Monthly Notices of the ROYAL ASTRONOMICAL SOCIETY

MNRAS **00,** 1 (2020) doi:10.1093/mnras/staa1082
Advance Access publication 2020 April 24

The flatness problem and the age of the Universe

Phillip Helbig ★

Institut d'Astrophysique et de Géophysique (Bât. B5c), Université de Liège, Quartier Agora, Allée du 6 Août, 19c, B-4000 Liège 1 (Sart-Tilman), Belgium

Accepted 2020 April 10. Received 2020 April 8; in original form 2020 February 21

ABSTRACT

Several authors have made claims, none of which has been rebutted, that the flatness problem, as formulated by Dicke and Peebles, is not really a problem but rather a misunderstanding. Nevertheless, the flatness problem is still widely perceived to be real. Most of the arguments against the idea of a flatness problem are based on the change with time of the density parameter Ω and normalized cosmological constant λ and, since the Hubble constant H is not considered, are independent of time-scale. An independent claim is that fine-tuning is required in order to produce a Universe which neither collapsed after a short time nor expanded so quickly that no structure formation could take place. I show that this claim does not imply that fine-tuning of the basic cosmological parameters is necessary, in part for similar reasons as in the more restricted flatness problem and in part due to an incorrect application of the idea of perturbing the early Universe in a gedankenexperiment; I discuss some typical pitfalls of the latter.

Key words: cosmological parameters – cosmology: miscellaneous – cosmology: theory.

1 INTRODUCTION AND PREVIOUS ARGUMENTS AGAINST THE FLATNESS PROBLEM

Here, I consider only ideal Friedmann (1922,1924) models, because historically fine-tuning claims have been discussed within the context of those models; also, the issues remain even in more-realistic models. In other words, the universe is considered to be homogeneous and isotropic and to consist of non-relativistic matter ('dust') and a cosmological constant (which does not vary in time). That is an acceptable approximation to our Universe at late times. At early times, radiation (or, more generally, relativistic matter as well) must be taken into account, but the arguments presented here are qualitatively the same whether that is included or neglected. Of course, essentially all problems can be solved by postulating appropriate initial conditions, but that is rightly seen as unsatisfying, although at some level some properties of the Universe might be due to nothing other than initial conditions. Note, however, that the flatness problem is different from another problem of classical cosmology, the isotropy or horizon problem. The latter does not exist, by definition, in an ideal Friedmann universe, while the point of the former is that *even given the fact* that the Universe is described by a Friedmann model (*why* that is the case is, of course, a different question), there is something puzzling about the *values* of the cosmological parameters which are observed.

For a homogeneous and isotropic ('Robertson–Walker') universe consisting of non-relativistic matter ('dust') of density ρ and the cosmological constant Λ (with dimension time^{-2}), the change in scale factor with time is described by the Friedmann equation

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3} + \frac{\Lambda R^2}{3} - kc^2, \tag{1}$$

with the dimensionless constant k equal to -1, 0, and $+1$ depending on spatial curvature (negative, vanishing or positive, respectively); R is the scale factor (with dimension length) of the universe, G is the gravitational constant, and c is the speed of light. It is useful to define the following terms:

$$\begin{aligned}
H &:= \frac{\dot{R}}{R} \\
\lambda &:= \frac{\Lambda}{3H^2} \\
\Omega &:= \frac{\rho}{\rho_{\mathrm{crit}}} && \equiv \frac{8\pi G\rho}{3H^2} \\
K &:= \Omega + \lambda - 1 \\
k &:= \mathrm{sign}(K) \\
q &:= \frac{-\ddot{R}R}{\dot{R}^2} && \equiv \frac{-\ddot{R}}{RH^2} \equiv \frac{\Omega}{2} - \lambda.
\end{aligned}$$

The Hubble constant H has the dimension time^{-1}; all other quantities defined above are dimensionless: the normalized cosmological constant λ, the density parameter Ω, the curvature parameter K, and the deceleration parameter q. ρ_{crit} is known as the critical density[1]

★ E-mail: helbig@astro.multivax.de, Phillip.Helbig@doct.uliege.be

[1]For $\lambda = 0$ and $k = 0$, $\rho = \rho_{\mathrm{crit}} = (3H^2)/(8\pi G)$. This density is 'critical' in the sense that, for $\lambda = 0$, a greater (lesser) density implies a positive (negative) curvature and a universe (assumed to be expanding now) which will collapse in the future (expand forever); similarly, for $k = 0$, a greater (lesser) density implies a negative (positive) cosmological constant and a universe (assumed to be expanding now) which will collapse in the future (expand forever). However, in the general case ($k \neq 0$ and $\lambda \neq 0$), ρ_{crit} does not have any special meaning, though Ω remains a useful parameter.

(see Helbig 2012 or Kayser, Helbig & Schramm 1997 for more details on this notation).

From the definitions above, it follows that

$$R = \frac{c}{H}\frac{k}{\sqrt{|\Omega + \lambda - 1|}} = \frac{c}{H}\frac{k}{\sqrt{|K|}} \tag{2}$$

for $k \neq 0$; for $k = 0$, the radius of curvature is infinite – the scale factor *at the present time* is then usually taken to be c/H_0.[2] Note that K is positive, if the curvature is positive. Often, Ω_k is defined as $-K$, so that the Friedman equation is $\Omega_m + \Omega_\lambda + \Omega_k = 1$ ($\Omega_m \equiv \Omega$, $\Omega_\lambda \equiv \lambda$).[3] Denoting the current epoch of observation with the suffix 0 and using the definitions above, one can write the Friedmann equation as

$$\dot{R}^2 = \dot{R}_0^2\left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0\right). \tag{3}$$

From this, one can calculate the age of the universe as a function of the scale factor

$$t = \int_0^R \frac{\mathrm{d}R}{\sqrt{\dot{R}_0^2\left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0\right)}}. \tag{4}$$

(Note that equation (4) can be inverted to give the scale factor as a function of time.) Using the definition of H, equation (4) can be written as

$$t = \frac{1}{H_0}\int_0^R \frac{\mathrm{d}R}{\sqrt{\left(\frac{\Omega_0 R_0^3}{R} + \lambda_0 R^2 - K_0 R_0^2\right)}}. \tag{5}$$

Alternatively, dividing the Friedmann equation, equation (3), by R^2 and factoring out R_0^{-2} on the right-hand side results in

$$\left(\frac{\dot{R}}{R}\right)^2 = \left(\frac{\dot{R}_0}{R_0}\right)^2\left(\frac{\Omega_0 R_0^3}{R^3} + \lambda_0 - \frac{K_0 R_0^2}{R^2}\right) \tag{6}$$

or, due to the definition of H, in

$$H^2 = H_0^2\left(\frac{\Omega_0 R_0^3}{R^3} + \lambda_0 - \frac{K_0 R_0^2}{R^2}\right), \tag{7}$$

which expresses the Hubble constant as a function of the scale factor. From the definitions above follows the dependence of λ on H,

$$\lambda = \lambda_0\left(\frac{H_0}{H}\right)^2. \tag{8}$$

Since the density depends on the scale factor,

$$\rho = \rho_0\left(\frac{R_0}{R}\right)^3, \tag{9}$$

Ω depends on it as well as on H,

$$\Omega = \Omega_0\left(\frac{H_0}{H}\right)^2\left(\frac{R_0}{R}\right)^3. \tag{10}$$

Note that H and R are related by equation (7). Thus, in an expanding universe, λ and Ω can increase with time only if H decreases.[4]

Although it had been discussed earlier (e.g. Dicke 1970), most treatments of the flatness problem can be traced back to the formulation of the problem by Dicke & Peebles (1979), who pointed out that a universe with $\Omega \neq 1$ is inherently unstable.[5] Many concluded from that that $\Omega = 1$ must hold exactly, which, if one assumes that $\Lambda = 0$ – which was common in the time after Dicke & Peebles (1979) until observations made it clear in the 1990s that $\Lambda > 0$ (at least when observations are interpreted within the context of Friedmann–Robertson–Walker (FRW)[6] models) – implies that our Universe must be the Einstein–de Sitter universe exactly or that some process, such as inflation, drove it very close to the Einstein–de Sitter universe.[7] Both the fine-tuning argument ('there must be some reason why $\Omega = 1$ to very high precision in the early Universe') and the instability argument ('even given that $\Omega = 1$ to very high precision in the early Universe, if Ω is not exactly 1, then it would be unlikely to observe $\Omega \approx 1$ today') have been shown to be wrong.

The fine-tuning argument is wrong basically because Ω is not the appropriate parameter to use (e.g. Cho & Kantowski 1994; Coule 1995; Evrard & Coles 1995; Coles & Ellis 1997; Kirchner & Ellis 2003; Adler & Overduin 2005; Gibbons & Turok 2008; Roukema & Blanlœil 2010; Helbig 2012; Holman 2018); that is most easily seen by studying the change in λ and Ω during the evolution of the universe as a dynamical system (e.g. Stabell & Refsdal 1966; Ehlers & Rindler 1989; Goliath & Ellis 1999; Uzan & Lehoucq 2001; Coley 2003; Wainwright & Ellis 2005), some such studies explicitly pointing out that that point of view demonstrates the lack of a flatness problem in classical cosmology (e.g. Kirchner & Ellis 2003; Lake 2005; Helbig 2012; Holman 2018). Note that Collins & Hawking (1973) claimed that the Anthropic Principle could solve the flatness problem, long before the popular formulation by Dicke & Peebles (1979) and before Carter (1974) publicly proposed the Anthropic

[2]In general, one can describe the change in time of the scale factor in relation to a fiducial, usually the current, scale factor; a is often defined as the relative scale factor R/R_0 and is thus dimensionless. Another common approach, which I use here, is to take the scale factor R to be the radius of curvature as given by equation (2) for $k \neq 0$. In this case, this works for any time t, not just t_0. For $k = 0$, R_0, the scale factor at a fiducial time (usually taken to be the present), is arbitrary but is often set to c/H_0. Note, however, that in general $R \neq c/H$, including the flat case where $R_0 = c/H_0$ ($R = c/H$ at all times in the special case of the relativistic equivalent of the Milne model with $\lambda = 0$ and $\Omega = 0$ and hence $k = -1$).

[3]I have long used K as defined above, as do Goliath & Ellis (1999), though Wainwright & Ellis (2005) define K with the opposite sign, using it as others use Ω_k. One also sees K used as I use k, (e.g. Ellis, Maartens & MacCallum 2012).

[4]This is an important point. Since all non-empty big-bang models begin their evolution arbitrarily close to the Einstein–de Sitter model with $\lambda = 0$ and $\Omega = 1$, large values of those parameters can be due *only* to a low value of the Hubble constant.

[5]They assumed that $\Lambda = 0$. If one replaces Ω with $\Omega + \lambda$, then some, but not all, of their arguments still hold. For example, if $\Omega + \lambda = 1$ exactly, then that holds for all time. On the other hand, the individual values of Ω and λ evolve with time (even though their sum is constant) in a flat universe (except in the cases of the Einstein–de Sitter universe with $\Lambda = 0$ and $\Omega = 1$ and the de Sitter universe with $\Lambda = 1$ and $\Omega = 0$); if $\Lambda < 0$ then both evolve to $(-)\infty$ at the time of maximum expansion (such universes always collapse in the future).

[6]One sometimes sees 'FLRW' instead of 'FRW', in order to include Lemaître. While I have great respect for Lemaître, Friedmann had discussed the full range of RW models based on GR, so FRW is a sufficient abbreviation in that context.

[7]Of course, even if there is no flatness problem in classical cosmology, it does not follow that inflation could not have happened.

Principle, though Hawking was aware of Carter's ideas (Williams 2007).[8]

Lake (2005) demonstrated that the instability argument does not hold for universes which expand forever because λ and Ω are large and the universe significantly non-flat only in the case that they are fine-tuned in the sense that $\alpha = \mathrm{sign}(K)(27\Omega^2\lambda)/(4K^3) \approx 1$. Note that that is the opposite of the claim that fine-tuning is required in order to have a flat universe (though, as noted above, that claim is false). Lake suggested that α (essentially the product of the square of the mass of a spatially closed universe and the cosmological constant), which has a fixed value throughout the life of the universe, is what should be used to characterize model universes. Adler & Overduin (2005) discussed various definitions of 'nearly flat', using essentially the same parameter as α used by Lake (2005), and arrived at the same conclusion, namely that a significantly non-flat universe implies a fine-tuning in α.

Helbig (2012) showed that, while λ and Ω become arbitrarily large in a universe which collapses, that is the case only during a relatively short (and special) time in the lifetime of the universe, thus a typical observer would not measure very large values.[9] (That holds for all universes which collapse except some with $\lambda > 0$, but in those cases, Lake's fine-tuning argument applies.) Of course, Ω approaches 0 for almost[10] all universes which expand forever, but the fact that Ω is not observed to be arbitrarily small is no more puzzling than the fact that we are, in some sense, infinitely close to the big bang if the Universe will expand forever.

Holman (2018) discussed in detail various questionable arguments and misconceptions regarding the flatness problem as well as different varieties of it. Although not a review per se, it is an excellent treatment of the flatness problem and misunderstandings of it, exploring some of the arguments against it, in particular the 'reverse-fine-tuning' argument of Lake (2005) and the fractional-time-scale argument of Helbig (2012), as well as related issues in a wider context. Also, Lewis & Barnes (2017), in a book-length discussion of fine-tuning in physics and cosmology, came to the conclusion that the flatness problem is mostly harmless. (That is significant since they otherwise point out several examples of fine-tuning.)

Even though the arguments mentioned above have been around for years or even decades, the argument of Dicke & Peebles (1979) is still found in its original form in modern textbooks (e.g. Ryden 2017; Longair & Smeenk 2019; Wright 2020)[11] and review articles (e.g. O'Raifeartaigh et al. 2018; Adams 2019). Even many observational astronomers are familiar with the flatness problem and see inflation as an attractive solution (e.g. Schmidt 1989; Sandage 1995).

2 TIME-SCALE ARGUMENTS

The first suggestion that the flatness problem could be avoided via a time-scale argument seems to be due to Tangherlini (1993), though not in the context of an FRW universe. Using a similar argument, as noted above, Helbig (2012) pointed out that, in a universe which will collapse, a typical observer would not observe large values of Ω and λ. The important point is the *relative* amount of time during which Ω and λ are $\gg 1$.

However, it is sometimes claimed, following Dicke (1970), 'that any deviations from flat geometry in the early universe would quickly escalate into a runaway open or closed universe, neither of which is observed' (O'Raifeartaigh et al. 2018, footnote 40, is a typical example). So even though it has been demonstrated that there is no flatness problem, i.e. a typical observer should not be puzzled by the fact that large values of Ω and λ are not observed – because the corresponding cosmological models are unlikely (Lake 2005) or because such values occur only during a short and special time in the history of the universe (Helbig 2012) – nor is some sort of fine-tuning necessary to explain the fact that $\Omega = 1$ to high precision in the early universe, nevertheless it is often claimed that Dicke had a valid point: even if there is no flatness problem in those senses, some sort of fine-tuning is necessary, because otherwise a short time after the big bang the universe would have collapsed or Ω would have evolved to a value $\ll 1$. Just as the argument of the relative time-scale shows why the tight-rope analogy (e.g. Coles 2009) is misleading, the quick-escalation claim is also wrong for essentially the same reason, namely the use of an inappropriate gedankenexperiment. That claim is somewhat different from those in the papers cited farther above, because those involve only relative times. To repeat, the rebuttals above claim that there is no flatness problem (in the instability-problem sense) based on relative times (i.e., the universe can be substantially non-flat, but only for a relatively short time) or no times at all (i.e., Lake's argument that a substantially non-flat universe requires fine-tuning), or rebut the fine-tuning argument (since all FRW models start arbitrarily close to the Einstein–de Sitter model, no fine-tuning is needed to explain why $\Omega = 1$ to high precision in the early universe). In contrast, claims referring to the age of the universe must involve the Hubble constant, whereas the papers cited farther above discuss only λ and Ω.

The argument is usually something like this:

> Imagine, shortly after the big bang, slightly increasing the density of the Universe; that would cause it to collapse after a very short time, perhaps only a few seconds or less.

Another version replaces 'density' by 'density parameter', i.e. Ω. Increasing the density while keeping the Hubble constant H fixed would also increase Ω, and vica versa. However, one could also increase Ω by keeping the density constant and decreasing H. That should already hint at the resolution: equation (1), the Friedmann

[8] Collins & Hawking (1973) is a classic example of application of the Anthropic Principle. Their paper has been cited often, but usually not in connection with the flatness problem or the Anthropic Principle. Examples like that can be used to claim that the Anthropic Principle can explain everything. Even if true, it does not follow that the Anthropic Principle *does* explain everything. On the other hand, if, as in the case of the flatness problem, another, and presumably better, explanation is found, it does not follow that *everything* explained by the Anthropic Principle *must* have another, and presumably better, explanation (see Barrow & Tipler 1988 for (much) more on the Anthropic Principle).

[9] Depending on how large one deems that α must be in order not to be fine-tuned, this argument is probably somewhat weaker than that of Lake (2005) in that it cannot explain why $K = 1$ to within at least 1 per cent or so, as is indicated by observations, but it still shows that the original flatness-problem argument (which was essentially the question why Ω is not 10^5, say) is incorrect.

[10] The exceptions are the extremely fine-tuned cases of a universe which asymptotically approaches the static Einstein universe (in that case the value of R has an upper limit which is reached after an infinite time) and the Einstein–de Sitter model, which always has $\Omega = 1$; the latter expands forever and has no upper limit on R, but $\dot{R}$, H, and ρ all approach 0. (Although, as discussed above, models near the Einstein–de Sitter model are not fine-tuned, the Einstein–de Sitter model itself is infinitely fine-tuned.)

[11] Note, however, that Smeenk (2019), in a different chapter of the same book, takes a more balanced view.

equation, is called the Friedmann equation because it is an equation; it makes no sense to imagine changing just *one* parameter. One would have to change at least two in order for the equation to remain valid. However, in general, such changes as in the gedankenexperiment above describe universes very different from our own, such as a closed universe with a mass of one kilogram. Yes, such a universe might collapse after a very short time, but that is irrelevant since it is not our Universe nor even a slight perturbation of it in any meaningful sense. In other words, if the Universe is somehow perturbed early on, and one wants to calculate all possible observational quantities today, one must specify as well which quantities – α, H, Ω, λ, ρ, Λ, K, the age of the Universe at the time of the perturbation, the mass of the Universe (for $k = +1$) – one regards as fixed and adjust at least one other such that the Friedmann equation is still valid; changing just one is not possible.

Since the usual objection is at best not well defined and at worse misleading or even wrong, one could leave it at that, but let us consider it more quantitatively.

Note that the age of the universe, equation (4), implicitly depends on H_0, via R_0; rewritten as equation (5), taking the definition of H_0 into account, that dependency is even more explicit. Thus, any discussion of the age of the universe as a function of the cosmological parameters must include the Hubble constant, explicitly or implicitly. Consider a finite universe with positive curvature[12] so that the mass of the universe is given by

$$M = \rho V \tag{11}$$

$$= \rho \times 2\pi^2 R^3. \tag{12}$$

Making use of the definitions of Ω and R, equation (2), it follows that

$$M = \frac{3H^2\Omega}{8\pi G} 2\pi^2 R^3 \tag{13}$$

$$= \frac{3H^2\Omega}{8\pi G} 2\pi^2 \left(\frac{c}{H} \frac{1}{\sqrt{|\Omega + \lambda - 1|}} \right)^3 \tag{14}$$

$$= \frac{3\pi c^3 \Omega}{4GH \left(\sqrt{|\Omega + \lambda - 1|} \right)^3}. \tag{15}$$

The mass of the universe is constant in time and is proportional to $\Omega/(H\sqrt{|K|}^3)$. Since the arguments of Lake (2005) and Helbig (2012) make it unlikely that an observer would measure values of Ω or K which are not of order 1, it is clear that a large (in terms of mass) universe implies a low Hubble constant, at least for a typical observer (i.e. one living at a likely time in a likely universe). On the other hand, the age of the universe is also inversely proportional to H. Thus, all else being equal, a universe which collapses after a second would have a mass about that of a globular cluster, clearly very different from our Universe. That a small perturbation (of course, properly carried out, not just changing one parameter as in the typical gedankenexperiment) in the early Universe can result in a universe so different than ours is merely another aspect of the fine-tuning problem, or rather the lack thereof: *all* FRW models are arbitrarily close to the Einstein–de Sitter universe early on. Of course, one could have a highly non-flat universe today with the same age as that of our Universe, simply by adjusting H_0 to give the required age, though that would imply, in the $k = +1$ case, a much less massive universe; also due to the arguments of Helbig (2012) and Lake (2005), the corresponding values of the cosmological parameters would occur only for a *relatively* short time during the lifetime of the universe or α would have to be fine-tuned to be ≈ 1.[13] So, at best, one could argue that *a highly non-flat universe* must be fine-tuned in order to be as old as our Universe, but that would also differ in terms of mass. As noted above, it is impossible to have a universe which differs from ours in only *one* respect. However, that is not the usual claim; the usual claim is that *our* Universe, which is *nearly flat*, must be fine-tuned in order to be as old as it is. That is not the case; our Universe is old essentially because it is massive. Lake (2005) argued that α, essentially the product of the square of the mass of the universe and Λ, should be thought of as the free parameter when ‘choosing a universe’. (Since $\alpha = 0$ for $\lambda = 0$ or $\Omega = 0$, one can use the corresponding non-zero factor as the free parameter in those cases, i.e. Λ or M.) It should be clear that a small perturbation to our Universe, caused by changing some parameters in the Friedmann equation at a time shortly after the big bang, should be small in terms of that parameter, which obviously does not lead to a vastly different age of the Universe. Note also that since the mass and the cosmological constant Λ are constant during the evolution of the universe, the time of the perturbation does not matter.

3 SUMMARY AND CONCLUSIONS

Since its original formulation by Dicke (1970) and the popularization by Dicke & Peebles (1979), especially after the idea of inflation became popular (e.g. Guth 1981; Linde 1982), many arguments were made, though largely ignored, which demonstrated that neither is fine-tuning in the early Universe necessary to explain the values of λ_0 and Ω_0 observed today, whatever they might be, nor is it puzzling that we do not observe values $\gg 1$ or $\ll 1$ for them. (As stressed by Holman 2018, those are two sides of the same coin.) Also, the argument that the early Universe must have been fine-tuned in order for it to last as long as it has is wrong since it is based on the impossible idea of modifying just *one* parameter in the early Universe. Even if the early Universe is ‘correctly perturbed’ in the sense of retaining the validity of the Friedmann equation, that argument is wrong since it is essentially a variation of the bogus instability argument. In addition, a universe with a significantly shorter time-scale than our Universe would be significantly different in other ways, and thus ruled out by weak-anthropic arguments (i.e. those using the weak form of the Anthropic Principle).

The arguments discussed above have been made mostly in the leading journals in the field, often by people well known and respected for other contributions, yet the argument of Dicke & Peebles (1979) is still often stated as an unquestionable fact. Perhaps the real flatness problem is the question as to why that is the case.

ACKNOWLEDGEMENTS

I thank the anonymous referee for helpful comments. This research has made use of NASA’s Astrophysics Data System Bibliographic Services.

[12] In this paper, I assume trivial topology. Also, for definiteness, I concentrate on the $k = +1$ case, though similar arguments are also possible for other values of k. Note that Di Valentino, Melchiorri & Silk (2020a,b) claim that the CMB as measured by *Planck* favours a closed Universe, though the situation is not completely clear since a closed Universe reduces the tension among some sets of observations but increases it among others, possibly indicating that assuming a flat Universe hides some discrepancies.

[13] Of course, if the universe collapses, there is the stipulation that the total lifetime must be more than twice the current age, assuming, as always in this paper, that the universe is expanding now.

REFERENCES

Adams F. C., 2019, Phys. Rep., 807, 1
Adler R. J., Overduin J. M., 2005, Gen. Rel. Grav., 37, 1491
Barrow J. D., Tipler F. J., 1988, The Anthropic Cosmological Principle. Oxford Univ. Press, Oxford
Carter B., 1974, in Longair M. S., ed., Confrontation of Cosmological Theories With Observational Data. Reidel Publishing Co., Dordrecht, p. 291
Cho H. T., Kantowski R., 1994, Phys. Rev. D, 50, 6144
Coles P., 2009, The Cosmic Tightrope, https://telescoper.wordpress.com/2009/05/03/the-cosmic-tightrope/
Coles P., Ellis G. F. R., 1997, Is the Universe Open or Closed? Cambridge Univ. Press, Cambridge
Coley A. A., 2003, Dynamical Systems and Cosmology. Astrophysics and Space Science Library Vol. 291. Springer Netherlands, Dordrecht
Collins C. B., Hawking S. W., 1973, ApJ, 180, 317
Coule D. H., 1995, Class. Quant. Grav., 12, 455
Dicke R. H., 1970, Gravitation and the Universe: Jayne Lectures for 1969. Amer. Phys. Soc., Philadelphia
Dicke R. H., Peebles P. J. E., 1979, in Hawking S. W., Israel W., eds, General Relativity: An Einstein Centenary Survey. Cambridge Univ. Press, Cambridge, UK, p. 504
Di Valentino E., Melchiorri A., Silk J., 2020a, Nat. Astr., 4, 196
Di Valentino E., Melchiorri A., Silk J., 2020b, Cosmic Discordance: Planck and luminosity distance data exclude LCDM , preprint (arXiv:2003.04935)
Ehlers J., Rindler W., 1989, MNRAS, 238, 503
Ellis G. F. R., Maartens R., MacCallum M. A. H., 2012, Relativistic Cosmology. Cambridge Univ. Press, Cambridge
Evrard G., Coles P., 1995, Class. Quant. Grav., 12, L93
Friedmann A. A., 1922, Zeitschr. Phys., 1, 377
Friedmann A. A., 1924, Zeitschr. Phys., 21, 326
Gibbons G. W., Turok N., 2008, Phys. Rev. D, 77, 063516
Goliath M., Ellis G. F. R., 1999, Phys. Rev. D, 2, 023502
Guth A. H., 1981, Phys. Rev. D, 23, 347
Helbig P., 2012, MNRAS, 421, 561
Holman M., 2018, Found. Phys., 48, 1617
Kayser R., Helbig P., Schramm T., 1997, A&A, 318, 680
Kirchner U., Ellis G. F. R., 2003, Class. Quant. Grav., 20, 1199
Lake K., 2005, Phys. Rev. Lett., 94, 201102
Lewis G. F., Barnes L. A., 2017, A Fortunate Universe. Cambridge Univ. Press, Cambridge
Linde A., 1982, Phys. Lett. B, 108, 389L
Longair M. S., Smeenk C., 2019, in Kragh H., Longair M. S., eds, The Oxford Handbook of the History of Modern Cosmology. Oxford Univ. Press, Oxford, p. 433
O'Raifeartaigh C., O'Keeffe M., Nahm W., Mitton S., 2018, Eur. Phys. J. H, 43, 73
Roukema B. F., Blanlœil V., 2010, Class. Quant. Grav., 27, 245001
Ryden B., 2017, Introduction to Cosmology, 2nd edn. Cambridge Univ. Press, Cambridge
Sandage A. R., 1995, in Binggeli B., Buser R., eds, The Deep Universe. Springer, Berlin, p. 1
Schmidt M., 1989, Interview of Maarten Schmidt by Alan Lightman on 1989 March 28, Niels Bohr Library & Archives, American Institute of Physics, College Park, MD, https://www.aip.org/history-programs/niels-bohr-library/oral-histories/33967
Smeenk C., 2019, in Kragh H., Longair M. S., eds, The Oxford Handbook of the History of Modern Cosmology. Oxford Univ. Press, Oxford, p. 517
Stabell R., Refsdal S., 1966, MNRAS, 132, 379
Tangherlini F. R., 1993, Il Nuovo Cimento, 108B, 1253
Uzan J.-P., Lehoucq R., 2001, Eur. J. Phys., 22, 371
Wainwright J., Ellis G. F. R., eds, 2005, Dynamical Systems in Cosmology. Cambridge Univ. Press, Cambridge
Williams B. R.-W., 2007, Master's thesis. Iowa State University, Ames, IA
Wright E. L., 2020, in Malkan M. A., Zuckerman B., eds, Origin and Evolution of the Universe. World Scientific, Singapore, p. 8

This paper has been typeset from a TEX/LATEX file prepared by the author.