the method of Z64 and DZ65 to the more realistic case that the beam is not completely empty, but only for the Einstein–de Sitter model. The empty-beam case is criticized as being too unrealistic, as there will always be some intergalactic matter; this will mean that there will always be a maximum in the angular-size distance. DS66 derive, in their equation (2), the second-order differential equation which is the basis for all further work in this field:

$$\ddot{z} - \frac{\dot{a}}{a}\dot{z} + 4\pi G\rho_g z = 0\,, \tag{5}$$

"which determines the linear distance $z(t)$ between rays", with $\rho_g = \alpha\rho$ (the subscript g refers to the *smooth* component, considered as a "***gas*** at zero pressure that fills all space uniformly" [my emphasis], the rest of the "matter being concentrated in discrete galaxies"); a is the scale factor and G the gravitational constant. Compared to Z64 and DZ65, they allow α (in the notation of KHS, η) to take an arbitrary value $0 \leq \alpha \leq 1$; η is thus completely general. The cosmological model is implicit in the term $\dot{a}/a$, in principle allowing one to study any cosmological model in which $\dot{a}/a$ can be calculated, but DS66 then restrict themselves to the Einstein–de Sitter model for the subsequent discussion, presenting a completely analytic solution for the angular-size distance for this cosmological model:

$$D^{\mathrm{A}} = \frac{1}{2\beta}\left((1+z)^{(\beta-\frac{5}{4})} - (1+z)^{(-\beta-\frac{5}{4})}\right) \tag{6}$$

(modern notation), where

$$\beta := \frac{1}{4}\sqrt{25 - 24\eta}\,. \tag{7}$$

DS66 point out that, for arbitrary $0 < \eta \leq 1$, the angular-size distance has a maximum at finite z and the angular-size distance goes to 0 for $z = \infty$. Also, the smaller the fraction of homogeneously distributed matter, *i.e.,* the smaller η, the higher the redshift of this maximum. The generalization of the differential equation of Z64 to an arbitrary value of η is obvious; less obvious is the relatively simple analytic solution for arbitrary η for the Einstein–de Sitter model; *cf.* KHS, equations (B18) & (B19).

Dyer & Roeder

Starting with an integral expression (including λ_0) for the angular-size distance in the $\eta = 0$ case, Dyer & Roeder [12], hereafter DR72, give analytic solutions (but now assuming $\lambda_0 = 0$, *i.e.,* the same assumptions as made by DZ65) for the three cases $\Omega_0 < 1$, $\Omega_0 = 1$, and $\Omega_0 > 1$ (though the first is actually not valid for $\Omega_0 = 0$); only the much simpler solutions for $\Omega_0 = 1$ (Z64) and $\Omega_0 = 0$ (ref. 9; see also Z64) were previously known. As was common at the time, instead of Ω_0, q_0 was used. In general, $q_0 = \Omega_0/2 - \lambda_0$, so that, for $\lambda_0 = 0$, $q_0 = \Omega_0/2$. The famous result of Etherington [16],

$$D^{\mathrm{L}} = (1 + z)^2 D^{\mathrm{A}}\,, \tag{8}$$

where D^{L} is the luminosity distance, is invoked to note that an empty beam leads to a lower apparent luminosity which leads one to underestimate q_0 if a completely homogeneous universe is assumed; their example has a real value of $q_0 = 1.82$ which, if calculated assuming a completely homogeneous universe, results in the value $q_0 = 1.40$. The exact numbers are not important; the point is that, to first order, the empty-beam distance is larger than in the standard (filled-beam) case, which is also the case for a lower value of q_0. But this is only to first order; with higher-redshift data, the two effects are not degenerate. It is also shown that, while the difference between the empty-beam distance and the standard distance is non-negligible, there is little difference between the Dyer–Roeder distance and that obtained by numerical integration in a corresponding Swiss-cheese model (see below).

Dyer & Roeder [17], hereafter DR73, can be seen as a combination of DZ65 and DS66, *i.e.,* Ω_0 and η are both arbitrary (though $\lambda_0 = 0$ is still assumed). For the general case, they derive a hypergeometric equation, and present explicit solutions for $\eta = 0$, $2/3$, and 1 as well as $\Omega_0 = 0$ (the second one being new). The special case $\eta = 1$ is the solution derived by Mattig [9] while that for $\eta = 0$ is that derived by DR72. New is a solution for $\eta = 2/3$, which is given for the luminosity distance. For $\Omega_0 = 1$, one has the solution derived by DS66, which is given for the angular-size distance. Differentiation of that equation leads to an expression for the maximum in the angular-size distance, showing that as η goes from 1 to 0, the redshift of this maximum goes from 1.25 to ∞. The point first made by Z64, that the maximum is due to matter in the beam, is emphasized. (Note, however, that an arbitrarily small η will lead to a maximum, though at arbitrarily large z.) They suggest comparing observations with calculations for each of the three values of η for which there is an analytic solution, given the lack of knowledge about intergalactic matter. Finally, as in DR72, they note that calculations for Swiss-cheese models (see below) — interestingly, including $\lambda_0 \neq 0$ — confirm that this is a good approximation, *i.e.,* "the mass deficiency in the beam is in general much more important than the gravitational-lens effect for reasonable deflectors", at least for "redshifts in the range of interest".

The distance for an empty or partially filled beam has become known as the Dyer–Roeder distance, although various aspects had been discussed by others before. This is probably due to the fact that the corresponding papers were published in a major English-language journal, used standard notation, and were more concerned with results than with theory. Dyer and Roeder were certainly responsible for putting the topic on the agenda of many astronomers. On the other hand, it seems unfair to neglect the pioneering efforts of Zel'dovich and his colleagues. Hence, in what follows I will refer to the distance calculated based on the above approximation as the *ZKDR* distance, a term introduced by Santos & Lima [18] and referring to Zel'dovich, Kantowski (see below), Dyer, and Roeder, though I take the 'D' to refer to Dashevskii as well, my criterion for being part of the acronym being having (co-)authored at least two papers on this topic, at least one of which was published within ten years of the first paper on this topic (Z64).

Theoretical foundations

Weinberg [19] pointed out that the standard distance formula, *e.g.,* assuming $\eta = 1$, must hold *on average* if lenses are transparent and there are no selection effects. This is due to flux conservation: the fact that almost all beams are under-dense and hence the average magnification[‡] is less than 1 is offset by the occasional strong-lensing event. Peacock [20] generalized the result of Weinberg [19] to arbitrary Ω_0. Dyer & Roeder [21] considered the effect of a finite source size in gravitational lensing, concluding that, all else being equal, η increases with the size of the source, since the larger the source, the more clumps will lie in front of it, and these will offset under-dense lines of sight. (In the limit, with the angular size equal to the whole sky, then obviously, ignoring absorption, η must be 1, since in this case the average density in the beam is equal to the average density in the universe. With realistic mass distributions, $\eta \approx 1$ is reached for much smaller angles.) The important quantity is not the size of the source *per se*, but rather the size of the source relative to the clumps; as already mentioned by Weinberg [19], one could think of η increasing with redshift since, due to structure formation, matter was more uniform at high redshift. The fact

[‡]Since gravitational lensing conserves surface brightness, magnification (increase in angular size) and amplification (increase in brightness) are proportional if the former is thought of as the increase in area rather than linear size; thus the two terms are sometimes used interchangeably, though strictly speaking, depending on context, one or the other might be more appropriate.

that the angular size of the beam also increases with redshift (the base of the cone is at the source; the apex at the observer) is an additional effect in the same direction.

Kibble & Lieu [22] also contributed significantly to the understanding of flux conservation in the context of the ZKDR distance. They showed analytically that, under very general conditions (including arbitrary shapes of clumps and strong lensing), the average *reciprocal* magnification in a clumpy universe is the same as that in a homogeneous universe, as long as the clumps are uncorrelated. An important distinction is whether one averages over a set of sources on the unperturbed celestial sphere, or whether one averages over all lines of sight. This is related to whether it is the mean magnification or the mean reciprocal magnification that is the same as in the homogeneous case.

Even if the *mean* magnification is 1, due to the skewness of the distribution, the *median* magnification is < 1. Clarkson *et al.* [23] pointed out that most narrow-beam lines of sight are significantly under-dense, even for beams as thick as 500 kpc. Although the basic idea of flux conservation is clear (and there are obvious caveats such as non-transparent matter), exact treatments can be very complicated and have led to confusion, much of which has been cleared up by Kaiser & Peacock [24]: Weinberg [19] is essentially right, though one needs to keep in mind the distinction between magnification and reciprocal magnification as discussed above in connection with Kibble & Lieu [22].

Kantowski [25], hereafter K69, took a somewhat different approach to that of Zel'dovich and those who expanded on his ideas, using Swiss-cheese models [26,27]. These are arguably less realistic than the approximation used in the papers discussed above, since in those models clumps of matter are surrounded by voids with $\rho = 0$. Since any spherical distribution of matter is, from outside the sphere, gravitationally indistinguishable from a point source of the same mass (even if it is not static)[§], by making clumps out of the matter removed from a surrounding void, the large-scale geometry and dynamics are not changed. Hence, such models are exact solutions of the Einstein field equations and the validity of approximations used to calculate the angular-size distance is not an issue (though, of course, one can question the validity of this approximation to the distribution of matter in our Universe). K69 calculated the apparent bolometric luminosity, which is inversely proportional to the square of the luminosity distance. Since the luminosity distance is larger than the angular-size distance by a factor of $(1 + z)^2$, it is easy to compare his results with those discussed above. K69 points out that in the case that most matter is in clumps (*i.e.,* $\eta \approx 0$), a real value of $q_0 = 2.2$ would, were one to wrongly assume the standard distance, appear as $q_0 = 1.5$. This foreshadows later work stressing the importance of taking inhomogeneities into account in classical observational cosmology, at least as long as a significant fraction of matter is in clumps and the Universe is similar to the approximations used to calculate distances in such a case.

[§]The conventional reference is to Birkhoff [28], though the corresponding theorem was actually proved earlier by Jebsen [29].

Dyer & Roeder[30] extended Swiss-cheese models to include cases where $\lambda_0 \neq 0$ and showed that the distances so computed correspond well to those based on previous work (DR72, DR73). Essentially, $\lambda_0 \neq 0$ affects the expansion history of the universe but nothing else. An important result is that the dependence of the distance–redshift relation on Ω_0 is decreased for $\eta \approx 0$, thus reducing the precision obtainable in practice. Dyer & Roeder[30] is interesting because it presents for the first time distance–redshift relations in a universe with arbitrary Ω_0, λ_0, and η. However, not only because the calculations are based on Swiss-cheese models, no closed formulae are given.

The fact that results are very similar to those based on the simpler assumptions of the ZKDR distance is encouraging, and provides justification for using the simpler approach. It could of course be the case that this approach is too simple for the real Universe, but in that case a Swiss-cheese model would also probably be too unrealistic. Fleury[31] demonstrated with completely analytic arguments the equivalence of the ZKDR distance and that calculated from a certain class of Swiss-cheese models at a well controlled level of approximation. This had been known for a long time based on comparisons of numerical results, but of course an analytic proof is very important.

Later work

Partially motivated by evidence for $\lambda_0 > 0$ (*e.g.,* refs. 32,33) as well as plans for higher-redshift observations of standard candles[34], it became necessary to calculate the ZKDR distance for $\lambda_0 \neq 0$. The only expression available for $\lambda_0 \neq 0$ was a complicated differential equation derived by Dyer & Roeder[30], but for Swiss-cheese models (see above). No closed solution was presented. Of course, it can be integrated numerically. However, it is rather cumbersome, and the terms do not have an obvious physical interpretation like those in the differential equations of Z64 and DS66. While it was appreciated that Swiss-cheese models are in some sense equivalent to the ZKDR distance derived *via* the Zel'dovich method, this was not shown strictly until much later[31]. Kayser[35] derived a differential equation for the angular-size distance in the style of Z64, DS66, and DR73, but for $0 \leq \eta \leq 1$ and arbitrary values of λ_0 and Ω_0, which he integrated numerically *via* standard but basic means. KHS saw a need for an efficient numerical implementation of that equation, which is the most general equation for the ZKDR distance under the standard assumptions that the universe is a (just slightly) perturbed FRW model (*i.e.,* no pressure, no dark energy more complicated than the cosmological constant, no back reaction, only Ricci (de)focussing; even today, there is no evidence that the first three are not excellent approximations, and the fourth is as well in many cases). Also, no efficient general implementation existed for the standard ($\eta = 1$) distance. Thus, a description of the differential equation derived by Kayser[35] and the efficient numerical implementation evolved to include a general description of various types of cosmological distances and a compendium of analytic solutions, probably the first time all this information had been presented in a uniform notation.

Kantowski, with collaborators, had returned to the topic of distance calculation in locally inhomogeneous cosmological models[36], coincidentally around the same time that I was writing the code for KHS. Although partially motivated by the m–z relation for type-Ia supernovae, further progress was made regarding the theory; in particular, Kantowski *et al.*[37] gave analytic expresssions using elliptic integrals for arbitrary λ_0, Ω_0, and $\eta = (0, 2/3, 1)$.

Testing the approximation

Unlike the Swiss-cheese model, the ZKDR distance is an approximation based on various assumptions. While it is reasonably clear that it must be correct in the appropriate limit (*i.e.,* the light propagates *very* far from all clumps, the fraction of mass in clumps is negligible so that it is clear that an FRW model is a good approximation, *etc.*), it is not immediately clear how good the approximation is in a more realistic scenario. One way to test this is to compare the ZKDR distance to an explicit numerical calculation, namely following photon trajectories through a mass distribution produced by a cosmological simulation. Several studies using simple numerical models for the cosmolgical distribution of matter[38–42] found good agreement between the ZKDR distance and an explicit numerical calculation. More-complicated models still found good agreement[43,44]; for a variety of cosmological models, the discrepancy was less than 1 per cent up to $z = 10$. Mörtsell[45] used essentially the same scheme to investigate the relation between η and the fraction of compact objects. By definition, $1 - \eta$ is the fraction of compact objects f_c in the pure ZKDR case, *i.e.,* only deamplification due to underdensity and no amplification due to gravitational lensing. As expected, taking lensing into account results in $1 - \eta < f_\mathrm{c}$. Interestingly, for a variety of cosmological models $((\Omega_0, \lambda_0) = (0.3, 0.6), (0.2, 0.0), (1.0, 0.0))$, for redshifts between 0 and 3, and for various models of the mass distribution (homogeneous and point masses, NFW profiles and point masses), the relation is approximated very well by $1 - \eta \approx 0.6 f_\mathrm{c}$. Similar results were found by Giblin *et al.*[46], who used a much more realistic model of the mass distribution, based on state-of-the-art simulations, "the first numerical cosmological study that is fully relativistic, non-linear and without symmetry"[47,48]. They stressed the scatter in the distance for a given redshift, which generally increases with redshift and is also dependent on the line of sight.

Classical cosmology

My definition of classical cosmology is the comparison with observations of a quantity the dependence of which on z depends on the values of H_0, λ_0, and Ω_0; such quantities depend on one or more types of distance. Since η influences the dependence of distances on z, it is clear that its effects must be considered when doing observational cosmology.

Magnitude–reshift relation

One of the most important advances in observational cosmology has been the application of the m–z relation to type-Ia supernovae. Goobar & Perlmutter[34] discussed the feasability of such a programme, and were later involved in the Supernova Cosmology Project, which reported measurements of λ_0 and Ω_0 based on 42 supernovae[49,50], a result confirmed and published slightly earlier by the High-z Supernova Search team[51,52]. While there had been hints, based on joint constraints from several cosmological tests, not only that the cosmological constant is positive but also that it has such a value that the Universe is currently accelerating[32,33], the m–z relation for type-Ia supernovae was the first cosmological test which, by itself, confirmed such a value for λ_0. (Contrary to some claims, this test does not 'directly' measure acceleration in any meaningful sense, even if one does not adopt the extreme view that all that is ever 'really' measured in observational astronomy, whether in imaging or in spectroscopy, are photon counts as a function of position on a detector.) Perlmutter *et al.*[49] also checked for the influence of η, using the FORTRAN code of KHS to compare the standard distance to that of two other models, one with $\eta = 0$ and the other with $\eta = \eta(\Omega_0)$, the latter based on the idea that all matter is in clumps for $\Omega_0 \leq 0.25$ and for $\Omega_0 \geq 0.25$ the fraction $0.25/\Omega_0$ is in clumps, thus $\eta = 0$ for $\Omega_0 \leq 0.25$, otherwise $\eta = 1 - 0.25/\Omega_0$. Their conclusion, based of course on their data at the time, is that significant differences occur only for models ruled out by other arguments, *i.e.,* $\Omega_0 > 1$.

Kantowski *et al.*[36], still using the soon-to-be-obsolete q_0-notation, had pointed out that η should be taken into account when discussing the m–z relation for type-Ia supernovae. They also presented an analytic solution for $\lambda_0 = 0$ but arbitrary Ω_0 and η, and introduced the parameter ν:

$$\eta = 1 - \frac{\nu(\nu+1)}{6}\,, \tag{9}$$

due to the fact that there are analytic solutions for certain integer values of ν.

Iwata & Yoo[53] assumed a flat universe and, taking Ω_0 from CMB measurements, then calculated $\eta(z)$ such that the cosmological parameters from the m–z relation for type-Ia supernovae agree; this was done for four different scenarios. This is complementary to the work of Helbig[54] (next paragraph) who, at almost exactly the same time, considered only constant η but for arbitrary FRW models, determining the value of η such that the m–z relation for type-Ia supernovae results in the same values for λ_0 and Ω_0 as those derived from the CMB.

Helbig[54] investigated the influence of η, noting that more and higher-redshift data had become available. While the data were not good enough to determine λ_0, Ω_0, and η simultaneously[¶], the constraints in the λ_0–Ω_0 plane depend strongly on η. Only by assuming $\eta \approx 1$ does one recover the concordance-cosmology values of $\lambda_0 \approx 0.7$ and $\Omega_0 \approx 0.3$. Since these values are now known to high precision independently of the m–z relation for type-Ia supernovae (*e.g.,* refs. 55–57), one can use the m–z relation for type-Ia supernovae to measure η. The result $\eta \approx 1$ agrees well with other tests to determine η from observations. (While no useful constraints are possible, the global maximum likelihood in the λ_0–Ω_0–η cube also indicates a high value of η.) Unknown to

¶This would imply the somewhat dubious assumption that η is independent of both redshift and the line of sight. Of course, more-realistic models could take such effects into account, but obviously the data would not be able to constrain them since even the simpler model with a constant η could not be constrained.

me at the time, very similar results, based on the same data, were obtained by Yang *et al.*[58], Bréton & Montiel[59], and, somewhat later, Li *et al.*[60] (the latter two restricted to a flat universe). While perhaps not surprising, it is of course important in science for results to be confirmed by others working independently. Although they investigated a wider range of models, when restricted to standard FRW models, the results of Dhawan *et al.*[61] are also consistent.

As mentioned above, for objects with an appreciable angular size, it is not surprising that $\eta \approx 1$, since clumps would have to be very big and thus few and far between in order that a relatively thick beam could be under-dense; surprising is that $\eta \approx 1$ holds for even the very thin beams of supernovae. The supernovae discussed above are at distances of up to a gigaparsec or so while the physical size of the visible supernovae is roughly the size of the Solar System, so the beams are very thin indeed.

Since the observations indicate that $\eta \approx 1$, one can ask whether this is true 'on average' as discussed by Weinberg[19], or whether each line of sight indicates $\eta \approx 1$. In the former case, one would expect a dispersion in the distance at high redshift. Indeed, the scatter does increase with redshift, but so do the observational uncertainties. Since their quotient is independent of redshift, this indicates that each line of sight indicates $\eta \approx 1$, in other words that all lines of sight fairly sample the mass distribution of the Universe[62].[‖]

One of the basic cosmological tests is the 'standard rod' test, *i.e.,* the comparison of the angular size as a function of redshift of an object of given size to the theoretical expection derived from the angular-size–redshift relation, which in turn depends on the the cosmological parameters. (By the same token, the calculation of the physical size from the observed angular size depends on the cosmological model, and on η.) Although a classic test, no useful constraints have been derived from it — except in the cases of the CMB and BAO, though here the corresponding physical lengths are so large that the ZKDR distance plays no role (*e.g.,* ref. 63) — primarily because of the difficulty in finding a standard rod. Nevertheless, some progress can be made. For example, using a large sample of milliarcsecond radio sources Alcaniz *et al.*[64], assuming a Gaussian prior $\Omega_0 = 0.35 \pm 0.07$ in a flat universe, found the best fit at $\Omega_0 = 0.35$ and $\eta = 0.8$ (with no prior on Ω_0, the results were $\Omega_0 = 0.2$ and $\eta = 1.0$), consistent with the results with respect to the m–z relation discussed above.

Gravitational lensing

In gravitational lensing, it is clear that the approximation of a completely homogeneous universe with regard to light propagation cannot be valid, since otherwise there would be no gravitational lensing. Perhaps for this reason, the ZKDR distance has been used more in gravitational lensing than in other fields.

Asada[65] assumed the validity of the ZKDR distance and used it to investigate how inhomogeneities affect observations of gravitational lenses, in particular bending angle, lensing statistics, and time delay. An interesting analytic

[‖]This does not imply that the Universe is effectively homogeneous, but rather that the distance calculated from redshift is approximately the same as that which would be calculated in an effectively homogeneous universe.

result is that all three combinations of distances** involved in these phenomena are monotonic with respect to the clumpiness for all combinations of λ_0, Ω_0, and source and lens redshifts. The clumpiness decreases the bending angle and number of strong-lensing events and increases the time delay. In the first two cases, decreasing η has the same effect as decreasing λ_0. In other words, using a value of η which is too large (such as the common assumption $\eta = 1$) would lead one to underestimate the value of λ_0.††

At almost the same time (publication was one month later) and completely independently, Helbig[66] investigated not the common gravitational-lensing topics mentioned above, but rather the correlation between image separation and source redshift, in a reply to the work of Park & Gott[67] who had noted a negative correlation. Helbig[66] showed that decreasing η has the same effect as decreasing $K := \lambda_0 + \Omega_0 - 1$ (*i.e.,* this effect is also monotonic in η); also, decreasing η reduces the differences between cosmological models characterized by λ_0 and Ω_0. The strong negative correlation reported by Park & Gott[67], though, seems to be based on an unclean data sample and also is not statistically significant.

The basic observational quantities in a strong (*e.g.,* multiple-image) gravitational lens system — angles, flux ratios — are dimensionless, except for the time delays between pairs of images[68]. This allows one to determine the Hubble constant from a measurement of the time delay, assuming a mass model for the lens. However, this is true only in the low-redshift limit; at higher redshift, the cosmological model plays a role[69]. The cosmological parameters Ω_0 and λ_0 are now known very well from cosmological tests other than gravitational-lensing time delays (*e.g.,* refs. 55–57); one could thus assume them to be exactly known and use observations related to cosmological distances to determine η (*e.g.,* ref. 54).Within the uncertainties as they were 35–40 years ago, for the angular-size distance, at low redshift the values of Ω_0 and λ_0 are more important, while η becomes more important at high redshift (see Fig. 2). Due to the different combination of angular-size distances, for lensing statistics the effect of η tends to cancel (*e.g.,* ref. 2) while in the case of gravitational-lensing time delays the importance of η is enhanced even at lower redshift (*e.g.,* refs. 70,71).

Kayser & Refsdal[70] illustrated this dramatically for several world models with $\lambda_0 = 0$, comparing the $\eta = 1$ and $\eta = 0$ cases. For the double quasar 0957+561[72], the cosmological correction factor (which gives the influence of the cosmological model compared to the limiting low-redshift case) was calculated for σ_0 values ranging from 0 to 2 (corresponding to $0 \leq \Omega_0 \leq 4$) with q_0 values of 1.0, 0.5, 0.0, and -1 ($\lambda_0 = \sigma_0 - q_0$). (Note that here, $\sigma = \Omega/2$, once a

**The combinations are $D_{\rm ds}/D_{\rm s}$, $D_{\rm d}D_{\rm ds}/D_{\rm s}$, and $D_{\rm d}D_{\rm s}/D_{\rm ds}$, respectively. The subscripts refer to the deflector (lens) and source. In the case of only one subscript, it is the second, the first being understood to refer to the observer. This is probably the most common notation. Other schemes explicitly write the first subscript when it refers to the observer as well, use 'l' instead of 'd' to refer to the lens (deflector), use capital letters, or some combination of these. The same subscripts are used to refer to the corresponding redshifts, *e.g.,* $z_{\rm s}$, though sometimes $z_{\rm d}$ is used in the sense of a variable and $z_{\rm l}$ to refer to the redshift of an explicit gravitational lens.

††Note that this is opposite to the effect in the m–z relation.

common notation, and has nothing to do with $\sigma = r/R$ in Fig. 1.) Helbig[71] repeated the exercise for arbitrary combinations of λ_0, Ω_0, and η, again showing the importance of η, which has become even more important now that the values of λ_0 and Ω_0 are so well known.

While the idea is simple in principle[68], in practice many details need to be taken into account when determining H_0 from gravitational-lens time delays (especially if the uncertainties should be small enough to be competitive with other methods), such as measuring the time delay itself and determining realistic uncertainties (*e.g.,* ref. 73) and constructing a realistic mass model for the lens (*e.g.,* refs. 74,75). At this level of detail, characterizing the density along the line of sight by a single parameter η, or even $\eta(z)$, is too coarse. Rather, one attempts to measure the mass distribution explicitly, by counting galaxies (*e.g.,* ref. 76) or using weak gravitational lensing (*e.g.,* ref. 77).

Weak gravitational lensing is normally defined as gravitational lensing without multiple images. If the source can be resolved, then information can be gleaned from the distortion of the image. In such a case, however, if the source is at a cosmological distance, $\eta \approx 1$ (because the distance implies a large physical extent near the source, averaging over the matter distribution, and because it appears that, at large redshift, distances behave as if $\eta \approx 1$, as noted above). Relevant for the ZKDR distance with respect to weak lensing is thus weak lensing of point sources.

Wang[78] pointed out that weak lensing leads to a non-Gaussian magnification distribution of standard candles at a given redshift, due to the fact that η can vary with direction. One can thus think of our Universe as a mosaic of cones centred on the observer, each with a different value of η, where there is a unique mapping between η and the magnification of a source. Of course, since the ZKDR distance depends on Ω_0 and λ_0 as well as η, different cosmological models can lead to very different magnification distributions for the same matter distribution.[‡‡]

Williams & Song[79] took the opposite approach: assuming that the standard distance ($\eta = 1$) is correct, they found that bright SNe are preferentially found behind regions (5–15 arcmin in radius) that are over-dense in the foreground due to $z \approx 0.1$ galaxies, the difference between brightest and faintest being about 0.3–0.4 mag. (In other words, the fact that bright supernovae are preferentially found behind over-dense regions indicates that the standard distance is incorrect.) The effect, significant at > 99 per cent, depends on the amount and distribution of matter along the line of sight to the sources but not on the details of the galaxy-biasing scheme.

In a very detailed work, Kainulainen & Marra[80] studied the effects of weak gravitational lensing caused by a stochastic distribution of dark-matter haloes, restricted to flat FRW models and examining those with $\Omega_0 = 0.28$ (close to

‡‡Note that her claim that Perlmutter *et al.*[49] "assumed a smooth universe" is somewhat misleading. While they did not consider a direction-dependent η, they did compare the extreme cases of $\eta = 1$ and $\eta = 0$ as well as the case of an Ω_0-dependet η (*i.e.,* galaxies assigned to clumps and the rest of the matter distributed smoothly, which implies an increase in η with increasing Ω_0), in all cases using the code of KHS.

the current concordance model) and $\Omega_0 = 1$ (the Einstein–de Sitter model) as representative examples. In particular, they calculated the difference between the distance in their model and the ZKDR distance for $\eta = 0.5$ and $\eta = 0$ for these two models, finding a maximum relative error of only 0.06 for the extreme case of the empty-beam Einstein–de Sitter model at $z = 1.6$ (the upper limit of their redshift range). This is yet another example of the proof of the validity of the assumptions underlying the ZKDR distance.

The situation today

The basis of observational cosmology is calculating the dependence of some observational quantity — usually related to some distance — on redshift for a variety of cosmological models, then determining the corresponding cosmological parameters *via* finding the model which gives the best fit to the data. Small-scale inhomogeneities can affect the relation between redshift and distance, thus it at least needs to be investigated whether results depend on the amount of inhomogeneity.

"We are now in the era of precision cosmology" is something that I have heard at many talks and read in many papers. At the same time, one rarely hears about the ZKDR distance today. True, one rarely heard of it 40 years ago, but it was not necessary then since there was essentially no high-redshift observational cosmology involving objects of essentially pointlike angular size. Nevertheless, even practical "astronomers at the telescope", as Sandage referred to himself and colleagues[1], were aware of the fact that one sees high-redshift objects by looking between foreground galaxies (due to selection effects or design), and thus the ZKDR distance should be used in those cases. While the main Supernova Cosmology Project paper[49] did look at the influence of η, and (correctly) concluded that it wasn't necessary to take into account with the data they had at the time, after their work such tests were either not done or if so not published until about 10 years later. Even then, it was not the teams doing supernova cosmology themselves but rather others re-analysing data from the literature. Perhaps the impression had been created that η didn't need to be taken into account, even though it should have been obvious that one should at least check when higher-redshift data became available. To be sure, many independent investigations had come to the conclusion that $\eta \approx 1$, not just on average, as is to be expected, at least under certain assumptions[19], but also for each individual line of sight, but that does not seem to be the reason for the neglect of η. Perhaps there was a desire to make a clean argument, especially as the results are based on one of *the* classical cosmological tests. Taking η into account weakens the constraints on λ_0 and Ω_0 (even though, in principle, such constraints might be more accurate even though less precise) and also makes clear the approximation — complete homogeneity — used. Of course, the CMB — like baryon acoustic oscillations (BAO) — has more parameters, but these are well understood; η itself is a very rough approximation; the more one thinks about it, the clearer it becomes how rough it is. While one can both see a variation of η among lines of sight as an additional source of uncertainty as well as, at least numerically, use $\eta = \eta(z)$, one doesn't know *a priori* which lines of sight are under-dense (or over-dense) nor the form of the function $\eta = \eta(z)$ (though, as mentioned above, it probably generally increases with z). Indeed, the current practice, for those who care about such details, is to try to *measure* η along a given line of sight and take the value explicitly into account (though this is often expressed as "measuring the convergence"), or even taking individual galaxies along the line of sight explicitly into consideration when modelling a gravitational-lens system (as opposed to just a main lens and some additional convergence and/or shear).

On the other hand, we have been lucky. It seems that, at least to the extent that inhomogeneity can be described by η, observations do indicate $\eta \approx 1$ (at least effectively), so in practice leaving it out of consideration doesn't make a very big difference. It didn't have to be that way. There is no question that the ZKDR distance is appropriate in a universe with a mass distribution corresponding to that on which the assumptions leading to the ZKDR distance are based. However, it seems that our Universe is not like that. Rather, it consists of a 'cosmic web', a large-scale structure of voids, sheets, filaments, and rich clusters of galaxies. There is probably no smooth component on cosmological scales, but a mixture of regions of density which are less than and others which are (very much) more than the average density. For an object distant enough that the inhomogeneity can appreciably affect the distance, the beam traverses such a large portion of the Universe that it is effectively a fair sample, so that the distance calculated from the redshift is roughly the same as the ZKDR distance with $\eta \approx 1$. Also, the classic cosmological tests such as the m–z relation have to some extent been superseded by the CMB, BAO, and weak lensing (of resolved galaxies) which, due to the larger angular scales, are much less sensitive to η. In fact, measuring λ_0 and Ω_0 *via* these other tests allows one to use the classic tests to measure η, rather than it being an additional (nuisance) parameter, which can provide some information on the distribution of dark matter.

Acknowledgements

Fig. 1 is taken from Helbig[8] and Fig. 2 from Kayser *et al.*[3]. This text is an 'executive summary' of and borrows heavily from my recent review of this topic[8] which can be consulted for more details.

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