EXPANSIONS IN CANTOR REAL BASES

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Abstract. We introduce and study series expansions of real numbers with an arbitrary Cantor real base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$, which we call $\boldsymbol{\beta}$-representations. In doing so, we generalize both representations of real numbers in real bases and through Cantor series. We show fundamental properties of $\boldsymbol{\beta}$-representations, each of which extends existing results on representations in a real base. In particular, we prove a generalization of Parry's theorem characterizing sequences of nonnegative integers that are the greedy $\boldsymbol{\beta}$-representations of some real number in the interval $[0,1)$. We pay special attention to periodic Cantor real bases, which we call alternate bases. In this case, we show that the $\boldsymbol{\beta}$-shift is sofic if and only if all quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 are ultimately periodic, where $\boldsymbol{\beta}^{(i)}$ is the i-th shift of the Cantor real base $\boldsymbol{\beta}$.

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1. Introduction

Cantor expansions of real numbers were originally introduced by Cantor in 1869 [3]. A real number $x \in [0,1)$ is represented via a base sequence $(b_n)_{n\in\mathbb{N}}$ of integers greater than or equal to 2 as follows:

$$x = \sum_{n=0}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} b_i} \tag{1.1}$$

where for each $n \in \mathbb{N}$, the digit a_n belongs to the integer interval $[\![0, b_n - 1]\!]$. If infinitely many digits a_n are nonzero, then the series (1.1) is called the Cantor series of x. Many studies are devoted to Cantor series, a large amount of which are concerned with the digit frequencies; see [5, 6, 7, 11] to cite just a few.

Representations of real numbers using a real base were first defined by Rényi in 1957 [12]. In this context, a real number $x \in [0,1)$ is represented via a real base β greater than 1 as follows:

$$x = \sum_{n=0}^{+\infty} \frac{a_n}{\beta^{n+1}} \tag{1.2}$$

where the digits a_n can be chosen by using several appropriate algorithms. The most commonly used algorithm is the greedy algorithm according to which for each $n \in \mathbb{N}$, $a_n = \lfloor \beta\, T_\beta{}^n(x) \rfloor$ where $T_\beta \colon [0,1) \mapsto [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor$. Expansions in a real base are extensively studied and we can only cite a few of the many possible references [2, 9, 10, 13].

This paper investigates series expansions of real numbers that are based on a sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1. We call such a base sequence $\boldsymbol{\beta}$ a Cantor real base, and we talk about $\boldsymbol{\beta}$-representations. In doing so, we generalize both representations of real numbers through Cantor series and real base representations of real numbers.

This paper has the following organization. In Section 2, we introduce the basic definitions and we give a characterization of those infinite words a over the alphabet $\mathbb{R}_{\geq 0}$ for which there exists a Cantor real base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$. In Section 3, we define the greedy $\boldsymbol{\beta}$-representations of real numbers, which we call the $\boldsymbol{\beta}$-expansions. Then we prove several fundamental properties of $\boldsymbol{\beta}$-representations, each of which extends existing results on real base representations. In Section 4, we introduce the quasi-greedy $\boldsymbol{\beta}$-expansion $d^*_{\boldsymbol{\beta}}(1)$ of 1 and show that $d^*_{\boldsymbol{\beta}}(1)$ is the lexicographically greatest $\boldsymbol{\beta}$-representation not ending in 0^ω of all real numbers in $[0, 1]$. In Section 5, we prove a generalization of Parry's theorem [10] characterizing those infinite words over $\mathbb{N}$ that are the greedy $\boldsymbol{\beta}$-representations of some real number in the interval $[0, 1)$. In Section 6, we introduce the notion of $\boldsymbol{\beta}$-shift. We are able to give a description of the $\boldsymbol{\beta}$-shift in full generality. In Section 7, which is the last and biggest section, we focus on the periodic Cantor real bases, which we call alternate bases. We first give a characterization of those infinite words a over the alphabet $\mathbb{R}_{\geq 0}$ for which there exists an alternate base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$. Then we obtain a characterization of the $\boldsymbol{\beta}$-expansion of 1 among all $\boldsymbol{\beta}$-representations of 1, which generalizes a result of Parry [10]. Finally, generalizing Bertrand-Mathis' theorem [2], we show that for any alternate base $\boldsymbol{\beta}$, the $\boldsymbol{\beta}$-shift is sofic if and only if all quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 are ultimately periodic, where $\boldsymbol{\beta}^{(i)}$ is the i-th shift of the Cantor real base $\boldsymbol{\beta}$.

2. Cantor real bases

Let $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ be a sequence of real numbers greater than 1 such that $\prod_{n\in\mathbb{N}} \beta_n = +\infty$. We call such a sequence $\boldsymbol{\beta}$ a *Cantor real base*, or simply a *Cantor base*. We define the *$\boldsymbol{\beta}$-value (partial) map* $\mathrm{val}_{\boldsymbol{\beta}} \colon (\mathbb{R}_{\geq 0})^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{val}_{\boldsymbol{\beta}}(a) = \sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{i=0}^{n} \beta_i} \tag{2.1}$$

for any infinite word $a = a_0a_1a_2\cdots$ over $\mathbb{R}_{\geq 0}$, provided that the series converges. A *$\boldsymbol{\beta}$-representation* of a nonnegative real number x is an infinite word $a \in \mathbb{N}^{\mathbb{N}}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$. In particular, if $\boldsymbol{\beta} = (\beta, \beta, \ldots)$, then for all $x \in [0, 1]$, a $\boldsymbol{\beta}$-representation of x is a β-representation of x as defined by Rényi [12]. In this case, we do not distinguish the notation $\boldsymbol{\beta}$ and β: we write val_β and we talk about β-representations, as usual. Also, any sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 that takes only finitely many values is a Cantor base since in this case, the condition $\prod_{n\in\mathbb{N}} \beta_n = +\infty$ is trivially satisfied.

We will need to represent real numbers not only in a fixed Cantor base $\boldsymbol{\beta}$ but also in all Cantor bases obtained by shifting $\boldsymbol{\beta}$. We define

$$\boldsymbol{\beta}^{(n)} = (\beta_n, \beta_{n+1}, \ldots) \quad \text{for all } n \in \mathbb{N}.$$

In particular $\boldsymbol{\beta}^{(0)} = \boldsymbol{\beta}$. We will also need to consider shifted infinite words. Let us denote by σ_A the *shift operator*

$$\sigma_A \colon A^{\mathbb{N}} \to A^{\mathbb{N}},\ a_0a_1a_2\cdots \mapsto a_1a_2a_3\cdots$$

over the alphabet A. Whenever there is no ambiguity on the alphabet, we simply denote the shift operator by σ. Throughout this text, if a is an infinite word then for all $n \in \mathbb{N}$, a_n designates its letter indexed by n, so that $a = a_0a_1a_2\cdots$.

The $\boldsymbol{\beta}$-representations of 1 will be of interest in what follows, in particular the greedy and the quasi-greedy expansions (see Sections 3 and 4). We start our study by providing a characterization of those infinite words a over the alphabet $\mathbb{R}_{\geq 0}$ for which there exists a Cantor real base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$.

When $\boldsymbol{\beta} = (\beta, \beta, \ldots)$, for any infinite word a over $\mathbb{N}$ satisfying some suitable conditions, the equation $\mathrm{val}_\beta(a) = 1$ admits a unique solution $\beta > 1$ (see [9, Corollary 7.2.10]). This classical result remains true for nonnegative real digits and weaker conditions on the infinite word a.

Lemma 1. *Let a be an infinite word over $\mathbb{R}_{\geq 0}$ such that $a_n \in O(n^d)$ for some $d \in \mathbb{N}$. There exists a real base β such that $\operatorname{val}_\beta(a) = 1$ if and only if $\sum_{n\in\mathbb{N}} a_n > 1$, in which case β is unique and $\beta \geq a_0$, and if moreover for all $n \in \mathbb{N}$, $a_n \leq a_0$, then $\beta \leq a_0 + 1$.*

Proof. If $\sum_{n\in\mathbb{N}} a_n \leq 1$ then for all real bases β, $\operatorname{val}_\beta(a) < 1$. Indeed, this is obvious if $a = 0^\omega$, and else $\operatorname{val}_\beta(a) < \sum_{n\in\mathbb{N}} a_n \leq 1$.

Now, suppose that $\sum_{n\in\mathbb{N}} a_n > 1$. Let $N \in \mathbb{N}$ be such that $\sum_{n=0}^N a_n > 1$. The function $f\colon [0,1) \to \mathbb{R}$, $x \mapsto \sum_{n\in\mathbb{N}} a_n x^{n+1}$ is well-defined, continuous, increasing and such that $f(0) = 0$ and that for all $x \in [0,1)$, $f(x) \geq \sum_{n=0}^N a_n x^{n+1}$. The function $g\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto \sum_{n=0}^N a_n x^{n+1}$ is continuous, increasing and such that $g(0) = 0$ and $g(1) > 1$. Therefore, there exists a unique $x_0 \in (0,1)$ such that $g(x_0) = 1$, and hence such that $f(x_0) \geq 1$. Now, there exists a unique $\gamma \in (0, x_0]$ such that $f(\gamma) = 1$. By setting $\beta = \frac{1}{\gamma}$, we get that $\beta \geq \frac{1}{x_0} > 1$ and $\operatorname{val}_\beta(a) = f\left(\frac{1}{\beta}\right) = 1$. Moreover, $\beta \geq a_0$ for otherwise $f\left(\frac{1}{\beta}\right) > f\left(\frac{1}{a_0}\right) \geq 1$.

If moreover for all $n \in \mathbb{N}$, $a_n \leq a_0$, then $\beta \leq a_0 + 1$ for otherwise we would have

$$\operatorname{val}_\beta(a) = \sum_{n\in\mathbb{N}} \frac{a_n}{\beta^{n+1}} < a_0 \sum_{n\in\mathbb{N}} \frac{1}{(a_0+1)^{n+1}} = 1.$$

□

No upper bound on the growth order of the digits a_n is needed in order to find a Cantor base $\boldsymbol{\beta}$ such that $\operatorname{val}_{\boldsymbol{\beta}}(a) = 1$.

Lemma 2. *Let a be an infinite word over $\mathbb{R}_{\geq 0}$ such that $\sum_{n\in\mathbb{N}} a_n = +\infty$. Then there exists a Cantor base $\boldsymbol{\beta}$ such that $\operatorname{val}_{\boldsymbol{\beta}}(a) = 1$.*

Proof. First of all, observe that the hypothesis implies that a does not end in 0^ω and that $\prod_{n\in\mathbb{N}}(a_n + 1) = +\infty$.

We define two sequences of nonnegative integers $(n_k)_{1\leq k\leq K}$ and $(\ell_k)_{1\leq k\leq K}$ where $K \in \mathbb{N} \cup \{+\infty\}$. The length K of these two sequences is the number of zero blocks in a, i.e. the factors of the form 0^ℓ which are neither preceded nor followed by 0 in a. Two cases stand out: either $K \in \mathbb{N}$ or $K = +\infty$. We describe the two cases at once. In order to do so, it should be understood that the parts of the definition where $k > K$ should just be ignored when $K \in \mathbb{N}$. Let n_1 denote the least $n \in \mathbb{N}$ such that $a_n = 0$ and let ℓ_1 denote the least $\ell \in \mathbb{N}$ such that $a_{n_1+\ell} > 0$. Then for $k \geq 2$, let n_k denote the least integer $n > n_{k-1} + \ell_{k-1}$ such that $a_n = 0$ and let ℓ_k denote the least $\ell \in \mathbb{N}$ such that $a_{n_k+\ell} > 0$. Thus, $(n_k)_{1\leq k\leq K}$ is the sequence of positions of appearance of the successive zero blocks in a and $(\ell_k)_{1\leq k\leq K}$ is the sequence of lengths of these blocks.

Next, for all $k \in [\![1, K]\!]$, we pick any α_k in the interval $(1, \sqrt[\ell_k]{a_{n_k+\ell_k} + 1})$. For all $n \in \mathbb{N}$, we define

$$\beta_n = \begin{cases} a_n + 1 & \text{if } n \in [\![0, n_1 - 1]\!] \text{ or } n \in \bigcup_{k=1}^K [\![n_k + \ell_k + 1, n_{k+1} - 1]\!] \\ \alpha_k & \text{if } n \in [\![n_k, n_k + \ell_k - 1]\!] \text{ for some } k \in [\![1, K]\!] \\ \frac{a_n+1}{\alpha_k^{\ell_k}} & \text{if } n = n_k + \ell_k \text{ for some } k \in [\![1, K]\!] \end{cases}$$

where we set $n_{K+1} = +\infty$ if $K \in \mathbb{N}$. In particular if $K = 0$, i.e. if for all $n \in \mathbb{N}$, $a_n > 0$, then for all $n \in \mathbb{N}$, $\beta_n = a_n + 1$.

Let us show that in any case, the obtained sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ is such that $\prod_{n\in\mathbb{N}} \beta_n = +\infty$ and $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$. By construction,

$$\prod_{n\in\mathbb{N}} \beta_n = \prod_{n=0}^{n_1-1} (a_n+1) \cdot \prod_{k=1}^{K} \left(\alpha_k^{\ell_k} \cdot \frac{a_{n_k+\ell_k}+1}{\alpha_k^{\ell_k}} \cdot \prod_{n=n_k+\ell_k+1}^{n_{k+1}-1} (a_n+1) \right) = \prod_{n\in\mathbb{N}} (a_n+1).$$

By induction we can show that

$$\sum_{n=0}^{n_k+\ell_k} \frac{a_n}{\prod_{i=0}^{n} \beta_i} = 1 - \frac{1}{\prod_{i=0}^{n_k+\ell_k} \beta_i} \quad \text{for all } k \in [\![1, K]\!].$$

If $K = +\infty$ then we obtain that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$ by letting k tend to infinity. Otherwise, $K \in \mathbb{N}$. Set $n_0 = -1$ and $\ell_0 = 0$. By induction again, we can show that

$$\sum_{n=n_K+\ell_K+1}^{m} \frac{a_n}{\prod_{i=n_K+\ell_K+1}^{n} \beta_i} = 1 - \frac{1}{\prod_{i=n_K+\ell_K+1}^{m} \beta_i} \quad \text{for all } m \in \mathbb{N}.$$

By letting m tend to infinity, we get

$$\mathrm{val}_{\boldsymbol{\beta}^{(n_K+\ell_K+1)}}(\sigma^{n_K+\ell_K+1}(a)) = 1.$$

Finally, we obtain

$$\begin{aligned}
\mathrm{val}_{\boldsymbol{\beta}}(a) &= \sum_{n=0}^{n_K+\ell_K} \frac{a_n}{\prod_{i=0}^{n} \beta_i} + \sum_{n=n_K+\ell_K+1}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i} \\
&= 1 - \frac{1}{\prod_{i=0}^{n_K+\ell_K} \beta_i} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(n_K+\ell_K+1)}}(\sigma^{n_K+\ell_K+1}(a))}{\prod_{i=0}^{n_K+\ell_K} \beta_i} \\
&= 1.
\end{aligned}$$

□

Proposition 3. *Let a be an infinite word over $\mathbb{R}_{\geq 0}$. There exists a Cantor base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$ if and only if $\sum_{n\in\mathbb{N}} a_n > 1$.*

Proof. Similarly as in the proof of Lemma 1, the condition $\sum_{n\in\mathbb{N}} a_n > 1$ is necessary. Now, suppose that $\sum_{n\in\mathbb{N}} a_n > 1$. If $\sum_{n\in\mathbb{N}} a_n = +\infty$ then we use Lemma 2. Otherwise, we have $1 < \sum_{n\in\mathbb{N}} a_n < +\infty$ and we apply Lemma 1. □

3. The greedy algorithm

For $x \in [0,1]$, a distinguished $\boldsymbol{\beta}$-representation $\varepsilon_0(x)\varepsilon_1(x)\varepsilon_2(x)\cdots$ is given thanks to the *greedy algorithm*:

- $\varepsilon_0(x) = \lfloor \beta_0 x \rfloor$ and $r_0(x) = \beta_0 x - \varepsilon_0(x)$
- $\varepsilon_n(x) = \lfloor \beta_n r_{n-1}(x) \rfloor$ and $r_n = \beta_n r_{n-1}(x) - \varepsilon_n(x)$ for $n \in \mathbb{N}_{\geq 1}$.

The obtained $\boldsymbol{\beta}$-representation of x is denoted by $d_{\boldsymbol{\beta}}(x)$ and is called the *$\boldsymbol{\beta}$-expansion* of x. Note that the n-th digit $\varepsilon_n(x)$ belongs to $\{0, \ldots, \lfloor \beta_n \rfloor\}$. We let $A_{\boldsymbol{\beta}}$ denote the (possibly infinite) alphabet $\{0, \ldots, \sup_{n\in\mathbb{N}} \lfloor \beta_n \rfloor\}$. The algorithm is called greedy since at each step it chooses the largest possible digit. Indeed, consider $x \in [0,1]$ and $\ell \in \mathbb{N}$, and suppose that the digits $\varepsilon_0(x), \ldots, \varepsilon_{\ell-1}(x)$ are already known. Then the digit $\varepsilon_\ell(x)$ is the largest element of $[\![0, \lfloor \beta_\ell \rfloor]\!]$ such that $\sum_{n=0}^{\ell} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i} \leq x$. Thus

$$x = \sum_{n=0}^{\ell} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i} + \frac{r_\ell(x)}{\prod_{i=0}^{\ell} \beta_i}$$

where $r_\ell(x) \in [0,1)$. Note that since a Cantor base satisfies $\prod_{n\in\mathbb{N}} \beta_n = +\infty$, the latter equality implies the convergence of the greedy algorithm and that $x = \mathrm{val}_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x))$. We let $D_{\boldsymbol{\beta}}$ denote the subset of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ of all $\boldsymbol{\beta}$-expansions of real numbers in the interval $[0,1)$:

$$D_{\boldsymbol{\beta}} = \{d_{\boldsymbol{\beta}}(x) \colon x \in [0,1)\}.$$

In what follows, the $\boldsymbol{\beta}$-expansion of 1 will play a special role. For the sake of clarity, we denote its digits by ε_n instead of $\varepsilon_n(1)$. We sometimes write $\varepsilon_{\boldsymbol{\beta},n}(x)$ and $\varepsilon_{\boldsymbol{\beta},n}$ instead of $\varepsilon_n(x)$ and ε_n when the Cantor base $\boldsymbol{\beta}$ needs to be emphasized. As previously mentioned, if $\boldsymbol{\beta} = (\beta,\beta,\ldots)$, then for all $x \in [0,1]$, the $\boldsymbol{\beta}$-expansion of x is equal to the usual β-expansion of x as defined by Rényi [12] and we write indistinctly $\boldsymbol{\beta}$ or β.

We can also express the obtained digits $\varepsilon_n(x)$ and remainders $r_n(x)$ thanks to the β_n-transformations. For $\beta > 1$, the β-transformation is the map

$$T_\beta \colon [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor .$$

Then for all $x \in [0,1)$ and $n \in \mathbb{N}$, we have

$$\varepsilon_n(x) = \left\lfloor \beta_n \big(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)\big) \right\rfloor \quad \text{and} \quad r_n(x) = T_{\beta_n} \circ \cdots \circ T_{\beta_0}(x).$$

Example 4. If there exists $n \in \mathbb{N}$ such that β_n is an integer (without any restriction on the other β_m), then $d_{\boldsymbol{\beta}^{(n)}}(1) = \beta_n 0^\omega$ where the ω notation means an infinite repetition.

Example 5. For $n \in \mathbb{N}$, let $\alpha_n = 1 + \frac{1}{2^{n+1}}$ and $\beta_n = 2 + \frac{1}{2^{n+1}}$. The sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$ is not a Cantor base since $\prod_{n\in\mathbb{N}} \alpha_n < +\infty$. If we perform the greedy algorithm on $x = 1$ for the sequence $\boldsymbol{\alpha}$, we obtain the sequence of digits 10^ω, which is clearly not an $\boldsymbol{\alpha}$-representation of 1. However, the sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ is indeed a Cantor base since $\prod_{n\in\mathbb{N}} \beta_n = +\infty$.

Example 6. Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$.

(1) Consider $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ the Cantor base defined by

$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$, where rep_2 is the function mapping any nonnegative integer to its 2-expansion. We get $\boldsymbol{\beta} = (\alpha,\beta,\beta,\alpha,\beta,\alpha,\alpha,\beta,\ldots)$ where the infinite word $\beta_0\beta_1\beta_2\cdots$ is the Thue-Morse word over the alphabet $\{\alpha,\beta\}$. We compute $d_{\boldsymbol{\beta}}(1) = 20010110^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 1010110^\omega$ and $d_{\boldsymbol{\beta}^{(2)}}(1) = 110^\omega$.

(2) Consider $\boldsymbol{\beta} = (\sqrt{13},\alpha,\beta,\alpha,\beta,\alpha,\beta,\ldots)$. It is easily checked that $d_{\boldsymbol{\beta}}(1) = 3(10)^\omega$ and that for all $m \in \mathbb{N}$, $d_{\boldsymbol{\beta}^{(2m+1)}}(1) = 2010^\omega$ and $d_{\boldsymbol{\beta}^{(2m+2)}}(1) = 110^\omega$.

We call an *alternate base* a periodic Cantor base, i.e. a Cantor base for which there exists $p \in \mathbb{N}_{\geq 1}$ such that for all $n \in \mathbb{N}$, $\beta_n = \beta_{n+p}$. In this case we simply note $\boldsymbol{\beta} = (\overline{\beta_0,\ldots,\beta_{p-1}})$ and the integer p is called the *length* of the alternate base $\boldsymbol{\beta}$. In what follows, most examples will be alternate bases and Section 7 will be specifically devoted to their study.

Example 7. Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the Golden Ratio and let $\boldsymbol{\beta} = (\overline{3,\varphi,\varphi})$. For all $m \in \mathbb{N}$, we have $d_{\boldsymbol{\beta}^{(3m)}}(1) = 30^\omega$, $d_{\boldsymbol{\beta}^{(3m+1)}}(1) = 110^\omega$ and $d_{\boldsymbol{\beta}^{(3m+2)}}(1) = 1(110)^\omega$.

Let us show that the classical properties of the β-expansion theory are still valid for Cantor bases. Some are just an adaptation of the related proofs in [9] but for the sake of completeness the details are written. From now on, unless otherwise stated, we consider a fixed Cantor base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$.

Proposition 8. *For all $x \in [0,1)$ and all $n \in \mathbb{N}$, we have*

$$\sigma^n \circ d_{\boldsymbol{\beta}}(x) = d_{\boldsymbol{\beta}^{(n)}} \circ T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x).$$

Proof. This is a straightforward verification. □

Lemma 9. *For all infinite words a over $\mathbb{N}$ and all $x \in [0,1]$, $a = d_{\boldsymbol{\beta}}(x)$ if and only if $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$ and for all $\ell \in \mathbb{N}$,*

$$\sum_{n=\ell+1}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i} < \frac{1}{\prod_{i=0}^{\ell} \beta_i}. \tag{3.1}$$

Proof. From the greedy algorithm, for all $x \in [0,1]$, $\mathrm{val}_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x)) = x$ and for all $\ell \in \mathbb{N}$,

$$\left(\sum_{n=\ell+1}^{+\infty} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i}\right) \prod_{i=0}^{\ell} \beta_i = \left(x - \sum_{n=0}^{\ell} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i}\right) \prod_{i=0}^{\ell} \beta_i = r_\ell(x) < 1.$$

Conversely, suppose that a is an infinite word over $\mathbb{N}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$ and such that for all $\ell \in \mathbb{N}$, (3.1) holds. Let us show by induction that for all $m \in \mathbb{N}$, $a_m = \varepsilon_m(x)$. From (3.1) for $\ell = 0$, we get that $x - \frac{a_0}{\beta_0} < \frac{1}{\beta_0}$. Thus, $\beta_0 x - 1 < a_0$. Since $\frac{a_0}{\beta_0} \le x$, we get that $a_0 \le \beta_0 x$. Therefore, $a_0 = \lfloor \beta_0 x \rfloor = \varepsilon_0(x)$. Now, suppose that $m \in \mathbb{N}_{\ge 1}$ and that for $n \in [\![0, m-1]\!]$, $a_n = \varepsilon_n(x)$. Then

$$a_m + \left(\sum_{n=m+1}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i}\right) \prod_{i=0}^{m} \beta_i = \varepsilon_m(x) + r_m(x).$$

By using (3.1) for $\ell = m$ and since $r_m(x) < 1$, we obtain that $a_m = \varepsilon_m(x)$. □

Proposition 10. *Let a be a $\boldsymbol{\beta}$-representation of some real number x in $[0,1]$. Then the following four assertions are equivalent.*

1. *The infinite word a is the $\boldsymbol{\beta}$-expansion of x.*
2. *For all $n \in \mathbb{N}_{\ge 1}$, $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$.*
3. *The infinite word $\sigma(a)$ belongs to $D_{\boldsymbol{\beta}^{(1)}}$.*
4. *For all $n \in \mathbb{N}_{\ge 1}$, $\sigma^n(a)$ belongs to $D_{\boldsymbol{\beta}^{(n)}}$.*

Proof. Since $\mathrm{val}_{\boldsymbol{\beta}}(a) = x \in [0,1]$, it follows from Lemma 9 that $a = d_{\boldsymbol{\beta}}(x)$ if and only if for all $\ell \in \mathbb{N}$, (3.1) holds. In order to obtain the equivalences between the first three items, it suffices to note that the greedy condition (3.1) can be rewritten as $\mathrm{val}_{\boldsymbol{\beta}^{(\ell+1)}}(\sigma^{\ell+1}(a)) < 1$. Clearly (4) implies (3). Finally we obtain that (3) implies (4) by iterating the implication (1) $\implies$ (3). □

Corollary 11. *An infinite word a over $\mathbb{N}$ belongs to $D_{\boldsymbol{\beta}}$ if and only if for all $n \in \mathbb{N}$, $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$.*

Proposition 12. *The $\boldsymbol{\beta}$-expansion of a real number $x \in [0,1]$ is lexicographically maximal among all $\boldsymbol{\beta}$-representations of x.*

Proof. Let $x \in [0,1]$ and $a \in \mathbb{N}^{\mathbb{N}}$ be a $\boldsymbol{\beta}$-representation of x. Proceed by contradiction and suppose that $a >_{\mathrm{lex}} d_{\boldsymbol{\beta}}(x)$. There exists $\ell \in \mathbb{N}$ such that $\varepsilon_0(x) \cdots \varepsilon_{\ell-1}(x) = a_0 \cdots a_{\ell-1}$ and $a_\ell > \varepsilon_\ell(x)$. Then

$$\sum_{n=\ell}^{+\infty} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i} = \sum_{n=\ell}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i} \ge \frac{\varepsilon_\ell(x)+1}{\prod_{i=0}^{\ell} \beta_i} + \sum_{n=\ell+1}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i}$$

and hence

$$\sum_{n=\ell+1}^{+\infty} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n}\beta_i} \geq \frac{1}{\prod_{i=0}^{\ell}\beta_i}$$

which is impossible by Lemma 9. □

Proposition 13. *The function* $d_{\boldsymbol{\beta}}\colon [0,1] \to {A_{\boldsymbol{\beta}}}^{\mathbb{N}}$ *is increasing:*

$$\forall x, y \in [0,1], \quad x < y \iff d_{\boldsymbol{\beta}}(x) <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(y).$$

Proof. Suppose that $d_{\boldsymbol{\beta}}(x) <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(y)$. There exists $\ell \in \mathbb{N}$ such that $\varepsilon_0(x)\cdots\varepsilon_{\ell-1}(x) = \varepsilon_0(y)\cdots\varepsilon_{\ell-1}(y)$ and $\varepsilon_\ell(x) < \varepsilon_\ell(y)$. By Lemma 9, we get

$$x = \sum_{n\in\mathbb{N}} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n}\beta_i} < \sum_{n=0}^{\ell-1} \frac{\varepsilon_n(y)}{\prod_{i=0}^{n}\beta_i} + \frac{\varepsilon_\ell(y)-1}{\prod_{i=0}^{\ell}\beta_i} + \frac{1}{\prod_{i=0}^{\ell}\beta_i} = \sum_{n=0}^{\ell} \frac{\varepsilon_n(y)}{\prod_{i=0}^{n}\beta_i} \leq y.$$

It follows immediately that $x < y$ implies $d_{\boldsymbol{\beta}}(x) <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(y)$. □

Corollary 14. *If a is an infinite word over $\mathbb{N}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \leq 1$, then $a \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$. In particular, $d_{\boldsymbol{\beta}}(1)$ is lexicographically maximal among all $\boldsymbol{\beta}$-representations of all real numbers in $[0,1]$.*

Proof. Let a be an infinite word over $\mathbb{N}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \leq 1$. By Propositions 12 and 13, $a \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(\mathrm{val}_{\boldsymbol{\beta}}(a)) \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$. □

Recall the property of the β-expansions stating that considering two bases α and β, $\alpha < \beta$ if and only if $d_\alpha(1) < d_\beta(1)$ [10]. The following proposition shows the generalization of a weaker version of this property in the case of Cantor bases.

Proposition 15. *Let $\boldsymbol{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ be two Cantor bases such that for all $n \in \mathbb{N}$, $\prod_{i=0}^{n}\alpha_i \leq \prod_{i=0}^{n}\beta_i$. Then for all $x \in [0,1]$, we have $d_{\boldsymbol{\alpha}}(x) \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(x)$.*

Proof. Let $x \in [0,1]$ and suppose to the contrary that $d_{\boldsymbol{\alpha}}(x) >_{\mathrm{lex}} d_{\boldsymbol{\beta}}(x)$. Thus, there exists $\ell \in \mathbb{N}$ such that $\varepsilon_{\boldsymbol{\alpha},0}(x)\cdots\varepsilon_{\boldsymbol{\alpha},\ell-1}(x) = \varepsilon_{\boldsymbol{\beta},0}(x)\cdots\varepsilon_{\boldsymbol{\beta},\ell-1}(x)$ and $\varepsilon_{\boldsymbol{\alpha},\ell}(x) > \varepsilon_{\boldsymbol{\beta},\ell}(x)$. From Lemma 9 and from the hypothesis, we obtain that

$$x \leq \sum_{n=0}^{\ell-1} \frac{\varepsilon_{\boldsymbol{\alpha},n}(x)}{\prod_{i=0}^{n}\beta_i} + \frac{\varepsilon_{\boldsymbol{\alpha},\ell}(x)-1}{\prod_{i=0}^{\ell}\beta_i} + \sum_{n=\ell+1}^{+\infty} \frac{\varepsilon_{\boldsymbol{\beta},n}(x)}{\prod_{i=0}^{n}\beta_i} < \sum_{n=0}^{\ell} \frac{\varepsilon_{\boldsymbol{\alpha},n}(x)}{\prod_{i=0}^{n}\beta_i} \leq \sum_{n=0}^{\ell} \frac{\varepsilon_{\boldsymbol{\alpha},n}(x)}{\prod_{i=0}^{n}\alpha_i} \leq x,$$

a contradiction. □

Corollary 16. *Let $\boldsymbol{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ be two Cantor bases such that for all $n \in \mathbb{N}$, $\alpha_n \leq \beta_n$. Then for all $x \in [0,1]$, we have $d_{\boldsymbol{\alpha}}(x) \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(x)$.*

It is not true that $d_{\boldsymbol{\alpha}}(1) <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$ implies that for all $n \in \mathbb{N}$, $\prod_{i=0}^{n}\alpha_i \leq \prod_{i=0}^{n}\beta_i$ as the following example shows. The same example shows that the lexicographic order on the Cantor bases is not sufficient either. Here, the term lexicographic order refers to the following order: $\boldsymbol{\alpha} < \boldsymbol{\beta}$ whenever there exists $\ell \in \mathbb{N}$ such that $\alpha_n = \beta_n$ for $n \in [\![0, \ell-1]\!]$ and $\alpha_\ell < \beta_\ell$.

Example 17. Let $\boldsymbol{\alpha} = \overline{(2+\sqrt{3}, 2)}$ and $\boldsymbol{\beta} = \overline{(2+\sqrt{2}, 5)}$. Then $d_{\boldsymbol{\alpha}}(1) = 31^\omega$ and $d_{\boldsymbol{\beta}}(1)$ starts with the prefix 32, hence $d_{\boldsymbol{\alpha}}(1) <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$.

4. Quasi-greedy expansions

A $\boldsymbol{\beta}$-representation is said to be *finite* if it ends with infinitely many zeros, and *infinite* otherwise. The *length* of a finite $\boldsymbol{\beta}$-representation is the length of the longest prefix ending in a non-zero digit. When a $\boldsymbol{\beta}$-representation is finite, we usually omit to write the tail of zeros.

When the $\boldsymbol{\beta}$-expansion of 1 is finite, we show how to modify it in order to obtain an infinite $\boldsymbol{\beta}$-representation of 1 that is lexicographically maximal among all infinite $\boldsymbol{\beta}$-representations of 1. The obtained $\boldsymbol{\beta}$-representation is denoted by $d^*_{\boldsymbol{\beta}}(1)$ and is called the *quasi-greedy $\boldsymbol{\beta}$-expansion* of 1. It is defined recursively as follows:

(4.1)
$$d^*_{\boldsymbol{\beta}}(1) = \begin{cases} d_{\boldsymbol{\beta}}(1) & \text{if } d_{\boldsymbol{\beta}}(1) \text{ is infinite} \\ \varepsilon_0 \cdots \varepsilon_{\ell-2}(\varepsilon_{\ell-1}-1)d^*_{\boldsymbol{\beta}^{(\ell)}}(1) & \text{if } d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{\ell-1} \text{ with } \ell \in \mathbb{N}_{\geq 1},\ \varepsilon_{\ell-1} > 0. \end{cases}$$

Example 18. Let $\boldsymbol{\beta} = (\overline{3, \varphi, \varphi})$ the alternate base already considered in Example 7. Then we directly have that for all $m \in \mathbb{N}$, $d^*_{\boldsymbol{\beta}^{(3m+2)}}(1) = d_{\boldsymbol{\beta}^{(3m+2)}}(1) = 1(110)^\omega$. In order to compute $d^*_{\boldsymbol{\beta}^{(3m)}}(1)$ and $d^*_{\boldsymbol{\beta}^{(3m+1)}}(1)$, we need to go through the definition several times. For all $m \in \mathbb{N}$, we compute $d^*_{\boldsymbol{\beta}^{(3m)}}(1) = 2d^*_{\boldsymbol{\beta}^{(3m+1)}}(1) = 210d^*_{\boldsymbol{\beta}^{(3m+3)}}(1) = 210d^*_{\boldsymbol{\beta}^{(3m)}}(1) = (210)^\omega$ and $d^*_{\boldsymbol{\beta}^{(3m+1)}}(1) = 10d^*_{\boldsymbol{\beta}^{(3m+3)}}(1) = 10(210)^\omega = (102)^\omega$.

Example 19. Let $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$ be an alternate base such that for all $i \in [\![0, p-1]\!]$, $\beta_i \in \mathbb{N}_{\geq 2}$. Then for all $i \in [\![0, p-1]\!]$, $d_{\boldsymbol{\beta}^{(i)}}(1) = \beta_i 0^\omega$ and
$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = ((\beta_i - 1) \cdots (\beta_{p-1} - 1)(\beta_0 - 1) \ldots (\beta_{i-1} - 1))^\omega.$$

When $\boldsymbol{\beta} = (\beta, \beta, \ldots)$, we recover the usual definition of the quasi-greedy β-expansion [4, 8]. In particular, it is easy to check that in this case, if $d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{\ell-1}$ with $\ell \in \mathbb{N}_{\geq 1}$ and $\varepsilon_{\ell-1} > 0$, then the quasi-greedy expansion is purely periodic and $d^*_{\boldsymbol{\beta}}(1) = (\varepsilon_0 \ldots \varepsilon_{\ell-2}(\varepsilon_{\ell-1} - 1))^\omega$. For arbitrary Cantor bases, the situation is more complicated and the quasi-greedy expansion can be not periodic.

Example 20. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. We compute $d_{\boldsymbol{\beta}}(1) = 201$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 11$. Then $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$ and $d^*_{\boldsymbol{\beta}}(1) = 200d^*_{\boldsymbol{\beta}^{(1)}}(1) = 200(10)^\omega$.

Moreover, even if the $\boldsymbol{\beta}$-expansion is finite, the quasi-greedy $\boldsymbol{\beta}$-representation can be infinite not ultimately periodic. Suppose that $d_{\boldsymbol{\beta}}(1)$ is finite and that an infinite quasi-greedy is involved during the computation of $d^*_{\boldsymbol{\beta}}(1)$. Let $n \in \mathbb{N}_{\geq 1}$ be the positive integer such that $d^*_{\boldsymbol{\beta}^{(n)}}(1)$ is the involved infinite expansion. Then $d^*_{\boldsymbol{\beta}}(1)$ is ultimately periodic if and only if so is $d^*_{\boldsymbol{\beta}^{(n)}}(1)$.

Example 21. Consider the Cantor base $\boldsymbol{\beta} = (3, \beta, \beta, \beta, \beta, \ldots)$ where $\beta = \sqrt{6}(2 + \sqrt{6})$. We get $d_{\boldsymbol{\beta}}(1) = 3$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = d_\beta(1)$ is infinite not ultimately periodic since β is a non-Pisot quadratic number [1]. Therefore, the quasi-greedy expansion $d^*_{\boldsymbol{\beta}}(1) = 2d^*_{\boldsymbol{\beta}^{(1)}}(1)$ is not ultimately periodic.

Proposition 22. *The quasi-greedy expansion $d^*_{\boldsymbol{\beta}}(1)$ is a $\boldsymbol{\beta}$-representation of* 1.

Proof. It is a straightforward verification. □

Proposition 23. *If a is an infinite word over $\mathbb{N}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) < 1$, then $a <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$. Furthermore, $d^*_{\boldsymbol{\beta}}(1)$ is lexicographically maximal among all infinite $\boldsymbol{\beta}$-representations of all real numbers in* $[0, 1]$.

Proof. If $d_{\boldsymbol{\beta}}(1)$ is infinite then the result follows from Corollary 14. Thus, we suppose that there exists $\ell \in \mathbb{N}_{\geq 1}$ such that $d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{\ell-1}$ and $\varepsilon_{\ell-1} > 0$.

First, let $a \in \mathbb{N}^{\mathbb{N}}$ be such that $\mathrm{val}_{\boldsymbol{\beta}}(a) < 1$ and suppose to the contrary that $a \geq_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$. By Corollary 14, $a <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$. Then $a_0 \cdots a_{\ell-2} = \varepsilon_0 \cdots \varepsilon_{\ell-2}$, $a_{\ell-1} = \varepsilon_{\ell-1} - 1$ and $\sigma^\ell(a) \geq_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(\ell)}}(1)$. Since

$$\begin{aligned}\mathrm{val}_{\boldsymbol{\beta}}(a) &= \sum_{n=0}^{\ell-2} \frac{\varepsilon_n}{\prod_{i=0}^n \beta_i} + \frac{\varepsilon_{\ell-1}-1}{\prod_{i=0}^{\ell-1}\beta_i} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(\ell)}}\big(\sigma^\ell(a)\big)}{\prod_{i=0}^{\ell-1}\beta_i} \\ &= 1 - \frac{1}{\prod_{i=0}^{\ell-1}\beta_i}\Big(1 - \mathrm{val}_{\boldsymbol{\beta}^{(\ell)}}\big(\sigma^\ell(a)\big)\Big),\end{aligned}$$

we get that $\mathrm{val}_{\boldsymbol{\beta}^{(\ell)}}\big(\sigma^\ell(a)\big) < 1$. By Corollary 14 again, $\sigma^\ell(a) <_{\mathrm{lex}} d_{\boldsymbol{\beta}^{(\ell)}}(1)$. Therefore $d_{\boldsymbol{\beta}^{(\ell)}}(1)$ must be finite and we obtain that $a = d^*_{\boldsymbol{\beta}}(1)$ by iterating the reasoning. But then $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$, a contradiction.

We now turn to the second part. Suppose that $a \in \mathbb{N}^{\mathbb{N}}$ does not end in 0^ω and is such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \leq 1$. Our aim is to show that $a \leq_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$. We know from Corollary 14 that $a \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$. Now, suppose to the contrary that $a >_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$. Then $a_0 \cdots a_{\ell-2} = \varepsilon_0 \cdots \varepsilon_{\ell-2}$, $a_{\ell-1} = \varepsilon_{\ell-1} - 1$, and $\sigma^\ell(a) >_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(\ell)}}(1)$. As in the first part of the proof, we obtain that $\mathrm{val}_{\boldsymbol{\beta}^{(\ell)}}(\sigma^\ell(a)) \leq 1$ and that $d_{\boldsymbol{\beta}^{(\ell)}}(1)$ must be finite. By iterating the reasoning, we obtain that $a = d^*_{\boldsymbol{\beta}}(1)$, a contradiction. □

5. Admissible sequences

In [10], Parry characterized those infinite words over $\mathbb{N}$ that belong to D_β. Such infinite words are sometimes called *β-admissible sequences*. Analogously, infinite word in $D_{\boldsymbol{\beta}}$ are said to be a *$\boldsymbol{\beta}$-admissible sequence*. In this section, we generalize Parry's theorem to Cantor bases.

Lemma 24. *Let a be an infinite word over $\mathbb{N}$ and for each $n \in \mathbb{N}$, let $b^{(n)}$ be a $\boldsymbol{\beta}^{(n)}$-representation of 1. Suppose that for all $n \in \mathbb{N}$, $\sigma^n(a) \leq_{\mathrm{lex}} b^{(n)}$. Then for all $k, \ell, m, n \in \mathbb{N}$ with $\ell \geq 1$, the following implication holds:*

$$a_k \cdots a_{k+\ell-1} <_{\mathrm{lex}} b^{(n)}_m \cdots b^{(n)}_{m+\ell-1} \implies \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(a_k \cdots a_{k+\ell-1}) \leq \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(b^{(n)}_m \cdots b^{(n)}_{m+\ell-1}). \tag{5.1}$$

Consequently, for all $k, m, n \in \mathbb{N}$, the following implication holds:

$$\sigma^k(a) <_{\mathrm{lex}} \sigma^m(b^{(n)}) \implies \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(\sigma^k(a)) \leq \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(\sigma^m(b^{(n)})). \tag{5.2}$$

Proof. Proceed by induction on ℓ. The base case $\ell = 1$ is clear. Let $\ell \geq 2$ and suppose that for all $\ell' < \ell$ and all $k, m, n \in \mathbb{N}$, the implication (5.1) is true. Now let $k, m, n \in \mathbb{N}$ and suppose that $a_k \cdots a_{k+\ell-1} <_{\mathrm{lex}} b^{(n)}_m \cdots b^{(n)}_{m+\ell-1}$. Two cases are possible.

Case 1: $a_k = b^{(n)}_m$. Then $a_{k+1} \cdots a_{k+\ell-1} <_{\mathrm{lex}} b^{(n)}_{m+1} \cdots b^{(n)}_{m+\ell-1}$ and by induction hypothesis, we obtain that $\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(a_{k+1} \cdots a_{k+\ell-1}) \leq \mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(b^{(n)}_{m+1} \cdots b^{(n)}_{m+\ell-1})$. Therefore

$$\begin{aligned}\mathrm{val}_{\boldsymbol{\beta}^{(k)}}(a_k \cdots a_{k+\ell-1}) &= \frac{a_k}{\beta_k} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(a_{k+1} \cdots a_{k+\ell-1})}{\beta_k} \\ &\leq \frac{b^{(n)}_m}{\beta_k} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(b^{(n)}_{m+1} \cdots b^{(n)}_{m+\ell-1})}{\beta_k} \\ &= \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(b^{(n)}_m \cdots b^{(n)}_{m+\ell-1}).\end{aligned}$$

Case 2: $a_k < b^{(n)}_m$. Since $\sigma^{k+1}(a) \le_{\text{lex}} b^{(k+1)}$ by hypothesis, we have

$$a_{k+1} \cdots a_{k+\ell-1} \le_{\text{lex}} b^{(k+1)}_0 \cdots b^{(k+1)}_{\ell-2}.$$

By induction hypothesis,

$$\text{val}_{\boldsymbol{\beta}^{(k+1)}}(a_{k+1} \cdots a_{k+\ell-1}) \le \text{val}_{\boldsymbol{\beta}^{(k+1)}}(b^{(k+1)}_0 \cdots b^{(k+1)}_{\ell-2}) \le 1.$$

Then

$$\begin{aligned}
\text{val}_{\boldsymbol{\beta}^{(k)}}(a_k \cdots a_{k+\ell-1}) &= \frac{a_k}{\beta_k} + \frac{\text{val}_{\boldsymbol{\beta}^{(k+1)}}(a_{k+1} \cdots a_{k+\ell-1})}{\beta_k} \\
&\le \frac{b^{(n)}_m - 1}{\beta_k} + \frac{\text{val}_{\boldsymbol{\beta}^{(k+1)}}(b^{(k+1)}_0 \cdots b^{(k+1)}_{\ell-2})}{\beta_k} \\
&\le \text{val}_{\boldsymbol{\beta}^{(k)}}(b^{(n)}_m \cdots b^{(n)}_{m+\ell-1}).
\end{aligned}$$

Thus, the implication (5.1) is proved. The implication (5.2) immediately follows. □

Lemma 25. *Let a be an infinite word over $\mathbb{N}$ and for each $n \in \mathbb{N}$, let $b^{(n)}$ be a $\boldsymbol{\beta}^{(n)}$-representation of 1. Suppose that for all $n \in \mathbb{N}$, $\sigma^n(a) <_{\text{lex}} b^{(n)}$. Then for all $n \in \mathbb{N}$, $\text{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$ unless there exists $\ell \in \mathbb{N}_{\ge 1}$ such that*

- $b^{(n)} = b^{(n)}_0 \cdots b^{(n)}_{\ell-1}$ *with* $b^{(n)}_{\ell-1} > 0$
- $a_n a_{n+1} \cdots a_{n+\ell-1} = b^{(n)}_0 \cdots b^{(n)}_{\ell-2}(b^{(n)}_{\ell-1} - 1)$
- $\text{val}_{\boldsymbol{\beta}^{(n+\ell)}}(\sigma^{n+\ell}(a)) = 1$

in which case $\text{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) = 1$.

Proof. Let $n \in \mathbb{N}$. By hypothesis, $\sigma^n(a) <_{\text{lex}} b^{(n)}$. So there exists $\ell \in \mathbb{N}_{\ge 1}$ such that $a_n \cdots a_{n+\ell-2} = b^{(n)}_0 \cdots b^{(n)}_{\ell-2}$ and $a_{n+\ell-1} < b^{(n)}_{\ell-1}$. By hypothesis, we also have $\sigma^{n+\ell}(a) <_{\text{lex}} b^{(n+\ell)}$. We get from Lemma 24 that

$$\text{val}_{\boldsymbol{\beta}^{(n+\ell)}}(\sigma^{n+\ell}(a)) \le \text{val}_{\boldsymbol{\beta}^{(n+\ell)}}(b^{(n+\ell)}) = 1.$$

Then

$$\begin{aligned}
\text{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) &= \text{val}_{\boldsymbol{\beta}^{(n)}}(a_n \cdots a_{n+\ell-2}) + \frac{a_{n+\ell-1}}{\prod_{i=n}^{n+\ell-1} \beta_i} + \frac{\text{val}_{\boldsymbol{\beta}^{(n+\ell)}}(\sigma^{n+\ell}(a))}{\prod_{i=n}^{n+\ell-1} \beta_i} \\
&\le \text{val}_{\boldsymbol{\beta}^{(n)}}(b^{(n)}_0 \cdots b^{(n)}_{\ell-2}) + \frac{b^{(n)}_{\ell-1} - 1}{\prod_{i=n}^{n+\ell-1} \beta_i} + \frac{1}{\prod_{i=n}^{n+\ell-1} \beta_i} \\
&= \text{val}_{\boldsymbol{\beta}^{(n)}}(b^{(n)}_0 \cdots b^{(n)}_{\ell-1}) \\
&\le 1.
\end{aligned}$$

Moreover, the equality holds throughout if and only if $b^{(n)} = b^{(n)}_0 \cdots b^{(n)}_{\ell-1}$, $a_{n+\ell-1} = b^{(n)}_{\ell-1} - 1$ and $\text{val}_{\boldsymbol{\beta}^{(n+\ell)}}(\sigma^{n+\ell}(a)) = 1$. The conclusion follows. □

The following theorem generalizes Parry's theorem for real bases [10].

Theorem 26. *An infinite word a over $\mathbb{N}$ belongs to $D_{\boldsymbol{\beta}}$ if and only if for all $n \in \mathbb{N}$, $\sigma^n(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.*

Proof. In view of Corollary 11, it suffices to show that the following two assertions are equivalent.

(1) For all $n \in \mathbb{N}$, $\text{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$.

(2) For all $n \in \mathbb{N}$, $\sigma^n(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.

The fact that (1) implies (2) follows from Proposition 23. Since any quasi-greedy expansion of 1 is infinite, we obtain that (2) implies (1) by Proposition 22 and Lemma 25. $\square$

Example 27. Let $\boldsymbol{\beta} = (\overline{3, \varphi, \varphi})$ be the alternate base already studied in Examples 7 and 18. Then $a = 210(110)^\omega$ is the $\boldsymbol{\beta}$-expansion of some $x \in (0,1)$. In fact, since $d^*_{\boldsymbol{\beta}^{(0)}}(1) = (210)^\omega$, $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (102)^\omega$ and $d^*_{\boldsymbol{\beta}^{(2)}}(1) = 1(110)^\omega$, by Theorem 26, there exists $x \in [0,1)$ such that $a = d_{\boldsymbol{\beta}}(x)$. We can compute that $a = d_{\boldsymbol{\beta}}(\text{val}_{\boldsymbol{\beta}}(a)) = d_{\boldsymbol{\beta}}\big(\frac{19+9\sqrt{5}}{3(7+3\sqrt{5})}\big)$.

We obtain a corollary characterizing the $\boldsymbol{\beta}$-expansions of a real number x in the interval $[0,1]$ among all its $\boldsymbol{\beta}$-representations.

Corollary 28. *A $\boldsymbol{\beta}$-representation a of some real number $x \in [0,1]$ is its $\boldsymbol{\beta}$-expansion if and only if for all $n \in \mathbb{N}_{\geq 1}$, $\sigma^n(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.*

Proof. Let $a \in \mathbb{N}^{\mathbb{N}}$ be such that $\text{val}_{\boldsymbol{\beta}}(a) \in [0,1]$. From Theorem 26, $\sigma(a)$ belongs to $D_{\boldsymbol{\beta}^{(1)}}$ if and only if for all $n \in \mathbb{N}_{\geq 1}$, $\sigma^n(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. The conclusion then follows from Proposition 10. $\square$

Example 29. Consider $\boldsymbol{\beta} = \big(\overline{\frac{16+5\sqrt{10}}{9}, 9}\big)$. Then $d_{\boldsymbol{\beta}}(1) = d^*_{\boldsymbol{\beta}}(1) = 34(27)^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 90^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = 834(27)^\omega$. For all $m \in \mathbb{N}_{\geq 1}$, we have $\sigma^{2m}(34(27)^\omega) <_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$ and $\sigma^{2m-1}(34(27)^\omega) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(1)}}(1)$ as prescribed by Corollary 28.

In comparison with the β-expansion theory, considering a Cantor base $\boldsymbol{\beta}$ and an infinite word a over $\mathbb{N}$, Corollary 28 does not give a purely combinatorial condition to check whether a is the $\boldsymbol{\beta}$-expansion of 1. More details will be given in Section 7, where we will see that even though an improvement of this result in the context of alternate bases can be proved, a purely combinatorial condition cannot exist. In particular, see Example 42.

6. The $\boldsymbol{\beta}$-shift

Let $S_{\boldsymbol{\beta}}$ denote the topological closure of $D_{\boldsymbol{\beta}}$ with respect to the prefix distance of infinite words: $S_{\boldsymbol{\beta}} = \overline{D_{\boldsymbol{\beta}}}$.

Proposition 30. *An infinite word a over $\mathbb{N}$ belongs to $S_{\boldsymbol{\beta}}$ if and only if for all $n \in \mathbb{N}$, $\sigma^n(a) \leq_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.*

Proof. Suppose that $a \in S_{\boldsymbol{\beta}}$. Then there exists a sequence $(a^{(k)})_{k\in\mathbb{N}}$ of $D_{\boldsymbol{\beta}}$ converging to a. By Theorem 26, for all $k, n \in \mathbb{N}$, we have $\sigma^n(a^{(k)}) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. By letting k tend to infinity, we get that for all $n \in \mathbb{N}$, $\sigma^n(a) \leq_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.

Conversely, suppose that for all $n \in \mathbb{N}$, $\sigma^n(a) \leq_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. For each $k \in \mathbb{N}$, let $a^{(k)} = a_0 \cdots a_k 0^\omega$. Then $\lim\limits_{k\to+\infty} a^{(k)} = a$ and for all $k, n \in \mathbb{N}$, $\sigma^n(a^{(k)}) \leq_{\text{lex}} \sigma^n(a) \leq_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. Since $d^*_{\boldsymbol{\beta}^{(n)}}(1)$ is infinite, for all $k, n \in \mathbb{N}$, $\sigma^n(a^{(k)}) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. By Theorem 26, we deduce that for all $k \in \mathbb{N}$, $a^{(k)} \in D_{\boldsymbol{\beta}}$. Therefore $a \in S_{\boldsymbol{\beta}}$. $\square$

Proposition 31. *Let $a, b \in S_{\boldsymbol{\beta}}$.*

(1) *If $a <_{\text{lex}} b$ then $\text{val}_{\boldsymbol{\beta}}(a) \leq \text{val}_{\boldsymbol{\beta}}(b)$.*
(2) *If $\text{val}_{\boldsymbol{\beta}}(a) < \text{val}_{\boldsymbol{\beta}}(b)$ then $a <_{\text{lex}} b$.*

Proof. Consider two sequences $(a^{(k)})_{k\in\mathbb{N}}$ and $(b^{(k)})_{k\in\mathbb{N}}$ of $D_{\boldsymbol{\beta}}$ such that $\lim_{k\to\infty} a^{(k)} = a$ and $\lim_{k\to\infty} b^{(k)} = b$. Suppose that $a <_{\text{lex}} b$. Then there exists $\ell \in \mathbb{N}_{\geq 1}$ such that $a_0 \cdots a_{\ell-1} = b_0 \cdots b_{\ell-1}$ and $a_\ell < b_\ell$. By definition of the prefix distance, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $a_0^{(k)} \cdots a_\ell^{(k)} = a_0 \cdots a_\ell$ and $b_0^{(k)} \cdots b_\ell^{(k)} = b_0 \cdots b_\ell$. Therefore, for all $k \geq K$, we have $a^{(k)} <_{\text{lex}} b^{(k)}$, and then by Proposition 13, $\operatorname{val}_{\boldsymbol{\beta}}(a^{(k)}) < \operatorname{val}_{\boldsymbol{\beta}}(b^{(k)})$. Since the function $\operatorname{val}_{\boldsymbol{\beta}}$ is continuous, by letting k tend to infinity, we obtain $\operatorname{val}_{\boldsymbol{\beta}}(a) \leq \operatorname{val}_{\boldsymbol{\beta}}(b)$. This proves the first item. The second item follows immediately. □

Further, we define

$$\Delta_{\boldsymbol{\beta}} = \bigcup_{n\in\mathbb{N}} D_{\boldsymbol{\beta}^{(n)}} \quad \text{and} \quad \Sigma_{\boldsymbol{\beta}} = \overline{\Delta_{\boldsymbol{\beta}}}.$$

Proposition 32. *The sets $\Delta_{\boldsymbol{\beta}}$ and $\Sigma_{\boldsymbol{\beta}}$ are both shift-invariant.*

Proof. Let a be an infinite word over $\mathbb{N}$ and $n \in \mathbb{N}$. It follows from Corollary 11 that if $a \in D_{\boldsymbol{\beta}^{(n)}}$ then $\sigma(a) \in D_{\boldsymbol{\beta}^{(n+1)}}$. Then, it is easily seen that if $a \in S_{\boldsymbol{\beta}^{(n)}}$ then $\sigma(a) \in S_{\boldsymbol{\beta}^{(n+1)}}$. □

Recall some definitions of symbolic dynamics. Let A be a finite alphabet. A subset of $A^{\mathbb{N}}$ is a *subshift* of $A^{\mathbb{N}}$ if it is shift-invariant and closed with respect to the topology induced by the prefix distance. In view of Proposition 32, the subset $\Sigma_{\boldsymbol{\beta}}$ of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ is a subshift, which we call the *$\boldsymbol{\beta}$-shift*. For a subset L of $A^{\mathbb{N}}$, we let $\operatorname{Fac}(L)$ (resp. $\operatorname{Pref}(L)$) denote the set of all finite factors (resp. prefixes) of all elements in L.

Proposition 33. *We have* $\operatorname{Fac}(D_{\boldsymbol{\beta}}) = \operatorname{Fac}(\Delta_{\boldsymbol{\beta}}) = \operatorname{Fac}(\Sigma_{\boldsymbol{\beta}})$.

Proof. By definition, $\operatorname{Fac}(D_{\boldsymbol{\beta}}) \subseteq \operatorname{Fac}(\Delta_{\boldsymbol{\beta}}) = \operatorname{Fac}(\Sigma_{\boldsymbol{\beta}})$. Let us show that $\operatorname{Fac}(D_{\boldsymbol{\beta}}) \supseteq \operatorname{Fac}(\Delta_{\boldsymbol{\beta}})$. Let $f \in \operatorname{Fac}(\Delta_{\boldsymbol{\beta}})$. There exist $n \in \mathbb{N}$ and $a \in D_{\boldsymbol{\beta}^{(n)}}$ such that $f \in \operatorname{Fac}(a)$. It follows from Corollary 11 that $0^n a$ belongs to $D_{\boldsymbol{\beta}}$. Therefore, $f \in \operatorname{Fac}(D_{\boldsymbol{\beta}})$. □

We define sets of finite words $X_{\boldsymbol{\beta},\ell}$ for $\ell \in \mathbb{N}_{\geq 1}$ as follows. If $d^*_{\boldsymbol{\beta}}(1) = t_0 t_1 \cdots$ then we let

$$X_{\boldsymbol{\beta},\ell} = \{t_0 \cdots t_{\ell-2} s \colon s \in [\![0, t_{\ell-1} - 1]\!]\}.$$

Note that $X_{\boldsymbol{\beta},\ell}$ is empty if and only if $t_{\ell-1} = 0$.

Proposition 34. *We have*

$$D_{\boldsymbol{\beta}} = \bigcup_{\ell_0 \in \mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta},\ell_0} \left(\bigcup_{\ell_1 \in \mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta}^{(\ell_0)},\ell_1} \left(\bigcup_{\ell_2 \in \mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta}^{(\ell_0+\ell_1)},\ell_2} \left(\quad \cdots \quad \right)\right)\right).$$

Proof. For the sake of conciseness, we let $X_{\boldsymbol{\beta}}$ denote the right-hand set of the equality. For $n \in \mathbb{N}$, write $d^*_{\boldsymbol{\beta}^{(n)}}(1) = t_0^{(n)} t_1^{(n)} \cdots$.

Let $a \in D_{\boldsymbol{\beta}}$. By Theorem 26, for all $n \in \mathbb{N}$, $\sigma^n(a) < d^*_{\boldsymbol{\beta}^{(n)}}(1)$. In particular, $a < d^*_{\boldsymbol{\beta}}(1)$. Thus, there exist $\ell_0 \in \mathbb{N}_{\geq 1}$ such that $t^{(0)}_{\ell_0-1} > 0$ and $s_0 \in [\![0, t^{(0)}_{\ell_0-1} - 1]\!]$ such that $a = t_0 \cdots t_{\ell_0-2} s_0 \sigma^{\ell_0}(a)$. Next, we also have $\sigma^{\ell_0}(a) < d^*_{\boldsymbol{\beta}^{(\ell_0)}}(1)$. Then there exist $\ell_1 \in \mathbb{N}_{\geq 1}$ such that $t^{(\ell_0)}_{\ell_1-1} > 0$ and $s_1 \in [\![0, t^{(\ell_0)}_{\ell_1-1} - 1]\!]$ such that $\sigma^{\ell_0}(a) = t_0^{(\ell_0)} \cdots t^{(\ell_0)}_{\ell_1-2} s_1 \sigma^{\ell_0+\ell_1}(a)$. We get that $a \in X_{\boldsymbol{\beta}}$ by iterating the process.

Now, let $a \in X_{\boldsymbol{\beta}}$. Then there exists a sequence $(\ell_k)_{k\in\mathbb{N}}$ of $\mathbb{N}_{\geq 1}$ such that $a = u_0 u_1 u_2 \cdots$ where for all $k \in \mathbb{N}$, $u_k \in X_{\boldsymbol{\beta}^{(\ell_0+\cdots \ell_{k-1})},\ell_k}$. By Theorem 26, in order to prove that $a \in D_{\boldsymbol{\beta}}$, it suffices to show that for all $n \in \mathbb{N}$, $\sigma^n(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. Let thus $n \in \mathbb{N}$. There exist $k \in \mathbb{N}$ and finite words x and y such that $u_k = xy$, $y \neq \varepsilon$ and $\sigma^n(a) = y u_{k+1} u_{k+2} \cdots$. Then $n = \ell_0 + \cdots + \ell_{k-1} + |x|$ and $\sigma^n(a) <_{\text{lex}} \sigma^{|x|}\big(d^*_{\boldsymbol{\beta}^{(\ell_0+\cdots \ell_{k-1})}}(1)\big)$. If $x = \varepsilon$ then we obtain $\sigma^n(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(\ell_0+\cdots\ell_{k-1})}}(1) = d^*_{\boldsymbol{\beta}^{(n)}}(1)$. Otherwise it follows from Corollary 28 that $\sigma^{|x|}\big(d_{\boldsymbol{\beta}^{(\ell_0+\cdots\ell_{k-1})}}(1)\big) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(\ell_0+\cdots\ell_{k-1}+|x|)}}(1) = d^*_{\boldsymbol{\beta}^{(n)}}(1)$, hence we get $\sigma^n(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$ as well. □

Corollary 35. *We have* $D_{\boldsymbol{\beta}} = \bigcup_{\ell\in\mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta},\ell} D_{\boldsymbol{\beta}^{(\ell)}}$.

Corollary 36. *Any prefix of* $d^*_{\boldsymbol{\beta}}(1)$ *belongs to* $\text{Pref}(D_{\boldsymbol{\beta}})$.

Proof. Write $d^*_{\boldsymbol{\beta}}(1) = t_0t_1t_2\cdots$ and let $\ell \in \mathbb{N}_{\geq 1}$. Since $d^*_{\boldsymbol{\beta}}(1)$ is infinite, there exists $k > \ell$ such that $t_{k-1} > 0$. Choose the least such k and let $s \in [\![0, t_{k-1}-1]\!]$. Then $t_0\cdots t_{\ell-1}0^{k-\ell-1}s$ belongs to $X_{\boldsymbol{\beta},k}$. The conclusion follows from Proposition 34. □

7. Alternate bases

Recall that an alternate base is a periodic Cantor base. The aim of this section is to discuss some results that are specific to these particular Cantor bases.

We start with a few elementary observations. First, the condition $\prod_{n\in\mathbb{N}}\beta_n = +\infty$ is trivially satisfied in the context of alternate bases since the sequence $(\beta_n)_{n\in\mathbb{N}}$ takes only finitely many values. Then, for an alternate base $\boldsymbol{\beta}$ of length p, the $\boldsymbol{\beta}$-value (2.1) of an infinite word a over $\mathbb{R}_{\geq 0}$ can be rewritten as

$$\text{val}_{\boldsymbol{\beta}}(a) = \sum_{n\in\mathbb{N}} \frac{a_n}{\big(\prod_{i=0}^{p-1}\beta_i\big)^{\lfloor \frac{n}{p}\rfloor}\prod_{i=0}^{n \bmod p}\beta_i}$$

or as

$$\text{val}_{\boldsymbol{\beta}}(a) = \sum_{m=0}^{+\infty} \frac{1}{\big(\prod_{i=0}^{p-1}\beta_i\big)^m} \sum_{j=0}^{p-1} \frac{a_{pm+j}}{\prod_{i=0}^{j}\beta_i}. \tag{7.1}$$

Further, the alphabet $A_{\boldsymbol{\beta}}$ is finite since $A_{\boldsymbol{\beta}} = \{0,\ldots,\max_{i\in[\![0,p-1]\!]}\lfloor\beta_i\rfloor\}$. Finally, note that a Cantor base of the form $(\beta,\beta,\ldots)$ is an alternate base of length 1, in which case, as already mentioned in Section 2, all definitions introduced so far coincide with those of Rényi [12] for real bases β.

In Proposition 3, we gave a characterization of those infinite words $a \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ for which there exists a Cantor base $\boldsymbol{\beta}$ such that $\text{val}_{\boldsymbol{\beta}}(a) = 1$. Here, we are interested in the stronger condition of the existence of an alternate base $\boldsymbol{\beta}$ satisfying $\text{val}_{\boldsymbol{\beta}}(a) = 1$.

Proposition 37. *Let a be an infinite word over $\mathbb{R}_{\geq 0}$ such that $a_n \in O(n^d)$ for some $d \in \mathbb{N}$ and let $p \in \mathbb{N}_{\geq 1}$. There exists an alternate base $\boldsymbol{\beta}$ of length p such that* $\text{val}_{\boldsymbol{\beta}}(a) = 1$ *if and only if* $\sum_{n\in\mathbb{N}} a_n > 1$. *If moreover $p \geq 2$, then there exist uncountably many such alternate bases.*

Proof. From Proposition 3, we already know that the condition $\sum_{n\in\mathbb{N}} a_n > 1$ is necessary. Now, suppose that $\sum_{n\in\mathbb{N}} a_n > 1$. If $p = 1$ then the result follows from Lemma 1. Suppose that $p \geq 2$. Consider any $(p-1)$-tuple $(\beta_1,\ldots,\beta_{p-1}) \in (\mathbb{R}_{>1})^{p-1}$. For all $\beta_0 > 1$, we can write $\text{val}_{\boldsymbol{\beta}}(a) = \text{val}_{\beta_0}(c)$ with $\boldsymbol{\beta} = (\overline{\beta_0,\beta_1,\ldots,\beta_{p-1}})$ and

$$c_m = \frac{1}{\big(\prod_{i=1}^{p-1}\beta_i\big)^m} \sum_{j=0}^{p-1} \frac{a_{pm+j}}{\prod_{i=1}^{j}\beta_i} \quad \text{for all } m \in \mathbb{N}.$$

Note that $c \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ and that $c_m \in O(m^d)$. By hypothesis, there exists $N \in \mathbb{N}$ such that $\sum_{n=0}^{N} a_n > 1$. Then

$$\sum_{m=0}^{\lfloor \frac{N}{p} \rfloor} c_m > \frac{\sum_{m=0}^{\lfloor \frac{N}{p} \rfloor} \sum_{j=0}^{p-1} a_{pm+j}}{\left(\prod_{i=1}^{p-1} \beta_i\right)^{\lfloor \frac{N}{p} \rfloor + 1}} \geq \frac{\sum_{n=0}^{N} a_n}{\left(\prod_{i=1}^{p-1} \beta_i\right)^{\lfloor \frac{N}{p} \rfloor + 1}}.$$

Therefore, any $(p-1)$-tuple $(\beta_1, \ldots, \beta_{p-1}) \in (\mathbb{R}_{>1})^{p-1}$ satisfying

$$\left(\prod_{i=1}^{p-1} \beta_i\right)^{\lfloor \frac{N}{p} \rfloor + 1} \leq \sum_{n=0}^{N} a_n$$

is such that $\sum_{m=0}^{\lfloor \frac{N}{p} \rfloor} c_m > 1$, and hence there exist uncountably many of them. For such a $(p-1)$-tuple, the infinite word c satisfies the hypothesis of Lemma 1, so there exists $\beta_0 > 1$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = \mathrm{val}_{\beta_0}(c) = 1$. □

7.1. **The greedy algorithm.** The greedy and the quasi-greedy $\boldsymbol{\beta}$-expansions of 1 enjoy specific properties whenever $\boldsymbol{\beta}$ is an alternate base. From now on, we let $\boldsymbol{\beta}$ be a fixed alternate base and we let p be its length.

Proposition 38. *The $\boldsymbol{\beta}$-expansion of 1 is not purely periodic.*

Proof. Suppose to the contrary that there exists $q \in \mathbb{N}_{\geq 1}$ such that for all $n \in \mathbb{N}$, $\varepsilon_n = \varepsilon_{n+q}$. By considering $\ell = \mathrm{lcm}(p, q)$, we get that $\boldsymbol{\beta}^{(\ell)} = \boldsymbol{\beta}$ and for all $n \in \mathbb{N}$, $\varepsilon_n = \varepsilon_{n+\ell}$. Therefore

$$1 = \mathrm{val}_{\boldsymbol{\beta}}\big(\varepsilon_0 \cdots \varepsilon_{\ell-1}\big) + \frac{1}{\prod_{i=0}^{\ell-1} \beta_i} = \mathrm{val}_{\boldsymbol{\beta}}\big(\varepsilon_0 \cdots \varepsilon_{\ell-2}(\varepsilon_{\ell-1} + 1)\big).$$

Thus $\varepsilon_0 \cdots \varepsilon_{\ell-2}(\varepsilon_{\ell-1} + 1)$ is a $\boldsymbol{\beta}$-representation of 1 lexicographically greater than $d_{\boldsymbol{\beta}}(1)$, which is impossible by Proposition 12. □

One might think at first that if for each $i \in [\![0, p-1]\!]$, $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic, then for $\beta = \prod_{i=0}^{p-1} \beta_i$, $d^*_{\beta}(1)$ must be ultimately periodic as well. This is not the case, as the following example shows. Moreover, the same example shows that the $\boldsymbol{\beta}$-expansion of 1 can be ultimately periodic with a period which is coprime with the length p of $\boldsymbol{\beta}$.

Example 39. Let $\boldsymbol{\beta} = \overline{(\sqrt{6}, 3, \frac{2+\sqrt{6}}{3})}$. It is easily checked that $d_{\boldsymbol{\beta}^{(0)}}(1) = 2(10)^{\omega}$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 3$ and $d_{\boldsymbol{\beta}^{(2)}}(1) = 11002$. But the product $\beta = \prod_{i=0}^{p-1} \beta_i = \sqrt{6}(2+\sqrt{6})$ is such that $d^*_{\beta}(1)$ is not ultimately periodic as explained in Example 21.

Proposition 40. *The quasi-greedy expansion $d^*_{\boldsymbol{\beta}}(1)$ is ultimately periodic if and only if either an ultimately periodic expansion is reached or only finite expansions are involved within the first p recursive calls to the definition of $d^*_{\boldsymbol{\beta}}(1)$.*

Proof. If there exists $n \in \mathbb{N}$ such that the infinite expansion $d^*_{\boldsymbol{\beta}^{(n)}}(1)$ is involved in the computation of $d^*_{\boldsymbol{\beta}}(1)$, then clearly $d^*_{\boldsymbol{\beta}}(1)$ is ultimately periodic if and only if so is $d^*_{\boldsymbol{\beta}^{(n)}}(1)$.

Now, suppose that only finite expansions are involved within p recursive calls to the definition of $d^*_{\boldsymbol{\beta}}(1)$. Then $d_{\boldsymbol{\beta}}(1)$ is finite. Thus, $d_{\boldsymbol{\beta}}(1) = \varepsilon_{\boldsymbol{\beta},0} \cdots \varepsilon_{\boldsymbol{\beta},\ell_0-1}$ with $\ell_0 \in \mathbb{N}_{\geq 1}$ and $\varepsilon_{\boldsymbol{\beta},\ell_0-1} > 0$. Then

$$d^*_{\boldsymbol{\beta}}(1) = \varepsilon_{\boldsymbol{\beta},0} \cdots \varepsilon_{\boldsymbol{\beta},\ell_0-2}(\varepsilon_{\boldsymbol{\beta},\ell_0-1} - 1) d^*_{\boldsymbol{\beta}^{(i_1)}}(1)$$

where $i_1 = \ell_0 \bmod p$. By hypothesis, $d_{\boldsymbol{\beta}^{(i_1)}}(1)$ is finite as well. Thus we have $d_{\boldsymbol{\beta}^{(i_1)}}(1) = \varepsilon_{\boldsymbol{\beta}^{(i_1)},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_1)},\ell_1-1}$ with $\ell_1 \in \mathbb{N}_{\geq 1}$ and $\varepsilon_{\boldsymbol{\beta}^{(i_1)},\ell_1-1} > 0$. Repeating the same argument, we obtain

$$d^*_{\boldsymbol{\beta}^{(i_1)}}(1) = \varepsilon_{\boldsymbol{\beta}^{(i_1)},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_1)},\ell_1-2}(\varepsilon_{\boldsymbol{\beta}^{(i_1)},\ell_1-1} - 1) d^*_{\boldsymbol{\beta}^{(i_2)}}(1)$$

where $i_2 = \ell_0 + \ell_1 \bmod p$. By continuing in the same fashion and by setting $i_0 = 0$, we obtain two sequences $(\ell_j)_{j \in [\![0,p-1]\!]}$ and $(i_j)_{j \in [\![0,p]\!]}$. Because for all $j \in [\![0,p]\!]$, we have $i_j \in [\![0,p-1]\!]$, there exist $j,k \in [\![0,p]\!]$ such that $j < k$ and $i_j = i_k$. Then $d^*_{\boldsymbol{\beta}}(1) = xy^\omega$ where

$$x = \varepsilon_{\boldsymbol{\beta}^{(i_0)},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_0)},\ell_0-2}(\varepsilon_{\boldsymbol{\beta}^{(i_0)},\ell_0-1} - 1) \ \cdots \ \varepsilon_{\boldsymbol{\beta}^{(i_{j-1})},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_{j-1})},\ell_{j-1}-2}(\varepsilon_{\boldsymbol{\beta}^{(i_{j-1})},\ell_{j-1}-1} - 1)$$

and

$$y = \varepsilon_{\boldsymbol{\beta}^{(i_j)},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_j)},\ell_j-2}(\varepsilon_{\boldsymbol{\beta}^{(i_j)},\ell_j-1} - 1) \ \cdots \ \varepsilon_{\boldsymbol{\beta}^{(i_{k-1})},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_{k-1})},\ell_{k-1}-2}(\varepsilon_{\boldsymbol{\beta}^{(i_{k-1})},\ell_{k-1}-1} - 1).$$

□

7.2. **Admissible sequences.** The condition given in Corollary 28 does not allow us to check whether a given $\boldsymbol{\beta}$-representation of 1 is the $\boldsymbol{\beta}$-expansion of 1 without effectively computing the quasi-greedy $\boldsymbol{\beta}$-expansion of 1, and hence the $\boldsymbol{\beta}$-expansion of 1 itself. The following proposition provides us with such a condition in the case of alternate bases, provided that we are given the quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 for $i \in [\![1,p-1]\!]$. Note that the shifted words starting in positions that are multiple of p are compared with the word a itself and not with the corresponding quasi-greedy expansions of 1 as in Corollary 28.

Proposition 41. *A $\boldsymbol{\beta}$-representation a of 1 is the $\boldsymbol{\beta}$-expansion of 1 if and only if for all $m \in \mathbb{N}_{\geq 1}$, $\sigma^{pm}(a) <_{\text{lex}} a$ and for all $m \in \mathbb{N}$ and $i \in [\![1,p-1]\!]$, $\sigma^{pm+i}(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(i)}}(1)$.*

Proof. The condition is necessary by Corollary 28 and since $d^*_{\boldsymbol{\beta}}(1) \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. Let us show that the condition is sufficient.

Let a be a $\boldsymbol{\beta}$-representation of 1 such that for all $m \in \mathbb{N}_{\geq 1}$, $\sigma^{pm}(a) <_{\text{lex}} a$ and for all $m \in \mathbb{N}$ and $i \in [\![1,p-1]\!]$, $\sigma^{pm+i}(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(i)}}(1)$. By Proposition 12, $a \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. By Theorem 26, if $a <_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$ then $\text{val}_{\boldsymbol{\beta}}(a) < 1$, which contradicts that a is a $\boldsymbol{\beta}$-representation of 1. Thus, $d^*_{\boldsymbol{\beta}}(1) \leq_{\text{lex}} a \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. If $d_{\boldsymbol{\beta}}(1)$ is infinite, then $a = d_{\boldsymbol{\beta}}(1)$ as desired. Now, suppose that $d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{\ell-1}$ with $\ell \in \mathbb{N}_{\geq 1}$ and $\varepsilon_{\ell-1} > 0$. Then $a_0 \cdots a_{\ell-2} = \varepsilon_0 \cdots \varepsilon_{\ell-2}$ and $a_{\ell-1} \in \{\varepsilon_{\ell-1} - 1, \varepsilon_{\ell-1}\}$. Since $\text{val}_{\boldsymbol{\beta}}(a) = 1$, if $a_{\ell-1} = \varepsilon_{\ell-1}$ then $a = d_{\boldsymbol{\beta}}(1)$. Therefore, in order to conclude, it suffices to show that $a_{\ell-1} \neq \varepsilon_{\ell-1} - 1$.

Suppose to the contrary that $a_{\ell-1} = \varepsilon_{\ell-1} - 1$. Then $d^*_{\boldsymbol{\beta}^{(\ell)}}(1) \leq_{\text{lex}} \sigma^\ell(a)$. By hypothesis, $\ell \equiv 0 \pmod{p}$. Therefore $d^*_{\boldsymbol{\beta}}(1) \leq_{\text{lex}} \sigma^\ell(a) \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. By repeating the same argument, we obtain that $a_\ell \cdots a_{2\ell-2} = \varepsilon_0 \cdots \varepsilon_{\ell-2}$ and $a_{2\ell-1} \in \{\varepsilon_{\ell-1} - 1, \varepsilon_{\ell-1}\}$. Since $\sigma^\ell(a) <_{\text{lex}} a$ by hypothesis, we must have $a_{2\ell-1} = \varepsilon_{\ell-1} - 1$. By iterating the argument, we obtain that $a = (\varepsilon_0 \cdots \varepsilon_{\ell-2}(\varepsilon_{\ell-1} - 1))^\omega$, contradicting that $\sigma^\ell(a) <_{\text{lex}} a$. □

When $p = 1$, Proposition 41 provides us with the purely combinatorial condition proved by Parry [10] in order to determine whether a given $\boldsymbol{\beta}$-representation of 1 is the $\boldsymbol{\beta}$-expansion of 1. However, when $p \geq 2$, we need to compute the quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 for every $i \in [\![1,p-1]\!]$ first. This might lead us to a circular computation, in which case the condition may seem not useful in practice. Indeed, suppose that $p = 2$ and that we are provided with a $\boldsymbol{\beta}$-representation a of 1 and a $\boldsymbol{\beta}^{(1)}$-representation b of 1. Then in order to check if $a = d_{\boldsymbol{\beta}}(1)$, we need to compute $d^*_{\boldsymbol{\beta}^{(1)}}(1)$, and hence $d_{\boldsymbol{\beta}^{(1)}}(1)$ first. But then, in order to check if $b = d_{\boldsymbol{\beta}^{(1)}}(1)$, we need to compute $d^*_{\boldsymbol{\beta}}(1)$, and hence $d_{\boldsymbol{\beta}}(1)$, which brings us back to the initial problem. Nevertheless, this condition can be useful to check if a specific

$\boldsymbol{\beta}$-representation of 1 is the $\boldsymbol{\beta}$-expansion of 1. For example, consider a $\boldsymbol{\beta}$-representation a of 1 such that for all $m \in \mathbb{N}_{\geq 1}$, $\sigma^{pm}(a) <_{\text{lex}} a$ and for all $m \in \mathbb{N}$ and $i \in [\![1, p-1]\!]$, $a_{pm+i} < \lfloor \beta_i \rfloor - 1$, then the infinite words a satisfies the hypothesis of Proposition 41 and a is the $\boldsymbol{\beta}$-expansion of 1.

We have seen that considering an infinite word a over $\mathbb{N}$ and a positive integer p, there may exist more than one alternate base $\boldsymbol{\beta}$ of length p such that $\operatorname{val}_{\boldsymbol{\beta}}(a) = 1$. Moreover, among all of these alternate bases, it may be that some are such that a is greedy and others are such that a is not. Thus, a purely combinatorial condition for checking whether a $\boldsymbol{\beta}$-representation is greedy cannot exist.

Example 42. Consider $a = 2(10)^\omega$. Then $\operatorname{val}_{\boldsymbol{\alpha}}(a) = \operatorname{val}_{\boldsymbol{\beta}}(a) = 1$ for both $\boldsymbol{\alpha} = (\overline{1+\varphi, 2})$ and $\boldsymbol{\beta} = (\overline{\frac{31}{10}, \frac{420}{341}})$. It can be checked that $d_{\boldsymbol{\alpha}}(1) = a$ and $d_{\boldsymbol{\beta}}(1) \neq a$.

Furthermore, an infinite word a over $\mathbb{N}$ can be greedy for more than one alternate base.

Example 43. The infinite word 110^ω is the expansion of 1 with respect to the three alternate bases φ, $(\overline{\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2}})$ and $(\overline{1.7, \frac{1}{0.7}})$.

At the opposite, it may happen that an infinite word a is a $\boldsymbol{\beta}$-representation of 1 for different alternate bases $\boldsymbol{\beta}$ but that none of these are such that a is greedy. As an illustration, by Proposition 38, for all purely periodic infinite words a over $\mathbb{N}$, all alternate bases $\boldsymbol{\beta}$ such that $\operatorname{val}_{\boldsymbol{\beta}}(a) = 1$ are such that a is not the $\boldsymbol{\beta}$-expansion of 1.

Example 44. The infinite word $(10)^\omega$ is a representation of 1 with respect to the three alternate bases considered in Example 43. However, the infinite words $(10)^\omega$ is purely periodic therefore, by Proposition 38, it is not the expansion of 1 in any alternate base.

7.3. **The $\boldsymbol{\beta}$-shift.** We define sets of finite words $Y_{\boldsymbol{\beta},h}$ for $h \in [\![0, p-1]\!]$ as follows. If $d^*_{\boldsymbol{\beta}}(1) = t_0 t_1 \cdots$ then we let

$$Y_{\boldsymbol{\beta},h} = \{t_0 \cdots t_{\ell-2} s \colon \ell \in \mathbb{N}_{\geq 1},\ \ell \bmod p = h,\ t_{\ell-1} > 0,\ s \in [\![0, t_{\ell-1} - 1]\!]\}.$$

Note that $Y_{\boldsymbol{\beta},h}$ is empty if and only if for all $\ell \in \mathbb{N}_{\geq 1}$ such that $\ell \bmod p = h$, $t_{\ell-1} = 0$. So, unlike the sets $X_{\boldsymbol{\beta},h}$ defined in Section 6, the sets $Y_{\boldsymbol{\beta},h}$ can be infinite. More precisely, $Y_{\boldsymbol{\beta},h}$ is infinite if and only if there exists infinitely many $\ell \in \mathbb{N}_{\geq 1}$ such that $\ell \bmod p = h$ and $t_{\ell-1} > 0$.

Proposition 45. *We have*

$$D_{\boldsymbol{\beta}} = \bigcup_{h_0=0}^{p-1} Y_{\boldsymbol{\beta},h_0} \left(\bigcup_{h_1=0}^{p-1} Y_{\boldsymbol{\beta}^{(h_0)},h_1} \left(\bigcup_{h_2=0}^{p-1} Y_{\boldsymbol{\beta}^{(h_0+h_1)},h_2} \left(\quad \cdots \quad \right)\right)\right).$$

Proof. It is easily seen that for all $h \in [\![0, p-1]\!]$,

$$\bigcup_{h=0}^{p-1} Y_{\boldsymbol{\beta},h} = \bigcup_{\ell \in \mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta},\ell}.$$

The conclusion follows from Proposition 34. □

Corollary 46. *We have* $D_{\boldsymbol{\beta}} = \displaystyle\bigcup_{h=0}^{p-1} Y_{\boldsymbol{\beta},h} D_{\boldsymbol{\beta}^{(h)}}$.

In the case where all $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ are ultimately periodic, we define an automaton $\mathcal{A}_{\boldsymbol{\beta}}$ over the finite alphabet $A_{\boldsymbol{\beta}}$. Let $d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} \cdots t_{m_i-1}^{(i)} \big(t_{m_i}^{(i)} \cdots t_{m_i+n_i-1}^{(i)}\big)^\omega$. The set of states is

$$Q = \big\{q_{i,j,k} \colon i, j \in [\![0, p-1]\!],\ k \in [\![0, m_i + n_i - 1]\!]\big\}.$$

The set I of initial states and the set F of final states are defined as

$$I = \{q_{i,i,0} \colon i \in [\![0, p-1]\!]\} \quad \text{and} \quad F = Q.$$

The (partial) transition function $\delta \colon Q \times A_{\boldsymbol{\beta}} \to Q$ of the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ is defined as follows. For each $i, j \in [\![0, p-1]\!]$ and each $k \in [\![0, m_i + n_i - 1]\!]$, we have

$$\delta(q_{i,j,k}, t_k^{(i)}) = \begin{cases} q_{i,(j+1) \bmod p, k+1} & \text{if } k \neq m_i + n_i - 1 \\ q_{i,(j+1) \bmod p, m_i} & \text{else} \end{cases}$$

and for all $s \in [\![0, t_k^{(i)} - 1]\!]$, we have

$$\delta(q_{i,j,k}, s) = q_{(j+1) \bmod p, (j+1) \bmod p, 0}.$$

Example 47. Let $\boldsymbol{\beta} = \overline{(\varphi^2, 3+\sqrt{5})}$. Then $d_{\boldsymbol{\beta}^{(0)}}(1) = 2(30)^\omega$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 5(03)^\omega$. The corresponding automaton $\mathcal{A}_{\boldsymbol{\beta}}$ is depicted in Figure 1. By removing the non-accessible

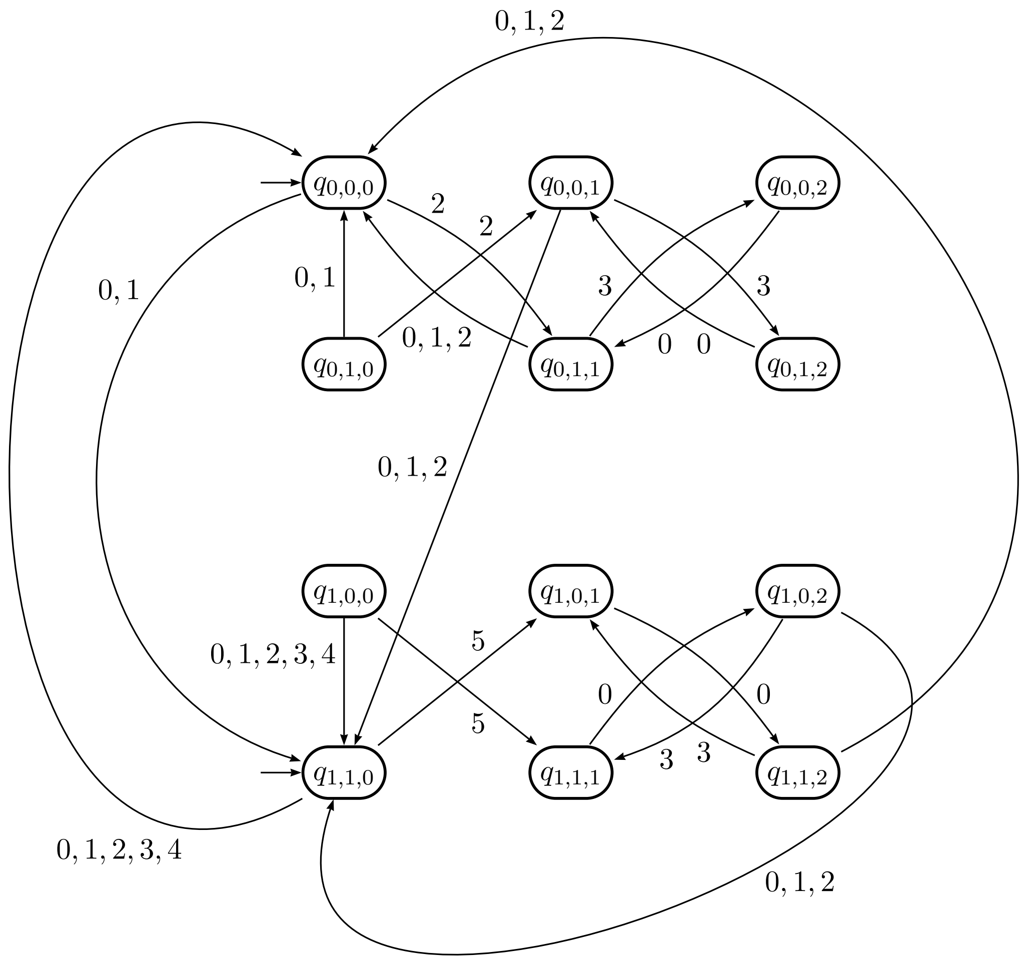

FIGURE 1. The automaton $\mathcal{A}_{\overline{(\varphi^2, 3+\sqrt{5})}}$.

states, we obtain the automaton of Figure 2.

The following result extends a result of Bertrand-Mathis for real bases [2]. Recall that a subshift S of $A^{\mathbb{N}}$ is called *sofic* if the language $\mathrm{Fac}(S) \subseteq A^*$ is accepted by a finite automaton.

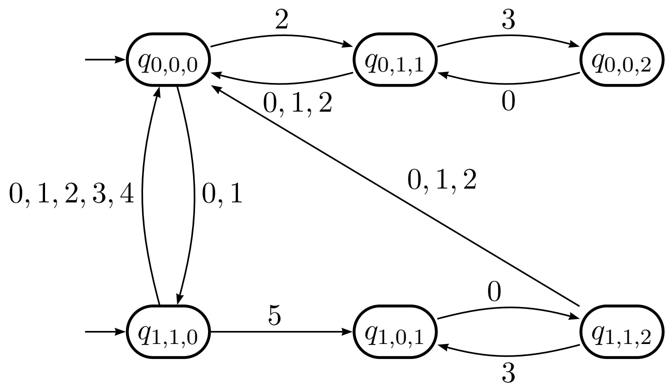

FIGURE 2. An accessible automaton accepting $\mathrm{Fac}(\Sigma_{\overline{(\varphi^2,3+\sqrt{5})}})$.

Theorem 48. *The $\boldsymbol{\beta}$-shift $\Sigma_{\boldsymbol{\beta}}$ is sofic if and only if for all $i \in [\![0,p-1]\!]$, $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic.*

Proof. Suppose that for all $i \in [\![0,p-1]\!]$, $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic. We show that the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ accepts the language $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$. From Propositions 32 and 33, we obtain that

$$\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}}) = \mathrm{Pref}(\Delta_{\boldsymbol{\beta}}) = \bigcup_{i=0}^{p-1} \mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}}). \tag{7.2}$$

Therefore, it suffices to show that for each $i \in [\![0,p-1]\!]$, the language accepted from the initial state $q_{i,i,0}$ is precisely $\mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. Let thus $i \in [\![0,p-1]\!]$.

First, consider a word w accepted from $q_{i,i,0}$. By Corollary 36, if w is a prefix of $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ then $w \in \mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. Otherwise, by construction of $\mathcal{A}_{\boldsymbol{\beta}}$, w starts with some $u \in Y_{\boldsymbol{\beta}^{(i)},h_0}$ where $h_0 = |u| \bmod p$. Moreover, the state reached after reading u from $q_{i,i,0}$ is $q_{j,j,0}$ where $j = (i+h_0) \bmod p$. We obtain that $w \in \mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}})$ by iterating the reasoning and by using Proposition 45.

Conversely, let $w \in \mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. By Proposition 45, we know that there exists $\ell \in \mathbb{N}$ and $h_0,\dots,h_\ell \in [\![0,p-1]\!]$ such that $w = u_0 \cdots u_{\ell-1} x$ with $u_k \in Y_{\boldsymbol{\beta}^{(i+h_0+\cdots h_{k-1})},h_k}$ for all $k \in [\![0,\ell-1]\!]$ and x is a (possibly empty) prefix of $d^*_{\boldsymbol{\beta}^{(i_\ell)}}(1)$ where $i_\ell = (i+h_0+\cdots+h_{\ell-1}) \bmod p$. By construction of $\mathcal{A}_{\boldsymbol{\beta}}$, by reading u_0 from the state $q^{(i)}_{i,0}$, we reach the state $q_{i_1,i_1,0}$ where $i_1 = (i+h_0) \bmod p$. Then, by reading u_1 from the latter state, we reach the state $q_{i_2,i_2,0}$ where $i_2 = (i+h_0+h_1) \bmod p$. By iterating the argument, after reading $u_0 \cdots u_{\ell-1}$, we end up in the state $q_{i_\ell,i_\ell,0}$. Since x is a prefix of $d^*_{\boldsymbol{\beta}^{(i_\ell)}}(1)$, it is possible to read x from the state $q_{i_\ell,i_\ell,0}$ in $\mathcal{A}_{\boldsymbol{\beta}}$. Since all states of $\mathcal{A}_{\boldsymbol{\beta}}$ are final, we obtain that w is accepted from $q_{i,i,0}$.

We turn to the necessary condition. Let

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} t_1^{(i)} \cdots \quad \text{for every } i \in [\![0,p-1]\!].$$

Suppose that $j \in [\![0,p-1]\!]$ is such that $d^*_{\boldsymbol{\beta}^{(j)}}(1)$ is not ultimately periodic. Our aim is to find an infinite sequence $(w^{(m)})_{m\in\mathbb{N}}$ of finite words over $A_{\boldsymbol{\beta}}$ such that for all distinct $m,n \in \mathbb{N}$, the words $w^{(m)}$ and $w^{(n)}$ are not right-congruent with respect to $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$. Recall that words x and y are not right-congruent with respect to a language L if $x^{-1}L \neq y^{-1}L$, i.e. if there exists some word z such that either $xz \in L$ and $yz \notin L$, or $xz \notin L$ and $yz \in L$.

If we succeed then we will know that the number of right-congruence classes is infinite and we will be able to conclude that $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$ is not accepted by a finite automaton.

We define a partition $(G_1, \ldots, G_q)$ of $[\![0, p-1]\!]$ as follows. Let $r = \mathrm{Card}\{d^*_{\boldsymbol{\beta}^{(i)}}(1) \colon i \in [\![0, p-1]\!]\}$ and let $i_1, \ldots, i_r \in [\![0, p-1]\!]$ be such that $d^*_{\boldsymbol{\beta}^{(i_1)}}(1), \ldots, d^*_{\boldsymbol{\beta}^{(i_r)}}(1)$ are pairwise distinct. Without loss of generality, we can suppose that $d^*_{\boldsymbol{\beta}^{(i_1)}}(1) >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(i_r)}}(1)$. Let $q \in [\![1, r]\!]$ be the unique index such that $d^*_{\boldsymbol{\beta}^{(i_q)}}(1) = d^*_{\boldsymbol{\beta}^{(j)}}(1)$. We set

$$G_s = \{i \in [\![0, p-1]\!] \colon d^*_{\boldsymbol{\beta}^{(i)}}(1) = d^*_{\boldsymbol{\beta}^{(i_s)}}(1)\} \quad \text{for } s \in [\![1, q-1]\!]$$

and

$$G_q = \{i \in [\![0, p-1]\!] \colon d^*_{\boldsymbol{\beta}^{(i)}}(1) \leq d^*_{\boldsymbol{\beta}^{(j)}}(1)\}.$$

For each $s \in [\![1, q-1]\!]$, we write $G_s = \{i_{s,1}, \ldots, i_{s,\alpha_s}\}$ where $i_{s,1} < \ldots < i_{s,\alpha_s}$ and we use the convention that $i_{s,\alpha_s+1} = i_{s+1,1}$ for $s \leq q-2$ and $i_{q-1,\alpha_{q-1}+1} = j$. Moreover, we let $g \in \mathbb{N}_{\geq 1}$ be such that for all $i, i' \in [\![0, p-1]\!]$ such that $d^*_{\boldsymbol{\beta}^{(i)}}(1) \neq d^*_{\boldsymbol{\beta}^{(i')}}(1)$, the length-$g$ prefixes of $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ and $d^*_{\boldsymbol{\beta}^{(i')}}(1)$ are distinct. Then, for $s \in [\![1, q-1]\!]$, we define C_s to be the least $c \in \mathbb{N}_{\geq 1}$ such that $t^{(i_s)}_{g-1+c} > 0$. Finally, let $N \in \mathbb{N}_{\geq 1}$ be such that $pN \geq \max\{g, C_1, \ldots, C_{q-1}\}$.

For all $m \in \mathbb{N}$, consider

$$w^{(m)} = \left(\prod_{s=1}^{q-1} \prod_{k=1}^{\alpha_s} t^{(i_s)}_0 \cdots t^{(i_s)}_{g-1} 0^{p(2N+1)-g+i_{s,k+1}-i_{s,k}}\right) t^{(j)}_0 \cdots t^{(j)}_{m-1}.$$

For all $m \in \mathbb{N}$, $s \in [\![1, q-1]\!]$ and $k \in [\![1, \alpha_s]\!]$, the factor $t^{(i_s)}_0 \cdots t^{(i_s)}_{g-1} 0^{p(2N+1)-g+i_{s,k+1}-i_{s,k}}$ has length $p(2N+1) + i_{s,k+1} - i_{s,k}$, and hence occurs at a position congruent to $i_{s,k} - i_{1,1}$ modulo p in $w^{(m)}$. Similarly, for all $m \in \mathbb{N}$, the factor $t^{(j)}_0 \cdots t^{(j)}_{m-1}$ occurs at a position congruent to $j - i_{1,1}$ modulo p in $w^{(m)}$. These observations will be crucial in what follows. The situation is illustrated in Figure 3.

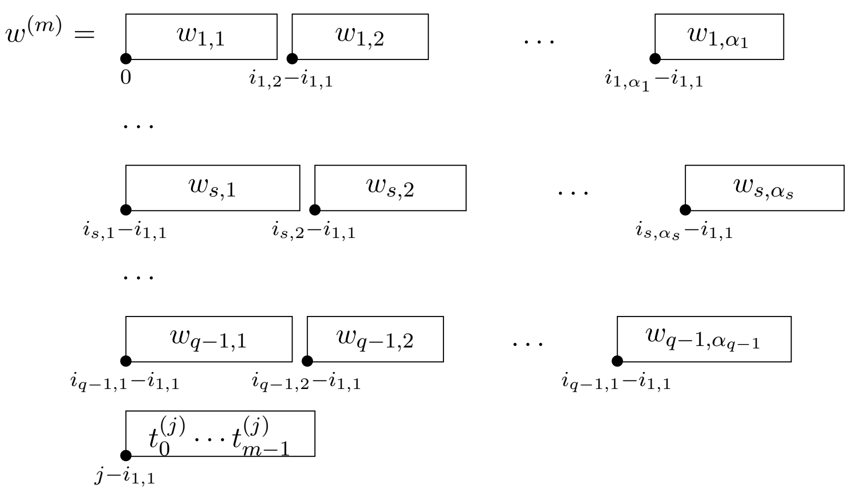

FIGURE 3. Factorization of the word $w^{(m)}$. The black dots designate the positions modulo p of the occurrences of the factors $w_{k,s}$ and $t^{(j)}_0 \cdots t^{(j)}_{m-1}$ in $w^{(m)}$, where $w_{k,s} = t^{(i_s)}_0 \cdots t^{(i_s)}_{g-1} 0^{p(2N+1)-g+i_{s,k+1}-i_{s,k}}$.

Now, let $m,n\in\mathbb{N}$ be distinct. Since $d^*_{\boldsymbol{\beta}^{(j)}}(1)$ is not ultimately periodic, $\sigma^m\big(d^*_{\boldsymbol{\beta}^{(j)}}(1)\big)\neq\sigma^n\big(d^*_{\boldsymbol{\beta}^{(j)}}(1)\big)$. Thus, there exists $\ell\in\mathbb{N}_{\geq 1}$ such that $t^{(j)}_m\cdots t^{(j)}_{m+\ell-2}=t^{(j)}_n\cdots t^{(j)}_{n+\ell-2}$ and $t^{(j)}_{m+\ell-1}\neq t^{(j)}_{n+\ell-1}$. Without loss of generality, we suppose that $t^{(j)}_{m+\ell-1}>t^{(j)}_{n+\ell-1}$. Let $z=t^{(j)}_m\cdots t^{(j)}_{m+\ell-1}$. Our aim is to show that $w^{(m)}z\in\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$ and $w^{(n)}z\notin\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$.

In order to obtain that $w^{(m)}z\in\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$, we show that $w^{(m)}z\in\mathrm{Pref}(D_{\boldsymbol{\beta}^{(i_{1,1})}})$. First, for all $s\in[\![1,q-1]\!]$ and $k\in[\![1,\alpha_s]\!]$, $t^{(i_s)}_0\cdots t^{(i_s)}_{g-1}0^{C_s}\in Y_{\boldsymbol{\beta}^{(i_{s,k})},(g+C_s)\bmod p}$. Second, for all $i\in[\![0,p-1]\!]$, $0\in Y_{\boldsymbol{\beta}^{(i)},1}$. Third, by Corollary 36, for all $h\in[\![0,p-1]\!]$, $t^{(j)}_0\cdots t^{(j)}_{m-1}z\in\mathrm{Pref}(Y_{\boldsymbol{\beta}^{(j)},h})$. The conclusion follows from Proposition 45.

In view of (7.2), in order to prove that $w^{(n)}z\notin\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$, it suffices to show that for all $i\in[\![0,p-1]\!]$, $w^{(n)}z\notin\mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. Proceed by contradiction and let $i\in[\![0,p-1]\!]$ and $w\in D_{\boldsymbol{\beta}^{(i)}}$ such that $w^{(n)}z$ is a prefix of w. By Theorem 26, for all $s\in[\![1,q]\!]$, the factor $t^{(i_s)}_0\cdots t^{(i_s)}_{g-1}0^{C_s}$ occurs at a position e in w such that $(i+e)\bmod p$ belongs to $G_1\cup\cdots\cup G_s$. For $s=1$, we obtain that for all $k\in[\![1,\alpha_1]\!]$, $(i+i_{1,k}-i_{1,1})\bmod p\in G_1$, and hence that

$$G_1=\{(i+i_{1,1}-i_{1,1})\bmod p,\dots,(i+i_{1,\alpha_1}-i_{1,1})\bmod p\}.$$

For $s=2$, we get that for all $k\in[\![1,\alpha_2]\!]$, $(i+i_{2,k}-i_{1,1})\bmod p\in G_1\cup G_2$. If $(i+i_{2,k}-i_{1,1})\bmod p\in G_1$ for some $k\in[\![1,\alpha_2]\!]$, then there exists $k'\in[\![1,\alpha_1]\!]$ such that $(i+i_{2,k}-i_{1,1})\bmod p=(i+i_{1,k'}-i_{1,1})\bmod p$, hence such that $i_{2,k}=i_{1,k'}$, which is impossible since G_1 and G_2 are pairwise disjoint. It follows that

$$G_2=\{(i+i_{2,1}-i_{1,1})\bmod p,\dots,(i+i_{2,\alpha_2}-i_{1,1})\bmod p\}.$$

By iterating the reasoning, we obtain that

$$G_s=\{(i+i_{s,1}-i_{1,1})\bmod p,\dots,(i+i_{s,\alpha_s}-i_{1,1})\bmod p\}\quad\text{for all } s\in[\![1,q-1]\!].$$

We finally get that $(i+j-i_{1,1})\bmod p$ belongs to G_q. Then $d^*_{\boldsymbol{\beta}^{((i+j-i_{1,1})\bmod p)}}(1)\leq_{\mathrm{lex}}d^*_{\boldsymbol{\beta}^{(j)}}(1)$. Let r be the position where the factor $t^{(j)}_0\cdots t^{(j)}_{n-1}$ occurs in $w^{(n)}$, and hence also in w since $w^{(n)}z$ is a prefix of w. We have seen that $r\equiv j-i_{1,1}\pmod p$. Since $w\in D_{\boldsymbol{\beta}^{(i)}}$, it follows from Theorem 26 that

$$\sigma^r(w)<_{\mathrm{lex}}d^*_{\boldsymbol{\beta}^{(i+r)}}(1)=d^*_{\boldsymbol{\beta}^{((i+j-i_{1,1})\bmod p)}}(1)\leq_{\mathrm{lex}}d^*_{\boldsymbol{\beta}^{(j)}}(1).$$

We have thus reached a contradiction since the factor $t^{(j)}_0\cdots t^{(j)}_{n-1}z$ is lexicographically greater than the length-$(n+\ell)$ prefix of $d^*_{\boldsymbol{\beta}^{(j)}}(1)$. □

Note that, in the classical case $p=1$, the previous proof is much shorter since $\mathrm{Fac}(\Sigma_\beta)=\mathrm{Pref}(D_\beta)$, and hence we can directly deduce that the words $t^{(j)}_0\cdots t^{(j)}_{m-1}$ and $t^{(j)}_0\cdots t^{(j)}_{n-1}$ (where in fact, $j=0$) are not right-congruent with respect to $\mathrm{Fac}(\Sigma_\beta)$.

Interestingly, some new phenomena occur in our extended framework when looking at subshifts of finite type. A subshift S of $A^{\mathbb{N}}$ is said to be of *finite type* if its minimal set of forbidden factors is finite. For $p=1$, it is well known that the β-shift is of finite type if and only if $d_\beta(1)$ is finite [2]. However, this result does not generalize to $p\geq 2$ as is illustrated by the following example.

Example 49. Consider the alternate base $\boldsymbol{\beta}=\overline{(\frac{1+\sqrt{13}}{2},\frac{5+\sqrt{13}}{6})}$ of Example 20. Then $d^*_{\boldsymbol{\beta}^{(0)}}(1)=200(10)^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1)=(10)^\omega$. We see that all words in $2(00)^*2$ are factors avoided by $\Sigma_{\boldsymbol{\beta}}$, so the $\boldsymbol{\beta}$-shift $\Sigma_{\boldsymbol{\beta}}$ is not of finite type.

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